Chapter 5

Finite-dimensional approximation on 4-manifolds

5.1 Construction of the relative Bauer–Furuta invariant

Let $(X, \mathfrak{t})$ be a compact spin^c 4-manifold with boundary Y. Take a Riemannian metric $\hat{g}$ of X such that a neighborhood of Y in X is isometric to $Y \times (-1, 0]$. We assume that the restriction $\mathfrak{s}$ of $\mathfrak{t}$ to Y is a torsion spin^c structure. Put

$$\mathcal{E}^{\pm}_{X,k} := \mathcal{H}^1(X) \times_{H^1(X;\mathbb{Z})} L^2_k(\Gamma(\mathbb{S}^{\pm})),$$
$$\mathcal{W}_{X,k} := B_X \times L^2_k(\Omega^1_{CC}(X)).$$

Here, $B_X = \mathrm{Pic}(X)$ and $\mathbb{S}^{\pm}$ are the spinor bundles on X and $\Omega^1_{CC}(X)$ is the space of 1-forms on X in double Coulomb gauge. See [23] for the double Coulomb gauge condition. Note that $\mathcal{E}^{\pm}_{X,k}$, $\mathcal{W}_{X,k}$ are Hilbert bundles over B_X. We have the Dirac operator

$$D_X : \mathcal{E}^{+}_{X,k} \to \mathcal{E}^{-}_{X,k-1}$$

on X, and as before, we can define the fiberwise norm $\|\cdot\|_k$ on $\mathcal{E}^{\pm}_{X,k}$ for each non-negative number k. Also we put

$$\mathcal{E}_{Y,k} := \mathcal{H}^1(Y) \times_{H^1(Y;\mathbb{Z})} L^2_k(\mathbb{S}),$$
$$\mathcal{W}_{Y,k} := B_Y \times L^2_k(\mathrm{im}\, d^*) \subset B_Y \times L^2_k(\Omega^1(Y)).$$

Here, $P_Y = \mathrm{Pic}(Y)$.

Proposition 5.1.1. *For $k, l \geq 0$, there are constants $R_{X,k}, R_{Y,l} > 0$ such that for any solution $x \in \mathcal{E}^{+}_{X,2} \oplus \mathcal{W}_{X,2}$ to the Seiberg–Witten equations on X and any Seiberg–Witten trajectory $\gamma : \mathbb{R}_{\geq 0} \to \mathcal{E}_{Y,2} \oplus \mathcal{W}_{Y,2}$ with finite energy and with*

$$r_Y(x) = \gamma(0),$$

we have

$$\|x\|_k \leq R_{X,k}, \quad \|\gamma(t)\|_l \leq R_{Y,l}$$

for all $t \in \mathbb{R}_{\geq 0}$. Here, r_Y stands for the restriction to the boundary Y.

See [23, Section 4] for this proposition.

Let D_Y be the family of Dirac operators on Y parameterized by B_Y. Assume that $\mathrm{ind}\, D_Y = 0$ in $K^1(B_Y)$. Choose a spectral system $\mathfrak{S}$. As usual, put

$$F_n = P_n \cap Q_n, \quad W_n = W_{P,n} \cap W_{Q,n}.$$

Then F_n, W_n are subbundles of $\mathcal{E}_{Y,0}$, $\mathcal{W}_{Y,0}$ with finite rank.

From now on, we assume that k is a half-integer and $k > 5$ so that we can use the results in Chapters 2 and 3. We consider the map

$$\begin{aligned} SW_{X,n}\colon & \mathcal{E}^+_{X,k} \oplus \mathcal{W}_{X,k} \\ & \to \left(\mathcal{E}^-_{X,k-1} \times L^2_{k-1}(\Omega^+(X))\right) \times ((P_n \oplus W_{P,n}) \cap L^2_{k-\frac{1}{2}}) \end{aligned} \tag{5.1.1}$$

defined by

$$SW_{X,n}(\hat{\phi}, \widehat{\omega}) = (D_X\hat{\phi} + \rho(\widehat{\omega})\hat{\phi}, F^+_{\hat{A}} - q(\hat{\phi}), \pi_{P_n} r_Y \hat{\phi}, \pi_{W_{P,n}} r_Y \widehat{\omega}).$$

