Chapter 3

Well-definedness

Here we show how changing the choices in the construction above affect the resulting space output.

3.1 Variation of approximations

First, we consider the change due to passing between different approximations. For this section, we fix a 3-manifold with spin^c structure $(Y, \mathfrak{s})$.

As before, let P_n, Q_n be spectral sections of $-D$, D with

$$(\mathcal{E}_0(D))_{-\infty}^{\mu_{n,-}} \subset P_n \subset (\mathcal{E}_0(D))_{-\infty}^{\mu_{n,+}},$$
$$(\mathcal{E}_0(D))_{\lambda_{n,+}}^{\infty} \subset Q_n \subset (\mathcal{E}_0(D))_{\lambda_{n,-}}^{\infty}.$$

We may assume that $|\mu_{n,+} - \mu_{n,-}|$ and $|\lambda_{n,+} - \lambda_{n,-}|$ are bounded. We call any such sequence of spectral sections a *good sequence of spectral sections*.

Fix half-integers $k_+, k_- > 5$. Put $\ell = \min\{k_+, k_-\}$.

Let $F_n = P_n \cap Q_n \subset (\mathcal{E}_0)_{\lambda_{n,-}}^{\mu_{n,+}}$, as before. Fix $\mathbb{H}$ to be the quaternion representation of Pin(2), and let $B = \mathrm{Pic}(Y)$ denote the Picard torus of Y. We write $I(\varphi, S)$ for the (parameterized) Conley index of a flow φ and isolated invariant set S; we will usually suppress S from the notation, and $I^u(\varphi, S)$ for the unparameterized version; see Appendix A.2. Finally, a further bit of notation for the statement of the following theorem. Let $\mathrm{Th}(E, Z)$, for a vector bundle $\pi\colon E \to Z$, denote the Thom construction of π.

Theorem 3.1.1. *Let $\eta_n^P\colon P_{n+1} \to P_n \oplus \mathbb{C}^{k_{P,n}}$ and $\eta_n^Q\colon Q_{n+1} \to Q_n \oplus \mathbb{C}^{k_{Q,n}}$ be vector-bundle isometries (with respect to the $k_\pm$-metric), where $\mathbb{C}^{k_{P,n}}$ and $\mathbb{C}^{k_{Q,n}}$ are the trivial bundles over B of rank $k_{P,n}$ and $k_{Q,n}$. Let $\eta_n^{W,+}\colon W_{n+1}^+ \to W_n^+ \oplus \mathbb{R}^{k_{W,+,n}}$ and $\eta_n^{W,-}\colon W_{n+1}^- \to W_n^- \oplus \mathbb{R}^{k_{W,-,n}}$ be another pair of isometries. Then there is an S^1-equivariant parameterized homotopy equivalence of Conley indices*

$$\eta_*\colon I(\varphi_{n+1}) \to \Sigma_B^{\mathbb{C}^{k_{Q,n}} \oplus \mathbb{R}^{k_{W,-,n}}} I(\varphi_n),$$

which is well defined up to homotopy for the induced map

$$\nu_! \eta_*\colon I^u(\varphi_{n+1}) \to \Sigma^{\mathbb{C}^{k_{Q,n}} \oplus \mathbb{R}^{k_{W,-,n}}} I^u(\varphi_n).$$

Furthermore, if $\mathfrak{s}$ is a self-conjugate spin^c *structure and instead $\eta_n^P\colon P_{n+1} \to P_n \oplus \mathbb{H}^{k_{\mathbb{H},P,n}}$ and $\eta_n^Q\colon Q_{n+1} \to Q_n \oplus \mathbb{H}^{k_{\mathbb{H},Q,n}}$, and the maps $\eta^{W,\pm}$ above are equivariant* *with respect to the C_2-action on W_{n+1}, W_n and $\widetilde{\mathbb{R}}^{k_{W,\pm,n}}$, then there is a well-defined, up to equivariant homotopy,* Pin(2)*-equivariant homotopy equivalence*

$$\nu_! \eta_* \colon I^u(\varphi_{n+1}) \to \Sigma^{\mathbb{H}^{k_{\mathbb{H},Q,n}} \oplus \widetilde{\mathbb{R}}^{k_{W,-,n}}} I^u(\varphi_n),$$

and similarly for the parameterized version.

The restriction η_* to the S^1-fixed point set $I(\varphi_{n+1})^{S^1}$ is a fiber-preserving homotopy equivalence to $\Sigma_B^{\mathbb{R}^{k_{W,-,n}}} I_n(\varphi)^{S^1}$.

More generally, without a selection of maps η_n° as above, there is an S^1-equivariant parameterized homotopy equivalence of Conley indices

$$\eta_* \colon I(\varphi_{n+1}) \to \Sigma_B^{Q_{n+1}/Q_n} \Sigma_B^{W_{n+1}^-/W_n^-} I^u(\varphi_n),$$

so that the induced, unparameterized map

$$\nu_! \eta_* \colon I^u(\varphi_{n+1}) \to \mathrm{Th}(Q_{n+1}/Q_n \oplus W_{n+1}^-/W_n^-, I^u(\varphi_n))$$

is well defined up to homotopy, as well as a similar statement for self-conjugate $\mathfrak{s}$.

Proof. By Lemma 3.1.2 below and invariance of the Conley index under deformations, there is a well-defined homotopy equivalence $\eta^1 \colon I^u(\varphi_{n+1}) \to I^u(\varphi_{n+1}^{\mathrm{split}})$, where $\varphi_{n+1}^{\mathrm{split}}$ is defined in Lemma 3.1.2 (and similarly for the parameterized version). Using the invariance of the Conley index under homeomorphism, we have a well-defined homotopy equivalence

$$\eta^2 \colon I(\varphi_{n+1}^{\mathrm{split}}) \to I(\varphi_{n+1}^{\mathrm{split},\eta}),$$

where $\varphi_{n+1}^{\mathrm{split},\eta}$ is defined in Lemma 3.1.9. Finally, by Lemma 3.1.9, the well-definedness of the Conley index (independent of a choice of index pair), and the definition of the Conley index (using our choice of index pair from Lemma 3.1.9), there is a well-defined homotopy equivalence

$$\eta^3 \colon I(\varphi_{n+1}^{\mathrm{split},\eta}) \to \Sigma_B^{Q_{n+1}/Q_n} \Sigma_B^{W_{n+1}^-/W_n^-} I(\varphi_n).$$

In the case that we have fixed trivializations, as above, of W_{n+1}^-/W_n^- and Q_{n+1}^-/Q_n^-, the target of η^3 is identified with

$$\Sigma_B^{\mathbb{C}^{k_{Q,n}} \oplus \mathbb{R}^{k_{W,-,n}}} I(\varphi_n).$$

Since the flows used to define the homotopy equivalences preserve the fibers of the S^1-fixed point sets (that is, $X(\phi)_H = 0$ if $\phi = 0$), we can see from the formulas for the maps f, g, F_λ, G_λ in the proof of [43, Theorem 6.2] that the restrictions of η^1, η^2, η^3 to the S^1-fixed point sets preserve the fibers.

The argument adapts immediately to the case in which there is a spin structure, and the theorem follows. ∎

Let $\Sigma_{n+1}^{\pm}$ be the $L^2_{k_+,k_-}$-orthogonal complement to P_n in P_{n+1} (resp. Q_n in Q_{n+1}). Similarly, let $\Sigma_{n+1}^{W,\pm}$ be the $L^2_{k_+,k_-}$-orthogonal complement to $W_n^{\pm}$ in $W_{n+1}^{\pm}$. Let $\Sigma_{n+1} = \Sigma_{n+1}^{+} \oplus \Sigma_{n+1}^{-}$ and $\Sigma_{n+1}^{W} = \Sigma_{n+1}^{W,+} \oplus \Sigma_{n+1}^{W,-}$. Then $F_{n+1} = F_n \oplus \Sigma_{n+1}$ and $W_{n+1} = W_n \oplus \Sigma_{n+1}^{W}$. Write $\pi_{\Sigma_{n+1}}$ for the projection to Σ_{n+1} with respect to the $L^2_{k_+,k_-}$-norm. We also write $\pi_{\Sigma_{n+1}^W}$ for the projection Σ_{n+1}^W with respect to the $L^2_{k_+,k_-}$-norm.

Let $\mathcal{X}_n$ be the approximate Seiberg–Witten vector field on $F_n \oplus W_n$, for all n, as defined in (2.3.10). Let R be large enough as in Theorem 2.3.3.

For a path $\gamma(t)$ in the total space of $F_{n+1} \oplus W_{n+1}$, we write $\gamma(t) = (\phi^{(1)}(t) + \sigma(t)) \oplus (\omega^{(1)}(t) + \omega^{(2)}(t))$, as an element in the fiber over $b(t) = p(\gamma(t))$, where $\phi^{(1)}(t)$ is an element of $(F_n)_{b(t)}$, $\sigma(t) \in (\Sigma_n)_{b(t)}$, $\omega^{(1)}(t) \in (W_n)_{b(t)}$ and $\omega^{(2)}(t) \in (\Sigma_n^W)_{b(t)}$.

We then write $\gamma(t) = (\phi^{(1)}(t), \sigma(t), \omega^{(1)}(t), \omega^{(2)}(t), b(t))$ to describe γ in terms of these coordinates. We also write $\phi_{n+1}(t)$ to refer to the path in the total space of F_{n+1} determined by $(\phi_{n+1}^{(1)}(t), \sigma_{n+1}(t), b(t))$.

Lemma 3.1.2. *Let $\mathcal{X}_n^{\text{split}}$ be the vector field on the total space of $(F_n \oplus \Sigma_n) \oplus (W_n \oplus \Sigma_n^W)$ defined by (3.1.1), where*

$$\gamma_{n+1}(t) = (\phi_{n+1}^{(1)}(t), \sigma_{n+1}(t), \omega_{n+1}^{(1)}(t), \omega_{n+1}^{(2)}(t), b_{n+1}(t))$$

and $\hat{\gamma}_{n+1}(t)$ is the path obtained by (fiberwise) projecting $\gamma_{n+1}(t)$ to $(F_n \oplus W_n)_{b_{n+1}(t)}$:

$$\begin{aligned}
\frac{d\phi_{n+1}^{(1)}}{dt}(t) &= -\chi\big\{(\nabla_{X_H}\pi_{F_n})\phi_{n+1}^{(1)}(t) + \pi_{F_n}\big(D\phi_{n+1}^{(1)}(t) + c_1(\hat{\gamma}_{n+1}(t))\big)\big\},\\
\frac{d\sigma_{n+1}}{dt}(t) &= -\chi\big\{(\nabla_{X_H}\pi_{\Sigma_{n+1}})\sigma_{n+1}(t) + \pi_{\Sigma_{n+1}}(D\sigma_{n+1}(t))\big\},\\
\frac{db_{n+1}}{dt}(t) &= -\chi X_H(\phi_{n+1}^{(1)}(t)),\\
\frac{d\omega_{n+1}^{(1)}}{dt}(t) &= -\chi\big\{*d\omega_{n+1}^{(1)}(t) + \pi_{W_n}c_2(\hat{\gamma}_{n+1}(t))\big\},\\
\frac{d\omega_{n+1}^{(2)}}{dt}(t) &= -\chi * d\omega_{n+1}^{(2)}(t).
\end{aligned} \tag{3.1.1}$$

Here, χ is the cut-off function in (2.3.10). Then, for n sufficiently large, there is a continuous family of vector fields $\mathcal{X}_{n+1}^{\tau}$ on (the total space of) $F_{n+1} \oplus W_{n+1}$ between $\mathcal{X}_{n+1}$ and $\mathcal{X}_{n+1}^{\text{split}}$, with associated flows φ_{n+1}^{τ}, so that A_{n+1} is an isolating neighborhood for all τ, where

$$\begin{aligned}
A_{n+1} = A_n^o &\times_B B_{k_+}(\Sigma_{n+1}^{+}; R) \times_B B_{k_-}(\Sigma_{n+1}^{-}; R)\\
&\times_B B_{k_+}(\Sigma_{n+1}^{W,+}; R) \times_B B_{k_-}(\Sigma_{n+1}^{W,-}; R),
\end{aligned}$$

where A_n^o is as A_n in the proof of Theorem 2.3.3.

Proof. This is an immediate consequence of Lemmas 3.1.3, 3.1.7 and 3.1.8. ■

We construct the homotopy $\mathcal{X}^\tau_{n+1}$, with associated flow $\varphi^\tau_{n+1,k_+,k_-}$, in stages.

