Chapter 4

Computation

In this chapter we provide a sample of calculations of the Seiberg–Witten Floer homotopy type.

4.1 Seiberg–Witten Floer homotopy type in reducible case

We will need the following lemma.

Lemma 4.1.1. *Let* $\varphi\colon M \times \mathbb{R} \to M$ *be a smooth flow on a smooth manifold* M *and* N *be a compact submanifold (with corners) of* M *with* $\dim M = \dim N$*. Assume that the following conditions are satisfied:*

(1) $\partial N = L_+ \cup L_-$*, where* L_+*,* L_- *are compact submanifolds (with corners) of* ∂N *with* $L_+ \cap L_- = \partial L_+ = \partial L_-$*.*

(2) *For* $x \in \operatorname{int}(L_+)$*, there is* $\varepsilon > 0$ *such that* $\varphi(x,t) \in \operatorname{int}(N)$ *for* $t \in (0,\varepsilon)$*.*

(3) *For* $x \in L_-$*, there is* $\varepsilon > 0$ *such that* $\varphi(x,t) \notin N$ *for* $t \in (0,\varepsilon)$*.*

Then N *is an isolating neighborhood and* (N, L_-) *is an index pair of* $\operatorname{inv}(N)$*. (See* [14] *for a similar statement.)*

Proof. By conditions (2) and (3), we have $\operatorname{inv}(N) \subset \operatorname{int}(N)$. It is easy to see that L_- is an exit set from the three conditions. Also, condition (3) implies that L_- is positively invariant in N. ■

Fix a spin^c 3-manifold $(Y, \mathfrak{s})$, along with a spectral system $\mathfrak{S}$, which we will usually suppress from the notation. Let $k_+, k_- > 5$ be half-integers with $|k_+ - k_-| \leq \frac{1}{2}$, $k = \min\{k_+, k_-\}$ and

$$\varphi_n = \varphi_{n,k_+,k_-}\colon (F_n \oplus W_n) \times \mathbb{R} \to F_n \oplus W_n$$

be the flow induced by the Seiberg–Witten equations.

Fix $R \gg 0$. Put

$$A_n(R) := (B_{k_+}(F_n^+; R) \times_B B_{k_-}(F_n^-; R)) \times_B (B_{k_+}(W_n^+; R) \times_B B_{k_-}(W_n^-; R)).$$

Let $I_n \to B = \operatorname{Pic}(Y)$ be the parameterized Conley index of $\operatorname{inv}(A_n(R), \varphi_n)$.

Theorem 4.1.2. *Assume that the following conditions are satisfied:*

(1) $\ker(D\colon \mathcal{E}_\infty \to \mathcal{E}_\infty) = 0$.

(2) *All solutions to the Seiberg–Witten equations* (2.3.4) *with finite energy are reducible.*

Let $\mathfrak{S}$ *be a spectral system such that* $P_0 = \mathcal{E}_0(D)^0_{-\infty}$. *Then for all* $n \gg 0$ *we have*

$$I_n \cong S_B^{F_n^- \oplus W_n^-},$$

as an S^1*-equivariant space, with the obvious projection to* B. *Hence the Seiberg–Witten Floer parameterized homotopy type is given by*

$$\mathcal{SWF}(Y, \mathfrak{s}, [\mathfrak{S}]) \cong \Sigma_B^{\mathbb{C}^{-D_n^2} \oplus \mathbb{R}^{-D_n^4}} I_n \cong S_B^0$$

in $PSW_{S^1,B}$. *Here,* $D_n^2 = \operatorname{rank} F_n$, $D_n^4 = \operatorname{rank} W_n^-$ *and* $PSW_{S^1,B}$ *is the category defined in Definition* A.1.9.

If the spin^c *structure is self-conjugate, the* Pin(2)*-Seiberg–Witten Floer parameterized homotopy type is given by*

$$\mathcal{SWF}^{\operatorname{Pin}(2)}(Y, \mathfrak{s}, [\mathfrak{S}]) \cong S_B^0$$

in $PSW_{\operatorname{Pin}(2),B}$.