Here, π_{P_n}, $\pi_{W_{P,n}}$ are the L^2-projection, where we have also written P_n for the total space of the spectral section P_n. We will take subbundles U_n, U'_n of $\mathcal{E}^+_{X,k}$, $\mathcal{E}^-_{X,k-1}$ with finite rank as follows. The operator

$$(D_X, \pi_{P_0} r_Y)\colon \mathcal{E}^+_{X,k} \to \mathcal{E}^-_{X,k-1} \oplus r^*_Y(P_0 \cap L^2_{k-\frac{1}{2}})$$

is Fredholm. (See [40], [28, Section 17.2] and Section 2.1.) Hence there is a fiberwise linear operator

$$\mathfrak{p}\colon \underline{\mathbb{C}}^m \to \mathcal{E}^-_{X,k-1} \oplus r^*_Y(P_0 \cap L^2_{k-\frac{1}{2}})$$

such that

$$(D_X, \pi_{P_0} r_Y) + \mathfrak{p} : \mathcal{E}^+_{X,k} \oplus \underline{\mathbb{C}}^m \to \mathcal{E}^-_{X,k-1} \oplus r^*_Y(P_0 \cap L^2_{k-\frac{1}{2}}) \tag{5.1.2}$$

is surjective. Here, $\underline{\mathbb{C}}^m = B_X \times \mathbb{C}^m$ is the trivial bundle over B_X.

Lemma 5.1.2. *For any n and any subbundle U' in $\mathcal{E}^-_{X,k-1}$, $U' \oplus r^*_Y F_n$ and the image of*

$$(D_X, \pi_{P_n} r_Y) + \mathfrak{p}\colon \mathcal{E}^+_{X,k} \oplus \underline{\mathbb{C}}^m \to \mathcal{E}^-_{X,k-1} \oplus r^*_Y(P_n \cap L^2_{k-\frac{1}{2}})$$

are transverse in $\mathcal{E}^-_{X,k-1} \oplus r^*_Y(P_n \cap L^2_{k-\frac{1}{2}})$.

Proof. Take any element (x', y) from $\mathcal{E}^-_{X,k-1} \oplus r^*_Y(P_n \cap L^2_{k-\frac{1}{2}})$. There is $(x, v) \in \mathcal{E}^+_{X,k} \oplus \underline{\mathbb{C}}^m$ such that

$$((D_X, \pi_{P_0} r_Y) + \mathfrak{p})(x, v) = (x', \pi_{P_0}(y)).$$

Note that

$$P_n \cap (P_0)^\perp = F^+_n.$$

We can write

$$(D_X, \pi_{P_n} r_Y + \mathfrak{p})(x, v) = \big((D_X, (\pi_{P_0} + \pi_{F^+_n}) r_Y) + \mathfrak{p}\big)(x, v) = (x', \pi_{P_0}(y) + z),$$

where $z = \pi_{F_n^+}(r_Y x) \in F_n^+ \subset F_n$. Hence

$$\begin{aligned}(x', y) &= (x', \pi_{P_0}(y) + z) + (0, \pi_{F_n^+}(y) - z)\\ &\in \operatorname{im}((D_X, \pi_{P_n} r_Y) + \mathfrak{p}) + F_n.\end{aligned}$$ ∎

Take a sequence of finite-dimensional subbundles U_n' of $\mathcal{E}_{X,k-1}^-$ such that $\pi_{U_n'} \to \mathrm{id}_{\mathcal{E}_{X,k-1}^-}$ strongly as $n \to \infty$ and put

$$U_n := ((D_X, \pi_{P_n} r_Y) + \mathfrak{p})^{-1}(U_n' \oplus r_Y^* F_n). \tag{5.1.3}$$

By Lemma 5.1.2, U_n are subbundles of $\mathcal{E}_{X,k}^+ \oplus \underline{\mathbb{C}}^m$. Note that

$$[U_n] - [U_n' \oplus r_Y^* F_n] - [\underline{\mathbb{C}}^m] = [\operatorname{ind}(D_X, P_n)] \in K(B_X).$$

Here, the right-hand side is the index bundle defined in [40, Section 6].