Lemma 3.1.3. *Let $\mathcal{X}^\tau_{n+1}$ for $\tau \in [0,1]$ be defined by*

$$\begin{aligned}
\frac{d\phi^{(1)}_{n+1}}{dt}(t) &= -\chi\big\{(\nabla_{X_H}\pi_{F_n})(\phi^{(1)}_{n+1}(t)+\sigma_{n+1}(t)) \\
&\qquad + (1-\tau)\pi_{F_n}\big(D\phi_{n+1}(t)+c_1(\gamma_{n+1}(t))\big) \\
&\qquad + \tau\pi_{F_n}\big(D(\phi^{(1)}_{n+1})+c_1(\hat{\gamma}_{n+1}(t))\big)\big\}, \\
\frac{d\sigma_{n+1}}{dt}(t) &= -\chi\big\{(\nabla_{X_H}\pi_{\Sigma_{n+1}})(\phi^{(1)}_{n+1}+\sigma_{n+1}(t)) \\
&\qquad + (1-\tau)\pi_{\Sigma_{n+1}}\big(D(\phi^{(1)}_{n+1}(t)+\sigma_{n+1}(t))+c_1(\gamma_{n+1}(t))\big) \\
&\qquad + \tau\pi_{\Sigma_{n+1}}D\sigma_{n+1}(t)\big\}, \\
\frac{db_{n+1}}{dt}(t) &= -\chi X_H(\phi_{n+1}(t)), \\
\frac{d\omega^{(1)}_{n+1}}{dt}(t) &= -\chi\big\{*d\omega^{(1)}_{n+1}(t)+\tau\pi_{W_n}c_2(\hat{\gamma}_{n+1}(t)) \\
&\qquad + (1-\tau)\pi_{W_n}c_2(\gamma_{n+1}(t))\big\}, \\
\frac{d\omega^{(2)}_{n+1}}{dt}(t) &= -\chi\big\{*d\omega^{(2)}_{n+1}(t)+(1-\tau)\pi_{\Sigma^W_{n+1}}c_2(\gamma_{n+1}(t))\big\}.
\end{aligned}$$

Here, χ is the cut-off function in (2.3.10). Then, for all $n \gg 0$, A_{n+1} is an isolating neighborhood of $\varphi^\tau_{n+1,k_+,k_-}$ for all $\tau \in [0,1]$.

Proof. The lemma is a consequence of Lemmas 3.1.4, 3.1.5 and 3.1.6. Indeed, let

$$A^o_n = (B_{k_+}(F^+_n;R)\times_B B_{k_-}(F^-_n;R))\times_B (B_{k_+}(W^+_n;R)\times_B B_{k_-}(W^-_n;R))$$

be as in the proof of Theorem 2.3.3. Suppose that

$$\operatorname{inv} A_{n+1} \not\subset \operatorname{int} A_{n+1},$$

for some $\tau_n \in [0,1]$, for all n. Then there is a sequence of finite-energy approximate trajectories $\gamma_{n+1}(t)$, for $\varphi^{\tau_{n+1}}_{n+1,k_+,k_-}$, so that $\gamma_{n+1}(0) \in \partial A_{n+1}$. There are four cases as in the proof of Theorem 2.3.3; we only treat the case that

$$\begin{aligned}
\gamma_{n+1}(0) \in (S_{k_+}&(F^+_{n+1};R)\times_B B_{k_-}(F^-_{n+1};R)) \\
&\times_B (B_{k_-}(W^+_{n+1};R)\times_B B_{k_-}(W^-_{n+1};R))
\end{aligned}$$

for all n, the other cases being similar.

As in the proof of Theorem 2.3.3, we have a lift

$$\tilde{\gamma}_{n+1} = (\tilde{\phi}_{n+1}, \omega_{n+1}) \colon \mathbb{R} \to \mathcal{H}^1(Y) \times L^2_{k_+,k_-}(\mathbb{S}) \times L^2_{k_+,k_-}(\operatorname{im} d^*)$$

with $p(\tilde{\gamma}_{n+1}(0)) \in \Delta$.

By Lemma 3.1.5 and Proposition 2.6.3, the sequence $\tilde{\gamma}$ has a subsequence converging, uniformly in $(\ell - 1)$-norm to some continuous map

$$\tilde{\gamma} \colon I \to \mathcal{H}^1(Y) \times L^2_{k_+-1,k_--1}(\mathbb{S}) \times L^2_{k_+-1,k_--1}(\operatorname{im} d^*).$$

By Lemma 3.1.6, $\tilde{\gamma}$ is a solution of the Seiberg–Witten equations. Finally, by Lemma 3.1.4, we obtain that the sequence $\tilde{\phi}^+_n(0)$ converged to $\tilde{\phi}^+(0)$ uniformly in $L^2_{k_+}$-norm, which is a contradiction. ■

Lemma 3.1.4. *Assume that we have a sequence of trajectories $\tilde{\gamma}_{n+1}$ as in the proof of Lemma 3.1.2, with in particular*

$$\begin{aligned}\gamma_{n+1}(0) \in (S_{k_+}(F^+_{n+1}; R) \times_B B_{k_-}(F^-_{n+1}; R)) \\ \times_B (B_{k_-}(W^+_{n+1}; R) \times_B B_{k_-}(W^-_{n+1}; R)).\end{aligned}$$

Then there is some R_1 so that

$$\|\phi^+_{n+1}(0)\|_{k_+ + \frac{1}{2}} < R_1,$$

for all n.

Proof. We emphasize only what must be changed from the proof of Lemma 2.7.2. We check the case where k_+ is an integer. We calculate

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big|_{t=0} \|\phi^+_{n+1}(t)\|^2_{k_+} \\
&\quad = \operatorname{Re}\big(\langle (\nabla_{X_H}(D')^{k_+})\phi^+_{n+1}(0), (D')^{k_+}\phi^+_{n+1}(0)\rangle_0 \\
&\qquad + \langle (\nabla_{X_H}\pi^+)\phi_{n+1}(0), \phi^+_{n+1}(0)\rangle_{k_+} \\
&\qquad - \langle (\nabla_{X_H}\pi_{F_{n+1}})\phi_{n+1}(0), \phi^+_{n+1}(0)\rangle_{k_+} \\
&\qquad - \langle (1-\tau)\pi_{F_n} D'\phi_{n+1}(0), \phi^+_{n+1}(0)\rangle_{k_+} \\
&\qquad + \langle ((1-\tau)\mathbb{A} - (1-\tau)\pi_{F_{n+1}})c_1(\gamma_{n+1}(0)), \phi^+_{n+1}(0)\rangle_{k_+} \\
&\qquad - \tau\langle \pi_{F_n} D(\phi^{(1)}_{n+1}(0)), \phi^+_{n+1}(0)\rangle_{k_+} - \langle \tau\pi_{F_n} c_1(\hat{\gamma}_{n+1}(0)), \phi^+_{n+1}(0)\rangle_{k_+} \\
&\qquad - \langle \pi_{\Sigma_{n+1}} D\sigma_{n+1}(0), \phi^+_{n+1}(0)\rangle_{k_+} \\
&\qquad - (1-\tau)\big\langle \pi_{\Sigma_{n+1}} D\big(\phi^{(1)}_{n+1}(0) + c_1(\gamma_{n+1}(0))\big), \phi^+_{n+1}(0)\big\rangle_{k_+}\big).
\end{aligned}$$

Following the argument of Lemma 2.7.2, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\Big|_{t=0}&\|\phi^+_{n+1}(t)\|^2_{k_+}\\ &\leq CR^3\|\phi^+_{n+1}(0)\|_{k_++\frac{1}{2}} - \langle \pi_{F_{n+1}} D'\phi_{n+1}(0), \phi^+_{n+1}(0)\rangle_{k_+}\\ &\quad + \tau\big(\langle \pi_{\Sigma_{n+1}} D\phi^{(1)}_{n+1}(0), \phi^+_{n+1}(0)\rangle_{k_+} + \langle \pi_{F_n} D\sigma_{n+1}(0), \phi^+_{n+1}(0)\rangle_{k_+}\big).\end{aligned}$$

But

$$\langle \pi_{F_{n+1}} D'\phi_{n+1}(t), \phi^+_{n+1}(t)\rangle_{k_+} = \|\phi^+_{n+1}(t)\|^2_{k_++\frac{1}{2}}.$$

Since $[D', \pi_{\Sigma_{n+1}}]$ is uniformly bounded, we obtain

$$\tau\langle \pi_{\Sigma_{n+1}} D\phi^{(1)}_{n+1}, \phi^+_{n+1}\rangle_{k_+} \leq CR^2$$

for some constant C independent of n.

A similar argument applies to $\langle \pi_{F_n} D\sigma_{n+1}, \phi^+_{n+1}\rangle_{k_+}$. The lemma then follows as did Lemma 2.7.2. ■

Lemma 3.1.5. *The sequence $(\tilde{\phi}_n, \omega_n)$ is equicontinuous in $L^2_{K_T,\ell-5,w}$-norm.*

Proof. This follows exactly as in the proof of Theorem 2.3.3. ■

By Proposition 2.6.3, any sequence which is equicontinuous in $L^2_{K_T,\ell-5,w}$-norm and bounded in ℓ-norm has a subsequence converging, uniformly in $\|\cdot\|_{\ell-1}$, to some continuous map $\tilde{\gamma}\colon I \to \mathcal{H}^1(Y) \times L^2_{\ell-1}(\mathbb{S}) \times L^2_{\ell-1}(\operatorname{im} d^*)$.

Lemma 3.1.6. *A limit $\tilde{\gamma}$ for the sequence (ϕ_n, ω_n) as above, is a solution of the Seiberg–Witten equations over $Y \times \mathbb{R}$.*

Proof. Take $T \in \mathbb{Z}_{>0}$ and $t \in [-T, T]$. We have

$$\begin{aligned}\tilde{\phi}_{n+1}(t) &- \tilde{\phi}_{n+1}(0)\\ &= \int_0^t \frac{d\tilde{\phi}_{n+1}}{ds}(s)\,ds\\ &= -\int_0^t Z_1 + Z_2 + Z_3 + \pi_{\tilde{F}_{n+1}}\big(D(\tilde{\phi}^{(1)}_{n+1}(t) + \sigma_{n+1}(t)) + c_1(\tilde{\gamma}_n(t))\big)\\ &\quad + X_H(\phi_{n+1}(s))\,ds,\end{aligned}$$

where

$$\begin{aligned}Z_1 &= (\nabla_{X_H(\phi_{n+1}(t))}\pi_{F_{n+1}})\tilde{\phi}_{n+1},\\ Z_2 &= -\tau\pi_{\Sigma_{n+1}} D\phi^{(1)}_{n+1} - \tau\pi_{F_n} D\sigma_{n+1}(t),\\ Z_3 &= -\tau\big(\pi_{\Sigma_{n+1}} c_1(\tilde{\gamma}_n(t)) + \pi_{F_n} c_1(\tilde{\gamma}_n(t)) - \pi_{F_n} c_1(\tilde{\tilde{\gamma}}_n(t))\big).\end{aligned}$$

It suffices to show that the Z_i terms approach 0 uniformly in $L^2_{K_T,\ell-5,w}$, and that

$$\begin{aligned}\pi_{\widetilde{F}_{n+1}}\big(D(\tilde{\phi}_{n+1}) + c_1(\tilde{\gamma}_{n+1}(t))\big) + X_H(\phi_{n+1}(t))\\ \to D(\tilde{\phi}(t)) + c_1(\tilde{\gamma}(t)) + X_H(\phi(t)),\end{aligned}$$

also in $L^2_{K_T,\ell-5,w}$. Indeed, if that is the case, then the limit of integrals on the right-hand side is well defined, and

$$\tilde{\phi}(t) - \tilde{\phi}(0) = -\int_0^t \big(D\tilde{\phi} + c_1(\tilde{\gamma}(t)) + X_H(\phi(s))\big)\, ds, \tag{3.1.2}$$

giving the conclusion of the lemma.

Exactly as in the proof of Theorem 2.3.3, we obtain that Z_1 converges to 0 uniformly in $L^2_{K_T,\ell-5,w}$.

To show that $\pi_{F_n} D\sigma_{n+1}(t) \to 0$ in $L^2_{K_T,\ell-5,w}$, we use an elementary observation about projection with respect to different norms. That is, if V is a finite-dimensional vector space with norms $\|\cdot\|_1$ and $\|\cdot\|_2$, then for a subspace $V' \subset V$ and projection Π_1 to V' with respect to $\|\cdot\|_1$, then $\|\Pi_1 x\|_2/\|x\|_2 \le \rho_1\rho_2$ for $x \in V$, where $\rho_2 = \sup_{x\in V^*}\{\|x\|_2/\|x\|_1\}$ and $\rho_1 = \sup_{x\in V}\{\|x\|_1/\|x\|_2\}$.

We say a collection of finite-dimensional vector spaces V_i with norms $\|\cdot\|_{1,i}$ and $\|\cdot\|_{2,i}$ is *controlled* if $\rho_{1,i}\rho_{2,i}$ is bounded above.

We claim that the orthogonal complement of F_n in $(\mathcal{E}^{\mu_{n,+}}_{\lambda_{n,-}})_a$, call it $F_n^\perp$, with norms given by the restriction of $L^2_{k_+,k_-}$ and $L^2_{k_+-1,k_--1}$ (respectively), is controlled. Indeed, $F_n^\perp$ is a subspace of $(\mathcal{E}^{\mu_{n,+}}_{\mu_{n,-}})_a$. On $(\mathcal{E}^{\mu_{n,+}}_{\mu_{n,-}})_a$, by definition we have $\rho_1\rho_2 < \mu_{n,+}/\mu_{n,-}$. By our condition on the growth of the $\mu_{n,\pm}$, we then have that $\rho_{1,n}\rho_{2,n}$ is bounded as a function of n.