To prove this, we need the following.

Proposition 4.1.3. *Assume that all solutions to* (2.3.4) *with finite energy are reducible. For any* $\varepsilon > 0$, *there is* n_0 *such that for* $n > n_0$ *we have*

$$\operatorname{inv}(A_n(R)) \subset A_n(\varepsilon).$$

Proof. Put

$$\delta_n := \max\left\{ \|\phi^+\|_{k_+} : (\phi, \omega) \in \operatorname{inv}(A_n(R)) \right\}.$$

Let

$$\gamma_n = (\phi_n, \omega_n) \colon \mathbb{R} \to A_n(R)$$

be approximate Seiberg–Witten trajectories with

$$\|\phi_n^+(0)\|_{k_+} = \delta_n.$$

Then we have

$$\left.\frac{d}{dt}\right|_{t=0} \|\phi_n^+(t)\|_{k_+}^2 = 0.$$

As we have seen before, after passing to a subsequence, γ_n converges to a Seiberg–Witten trajectory γ with finite energy. By assumption, γ is reducible and we can write $\gamma = (0, \omega)$. As in Lemma 2.7.2, we can show that there is a constant $C > 0$ such that $\|\phi_n^+(0)\|_{k_+ + \frac{1}{2}} < C$ for all n. By the Rellich lemma, $\phi_n^+(0)$ converges to 0 in L_k^2. Therefore $\delta_n \to 0$.

Similarly,

$$\max\left\{\|\phi^-\|_{k_-} : (\phi,\omega) \in \mathrm{inv}(A_n(R))\right\},$$
$$\max\left\{\|\omega^+\|_{k_+} : (\phi,\omega) \in \mathrm{inv}(A_n(R))\right\},$$
$$\max\left\{\|\omega^-\|_{k_-} : (\phi,\omega) \in \mathrm{inv}(A_n(R))\right\}$$

go to 0 as $n \to 0$. ■

Proof of Theorem 4.1.2. Fix a small positive number ε with $\varepsilon^2 \ll \varepsilon$ and choose $n \gg 0$. By the proposition,

$$\mathrm{inv}(A_n(R)) \subset A_n(\varepsilon).$$

Put

$$\begin{aligned}
L_{n,-}(\varepsilon) = {} & (B_{k_+}(F_n^+;\varepsilon) \times_B S_{k_-}(F_n^-;\varepsilon)) \times_B (B_{k_+}(W_n^+;\varepsilon) \times_B B_{k_-}(W_n^-;\varepsilon)) \\
& \bigcup (B_{k_+}(F_n^+;\varepsilon) \times_B B_{k_-}(F_n^-;\varepsilon)) \times_B (B_{k_+}(W_n^+;\varepsilon) \times_B S_{k_-}(W_n^-;\varepsilon)), \\
L_{n,+}(\varepsilon) = {} & (S_{k_+}(F_n^+;\varepsilon) \times_B B_{k_-}(F_n^-;\varepsilon)) \times_B (B_{k_+}(W_n^+;\varepsilon) \times_B B_{k_-}(W_n^-;\varepsilon)) \\
& \bigcup (B_{k_+}(F_n^+;\varepsilon) \times_B B_{k_-}(F_n^-;\varepsilon)) \times_B (S_{k_+}(W_n^+;\varepsilon) \times_B B_{k_-}(W_n^-;\varepsilon)).
\end{aligned}$$

Then we have

$$\begin{aligned}
&\partial A_n(\varepsilon) = L_{n,-}(\varepsilon) \cup L_{n,+}(\varepsilon), \\
&L_{n,-}(\varepsilon) \cap L_{n,+}(\varepsilon) = \partial L_{n,-}(\varepsilon) = \partial L_{n,+}(\varepsilon).
\end{aligned}$$

We will show that the pair $(A_n(\varepsilon), L_{n,-}(\varepsilon))$ is an index pair. It is enough to check that $A_n(\varepsilon)$, $L_{n,-}(\varepsilon)$, $L_{n,+}(\varepsilon)$ satisfy conditions (2), (3) in Lemma 4.1.1. We consider the case when $k_+ \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$.