Choose finite-dimensional subbundles

$$V_n' = B_X \times V_{n,0}'$$

of $B_X \times L_{k-1}^2(\Omega^+(X))$ with $\pi_{V_n'} \to \mathrm{id}_{B_X \times L_{k-1}^2(\Omega^+(X))}$ strongly as $n \to \infty$ and put

$$V_n := (d^+, \pi_{W_{P,n}} r_Y)^{-1}(V_n' \oplus W_n) \subset \mathcal{W}_{X,k}.$$

We consider the maps

$$\begin{aligned} SW_{X,n,\mathfrak{p}} &:= (D_X, d^+) + \mathfrak{p} + \pi_{U_n' \oplus V_n'} c_X \colon U_n \oplus V_n \to U_n' \oplus V_n',\\ \widetilde{SW}_{X,n,\mathfrak{p}} &:= (SW_{X,n,\mathfrak{p}}, \pi_{P_n} r_Y, \pi_{W_{P,n}} r_Y, \mathrm{id}_{\underline{\mathbb{C}}^m})\colon \\ &\qquad U_n \oplus V_n \to U_n' \oplus V_n' \oplus r_Y^*(F_n \oplus W_n) \oplus \underline{\mathbb{C}}^m, \end{aligned} \tag{5.1.4}$$

where

$$c_X(\hat{\phi}, \hat{\omega}) = (\rho(\hat{\omega})\hat{\phi}, F_{\hat{A}_0}^+ + q(\hat{\phi}))$$

for a fixed connection $\hat{A}_0$ on X. Fix positive numbers R, R' with $0 \ll R' \ll R$. Put

$$A_n := (B_{k-\frac{1}{2}}(F_n^+; R) \times_{B_Y} B_k(F_n^-; R)) \times_{B_Y} (B_{k-\frac{1}{2}}(W_n^+; R) \times_{B_Y} B_k(W_n^-; R)).$$

Here, $B_{k-\frac{1}{2}}(F_n^+; R)$ is the ball in F_n^+ of radius R with respect to $L_{k-\frac{1}{2}}^2$, and similarly for $B_k(F_n^-; R)$, $B_{k-\frac{1}{2}}(W_n^+; R)$, $B_k(W_n^-; R)$. Note that we take different norms $L_{k-\frac{1}{2}}^2$ and L_k^2 for F_n^+, W_n^+ and F_n^-, W_n^-. By Theorem 2.3.3, for $n \gg 0$, A_n is an isolating neighborhood of the flow $\varphi_{n,k-\frac{1}{2},k}$, for suitable k. For $\varepsilon > 0$, we define compact subsets $K_{n,1}(\varepsilon)$, $K_{n,2}(\varepsilon)$ of A_n by

$$\begin{aligned} K_{n,1}(\varepsilon) := \big\{ y \in A_n : \exists(\hat{\phi}, v, \hat{\omega}) \in B_k(U_n \oplus V_n; R'),\ (\hat{\phi}, v) \in U_n \subset \mathcal{E}_{X,k}^+ \oplus \underline{\mathbb{C}}^m,\\ \hat{\omega} \in V_n,\ \|(SW_{X,n,\mathfrak{p}}, \mathrm{id}_{\mathbb{C}^m})(\hat{\phi}, v, \hat{\omega})\|_{k-1} \le \varepsilon,\\ y = \pi_{P_n \oplus W_{-\infty}^{\mu_n}} r_Y(\hat{\phi}, \hat{\omega}) \big\}, \end{aligned}$$

and

$$\begin{aligned} K_{n,2}(\varepsilon) := \big\{ y \in A_n : \exists(\hat{\phi}, v, \hat{\omega}) \in \partial B_k(U_n \oplus V_n; R'),\\ \|(SW_{X,n,\mathfrak{p}}, \mathrm{id}_{\mathbb{C}^m})(\hat{\phi}, v, \hat{\omega})\|_{k-1} \le \varepsilon,\\ y = \pi_{P_n \oplus W^{\mu_n}_{-\infty}} r_Y(\hat{\phi}, \hat{\omega})\big\}\\ \cup (\partial A_n \cap K_{n,1}(\varepsilon)). \end{aligned}$$

Here,

$$\|(SW_{X,n,\mathfrak{p}}, \mathrm{id}_{\mathbb{C}^m})(\hat{\phi}, v, \hat{\omega})\|_{k-1} = \|SW_{X,n,\mathfrak{p}}(\hat{\phi}, \hat{\omega})\|_{k-1} + \|v\|.$$

We will show that we can find a regular index pair containing $(K_{1,n}(\varepsilon), K_{2,n}(\varepsilon))$. See Appendix A.2 for the definition of a regular index pair.