We claim that $\pi_{F_n} D\sigma_{n+1}(t) \to 0$ in $L^2_{k_+-2,k_--2}$. Indeed, $\sigma_{n+1}(t)$ converges to 0 weakly in $L^2_{k_+,k_-}$ by definition and $\sigma_{n+1}(t)$ converges strongly to 0 in $L^2_{k_+-1,k_--1}$. Then $D\sigma_{n+1}(t)$ converges to 0 in $L^2_{k_+-2,k_--2}$. Finally, π_{F_n} is a bounded family of operators in $L^2_{k_+-2,k_--2}$ by the above argument, giving the claim. As a consequence, we also have convergence in $L^2_{K_T,\ell-5,w}$.

To show that $\pi_{\Sigma_{n+1}} D\phi^{(1)}_{n+1}$ converges to 0, we note that by Proposition 2.4.2,

$$\|[D, \pi_{\Sigma_{n+1}}]\colon L^2_j \to L^2_j\| \le C$$

for some constant C independent of n, for all half-integers $j \le k_+$. Moreover, we have $\pi_{\Sigma_{n+1}}\phi^{(1)}_{n+1} = 0$, and so we need only show that the sequence $[\pi_{\Sigma_{n+1}}, D]\phi^{(1)}_{n+1}$ converges to zero. Given the bound on $\sigma D\phi^{(1)}_{n+1}$ from the bound on the commutator $[D, \pi_{\Sigma_{n+1}}]$ above, and using the definition of the norms involved, we see that $\pi_{\Sigma_{n+1}} D\phi^{(1)}_{n+1} \to 0$ in $L^2_{\ell-1}$-norm.

A very similar argument shows that $\pi_{\Sigma_{n+1}} c_1(\gamma_n(t)) \to 0$ in $L^2_{K_T,\ell-5,w}$, and also that $\pi_{F_n} c_1(\gamma_n(t))$ and $\pi_{F_n} c_1(\hat{\gamma}_n(t))$ converge to $c_1(\gamma(t))$ in $L^2_{K_T,\ell-5,w}$, so that $Z_3 \to 0$.

A similar argument also shows the convergence in (3.1.2), and the proof is complete. ∎

For $\tau \in [1,2]$, define a flow $\varphi^{\tau}_{n+1,k_+,k_-}$ on $F_{n+1} \oplus W_{n+1}$ by

$$\begin{aligned}
\frac{d\phi^{(1)}_{n+1}}{dt}(t) = -\chi\big\{&(2-\tau)(\nabla_{X_H(\phi_{n+1}(t))}\pi_{F_n})(\phi_{n+1}(t)) \\
&+ \big(\pi_{F_n} D\phi^{(1)}_{n+1}(t) + c_1(\hat{\gamma}_{n+1}(t))\big) \\
&+ (\tau-1)(\nabla_{X_H(\phi_{n+1}(t))}\pi_{F_n})\phi^{(1)}_{n+1}(t)\big\}, \\
\frac{d\sigma_{n+1}}{dt}(t) = -\chi\big\{&(2-\tau)(\nabla_{X_H(\phi_{n+1}(t))}\pi_{\Sigma_{n+1}})(\phi^{(1)}_{n+1}(t) + \sigma_{n+1}(t)) \\
&+ (\tau-1)(\nabla_{X_H(\phi_{n+1}(t))}\pi_{\Sigma_{n+1}})\sigma_{n+1}(t) + \pi_{\Sigma_{n+1}} D\sigma_{n+1}(t)\big\},
\end{aligned}$$

with the other terms unchanged. Inspection shows that the total space of $F_{n+1} \oplus W_{n+1}$ is preserved by the flow.

Lemma 3.1.7. *For $n \gg 0$, for all $\tau \in [1,2]$, A_{n+1} is an isolating neighborhood for $\varphi^{\tau}_{n+1,k_+,k_-}$.*

Proof. We highlight only the difference in the argument compared to the proof of Lemma 3.1.3. We have a sequence of trajectories

$$\gamma_{n+1}(t) = (\phi^{(1)}_{n+1}(t), \sigma_{n+1}(t), \omega_{n+1}(t))$$

exactly as in that argument. We assume that

$$\begin{aligned}
\gamma_{n+1}(0) \in (&S_{k_+}(F^+_{n+1}; R) \times_B B_{k_-}(F^-_{n+1}; R)) \\
&\times_B (B_{k_-}(W^+_{n+1}; R) \times_B B_{k_-}(W^-_{n+1}; R))
\end{aligned}$$

for all n; the other cases are similar. The proofs of the analogs of Lemma 3.1.5 and Lemma 3.1.6 are unchanged, and we obtain that a lift $\tilde{\gamma}_n$ of γ_n to the universal covering converges in $L^2_{K_T,\ell-5,w}$-norm to a solution $\tilde{\gamma}(t)$ of the Seiberg–Witten equations. We need only prove an analog of Lemma 3.1.4, that $\|\phi^+_{n+1}\|_{k_++\frac{1}{2}}$ is bounded independent of τ, n. Suppose this is false, that is, that

$$\|\phi^{(1),+}_{n+1}(0) + \sigma^+_{n+1}(0)\|_{k_++\frac{1}{2}} \to \infty.$$

Then we study (for the case $k_+ \in \mathbb{Z}$, the other case being similar)

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big|_{t=0}\|\phi^{(1),+}_{n+1}(t) + \sigma^+_{n+1}(t)\|^2_{k_+} \\
&\quad = \mathrm{Re}\big(\langle(\nabla_{X_H}\pi^+)\phi^{(1)}_{n+1}(0), \phi^{(1),+}_{n+1}(0)\rangle_{k_+} \\
&\qquad + \langle(\nabla_{X_H}(D')^{k_+})\phi^{(1),+}_{n+1}(0), (D')^{k_+}\phi^{(1),+}_{n+1}(0)\rangle_0 \\
&\qquad - \langle\pi_{F_{n+1}}D'\phi^{(1),+}_{n+1}(0), \phi^{(1),+}_{n+1}(t)\rangle_{k_+} \\
&\qquad + \langle(\mathbb{A} - \pi_{F_{n+1}})c_1(\hat{\gamma}_{n+1}(0)), \phi^{(1),+}_{n+1}(0)\rangle_{k_+} \\
&\qquad - \langle(\nabla_{X_H}\pi_{F_{n+1}})\phi^{(1),+}_{n+1}(0), \phi^{(1),+}_{n+1}(0)\rangle_{k_+} \\
&\qquad - (2-\tau)\langle(\nabla_{X_H}\pi_{F_n})\sigma_{n+1}(0), \phi^{(1),+}_{n+1}(0)\rangle_{k_+} \\
&\qquad + \langle(\nabla_{X_H}(D')^{k_+})\sigma_{n+1}(0), (D')^{k_+}\sigma^{(1),+}_{n+1}(0)\rangle_0 \\
&\qquad + \langle(\nabla_{X_H}\pi^+)\sigma_{n+1}(0), \sigma^+_{n+1}(0)\rangle_{k_+} - \langle\pi_{\Sigma_{n+1}}D'\sigma_{n+1}(0), \sigma^+_{n+1}(t)\rangle_{k_+} \\
&\qquad - \langle(\nabla_{X_H}\pi_{\Sigma_{n+1}})\sigma_{n+1}(0), \sigma^+_{n+1}(0)\rangle_{k_+} \\
&\qquad - (2-\tau)\langle(\nabla_{X_H}\pi_{\Sigma_{n+1}})\phi^{(1),+}_{n+1}(0), \sigma^+_{n+1}(0)\rangle_{k_+}\big).
\end{aligned} \tag{3.1.3}
$$

All of these terms can be dealt with as in the proof of Lemma 3.1.4, with the exception of

$$
\begin{aligned}
&-(2-\tau)\,\mathrm{Re}\langle(\nabla_{X_H}\pi_{F_n})\sigma_{n+1}(0), \phi^{(1),+}_{n+1}(0)\rangle_{k_+} \\
&\qquad -(2-\tau)\,\mathrm{Re}\langle(\nabla_{X_H}\pi_{\Sigma_{n+1}})\phi^{(1)}_{n+1}(0), \sigma^+_{n+1}(0)\rangle_{k_+}.
\end{aligned}
$$

To bound this term, consider the expression $\langle\phi^{(1),+}_{n+1}(t), \sigma^+_{n+1}(t)\rangle_{k_+}$ as a function of t. By definition, this is zero, but expanding its derivative gives

$$
\begin{aligned}
0 &= \mathrm{Re}\langle(\nabla_{X_H}\pi^+)\phi^{(1)}_{n+1}(t), \sigma^+_{n+1}(t)\rangle_{k_+} \\
&\quad + \mathrm{Re}\langle(\nabla_{X_H}(D')^{k_+})\phi^{(1),+}_{n+1}(t), (D')^{k_+}\sigma^+_{n+1}(t)\rangle_0 \\
&\quad + \mathrm{Re}\langle\phi^{(1),+}_{n+1}(t), (\nabla_{X_H}\pi^+)\sigma_{n+1}(t)\rangle_{k_+} \\
&\quad + \mathrm{Re}\langle(D')^{k_+}\phi^{(1),+}_{n+1}(t), (\nabla_{X_H}(D')^{k_+})\sigma^+_{n+1}(t)\rangle_0 \\
&\quad + \mathrm{Re}\langle(\nabla_{X_H}\pi_{F_n})\phi^{(1)}_{n+1}(t), \sigma^+_{n+1}(t)\rangle_{k_+} \\
&\quad + \mathrm{Re}\langle\phi^{(1),+}_{n+1}(t), \nabla_{X_H}\pi_{\Sigma_{n+1}}\sigma_{n+1}(t)\rangle_{k_+}.
\end{aligned} \tag{3.1.4}
$$

Recall that

$$
\begin{aligned}
\pi_{\Sigma_{n+1}}(\nabla_{X_H}\pi_{F_n})\phi^{(1)}_{n+1} &= -\pi_{\Sigma_{n+1}}(\nabla_{X_H}\pi_{\Sigma_{n+1}})\phi^{(1)}_{n+1}, \\
\pi_{F_n}(\nabla_{X_H}\pi_{F_n})\sigma_{n+1} &= -\pi_{F_n}(\nabla_{X_H}\pi_{\Sigma_{n+1}})\sigma_{n+1}.
\end{aligned}
$$

Then (3.1.4), also using the estimates from the proof of Lemma 2.7.2, becomes

$$
\big|\langle(\nabla_{X_H}\pi_{F_n})\phi^{(1)}_{n+1}(t), \sigma^+_{n+1}(t)\rangle_{k_+} + \langle\phi^{(1),+}_{n+1}(t), (\nabla_{X_H}\pi_{\Sigma_{n+1}})\sigma_{n+1}(t)\rangle_{k_+}\big| \le CR^2.
$$

Then, using (3.1.3), we have

$$\frac{1}{2}\frac{d}{dt}\Big|_{t=0}\|\phi^{(1),+}_{n+1}(t)+\sigma^{+}_{n+1}(t)\|^2_{k_+} \le CR^3\|\phi^{(1),+}_{n+1}(0)\|_{k_++\frac{1}{2}} \\ - \mathrm{Re}\langle \pi_{\Sigma_{n+1}} D'\sigma_{n+1}(0), \sigma^{+}_{n+1}(0)\rangle_{k_+} \\ - \mathrm{Re}\langle \pi_{F_{n+1}} D'\phi^{(1)}_{n+1}(0), \phi^{(1),+}_{n+1}(t)\rangle_{k_+} + C.$$

The argument from Lemma 2.7.2 gives

$$0 = \frac{1}{2}\frac{d}{dt}\Big|_{t=0}\|\phi^{(1),+}_{n+1}(t)+\sigma^{+}_{n+1}(t)\|^2_{k_+} \\ \le CR^3\|\phi^{(1),+}_{n+1}(0)\|_{k_++\frac{1}{2}} - \|\phi^{(1),+}_{n+1}(0)\|^2_{k+\frac{1}{2}} - \|\sigma^{+}_{n+1}(0)\|^2_{k+\frac{1}{2}} + C.$$

Thus, $\|\phi^{(1),+}_{n+1}(0)+\sigma^{+}_{n+1}(0)\|_{k+\frac{1}{2}}$ is bounded. The proof of Lemma 3.1.7 then follows exactly as Theorem 2.3.3. ∎

Finally, for $\tau \in [2,3]$, set

$$\frac{d\phi^{(1)}_{n+1}(t)}{dt} = -\chi\{(3-\tau)(\nabla_{X_H(\phi_{n+1}(t))}\pi_{F_{n+1}})\phi^{(1)}_{n+1}(t) \\ + (\tau-2)(\nabla_{X_H(\phi_{n+1}(t))}\pi_{F_{n+1}})\phi^{(1)}_{n+1}(t) \\ + \pi_{F_n}\big(D\phi^{(1)}_{n+1}(t) + c_1(\hat{\gamma}_{n+1}(t))\big) \\ + (\tau-2)(\nabla_{X_H(\phi_{n+1}(t))}\pi_{F_n})\phi^{(1)}_{n+1}(t)\},$$

$$\frac{d\sigma_{n+1}}{dt}(t) = -\chi\{(3-\tau)(\nabla_{X_H(\phi_{n+1}(t))}\pi_{\Sigma_{n+1}})\sigma_{n+1}(t) \\ + (\tau-2)(\nabla_{X_H(\phi^{(1)}_{n+1}(t))}\pi_{\Sigma_{n+1}})\sigma_{n+1}(t)\},$$

$$\frac{db}{dt}(t) = -\chi\{(3-\tau)X_H(\phi_{n+1}(t)) + (\tau-2)X_H(\phi^{(1)}_{n+1}(t))\},$$

with the other terms unchanged. Note that it is clear that these equations preserve the total space of $F_{n+1} \oplus W_{n+1}$.