Take an approximate Seiberg–Witten trajectory

$$\gamma = (\phi,\omega)\colon (-\delta,\delta) \to F_n \oplus W_n$$

for a small positive number δ.

Assume that

$$\|\phi^+(0)\|_{k_+} = \varepsilon.$$

We have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\Big|_{t=0} \|\phi^+(t)\|_{k_+} &= \frac{1}{2}\frac{d}{dt}\Big|_{t=0} \langle |D|^{k_+ + \frac{1}{2}} \pi^+ \phi(t), |D|^{k_+ - \frac{1}{2}} \pi^+ \phi(t)\rangle_0 \\
&= \langle (\nabla_{X_H} |D|^{k_+ + \frac{1}{2}}) \phi^+(0), |D|^{k_+ - \frac{1}{2}} \phi^+(0)\rangle_0 \\
&\quad + \langle |D|^{k_+ + \frac{1}{2}} \phi^+(0), (\nabla_{X_H} |D|^{k_+ - \frac{1}{2}}) \phi^+(0)\rangle_0 \\
&\quad + \langle (\nabla_{X_H} \pi^+) \phi(0), \phi^+(0)\rangle_{k_+} + \left\langle \frac{d\phi}{dt}(0), \phi^+(0)\right\rangle_{k_+}.
\end{aligned}$$

Note that

$$\|X_H(\phi)\| = \|q(\phi)_{\mathcal{H}}\| \leq C\varepsilon^2.$$

Hence we have

$$\begin{aligned}
\left|\langle(\nabla_{X_H}|D|^{k_+ + \frac{1}{2}})\phi^+(0), |D|^{k_+ - \frac{1}{2}}\phi^+(0)\rangle_0\right| &\leq C\varepsilon^4, \\
\left|\langle|D|^{k_+ + \frac{1}{2}}\phi^+(0), (\nabla_{X_H}|D|^{k_+ - \frac{1}{2}})\phi^+(0)\rangle_0\right| &\leq C\varepsilon^4, \\
\left|\langle(\nabla_{X_H}\pi^+)\phi(0), \phi^+(0)\rangle_{k_+}\right| &\leq C\varepsilon^4,
\end{aligned}$$

by Proposition 2.5.5 and Lemma 2.5.8. Recall that $\pi^+ = 1 - \pi_{P_0}$, where π_{P_0} is the L^2-projection onto P_0. We have

$$\begin{aligned}
\left\langle\frac{d\phi}{dt}(0), \phi^+(0)\right\rangle_{k_+} &= -\langle(\nabla_{X_H}\pi_{F_n})\phi(0), \phi^+(0)\rangle_{k_+} - \langle\pi_{F_n}D\phi(0), \phi^+(0)\rangle_{k_+} \\
&\quad - \langle\pi_{F_n}c_1(\gamma(0)), \phi^+(0)\rangle_{k_+}
\end{aligned}$$

and

$$\begin{aligned}
&\langle(\nabla_{X_H}\pi_{F_n})\phi(0), \phi^+(0)\rangle_{k_+} = 0, \\
&\langle\pi_{F_n}D\phi(0), \phi^+(0)\rangle_{k_+} = \langle D\phi(0), \phi^+(0)\rangle_{k_+} \geq C\varepsilon^2, \\
&|\langle\pi_{F_n}c_1(\gamma(0)), \phi^+(0)\rangle_{k_+}| \leq C\varepsilon^3.
\end{aligned}$$

Here we have used Lemma 2.5.11 for the first equality. Therefore

$$\frac{d}{dt}\Big|_{t=0}\|\phi^+(t)\|_{k_+}^2 \leq -C\varepsilon^2 + C\varepsilon^3 < 0.$$