Proposition 5.1.3. *There is an $\varepsilon_0 > 0$ such that if $0 < \varepsilon < \varepsilon_0$, for n large, we can find a regular index pair (N_n, L_n) of* $\mathrm{inv}(A_n; \varphi_{n,k-\frac{1}{2},k})$ *with*

$$K_{n,1}(\varepsilon) \subset N_n \subset A_n, \quad K_{n,2}(\varepsilon) \subset L_n.$$

Proof. We write φ_n for $\varphi_{n,k-\frac{1}{2},k}$. We denote by $A_n^{[0,\infty)}$ the set

$$\big\{ y \in A_n : \forall t \in [0,\infty),\ \varphi_n(y,t) \in A_n \big\}.$$

By [35, Theorem 4], it is sufficient to prove the following for n large and ε small:

(i) if $y \in K_{n,1}(\varepsilon) \cap A_n^{[0,\infty)}$ then we have $\varphi_n(y,t) \notin \partial A_n$ for all $t \in [0,\infty)$,

(ii) $K_{n,2}(\varepsilon) \cap A_n^{[0,\infty)} = \emptyset$.

Furthermore, any index pair as constructed by [35, Theorem 4] may be thickened to give a regular index pair still satisfying the conditions of the proposition. See [47, Remark 5.4].

Note that for $y \in K_{n,1}(\varepsilon)$ we have

$$\|y^+\|_{k-\frac{1}{2}} < R \tag{5.1.5}$$

for all n since the restriction $L^2_k(X) \to L^2_{k-\frac{1}{2}}(Y)$ is bounded and $R' \ll R$.

First, we will prove that (i) holds for n large and ε small. Assume that this is not true. Then there is a sequence $\varepsilon_n \to 0$ such that after passing to a subsequence, we have $y_n \in A_n$, $(\hat{\phi}_n, v_n, \hat{\omega}_n) \in B_k(U_n \oplus V_n; R')$, $t_n \in [0,\infty)$ with

$$\begin{aligned} &y_n = \pi_{P_n \oplus W_{P,n}} r_Y(\hat{\phi}_n, \hat{\omega}_n),\\ &\|SW_{X,n,\mathfrak{p}}(\hat{\phi}_n, \omega_n)\|^2_{k-1} + \|v_n\|^2 \le \varepsilon_n^2,\\ &\varphi_n(y_n, [0,\infty)) \subset A_n,\\ &\varphi_n(y_n, t_n) \in \partial A_n. \end{aligned}$$

Note that $v_n \to 0$. Let

$$\gamma_n = (\phi_n, \omega_n) \colon [0, \infty) \to F_n \oplus W_n$$

be the approximate Seiberg–Witten trajectory defined by

$$\gamma_n(t) = \varphi_n(y_n, t).$$

After passing to a subsequence, one of the following holds for all n:

(a) $\phi_n^+(t_n) \in S_{k-\frac{1}{2}}(F_n^+; R)$,
(b) $\phi_n^-(t_n) \in S_k(F_n^-; R)$,
(c) $\omega_n^+(t_n) \in S_{k-\frac{1}{2}}(W_n^+; R)$,
(d) $\omega_n^-(t_n) \in S_k(W_n^-; R)$.

Note that in cases (a) and (c), we have $t_n > 0$ because of (5.1.5).

As in the proof of Theorem 2.3.3, we can show that there is a Seiberg–Witten trajectory

$$\gamma = (\phi, \omega) \colon [0, \infty) \to \mathcal{E}_{Y,k-\frac{3}{2},k-1} \oplus \mathcal{W}_{Y,k-\frac{3}{2},k-1}$$

such that after passing to a subsequence, γ_n converges to γ uniformly in $L^2_{k-\frac{3}{2}}$ on each compact set in $[0, \infty)$. Also, after passing to a subsequence, $(\hat{\phi}_n, \widehat{\omega}_n)$ converges to a solution $(\hat{\phi}, \widehat{\omega})$ to the Seiberg–Witten equations on X uniformly in L^2_{k-1} on each compact set in the interior of X. We have

$$r_Y(\hat{\phi}, \widehat{\omega}) = \gamma(0).$$

Assume that case (a) happens for all n. As mentioned, $t_n > 0$. Hence we have

$$\frac{d}{dt}\bigg|_{t=t_n} \|\phi_n^+(t)\|^2_{k-\frac{1}{2}} = 0.$$

As in Lemma 2.7.2, we can show that there is $C > 0$ such that $\|\phi_n^+(t_n)\|_k < C$ for all n. After passing to a subsequence, $t_n \to t_\infty \in \mathbb{R}_{\geq 0}$ or $t_n \to \infty$. First assume that $t_n \to t_\infty$. By the Rellich lemma, $\phi_n^+(t_n)$ converges in $L^2_{k-\frac{1}{2}}$ strongly. This implies that

$$\|\phi^+(t_\infty)\|_{k-\frac{1}{2}} = R,$$

which contradicts Proposition 5.1.1.