Lemma 3.1.8. *For $n \gg 0$, for all $\tau \in [2,3]$, A_{n+1} is an isolating neighborhood for $\varphi^{\tau}_{n+1,k_+,k_-}$.*

Proof. This claim is a consequence of the arguments used in Lemma 3.1.3 and 3.1.7, and there are no new difficulties. ∎

Write $B(Q_{n+1}/Q_n, R)$ for the R-disk bundle of Q_{n+1}/Q_n over $\mathrm{Pic}(Y)$, etc.

Lemma 3.1.9. *Say that (A_n^o, L_n) is an index pair for $\mathcal{X}_n$, for some L_n, of $\mathcal{X}_n$ on $F_n \oplus W_n$. Then $(\tilde{A}_{n+1}, \tilde{L}_{n+1})$ is an index pair for $\mathcal{X}_{n+1}^{\mathrm{split}}$, where*

$$\begin{aligned}\tilde{A}_{n+1} = A_n^o &\times_B (B(P_{n+1}/P_n \oplus W_{n+1}^+/W_n^+, R)) \\ &\times_B (B(Q_{n+1}/Q_n \oplus W_{n+1}^-/W_n^-, R)),\end{aligned}$$

for some R sufficiently large, and

$$\begin{aligned}\tilde{L}_{n+1} = L_n^o &\times_B (B(P_{n+1}/P_n \oplus W_{n+1}^+/W_n^+, R)) \\ &\times_B (\partial B(Q_{n+1}/Q_n \oplus W_{n+1}^-/W_n^-, R)).\end{aligned}$$

Proof. It follows from Lemma 3.1.2 that $\mathrm{inv}(\tilde{A}_n \setminus \tilde{L}_n) \subset \mathrm{int}(\tilde{A}_n \setminus \tilde{L}_n)$.

We next check that $\tilde{L}_n$ is positively invariant in $\tilde{A}_n$. Write

$$(\phi_{n+1}^{(1)}(t), \omega_{n+1}^{(1)}(t), \zeta_{n+1}(t))$$

in

$$\begin{aligned}(F_n \oplus W_n) &\times_B (B(P_{n+1}/P_n \oplus W_{n+1}^+/W_n^+, R)) \\ &\times_B (B(Q_{n+1}/Q_n \oplus W_{n+1}^-/W_n^-, R))\end{aligned}$$

for a trajectory of $\varphi_{n+1,k_+,k_-}^{\mathrm{split}}$. The flow on the $\mathcal{F}_n \times_B \mathcal{W}_n$-factor is independent of position on the $(B(P_{n+1}/P_n \oplus W_{n+1}^+/W_n^+, R)) \times_B (B(Q_{n+1}/Q_n \oplus W_{n+1}^-/W_n^-, R))$ factor, and in particular, if $(\phi_{n+1}^{(1)}(T_0), \omega_{n+1}^{(1)}(T_0)) \in L_n$, then $(\phi_{n+1}^{(1)}(t), \omega_{n+1}^{(1)}(t)) \in L_n$ for all $t \geq T_0$, by our assumption on L_n.

We must then show that if $\zeta_{n+1}(T_0) \in \partial B(Q_{n+1}/Q_n \oplus W_{n+1}^-/W_n^-, R)$, then

$$\zeta_{n+1}(t) \in \partial B(Q_{n+1}/Q_n \oplus W_{n+1}^-/W_n^-, R_1),$$

or exits $\tilde{A}_{n+1}$, for all $t \geq T_0$, if n is large enough. We regard the path $(\phi_{n+1}^{(1)}(t), \omega_{n+1}^{(1)}(t))$ as fixed, and $\zeta_{n+1}(t)$ as a trajectory of a vector field on the boundary $\partial B(Q_{n+1}/Q_n \oplus W_{n+1}^-/W_n^-, R_1)$.

Write $\zeta_{n+1}(t) = (b(t), \zeta_{n+1}^{(1),+}, \zeta_{n+1}^{(1),-}, \zeta_{n+1}^{(2),+}, \zeta_{n+1}^{(2),-})$, as a section of

$$V_n(R_1) = (B(P_{n+1}/P_n \oplus W_{n+1}^+/W_n^+, R_1)) \times_B (B(Q_{n+1}/Q_n \oplus W_{n+1}^-/W_n^-, R_1)).$$

We may, and do, assume without loss of generality that $T_0 = 0$. Then if $(\zeta_{n+1}^{(1),-}, \zeta_{n+1}^{(2),-}) \in \partial B(Q_{n+1}/Q_n \oplus W_{n+1}^-/W_n^-, R)$, either $\zeta_{n+1}^{(1),-}$ or $\zeta_{n+1}^{(2),-}$ has $\|\zeta_{n+1}^{(i),-}\|_{k_-} \geq R_1/2$. Assume $i = 1$, the other case being similar.

Recall that $(\phi_{n+1}^{(1)}(t), \omega_{n+1}^{(1)}(t), \zeta_{n+1}(t))$ is equivalent to a trajectory

$$\gamma_{n+1}(t) = (\phi_{n+1}^{(1)}(t), \sigma_{n+1}(t), \omega_{n+1}(t))$$

of $\mathcal{X}_{n+1}^{\mathrm{split}}$ on $F_{n+1} \oplus W_{n+1}$.

We consider

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big|_{t=0}\|\zeta_{n+1}^{(1),-}(t)\|_{k-}^2 \\
&\quad = \frac{1}{2}\frac{d}{dt}\Big|_{t=0}\|\sigma_{n+1}^{-}(t)\|_{k-}^2 \\
&\quad = \langle -(\nabla_{X_H}\pi_{\Sigma_{n+1}})\sigma_{n+1}(0) - \pi_{\Sigma_{n+1}} D'\sigma_{n+1}(0), \sigma_{n+1}^{-}(0)\rangle_{k-} \\
&\qquad - \langle \nabla_{X_H}(D')^{k-}\sigma_{n+1}(0), (D')^{k-}\sigma_{n+1}^{-}\rangle_0 + \langle (\nabla_{X_H}\pi^{-})\sigma_{n+1}(0), \sigma_{n+1}^{-}(0)\rangle_{k-} \\
&\quad \le CR^2 - \langle \pi_{\Sigma_{n+1}} D'\sigma_{n+1}(0), \sigma_{n+1}^{-}(0)\rangle_{k-} \\
&\quad = CR^2 - \|\sigma_{n+1}^{-}(0)\|_{k-+\frac{1}{2}}^2 .
\end{aligned}$$

Note that we have used that n can be taken sufficiently large that Σ_{n+1} is perpendicular to the image of $\mathbb{A}$.

Now, by definition of Σ_{n+1}, we have

$$\frac{\|\sigma_{n+1}^{-}(0)\|_{k-+\frac{1}{2}}^2}{\|\sigma_{n+1}^{-}(0)\|_{k-}^2} \to \infty$$

as $n \to \infty$.

Thus, if $\|\sigma_{n+1}^{-}(0)\|_{k-} \ge R/2$, we have that $\|\zeta_{n+1}^{(1),-}(t)\|_{k-}$ is always increasing at $t = 0$ (similarly, $\|\zeta_{n+1}^{(1),+}(t)\|_{k+}$ is decreasing at $t = 0$).

This shows that $\tilde{L}_{n+1}$ is positively invariant in $\tilde{A}_{n+1}$. It follows similarly that $\tilde{L}_{n+1}$ is an exit set. ∎

3.2 Spin^c structure for family of manifolds

Since we consider a family of spin^c 3-manifolds to show that the Conley index for the flow φ_n is independent of the choice of Riemannian metric of Y in Section 3.3, we will give the definition of spin^c structure for a family of Riemannian manifolds.

Take an n-dimensional real, oriented vector space V and an inner product g on V. We denote by $Fr(V, g)$ the space of orthonormal bases of (V, g) compatible with the orientation. Choose another inner product h on V. We define an isomorphism between $Fr(V, g)$ and $Fr(V, h)$. For $\{e_i\}_{i=1}^n \in Fr(V, g)$, put

$$h_{ij} = h(e_i, e_j) \in \mathbb{R}.$$

Then the matrix $H = (h_{ij})_{i,j=1,\dots,n}$ is symmetric and positive definite. We have the square root $\sqrt{H}$ of H defined as follows. Since H is symmetric and positive definite, we have the eigenspace decomposition

$$\mathbb{R}^n = \bigoplus_{i=1}^{r} V_{\lambda_i},$$

where $\lambda_i > 0$ are the distinct eigenvalues of H, and V_{λ_i} are the eigenspaces. Define $\sqrt{H}$ to be the matrix corresponding to the linear map $\mathbb{R}^n \to \mathbb{R}^n$ defined by $v \mapsto \sqrt{\lambda_i} v$ for $v \in V_{\lambda_i}$. Define a basis $f_1, \ldots, f_n$ of V by

$$(f_1 \ldots f_n) = (e_1 \ldots e_n)\sqrt{H}^{-1}.$$

We can see that $f_1, \ldots, f_n$ are an orthonormal basis with respect to h. So we get a map

$$Fr(V, g) \to Fr(V, h). \tag{3.2.1}$$

Take $G \in SO(n)$ and put

$$(e'_1 \ldots e'_n) = (e_1 \ldots e_n)G, \quad H' = (h(e'_i, e'_j))_{i,j=1,\ldots,n}.$$

It is easy to see that

$$H' = G^{-1}HG, \quad \sqrt{H'} = G^{-1}\sqrt{H}G.$$

This implies that the map (3.2.1) is an $SO(n)$-equivariant isomorphism.

For an oriented smooth Riemannian n-manifold (X, g), let $P_{X,g}$ be the principal $SO(n)$-bundle of oriented, orthonormal frames in TX. Recall that a spin^c structure of (X, g) is a pair of a principal Spin$^c(n)$ bundle $\tilde{P}_X$ on X and a smooth map $\xi\colon \tilde{P}_X \to P_{X,g}$ such that the diagram

$$\begin{tikzcd} \tilde{P}_X \arrow[rr, "\xi"] \arrow[dr] & & P_{X,g} \arrow[dl] \\ & X & \end{tikzcd}$$

commutes, and for $p \in \tilde{P}_X$ and $s \in \mathrm{Spin}^c(n)$ we have

$$\xi(p \cdot s) = \xi(p) \cdot \pi(s).$$

Here, $\pi\colon \mathrm{Spin}^c(n) \to SO(n)$ is the projection.

Take another Riemannian metric h on X. The $SO(n)$-equivariant isomorphism (3.2.1) induces an isomorphism

$$P_{X,g} \cong P_{X,h} \tag{3.2.2}$$

of principal bundles. Hence a spin^c structure $(\tilde{P}_X, \xi)$ of (X, g) naturally defines a spin^c structure of (X, h).

A locally trivial family of spin^c manifolds over a topological space L is a tuple $(E, G, \tilde{P}_E, \xi)$. The first component E stands for a locally trivial fiber bundle

$$X \to E \to L$$

over L with fiber X. For each $\ell \in L$ we have an open neighborhood U_ℓ of ℓ and a trivialization

$$E|_{U_\ell} \cong U_\ell \times E_\ell.$$

Here, E_ℓ is the fiber of E over ℓ. The second component G is a fiberwise Riemannian metric of E. Let P_E be the principal $SO(n)$-bundle on E whose fiber over ℓ is the principal $SO(n)$-bundle of oriented, orthonormal frames in TE_ℓ. Note that the local trivialization of E on U_ℓ and the isomorphism (3.2.2) induce an isomorphism

$$P_E|_{U_\ell} \cong U_\ell \times P_{E_\ell}$$

of principal bundles. The third component $\tilde{P}_E$ is a principal $\mathrm{Spin}^c(n)$ bundle over E. The fourth component ξ is a smooth map

$$\tilde{P}_E \to P_E$$

such that the diagram

$$\begin{tikzcd} \tilde{P}_E \arrow[rr, "\xi"] \arrow[dr] & & P_E \arrow[dl] \\ & E & \end{tikzcd}$$

commutes and $\xi(p, \cdot s) = \xi(p) \cdot \pi(s)$ for $p \in \tilde{P}_E$ and $s \in \mathrm{Spin}^c(n)$. Moreover, we assume that $\tilde{P}_E$ is locally trivial. That is, for each $\ell \in L$ there is an isomorphism

$$\tilde{P}_E|_{U_\ell} \cong U_\ell \times (\tilde{P}_E|_{E_\ell})$$

of principal bundles such that the following diagram commutes:

$$\begin{tikzcd} \tilde{P}_E|_{U_\ell} \arrow[r, "\cong"] \arrow[d, "\xi"'] & U_\ell \times (\tilde{P}_E|_{E_\ell}) \arrow[d, "\mathrm{id}_{U_\ell} \times \xi"] \\ P_E|_{U_\ell} \arrow[r, "\cong"'] & U_\ell \times P_{E_\ell}. \end{tikzcd}$$

3.3 Independence of metric

In this section we prove that the approximate Seiberg–Witten flow defined in (2.3.10) varies continuously as we vary the 3-manifold.