Assume that

$$\|\phi^-(0)\|_{k_-} = \varepsilon.$$

A similar calculation shows that

$$\frac{d}{dt}\Big|_{t=0}\|\phi^-(t)\|_{k_-}^2 > 0.$$

Similarly, if $\|\omega^+(0)\|_{k_+} = \varepsilon$ then $\frac{d}{dt}\big|_{t=0}\|\omega^+(t)\|_{k_+}^2 < 0$, and if $\|\omega^-(0)\|_{k_-} = \varepsilon$ then $\frac{d}{dt}\big|_{t=0}\|\omega^-(t)\|_{k_-}^2 > 0$. From these, it is easy to see that conditions (2), (3) in Lemma 4.1.1 are satisfied and we can apply Lemma 4.1.1 to conclude that the pair $(A_n(\varepsilon), L_n(\varepsilon))$ is an index pair.

Therefore we have

$$I_n = A_n(\varepsilon) \cup_{p_B} L_{n,-}(\varepsilon) \cong S_B^{F_n^- \oplus W_n^-}. \qquad \blacksquare$$

4.2 Examples

Example 4.2.1. Suppose that Y has a positive scalar curvature metric. Then the conditions of Theorem 4.1.2 are satisfied.

Example 4.2.2. Let Y be a nontrivial flat torus bundle over S^1 which is not the Hantzsche–Wendt manifold. Then Y has a flat metric and $b_1(Y) = 1$. Take a torsion spin^c structure $\mathfrak{s}$ of Y. All solutions to the unperturbed Seiberg–Witten equations on Y are reducible solutions $(A, 0)$ with $F_A = 0$. Also, all finite energy solutions to the unperturbed Seiberg–Witten equations on $Y \times \mathbb{R}$ are the reducible solutions $(\pi_Y^* A, 0)$, where A are the flat spin^c connections on Y and $\pi_Y : Y \times \mathbb{R} \to Y$ is the projection. Hence condition (2) of Theorem 4.1.2 is satisfied.

By [28, Lemma 37.4.1], if $\mathfrak{s}$ is not the torsion spin^c structure corresponding to the 2-plane field tangent to the fibers, condition (1) of Theorem 4.1.2 is satisfied.

We consider the sphere bundle of a complex line bundle over a surface Σ. We will make use of results from [42, 44] and [24, Section 8].

Let Σ be a closed, oriented surface of genus g and $p: N_d \to \Sigma$ be the complex line bundle on Σ of degree d. We will consider the sphere bundle $Y = S(N_d)$. We have

$$H^2(Y; \mathbb{Z}) \cong \mathbb{Z}^{2g} \oplus (\mathbb{Z}/d\mathbb{Z}).$$

The direct summand $\mathbb{Z}/d\mathbb{Z}$ corresponds to the image

$$\mathrm{Pic}^t(\Sigma)/\mathbb{Z}[N_d] \xrightarrow{p^*} \mathrm{Pic}^t(Y) \xrightarrow{c_1} H^2(Y; \mathbb{Z}),$$

where $\mathrm{Pic}^t(\Sigma)$ is the set of isomorphism classes of topological complex line bundles on Σ.

Fix a torsion spin^c structure $\mathfrak{s}$. We consider a metric

$$g_{Y,r} = (r\eta)^{\otimes 2} \oplus g_\Sigma$$

on Y for $r > 0$. Here, $i\eta \in i\Omega^1(Y)$ is a constant-curvature connection 1-form of $S(N_d)$. Following [42, 44], we take the connection ∇^0 on TY which is trivial in the fiber direction and is equal to the pullback of the Levi-Civita connection on Σ on $\ker \eta$. For $a \in \mathcal{H}^1(Y)$, let $D_{r,a}$ be the Dirac operator induced by ∇^0. We have

$$D_{r,a} = D_a + \delta_r,$$

where $\delta_r = \frac{1}{2} r d$. See [42, Section 5.1] and [44, Section 2.1]. The family $\{D_{r,a}\}_{a \in \mathcal{H}^1(Y)}$ induces an operator