To make this precise, let $\mathcal{F}$ be a locally trivial family of spin^c metrized 3-manifolds with compact base space L, so that L is a CW complex. See Section 3.2 for the definition of a locally trivial family of spin^c metrized manifolds. Note that associated to $\mathcal{F}$ there is also a bundle over L, $\mathrm{Pic}(\mathcal{F})$, whose fiber is the Picard-bundle at $\ell \in L$.

Suppose that we are given a sequence of continuously varying spectral sections $P_{n,\ell}$, $Q_{n,\ell}$ for $\ell \in L$ so that the $P_{n,\ell}$, $Q_{n,\ell}$ are good as at the beginning of Chapter 2, with $F_{n,\ell} = P_{n,\ell} \cap Q_{n,\ell}$ as a fiber bundle over (the total space of) $\tilde{L}$. Let φ_{n,ℓ,k_+,k_-} be the flow defined by projection onto $F_{n,\ell}$. Here, unlike in the case of a single 3-manifold, the flow preserves fibers of $F_{n,\ell}$ over L (though the flow can of course move over $\tilde{L}_\ell$, the fiber of $\tilde{L} \to L$).

There is one subtlety in that now the eigenvalues of $*d$ may vary in the family $\mathcal{F}$. In particular, we will assume the existence of increasing spectral sections $W_{P,n}$ for $-*d$, and increasing spectral sections $W_{Q,n}$ for $*d$, satisfying the analogs of (2.3.6)–(2.3.7), and set $W_n = W_{P,n} \cap W_{Q,n}$. With this notation fixed, we define W_n^+ and W_n^- as before.

Theorem 3.3.1. *Let $\mathcal{F}$, with compact base L, be a family of* spin^c *metrized* 3*-manifolds, with fiber $\mathcal{F}_b$ for $b \in L$. Let k_+, k_- be half-integers with $k_\pm > 5$ and with $|k_+ - k_-| \le \frac{1}{2}$. Fix a positive number R with $R > R_{k_+,k_-}$ for some R_{k_+,k_-}. Then*

$$(B_{k_+}(F_n^+; R) \times_B B_{k_-}(F_n^-; R)) \times_B (B_{k_+}(W_n^+; R) \times_B B_{k_-}(W_n^-; R))$$

is an isolating neighborhood of the flow φ_{n,ℓ,k_+,k_-} for $n \gg 0$. Here, $B_{k_\pm}(F_n^\pm; R)$ are the disk bundle of $F_n^\pm$ of radius R in $L^2_{k_\pm}$ and $B_{k_+}(F_n^+; R) \times_B B_{k_-}(F_n^-; R)$ is the fiberwise product.

The proof of this theorem differs from the proof of Theorem 2.3.3 only in notation, so we will not write out the details.

In particular, we have the following corollary.

Corollary 3.3.2. *Let $(Y, \mathfrak{s})$ be a* spin^c *manifold, with metrics g_0, g_1, and fix a family of good spectral sections $P_{n,0}$, $Q_{n,0}$ over (Y, g_0). Choose a family of metrics g_t connecting g_0 to g_1. Then there exists a family of spectral sections $P_{n,t}$, $Q_{n,t}$ extending $P_{n,0}$, $Q_{n,0}$ and so that the flow $\varphi_{n,0,k_+,k_-}$ on $F_{n,0}$ extends to a continuously varying flow φ_{n,t,k_+,k_-} on $F_{n,t}$, so that*

$$(B_{k_+}(F_n^+; R) \times_B B_{k_-}(F_n^-; R)) \times_B (B_{k_+}(W_n^+; R) \times_B B_{k_-}(W_n^-; R))$$

is an isolating neighborhood of the flow φ_{n,t,k_+,k_-} for $n \gg 0$ and all $t \in [0, 1]$. In particular, $I(\varphi_{n,0,k_+,k_-})$ is canonically, up to homotopy equivalence, identified with $I(\varphi_{n,1,k_+,k_-})$.

Proof. The claim about the existence of the extended spectral sections follows from the homotopy description of spectral sections and the fact that $[0, 1]$ is contractible. The claim on isolating neighborhoods is a consequence of Theorem 3.3.1. The well-definedness of the Conley index follows from the continuity property of the Conley index. ∎

3.4 Variation of Sobolev norms

Proposition 3.4.1. *Let (k_+^1, k_-^1) and (k_+^2, k_-^2) be pairs of half-integers > 5, with $|k_+^i - k_-^i| \le \frac{1}{2}$ for $i = 1, 2$. Fix R sufficiently large. Then there exists a family of flows φ_n^τ for $\tau \in [0, 1]$ so that*

$$(B_{g_+^\tau}(F_n^+; R) \times_B B_{g_-^\tau}(F_n^-; R)) \times_B (B_{g_+^\tau}(W_n^+; R) \times_B B_{g_-^\tau}(W_n^-; R))$$

is a family of isolating neighborhoods, where $g_\pm^\tau$ is the interpolated metric (defined below), and where $\varphi_n^0 = \varphi_{n,k_+^1,k_-^1}$ and $\varphi_n^1 = \varphi_{n,k_+^2,k_-^2}$. In particular, there is a homotopy equivalence

$$I(\varphi_{n,k_+^1,k_-^1}) \to I(\varphi_{n,k_+^2,k_-^2}),$$

suppressing the spectral section choices from the notation. The restriction to the S^1-fixed point set is a fiber-preserving homotopy equivalence.

Proof. Define the *interpolated metric* g^τ by

$$g^\tau(x, y) := \langle x, y\rangle_{k_\pm^\tau} := (1-\tau)\langle x, y\rangle_{k_+^1,k_-^1} + \tau\langle x, y\rangle_{k_+^2,k_-^2}.$$

We abuse notation and also write g^τ for the restriction of g^τ to subbundles, including $F_n^\pm$ and $W_n^\pm$.

The equation (2.7.1) defines a flow φ_n^τ, with π_{F_n}, π_{W_n} replaced appropriately. Hypothesis (2.7.2) continues to hold, with the subscripts $k_\pm$ replaced with $k_\pm^\tau$. Write $\pi_{F_n}^\tau$ for projection with respect to g^τ.

As usual, we will assume for a contradiction that

$$y_{n,0}^{\tau_n} = (\phi_{n,0}^{\tau_n}, \omega_{n,0}^{\tau_n}) \in \operatorname{inv} A_n \cap \partial A_n.$$

Let us treat the case that

$$\phi_{n,0}^{\tau_n} \in S_{g_+^\tau}(F_n^+; R) \in \operatorname{inv} A_n \cap \partial A_n,$$

where $S_{g_+^\tau}(V, R)$, for V a vector bundle over B, is the R-sphere bundle.

Exactly as in the proof of Theorem 2.3.3, we can extract a sequence of approximate solutions $\tilde{\gamma}_n^{\tau_n} = (\tilde{\phi}_n^{\tau_n}, \omega_n^{\tau_n})$, for $t \in [-T, T]$, with T fixed. To see this, we need to control $\frac{d\tilde{\phi}_n^\tau}{dt}$ in $(K_T, \ell - 5, w)$-norm. This amounts to generalizing Proposition 2.6.1 to the following situation.

Proposition 3.4.2. *Let k_+, k_- be half-integers, with $k_\pm > 5$, and also set $\ell = \min_{i=1,2}\{k_+^i, k_-^i\}$. Then*

$$\sup_{v \in B(TB;1)} \|\nabla_v \pi_{P_n}^\tau : L_{k^\tau}^2 \to L_{\ell-5,w}^2\| \to 0,$$

uniformly in τ.

This proposition holds because the natural modification of the estimate at the end of Corollary 2.5.2 holds.

Then the sequence $\tilde{\gamma}_n^{\tau_n}(t)$ converges to a map

$$\tilde{\gamma}\colon [-T,T] \to \mathcal{H}^1(Y) \times L^2_{\ell-1}(\mathbb{S}) \times L^2_{\ell-1}(\operatorname{im} d^*).$$

To verify that $\tilde{\gamma}$ solves the Seiberg–Witten equations, we observe that

$$(\nabla_{X_H} \pi^{\tau_n}_{F_n})\tilde{\phi}_n(s) \to 0$$

in $L^2_{K_T,\ell-5,w}$-norm, as follows from Proposition 3.4.2.

We have

$$\begin{aligned} \|\pi^{\tau_n}_{F_n} D\phi_n - D\phi_n\|_{\ell-2} &= \|\pi^{\tau_n}_{F_n} D\phi_n - D\phi_n + D\phi_n - D\phi_n\|_{\ell-2} \\ &\le \|[\pi^{\tau_n}_{F_n}, D]\phi_n\|_{\ell-2} + \|D\phi_n - D\phi\|_{\ell-2}. \end{aligned}$$

The first term drops out, using the rule of a sequence of controlled vector spaces, and we obtain that $\pi^{\tau_n}_{F_n} D\phi_n$ converges to $D\phi$ uniformly in $L^2_{\ell-2}$ on $[-T,T]$. By the proof of Lemma 2.7.1, the limit $\tilde{\gamma}$ is a solution of the Seiberg–Witten equations. The proof from this point follows along the same lines as Theorem 2.3.3. ■

3.5 The Seiberg–Witten invariant

In this section we repackage the construction of $\mathcal{SWF}_{[n]}(Y,\mathfrak{s})$ to take account of the choices made in the construction.

Definition 3.5.1. A 3-*manifold spectral system* (abbreviated as just a *spectral system*) for a family $\mathcal{F}$ of metrized spin^c 3-manifolds, with fiber $(Y,\mathfrak{s})$, is a tuple

$$\mathfrak{S} = \big(\mathbf{P}, \mathbf{Q}, \mathbf{W}_P, \mathbf{W}_Q, \{\eta^P_n\}_n, \{\eta^Q_n\}, \{\eta^{W_P}_n\}_n, \{\eta^{W_Q}_n\}_n\big), \tag{3.5.1}$$

where $\mathbf{P} = \{P_n\}_n$ (for $n \ge 0$) is a sequence of good (increasing) spectral sections of the Dirac operator $-D$; similarly, $\mathbf{Q} = \{Q_n\}_n$ is a sequence of good increasing spectral sections of D parameterized by $\operatorname{Pic}(\mathcal{F})$. The $\mathbf{W}_P = \{W_{P,n}\}_n$ are good spectral sections of the operator $-{*d}$; similarly, $\mathbf{W}_Q = \{W_{Q,n}\}_n$ are good spectral sections of $*d$. We require $W_{P,0}$ to be the sum of all negative eigenspaces of $*d$, as we may, since the nullspace of $*d$, acting on the bundle $L^2_k(\operatorname{im} d^*)$, is trivial, and similarly $W_{Q,0}$ will be the sum of positive eigenspaces. The η_n are exactly as in Theorem 3.1.1.

We have not established that there exist good sequences of spectral sections for $*d$ for all families $\mathcal{F}$. However, they exist in many situations, as for example when the family $\mathcal{F}$ is obtained as a mapping torus of a self-diffeomorphism preserving the fiber metric. In this case, $\mathcal{F}$ is a family over S^1 and the eigenvalues of $*d$ are constant functions on S^1. More generally, if there is a neighborhood U of b for each $b \in L$ such that $\mathcal{F}$ has a local trivialization $\mathcal{F}|_U \cong U \times Y$ preserving the fiber metric, then the eigenvalues of $*d$ are constants. So we have a good sequence of spectral sections of $*d$.

Definition 3.5.2. The *unparameterized Seiberg–Witten Floer spectrum*

$$\mathbf{SWF}^u(\mathcal{F}, \mathfrak{S}, k_+, k_-)$$

of a family $\mathcal{F}$ as in Definition 3.5.1 associated to a spectral system $\mathfrak{S}$, and $k_\pm$ half-integers with $k_\pm > 5$ and $|k_+ - k_-| \leq 1/2$, is the (partially defined) equivariant spectrum, whose sequence of spaces is defined as follows.

Let $\mathfrak{S}$ be a spectral system with components as named in (3.5.1). Let

$$D_n = (\dim(P_n - P_0), \dim(Q_n - Q_0), \dim(W_{P,n} - W_{P,0}), \dim(W_{Q,n} - W_{Q,0})),$$

whose components we denote D_n^ℓ for $\ell = 1, \ldots, 4$. Recall (cf. Appendix A.3) that we must assign, for a certain collection of representations, a space to each representation, together with structure maps. The spaces in the Seiberg–Witten Floer spectrum are most naturally defined at those representations $\mathbb{C}^{D_n^2} \oplus \mathbb{R}^{D_n^4}$; in order to define the spectra at other levels, we extrapolate from the definitions at these levels; see also Remark 3.5.13.

Let $\mathbb{N}_0$ be the set of nonnegative integers. For $(i_1, i_2) \in \mathbb{N}_0^2$ sufficiently large, let $A(i_1, i_2) = (A(i_1, i_2)_1, A(i_1, i_2)_2)$ denote the largest pair (D_n^2, D_n^4) among pairs (D_j^2, D_j^4) for which $(D_j^2, D_j^4) \leq (i_1, i_2)$. We can write

$$A(i_1, i_2) = (D^2_{n(i_1,i_2)}, D^4_{n(i_1,i_2)})$$

for some $n(i_1, i_2) \in \mathbb{N}_0$. Set $\mathbf{SWF}^u_{i_1,i_2}(\mathcal{F}, \mathfrak{S}, k_+, k_-)$ to be

$$\Sigma^{\mathbb{C}^{i_1 - A(i_1,i_2)_1} \oplus \mathbb{R}^{i_2 - A(i_1,i_2)_2}} \mathcal{SWF}^u_{[n(i_1,i_2)]}(\mathcal{F}, \mathfrak{S}, k_+, k_-).$$

Here, $\mathcal{SWF}^u_{[n(i_1,i_2)]}(\mathcal{F}, \mathfrak{S}, k_+, k_-)$ is the (unparameterized) Conley index with respect to the flow $\varphi_{n(i_1,i_2),k_+,k_-}$. If (i_1, i_2) is not sufficiently large, let $\mathbf{SWF}^u_{i_1,i_2}(\mathcal{F}, \mathfrak{S}, k_+, k_-)$ be a point. Define the transition map

$$\sigma_{(i,j),(i+1,j)} \colon \Sigma^{\mathbb{C}} \mathbf{SWF}^u_{i,j} \to \mathbf{SWF}^u_{i+1,j},$$

where $i + 1 \neq D_n^2$ for any n, as the identity (with the $\mathbb{C}$ factor contributing to the leftmost factor of $\Sigma^{\mathbb{C}^{i_1 - A(i_1,i_2)_1}}$), and similarly for transitions in the real coordinate. If $i + 1 = D_n^2$ for some n, we use the $(\eta_n)_*$ as defined in Theorem 3.1.1. Note that the $(\eta_n)_*$ are only well defined up to homotopy; we choose representatives in the homotopy class.

In the event that the family has a self-conjugate spin^c structure, and so that the spectral section $\mathfrak{S}$ is preserved by j, we use $\mathbb{H}$ instead of $\mathbb{C}$ above, as appropriate, so that $\mathbf{SWF}^u$ is indexed on the Pin(2)-universe described in Appendix A.1. To be more specific, we write $\mathbf{SWF}^{u,\mathrm{Pin}(2)}(\mathcal{F}, \mathfrak{S})$ for the Pin(2)-spectrum invariant. In particular, $\mathbf{SWF}^{u,\mathrm{Pin}(2)}_{i,j}$, viewed as an S^1-space, is identified with $\mathbf{SWF}^u_{2i,j}$.

We will often suppress some arguments of $\mathbf{SWF}^u$ from the notation where they are clear from context.

At the point-set level, there is a choice of index pairs (at each level (i_1, i_2)) involved in Definition 3.5.2. However, the space $\mathcal{SWF}^u_{[n]}(\mathcal{F}, \mathfrak{S}, k_+, k_-)$ is well defined up to canonical homotopy, since the Conley index forms a connected simple system, Theorem A.2.3.

Remark 3.5.3. We would be able to repeat Definition 3.5.2 in the parameterized setting, replacing the spectrum $\mathbf{SWF}^u$ with a parameterized spectrum $\mathbf{SWF}$, except that it is not known that the parameterized Conley index forms a connected simple system in $\mathcal{K}_{G,B}$, the category considered in Appendix A.

The spaces $\mathbf{SWF}^u_{(i_1,i_2)}(\mathcal{F})$ for (i_1, i_2) not a pair (D^2_n, D^4_n), for some n, seem to have rather an awkward definition, because they do not naturally represent the Conley index of some fixed flow. However, they may be viewed as the Conley indices of a split flow on $\underline{V} \times_{\mathrm{Pic}(\mathcal{F})} \mathcal{SWF}_{[n]}(\mathcal{F})$, for $V = \mathbb{C}^{i_1 - D^2_n} \oplus \mathbb{R}^{i_2 - D^4_n}$ a vector space equipped with a linear (repelling) flow.

More generally, associated to a spectral system $\mathfrak{S}$, we define the virtual dimension of the vector bundle $F_n \oplus W_n$ as

$$D_n = (\dim(P_n - P_0), \dim(Q_n - Q_0), \dim(W_n^+), \dim(W_n^-)).$$

We write $\mathfrak{S}(\vec{i})$ for the vector bundle of virtual dimension $\vec{i} = (i_1, i_2, i_3, i_4)$. If the spectral section does not produce a vector bundle in that virtual dimension, we define

$$\mathfrak{S}(i_1, i_2, i_3, i_4) = \underline{V} \oplus F_n \oplus W_n,$$

where $F_n \oplus W_n$ is the largest vector bundle coming from $\mathfrak{S}$ with virtual dimension at most (i_1, i_2, i_3, i_4), and where we define $\underline{V}$ to be the trivial S^1 (or Pin(2), as appropriate) vector bundle with dimension $(i_1, i_2, i_3, i_4) - D_n$. When we need to distinguish between the contributions of $F_n \oplus W_n$ and $\underline{V}$ to $\mathfrak{S}(\vec{i})$, we call $F_n \oplus W_n$ the *geometric* bundle, and $\underline{V}$ the *virtual* bundle.

We can treat $\mathfrak{S}(i_1, i_2, i_3, i_4)$ as a vector bundle with a split flow, as discussed above; its unparameterized Conley index is (canonically, up to homotopy) homotopy equivalent to $\mathcal{SWF}^u_{(i_2,i_4)}(\mathcal{F}, \mathfrak{S})$.

Let

$$\underline{V}(\vec{i}, \vec{j}) = \underline{\mathbb{C}}^{j_1 - i_1} \oplus \underline{\mathbb{C}}^{j_2 - i_2} \oplus \underline{\mathbb{R}}^{j_3 - i_3} \oplus \underline{\mathbb{R}}^{j_4 - i_4},$$

viewed as a vector bundle with linear flow, outward in the even factors, inward in the odd factors. Note that for any $\vec{j} \geq \vec{i}$ (that is, $j_1 \geq i_1, \ldots, j_4 \geq i_4$), there is a vector bundle morphism

$$\underline{V}(\vec{i}, \vec{j}) \oplus \mathfrak{S}(\vec{i}) \to \mathfrak{S}(\vec{j}), \tag{3.5.2}$$

as follows. Indeed, if $A(\vec{i}) = A(\vec{j})$, then (3.5.2) is defined by

$$\begin{aligned}\underline{V}(\vec{i}, \vec{j}) \oplus (\underline{V}(D_n, \vec{i}) \oplus F_n \oplus W_n) &= (\underline{V}(\vec{i}, \vec{j}) \oplus \underline{V}(D_n, \vec{i})) \oplus F_n \oplus W_n \\ &\to \underline{V}(D_n, \vec{j}) \oplus F_n \oplus W_n.\end{aligned}$$

If $\vec{j} = D_{n+1}$ and $\vec{i} = D_n$, the morphism (3.5.2) is just the structure map involved in the definition of a spectral system. For more general $\vec{j}, \vec{i}$, the morphism (3.5.2) is the composite coming from the sequence $\vec{i} \to D_{n_1} \to \cdots \to D_{n_k} = A(\vec{j}) \to \vec{j}$, where the rightmost factors of $\underline{V}(\vec{i}, \vec{j})$ are used first.

Similarly, we define $P(i_1) = \underline{\mathbb{C}}^{i_1 - D^1_{A(i_1)}} \oplus P_{A(i_1)}$, etc.

Definition 3.5.4. We call two spectral systems $\mathfrak{S}_1$ and $\mathfrak{S}_2$ for the same family $\mathcal{F}$ *equivalent* if there exists a collection of bundle isomorphisms,

$$\Phi_{P,i} \colon P^1(i) \to P^2(i),$$

and similarly for Q, W_P, W_Q, for all i sufficiently large, satisfying the following conditions. First, there exists some sufficiently large n, so that the $\Phi_{P,i}$ (respectively $\Phi_{Q,i}$ etc.), as i becomes large, must preserve the subbundles P^j_n for $j = 1, 2$ (similarly for Q^j_n etc.). (Indeed, for $\vec{i}$ sufficiently large, P^1_n (respectively Q^1_n etc.) will be contained in the geometric bundles of $P^2(i)$ (respectively $Q^2(i)$ etc.).)

Second, the Φ_i must be compatible with the structure maps of $\mathfrak{S}_1$, $\mathfrak{S}_2$ in that the following square commutes (as well as its analogs):

$$\begin{tikzcd} \underline{V} \oplus P^1(i) \arrow[r, "\mathrm{id} \oplus \Phi_{P,i}"] \arrow[d, "\eta"'] & \underline{V} \oplus P^2(i) \arrow[d, "\eta"] \\ P^1(j) \arrow[r, "\Phi_{P,j}"'] & P^2(j). \end{tikzcd}$$

We do not require the isomorphisms Φ_i (etc.) to preserve all of the P^j_n as n varies.

Note that a morphism of spectral systems as in Definition 3.5.4 also induces maps

$$\Phi_{\vec{i}} \colon \mathfrak{S}_1(\vec{i}) \to \mathfrak{S}_2(\vec{i})$$

for $\vec{i}$ sufficiently large, which preserve the subbundles $F^1_n \oplus W^1_n$ (which lie in $\mathfrak{S}_2(\vec{i})$ for $\vec{i}$ sufficiently large naturally), for some fixed large n, for $\vec{i}$ sufficiently large. There is also a commutative square:

$$\begin{tikzcd}
\underline{V} \oplus \mathfrak{S}_1(\vec{i}) \arrow[r, "\mathrm{id}\oplus\Phi_{\vec{i}}"] \arrow[d, "\eta"'] & \underline{V} \oplus \mathfrak{S}_2(\vec{i}) \arrow[d, "\eta"] \\
\mathfrak{S}_1(\vec{j}) \arrow[r, "\Phi_{\vec{j}}"'] & \mathfrak{S}_2(\vec{j}).
\end{tikzcd}$$

Proposition 3.5.5. *For $\mathcal{F}$ a family of* spin^c 3*-manifolds, n sufficiently large and $\Phi\colon \mathfrak{S}_1 \to \mathfrak{S}_2$ an equivalence of spectral systems, there is a homotopy equivalence, well defined up to homotopy,*

$$\Phi^u_{n,*} : \mathcal{SWF}^u_{[n]}(\mathcal{F}, \mathfrak{S}_1) \to \mathcal{SWF}^u_{[n]}(\mathcal{F}, \mathfrak{S}_2).$$

In fact, there is a fiberwise-deforming homotopy equivalence,

$$\Phi_{n,*}\colon \mathcal{SWF}_{[n]}(\mathcal{F}, \mathfrak{S}_1) \to \mathcal{SWF}_{[n]}(\mathcal{F}, \mathfrak{S}_2),$$

so that $\Phi^u_{n,*} = \nu_!\Phi_{n,*}$. Here, ν is the map $\mathrm{Pic}(\mathcal{F}) \to *$ *sending* $\mathrm{Pic}(\mathcal{F})$ *to a point, and $\nu_!$ is defined as in Appendix A. (Note that $\Phi_{n,*}$ is not claimed to be well defined.) Analogous statements hold for* $\mathrm{Pin}(2)$*-equivariant spectral sections.*

Proof. We consider the pullback of the flow φ_2 on $\mathfrak{S}_2(\vec{i})$ by the morphism (for some large $\vec{i}$)

$$\Phi_{\vec{i}}\colon \mathfrak{S}_1(\vec{i}) \to \mathfrak{S}_2(\vec{i}),$$

defining a flow on $\mathfrak{S}_1(\vec{i})$. Following the proof of Theorem 3.1.1, we see that there is a well-defined, up to homotopy, deformation of $\Phi^*_{\vec{i}}\varphi_2$ to φ_1. Deformation invariance of the Conley index gives a fiberwise-deforming homotopy equivalence

$$I(\varphi_1) \to I((\Phi_{\vec{i}})^*\varphi_2) \cong I(\varphi_2),$$

where the isomorphism is canonical (at the point-set level). Passing to the unparameterized Conley index, the morphism

$$I^u(\varphi_1) \to I^u((\Phi_{\vec{i}})^*\varphi_2)$$

is canonical (up to homotopy). This gives the proposition. ∎

We write $[\mathfrak{S}]$ for the equivalence class of a spectral system $\mathfrak{S}$.

Remark 3.5.6. As usual, if Conjecture A.2.4 holds, then $\Phi_{n,*}$ appearing in Proposition 3.5.5, is well defined.

Theorem 3.5.7. *The equivariant parameterized stable homotopy type of*

$$\Sigma_B^{\mathbb{C}^{-D_n^2}\oplus\mathbb{R}^{-D_n^4}} \mathcal{SWF}_{[n]}(\mathcal{F}, [\mathfrak{S}])$$

is independent of the choices in its construction. That is, it is independent of

(1) *the choice of* k_+, k_-,

(2) *the element* $n \gg 0$,

(3) *a choice of spectral system* $\mathfrak{S}$ *representing the equivalence class* $[\mathfrak{S}]$,

(4) *the family of metrics on* $\mathcal{F}$.

Here, $\Sigma_B^{\mathbb{C}^{-D_n^2}\oplus\mathbb{R}^{-D_n^4}}$ *stands for the desuspension by* $\mathbb{C}^{D_n^2} \oplus \mathbb{R}^{D_n^4}$ *in the category* $PSW_{S^1,B}$. *See Appendix A.1.*

If the spin^c *structure is self-conjugate, a similar statement holds for*

$$\Sigma_B^{\mathbb{H}^{-D_n^2}\oplus\tilde{\mathbb{R}}^{-D_n^4}} \mathcal{SWF}_{[n]}(\mathcal{F}, [\mathfrak{S}]).$$

Proof. Proposition 3.5.5 addresses changes in the spectral section. Proposition 3.4.1 addresses varying of $k_\pm$. The choice of n was handled in Theorem 3.1.1, and the metric was addressed in Theorem 3.3.1. ■

Definition 3.5.8. The *Seiberg–Witten Floer parameterized homotopy type*

$$\mathcal{SWF}(\mathcal{F}, [\mathfrak{S}])$$

is defined as the class of

$$\Sigma_B^{\mathbb{C}^{-D_n^2}\oplus\mathbb{R}^{-D_n^4}} \mathcal{SWF}_{[n]}(\mathcal{F}, [\mathfrak{S}]),$$

for any n.