$$D_r : \mathcal{E}_\infty \to \mathcal{E}_\infty.$$

We consider the perturbed Seiberg–Witten equations for $\gamma = (\phi, \omega)\colon \mathbb{R} \to \mathcal{E}_\infty \times \operatorname{im} d^*$:

$$\begin{aligned} \Big(\frac{d\phi}{dt}(t)\Big)_H &= -D_r\phi(t) - c_1(\gamma(t)), \\ \Big(\frac{d\phi}{dt}(t)\Big)_V &= -X_H(\phi(t)), \\ \frac{d\omega}{dt}(t) &= -*d\omega(t) - c_2(\gamma(t)). \end{aligned} \tag{4.2.1}$$

These equations are the gradient flow equation of the perturbed Chern–Simons–Dirac functional

$$CSD_r(\phi, \omega) = CSD(\phi, \omega) + \delta_r \|\phi\|^2_{L^2}.$$

The term $\delta_r \|\phi\|^2_{L^2}$ is a tame perturbation. See [28, p. 171]. We can apply Theorem 2.3.3 to the perturbed Seiberg–Witten equations (4.2.1).

The following is a direct consequence of [42, Corollary 5.17 and Theorem 5.19]. See also [44, Section 3.2] and [24, Proposition 8.1, Section 8.2].

Proposition 4.2.3. *Let $\mathfrak{s}_0$ be the* spin^c *structure of Y with spinor bundle $\mathbb{S} = p^*K_\Sigma^{-1} \oplus \underline{\mathbb{C}}$. Denote by L_q the flat complex line bundle on Y with $c_1 \equiv q \bmod d$ in* Tor $H^2(Y;\mathbb{Z})$. *Put $\mathfrak{s}_q := \mathfrak{s}_0 \otimes L_q$. Assume that $0 < g < d$. Then for $q \in \{g, g+1, \dots, d-1\}$, all critical points of the functional CSD_r associated with $\mathfrak{s}_q$ are reducible and nondegenerate.*

Note that this proposition implies that $\ker D_r = 0$ and hence we have a natural spectral section P_0 of D_r:

$$P_0 = (\mathcal{E}_0(D_r))^0_{-\infty}.$$

The following proposition is proved in [24, proof of Theorem 7.5].

Proposition 4.2.4. *Under the same assumption as Proposition 4.2.3, any gradient trajectory of CSD_r (that is, a solution to (4.2.1)) with finite energy is reducible.*

We can apply the proof of Theorem 4.1.2 to the perturbed Seiberg–Witten equations (4.2.1) to show the following.

Theorem 4.2.5. *Take $q \in \{g, g+1, \dots, d-1\}$. Let $\mathfrak{S}$ be a spectral system with $P_0 = \mathcal{E}_0(D_r)^0_{-\infty}$. In the above notation, for r small, we have*

$$I_n \cong S_B^{F_n^- \oplus W_n^-}.$$

Therefore we have

$$\mathcal{SWF}(Y, \mathfrak{s}_q, [\mathfrak{S}]) \cong S^0_B$$

in $PSW_{S^1,B}$. If $\mathfrak{s}$ is self-conjugate,

$$\mathcal{SWF}^{\mathrm{Pin}(2)}(Y, \mathfrak{s}_q, [\mathfrak{S}]) \cong S^0_B$$

in $PSW_{\mathrm{Pin}(2),B}$.

Dai and the authors [17] computed the Seiberg–Witten Floer stable homotopy type for almost rational plumbed 3-manifolds which have $b_1 = 0$. The computation is based on surgery exact triangles in [48]. If we establish a surgery exact triangle for the Seiberg–Witten Floer stable homotopy type $\mathcal{SWF}(Y, \mathfrak{s}, \mathfrak{S})$ defined in this memoir, it would be possible to compute for more 3-manifolds with $b_1 > 0$.