When the spin^c structure is self-conjugate, the Pin(2)-*Seiberg–Witten Floer parameterized homotopy type* $\mathcal{SWF}^{\mathrm{Pin}(2)}(\mathcal{F}, [\mathfrak{S}])$ is defined as the class of

$$\Sigma_B^{\mathbb{H}^{-D_n^2}\oplus\tilde{\mathbb{R}}^{-D_n^4}} \mathcal{SWF}_{[n]}(\mathcal{F}, [\mathfrak{S}]).$$

Recall from Appendix A.3 that a *weak* morphism of spectra is a (collection of) maps that is only defined in sufficiently high degrees (this is also the case for ordinary morphisms in Adams' [2] category of spectra).

Theorem 3.5.9. *For* $\mathcal{F}$ *a family of* spin^c 3*-manifolds, and* $\Phi\colon \mathfrak{S}_1 \to \mathfrak{S}_2$ *an equivalence of spectral systems, there is a weak morphism which is a homotopy equivalence (see Appendix A.3), well defined up to homotopy:*

$$\Phi_*\colon \mathbf{SWF}^u(\mathcal{F}, \mathfrak{S}_1) \to \mathbf{SWF}^u(\mathcal{F}, \mathfrak{S}_2).$$

That is, the collection of spectra

$$\mathbf{SWF}^u(\mathcal{F},[\mathfrak{S}]) = \{\mathbf{SWF}^u(\mathcal{F},\mathfrak{S})\}_{\mathfrak{S}}$$

forms a connected simple system in spectra, if $\mathcal{F}$ admits a spectral system.

Proof. First, independence of $\mathbf{SWF}^u(\mathcal{F},[\mathfrak{S}])$ from the choice of Sobolev norms was handled in Proposition 3.4.1. Moreover, variation of metric, for a particular level $\mathcal{SWF}^u_{[n]}(\mathcal{F},[\mathfrak{S}])$, was handled in Theorem 3.3.1. We then need only show that an equivalence of spectral systems induces a well-defined, up to homotopy, morphism

$$\mathbf{SWF}^u(\mathcal{F},\mathfrak{S}_1) \to \mathbf{SWF}^u(\mathcal{F},\mathfrak{S}_2).$$

For this, we use Proposition 3.5.5 to define the maps levelwise, and we need only show that the following square homotopy commutes (the squares involving other vector bundles $\mathfrak{S}(i_1,i_2,i_3,i_4)$ are straightforward):

$$\begin{tikzcd}
\Sigma^{V_n}\mathcal{SWF}^u_{[n]}(\mathcal{F},\mathfrak{S}_1) \arrow[r, "\mathrm{id}\wedge\Phi_{n,*}"] \arrow[d, "\eta_{n,*}"'] & \Sigma^{V_n}\mathcal{SWF}^u_{[n]}(\mathcal{F},\mathfrak{S}_2) \arrow[d, "\eta_{n,*}"] \\
\mathcal{SWF}^u_{[n+1]}(\mathcal{F},\mathfrak{S}_1) \arrow[r, "\Phi_{n+1,*}"'] & \mathcal{SWF}^u_{[n+1]}(\mathcal{F},\mathfrak{S}_2).
\end{tikzcd}$$

Here, $V_n = \mathbb{C}^{D^2_{n+1}-D^2_n} \oplus \mathbb{R}^{D^4_{n+1}-D^4_n}$. This is a consequence of the two composites involved being Conley-index continuation maps associated to deformations of the flow. Observe that the composite deformations are related to each other by a deformation of deformations. By [47, Section 6.3], the square homotopy commutes (the necessary adjustments of Salamon's argument for equivariance are straightforward). ∎

As usual, subject to Conjecture A.2.4, Theorem 3.5.9 would hold in the parameterized case.

Moreover, it is easy to determine when two spectral systems are equivalent, as follows.

Lemma 3.5.10. *The set of spectral systems for a family $\mathcal{F}$ of* spin^c 3*-manifolds up to equivalence, if nonempty, is affine equivalent to* $K(\mathrm{Pic}(\mathcal{F})) \times K(\mathrm{Pic}(\mathcal{F}))$*, where the difference of systems $\mathfrak{S}_1$, $\mathfrak{S}_2$ is sent to* $([P^1_0 - P^2_0],[Q^1_0 - Q^2_0])$.

Proof. By its construction, an equivalence of spectral systems is determined by its value $(\Phi_{P,i},\Phi_{Q,i},\Phi_{W_P,i},\Phi_{W_Q,i})$ for any sufficiently large i. In the positive spectral section part of the spinor coordinate, to construct an equivalence $\mathfrak{S}_1 \to \mathfrak{S}_2$ it is sufficient (and necessary) to construct an isomorphism $P^1(i) - P^1_n \to P^2(i) - P^1_n$ for some i large, relative to a fixed (large) n. By definition, $P^1(i) - P^1_n$ is canonically some number of copies of $\underline{\mathbb{C}}$, and so such an isomorphism exists if and only if

$$[P^2(i) - P^1_n] = [\underline{\mathbb{C}}^{\dim(P^1(i)-P^1_n)}].$$

This condition is satisfied exactly when $[P^1_0 - P^2_0] = 0 \in K(\mathrm{Pic}(\mathscr{F}))$, as needed.

The 1-form coordinate is handled similarly, but the bundles $W^{\pm}_n$ there are always trivial. ■

In particular, we note that there is a canonical choice, subject to a choice of Q_0, and up to adding trivial bundles, of a spectral section P_0, by requiring $P_0 - Q_0$ trivializable. We call these *normal* spectral sections; the set of equivalence classes of such is affine equivalent to $K(\mathrm{Pic}(Y))$, as above.

Definition 3.5.11. An (S^1-equivariant) *Floer framing* is an equivalence class of normal spectral sections. A Pin(2)-equivariant Floer framing is a (Pin(2))-equivalence class of normal spectral sections. Here, a Pin(2)-equivalence of (Pin(2)-equivariant) spectral sections is a collection of isomorphisms as in Definition 3.5.4 that are Pin(2)-equivariant.

There are various extensions of Lemma 3.5.10. Let us state a Pin(2)- equivariant version of the lemma.

Lemma 3.5.12. *The set of* Pin(2)*-spectral systems for a family* $\mathscr{F}$ *of* spin^c 3*-manifolds up to equivalence, if nonempty, is affine equivalent to*

$$KQ(\mathrm{Pic}(\mathscr{F})) \times KQ(\mathrm{Pic}(\mathscr{F})),$$

where the difference of systems $\mathfrak{S}_1$, $\mathfrak{S}_2$ *is sent to* $([P^1_0 - P^2_0], [Q^1_0 - Q^2_0])$*. Here, KQ is the quaternionic K-theory defined in* [19, 33].

Remark 3.5.13. We can define the spectrum $\mathbf{SWF}^u_{i_1,i_2}$ in a little different way. Fix a sufficiently large integer n and put

$$\mathbf{SWF}^u_{i_1,i_2} = \Sigma^{\mathbb{C}^{i_1-D^2_n} \oplus \mathbb{R}^{i_2-D^4_n}} \mathcal{SWF}^u_{[n]}$$

for $(i_1, i_2) \in \mathbb{N}^2_0$ with $i_1, i_2 \geq n$. The transition maps

$$\sigma_{(i_1,i_2),(i_1+1,i_2)} \colon \Sigma^{\mathbb{C}} \mathbf{SWF}^u_{i_1,i_2} \to \mathbf{SWF}^u_{i_1+1,i_2},$$
$$\sigma_{(i_1,i_2),(i_1,i_2+1)} \colon \Sigma^{\mathbb{R}} \mathbf{SWF}^u_{i_1,i_2} \to \mathbf{SWF}^u_{i_1,i_2+1}$$

are defined to be the identities. This spectrum is homotopy equivalent to the previous one.

In the previous definition of $\mathbf{SWF}^u$, we introduced $A(i_1, i_2)$, which allows us to avoid choosing a large integer n. This makes the definition of $\mathbf{SWF}^u$ more natural.

In the construction of $\mathcal{SWF}_{[n]}(\mathcal{F}, \mathfrak{S})$, we have a frame of the orthogonal complement of Q_n in Q_{n+1}. Using the frame, we have

$$\mathcal{SWF}_{[n+1]}(\mathcal{F}, \mathfrak{S}) \cong \Sigma_B^{\mathbb{C}^{k_{Q,n}} \oplus \mathbb{R}^{k_{W,-,n}}} \mathcal{SWF}_{[n]}(\mathcal{F}, \mathfrak{S}).$$

More generally, we can choose spectral sections Q_n such that the orthogonal complement of Q_n in Q_{n+1} does not necessarily have a frame. In this case, we have

$$\mathcal{SWF}_{[n+1]}(\mathcal{F}, \mathfrak{S}) \cong \Sigma_B^{(Q_{n+1}/Q_n) \oplus \mathbb{R}^{k_{W,-,n}}} \mathcal{SWF}_{[n]}(\mathcal{F}, \mathfrak{S}),$$

where Q_{n+1}/Q_n may not be trivialized. See Theorem 3.1.1. We can still define the Seiberg–Witten Floer stable homotopy type in a suitable stable homotopy category. The category is defined by taking R, W to be finite-dimensional, virtual G-vector bundles over B in Definition A.1.9, so that we can take desuspensions by nontrivial vector bundles. The Seiberg–Witten Floer stable homotopy type is defined to be the class of

$$\Sigma_B^{-(Q_n/Q_0) \oplus \mathbb{R}^{-D_n^4}} \mathcal{SWF}_{[n]}(\mathcal{F}, \mathfrak{S})$$

in the category, where n is a fixed large integer.

3.6 Elementary properties of $\mathcal{SWF}(Y, \mathfrak{s})$

Here we collect a few results about $\mathcal{SWF}(Y, \mathfrak{s})$ that follow almost directly from the definitions. We work only for a single $(Y, \mathfrak{s})$, but similar results hold in families.

Proposition 3.6.1. *The total space of $\mathcal{SWF}^u_{[n]}(Y, \mathfrak{s})$ has the homotopy type of a finite S^1-CW complex; respectively, the total space of $\mathcal{SWF}^{u,\mathrm{Pin}(2)}_{[n]}(Y, \mathfrak{s})$, when defined, is a finite* Pin(2)*-CW complex. As a consequence, for $G = S^1$ or* Pin(2)*, the Seiberg–Witten Floer spectrum $\mathbf{SWF}^{u,G}(Y, \mathfrak{s}, \mathfrak{S})$ is a finite G-CW spectrum.*

Proof. For this, we need to consider perturbations of the Seiberg–Witten equations. Recall the notion of cylinder functions from [28, Chapter 11]. As in [24, Definition 2.1], given a sequence of $\{C_j\}_{j=1}^{\infty}$ of positive real numbers and cylinder functions $\{\hat{f}_j\}_{j=1}^{\infty}$, let $\mathcal{P}$ be the Banach space

$$\mathcal{P} = \left\{ \sum_{j=1}^{\infty} \eta_j \hat{f}_j : \eta_j \in \mathbb{R}, \sum_{j=1}^{\infty} C_j |\eta_j| < \infty \right\}$$

with norm defined by $\| \sum_{j=1}^{\infty} \eta_j \hat{f}_j \| = \sum_{j=1}^{\infty} |\eta_j| C_j$. The elements of $\mathcal{P}$ are called *extended cylinder functions*.

For f an extended cylinder function, let grad $f = \mathfrak{q}$ be the L^2-gradient over $L^2_k(\mathbb{S}) \times \mathcal{H}^1(Y) \times L^2_k(\operatorname{im} d^*)$ of f. We write $(\mathfrak{q}_V, \mathfrak{q}_H, \mathfrak{q}_W)$ for the vertical, horizontal and 1-form components of $\mathfrak{q}$. Define the perturbed Seiberg–Witten equations by the downward gradient flow of $\mathcal{L} + f$, explicitly:

$$\begin{aligned} \frac{d\phi}{dt} &= -D_a\phi(t) - c_1(\gamma(t)) - \mathfrak{q}_V, \\ \frac{da}{dt} &= -X_H(\phi) - \mathfrak{q}_H, \\ \frac{d\omega}{dt} &= - * d\omega - c_2(\gamma(t)) - \mathfrak{q}_W. \end{aligned} \tag{3.6.1}$$

We may perform finite-dimensional approximation with the perturbed Seiberg–Witten equations in place of (2.3.2) (with the same spectral sections as for the unperturbed equations). It is straightforward but tedious to check that the proof of Theorem 2.3.3 holds also for (3.6.1), for k-extended cylinder functions f, where $k \geq \max\{k_+, k_-\} + \frac{1}{2}$. The key points are [24, Proposition 2.2] and [32, Lemma 4.10].

Moreover, for a family of perturbations, the analog of Theorem 2.3.3 continues to hold, by a similar argument. In particular, it is a consequence that $\mathcal{SWF}^u_{[n]}(Y, \mathfrak{s})$ is well defined up to canonical equivariant homotopy, independent of perturbation.

Finally, the space of perturbations $\mathcal{P}$ attains transversality for the Seiberg–Witten equations, in the sense that for a generic perturbation from $\mathcal{P}$, there are finitely many (all nondegenerate) stationary points for the perturbed formal gradient flow.

In particular, using the attractor–repeller sequence for the Conley index, together with the fact that the Conley index for a single nondegenerate critical point is a sphere, we observe that the Conley index $I^u(\varphi_{n,k_+,k_-})$ for n large is a finite G-CW complex. ∎

Proposition 3.6.2. *For $(Y, \mathfrak{s})$ a spin^c, oriented closed 3-manifold, and $\mathfrak{S}$ a spectral system, we have*

$$\mathcal{SWF}^u(Y, \mathfrak{s}, \mathfrak{S})^\vee \simeq \mathcal{SWF}^u(-Y, \mathfrak{s}, \mathfrak{S}^\vee),$$

where the spectral system $\mathfrak{S}^\vee$ is obtained by reversing the roles of P_n and Q_n in $\mathfrak{S}$.

Proof. This follows from the Spanier–Whitehead duality for the Conley index, Theorem A.2.8. ∎

Note that it would be desirable in Proposition 3.6.2 to have a similar result in the parameterized setting; the analog of Theorem A.2.8 in the parameterized setting has not been established, but would suffice.

Using the latter parts of Theorem 3.1.1, we have the following corollary.

Corollary 3.6.3. *The homotopy type of $\mathcal{SWF}_{[n]}(Y, \mathfrak{s}, \mathfrak{S})$ is independent of the spectral sections P_n for n large. That is, instead of $\mathcal{SWF}_{[n]}(Y, \mathfrak{s}, \mathfrak{S})$ depending on a* *choice in a set affine equivalent to* $K(\mathrm{Pic}(Y))\times K(\mathrm{Pic}(Y))$, $\mathcal{SWF}_{[n]}(Y,\mathfrak{s},\mathfrak{S})$ *is determined by a (relative) class in* $K(\mathrm{Pic}(Y))$.

Further,

$$\mathcal{SWF}_{[n]}(Y,\mathfrak{s},\mathfrak{S}_1)\simeq\Sigma_B^{\mathfrak{S}_1-\mathfrak{S}_2}\mathcal{SWF}_{[n]}(Y,\mathfrak{s},\mathfrak{S}_2),$$

where $\mathfrak{S}_1-\mathfrak{S}_2$ *is the bundle defined by Lemma* 3.5.10, *and where suspension is defined as in Remark* A.1.8.

We can now prove some of the results from the introduction.

Proof of Theorem 1.1.1. By [30], the vanishing of the triple-cup product on $H^1(Y;\mathbb{Z})$ implies that the family index of the Dirac operator on Y is trivial. Using this, fix a Floer framing $\mathfrak{P}$. In that case, Theorems 3.5.7 and 3.5.9 imply that $\mathcal{SWF}(Y,\mathfrak{s},\mathfrak{P})$ and $\mathbf{SWF}(Y,\mathfrak{s},\mathfrak{P})$ are well defined.

Proposition 3.6.1 gives the claim about finite CW structures.

Finally, when $b_1(Y)=0$, the relationship with $SWF(Y,\mathfrak{s})$ is immediate from the definition of $\mathcal{SWF}(Y,\mathfrak{s},\mathfrak{P})$, since the collection of linear subspaces used in the construction of $SWF(Y,\mathfrak{s})$ defines a spectral system as in Definition 3.5.1. ■

Proof of Theorem 1.3.2. The argument is completely parallel to the proof of Theorem 1.1.1. ■

Finally, we address the claims in the introduction about complex oriented cohomology theories. We start by reviewing the definition of an E-orientation of a vector bundle, where E is a multiplicative cohomology theory (see [3] for a discussion of orientability[1]). Indeed, let $V\to X$ be a topological vector bundle of rank m. Then an E-orientation is a class

$$u\in\widetilde{E}^m(\mathrm{Th}(V)),$$

so that, for all $x\in X$ and $i_x\colon S^m\to V$, the map associated to inclusion of a fiber over x, i_x^*u is a unit in $\widetilde{E}^m(S^m)=\widetilde{E}^0(S^0)$ (the latter equality being the suspension isomorphism of the cohomology theory E).

Recall that a cohomology theory E is *complex oriented* if it is oriented on all complex vector bundles. There is a universal such cohomology theory, complex cobordism MU, in the sense that for any complex-oriented cohomology theory E, there is a map of ring spectra $MU\to E$ inducing the orientation on E.

The utility of a complex-oriented cohomology theory E for studying the stable homotopy type $\mathcal{SWF}(Y,\mathfrak{s},\mathfrak{S}_1)$ is as follows. By Theorem 3.1.1, we have, by changing the spectral system $\mathfrak{S}_1$ to $\mathfrak{S}_2$, that there is an (S^1-equivariant) parameterized equivalence

$$\mathcal{SWF}(Y,\mathfrak{s},\mathfrak{S}_1)\to\Sigma^{\mathfrak{S}_1-\mathfrak{S}_2}\mathcal{SWF}(Y,\mathfrak{s},\mathfrak{S}_2). \tag{3.6.2}$$

[1] nLab also has a nice discussion, which our presentation follows.

In Chapter 6, after having considered the 4-dimensional invariant, we will introduce a number $n(Y, \mathfrak{s}, g, P_0)$ associated to a spectral section P_0 of the Dirac operator over Y, and a metric g on $(Y, \mathfrak{s})$. By its construction, $n(Y, \mathfrak{s}, g, P_0) = n(Y, \mathfrak{s}, g, [\mathfrak{S}])$ is an invariant of a spectral system up to equivalence $[\mathfrak{S}]$, and its main property is that it changes appropriately to counteract the shift in (3.6.2). That is,

$$n(Y, \mathfrak{s}, g, [\mathfrak{S}_1]) - n(Y, \mathfrak{s}, g, [\mathfrak{S}_2]) = \dim[\mathfrak{S}_1 - \mathfrak{S}_2],$$

as follows immediately from (6.2.1).

For E an S^1-equivariant cohomology theory, let

$$FE^*(Y, \mathfrak{s}, \mathfrak{S}_1) = \widetilde{E}^{*-2n(Y,\mathfrak{s},g,\mathfrak{S}_1)}(\nu_! \mathcal{SWF}(Y, \mathfrak{s}, \mathfrak{S}_1)).$$

We call $FE^*(Y, \mathfrak{s}, \mathfrak{S}_1)$ the *Floer E-cohomology* of the tuple $(Y, \mathfrak{s}, \mathfrak{S}_1)$.

More generally, we can also consider the notion of an *equivariant complex orientation*. This is more complicated to state; we follow [12] for the definition of equivariant complex orientability. That is, let A be an abelian compact Lie group, and fix a complete complex A-universe $\mathcal{U}$ (see Appendix A). A multiplicative equivariant cohomology theory $E^*_A(\cdot)$ is called *complex stable* if there are suspension isomorphisms:

$$\sigma_V \colon \widetilde{E}^n_A(X) \to \widetilde{E}^{n+\dim V}_A((V^+) \wedge X)$$

for all complex (finite-dimensional) A-representations V in $\mathcal{U}$. The natural transitivity condition on the σ_V is required, and the map σ_V is required to be given by multiplication by an element of $\widetilde{E}^{\dim V}(V^+)$ (necessarily a generator). A *complex orientation* of a complex stable theory E_A is a cohomology class $x(\varepsilon) \in E^*_A(\mathbb{C}P(\mathcal{U}, \mathbb{C}P(\varepsilon)))$ that restricts to a generator of

$$E^*_A(\mathbb{C}P(\alpha \oplus \varepsilon), \mathbb{C}P(\varepsilon)) \cong \widetilde{E}^*_A(S^{\alpha^{-1}}),$$

for all 1-dimensional representations α.

Building on the equivalence (3.6.2), we have the following claim.

Theorem 3.6.4. *Let E be an equivariant complex-oriented (nonparameterized) homology theory. Then, for any two spectral systems $\mathfrak{S}_1$, $\mathfrak{S}_2$, there is a canonical isomorphism*

$$\widetilde{E}^*(\nu_! \mathcal{SWF}(Y, \mathfrak{s}, \mathfrak{S}_1)) \to \widetilde{E}^*(\nu_! \Sigma^{\mathfrak{S}_2 - \mathfrak{S}_1} \mathcal{SWF}(Y, \mathfrak{s}, \mathfrak{S}_2)).$$

In particular, $FE^*(Y, \mathfrak{s}, \mathfrak{S}_1)$ is independent of $\mathfrak{S}_1$, and defines an invariant $FE^*(Y, \mathfrak{s})$.

Proof. The theorem is a consequence of the fact that, for an ex-space (X, r, s) over a base B, and a complex m-dimensional vector bundle V over B, with ν as usual the basepoint map $B \to *$,

$$\nu_! \Sigma^V_B X = \mathrm{Th}(r^* V). \tag{3.6.3}$$

This equality is a direct exercise in the definitions. In fact, if (X, r, s) is an S^1-ex-space, with base B on which S^1 acts trivially, the equality also holds at the level of S^1-spaces, where V is an S^1-equivariant vector bundle over B, inherited from its complex structure (so that the pullback r^*V is an S^1-equivariant vector bundle over the S^1-space X).

We have by (3.6.2),

$$\widetilde{E}^*(\nu_! \mathcal{SWF}(Y,\mathfrak{s},\mathfrak{S}_1)) = \widetilde{E}^*(\nu_! \Sigma^{\mathfrak{S}_1-\mathfrak{S}_2} \mathcal{SWF}(Y,\mathfrak{s},\mathfrak{S}_2)).$$

By (3.6.3),

$$\widetilde{E}^*(\nu_! \mathcal{SWF}(Y,\mathfrak{s},\mathfrak{S}_1)) = \widetilde{E}^*\big(\mathrm{Th}(r^*(\mathfrak{S}_1 - \mathfrak{S}_2))\big),$$

where r is the restriction map of the ex-space $\mathcal{SWF}(Y,\mathfrak{s},\mathfrak{S}_2)$. However, the complex orientation on E induces an isomorphism,

$$\widetilde{E}^*\big(\mathrm{Th}(r^*(\mathfrak{S}_1 - \mathfrak{S}_2))\big) \to \widetilde{E}^{*-2\dim(\mathfrak{S}_1-\mathfrak{S}_2)}(\mathcal{SWF}(Y,\mathfrak{s},\mathfrak{S}_2)),$$

which is exactly what we needed (the last isomorphism above, in the equivariant case, follows from the construction of Thom classes in [12, Theorem 6.3]).

The last claim of the theorem is then a consequence of the definition of FE^*. ∎

The most important equivariant complex orientable cohomology theory for us will be equivariant complex cobordism MU_G, defined by tom Dieck [50] for a compact Lie group G. It turns out, if G is abelian, that MU_G is the universal G-equivariant complex oriented cohomology theory, in the sense that any equivariant complex oriented cohomology theory E_G accepts a unique ring map of ring spectra $MU_G \to E_G$ so that the orientation on E_G is the image of the canonical orientation on MU_G. See [12].

We define $FMU^*(Y,\mathfrak{s})$ and $FMU^*_{S^1}(Y,\mathfrak{s})$ by

$$FMU^*(Y,\mathfrak{s}) = \widetilde{MU}^{*-2n(Y,\mathfrak{s},g,\mathfrak{S})}(\nu_! \mathcal{SWF}(Y,\mathfrak{s},\mathfrak{S})),$$
$$FMU^*_{S^1}(Y,\mathfrak{s}) = \widetilde{MU}^{*-2n(Y,\mathfrak{s},g,\mathfrak{S})}_{S^1}(\nu_! \mathcal{SWF}(Y,\mathfrak{s},\mathfrak{S})),$$

for some spectral sections $\mathfrak{S}$. By Theorem 3.6.4 and the complex orientation on MU and MU_{S^1}, these are well defined independent of a choice of $\mathfrak{S}$, and this proves Theorem 1.2.1.

For a spin structure $\mathfrak{s}$, we have the Pin(2)-equivariant Seiberg–Witten Floer stable homotopy type $\mathcal{SWF}^{\mathrm{Pin}(2)}(Y,\mathfrak{s},\mathfrak{S})$. To define Pin(2)-equivariant cohomology theory $FMU^*_{\mathrm{Pin}(2)}(Y,\mathfrak{s})$, we need to show that

$$\widetilde{MU}^{*-2n(Y,\mathfrak{s},\mathfrak{S})}_{\mathrm{Pin}(2)}(\nu_! \mathcal{SWF}^{\mathrm{Pin}(2)}(Y,\mathfrak{s},\mathfrak{S}))$$

is independent of the choice of $\mathfrak{S}$, which requires an orientation on $\widetilde{MU}^*_{\mathrm{Pin}(2)}$. But we cannot apply the argument in [12] to $\widetilde{MU}^*_{\mathrm{Pin}(2)}$ since Pin(2) is not abelian. We do not discuss orientations on $\widetilde{MU}^*_{\mathrm{Pin}(2)}$ in this memoir.