Next we consider the case $t_n \to \infty$. Let

$$\underline{\gamma}_n = (\underline{\phi}_n, \underline{\omega}_n) \colon [-t_n, \infty) \to F_n \oplus W_n$$

be the approximate Seiberg–Witten trajectory defined by

$$\underline{\gamma}_n(t) := \varphi_n(y_n, t + t_n).$$

As before, we can show that there is a Seiberg–Witten trajectory

$$\underline{\gamma}\colon \mathbb{R} \to \mathcal{E}_{Y,k-\frac{3}{2},k-1} \oplus \mathcal{W}_{Y,k-\frac{3}{2},k-1}$$

such that after passing to a subsequence, $\underline{\gamma_n}$ converges to $\underline{\gamma}$ uniformly in $L^2_{k-\frac{3}{2}}$ on each compact set in $\mathbb{R}$. As before we can show that the sequence $\|\underline{\phi}_n^+(0)\|_k$ is bounded and hence $\underline{\phi}_n^+(0)$ converges to $\underline{\phi}^+(0)$ in $L^2_{k-\frac{1}{2}}$ strongly. Therefore $\|\underline{\phi}^+(0)\|_{k-\frac{1}{2}} = R$, which contradicts Proposition 2.3.2. Thus (a) cannot happen.

Let us consider the case when (b) holds for all n. We have

$$\frac{d}{dt}\bigg|_{t=t_n} \|\phi_n^-(t)\|_k^2 \le 0.$$

As in the proof of Lemma 2.7.3,

$$\begin{aligned} 0 &\ge \frac{d}{dt}\bigg|_{t=t_n} \|\phi_n^-(t)\|_k^2 \\ &\ge -\langle D'\phi_n^-(t_n), \phi_n^-(t_n)\rangle_k - CR^2\|\phi_n^-(t_n)\|_{k+\frac{1}{2}} - CR^2 \\ &= \|\phi_n^-(t_n)\|^2_{k+\frac{1}{2}} - CR^2\|\phi_n^-(t_n)\|_{k+\frac{1}{2}} - CR^2. \end{aligned}$$

This implies that the sequence $\|\phi_n^-(t_n)\|_{k+\frac{1}{2}}$ is bounded and there is a subsequence such that $\phi_n^-(t_n)$ converges in L^2_k strongly. We have a contradiction as before.

In the case when (c) or (d) holds for all n, we have a contradiction similarly. We have proved that (i) holds for n large and ε small.

Next we will prove that (ii) holds for n large and ε small. If this is not true, there is a sequence $\varepsilon_n \to 0$ such that after passing to a subsequence, one of the following cases holds for all n:

(a) We have $(\hat{\phi}_n, v_n, \hat{\omega}_n) \in \partial B_k(U_n \oplus V_n; R')$, $y_n \in A_n^{[0,\infty)}$ with

$$\|SW_{X,n,\mathfrak{p}}(\hat{\phi}_n, \hat{\omega}_n)\|_{k-1} + \|v_n\| \le \varepsilon_n, \quad y_n = \pi_{P_n \oplus W_{P,n}} r_Y(\hat{\phi}_n, \hat{\omega}_n).$$

(b) We have $(\hat{\phi}_n, v_n, \hat{\omega}_n) \in B_k(U_n \oplus V_n; R')$, $y_n \in \partial A_n \cap A_n^{[0,\infty)}$ with

$$\|SW_{X,n,\mathfrak{p}}(\hat{\phi}_n, \hat{\omega}_n)\|_{k-1} + \|v_n\| \le \varepsilon_n, \quad y_n = \pi_{P_n \oplus W_{P,n}} r_Y(\hat{\phi}_n, \hat{\omega}_n).$$

First we consider the case (a). Let

$$\gamma_n = (\phi_n, \omega_n)\colon [0, \infty) \to F_n \oplus W_n$$

be the approximate Seiberg–Witten trajectory defined by

$$\gamma_n(t) = \varphi_n(y_n, t).$$

As before, there is a Seiberg–Witten trajectory

$$\gamma = (\phi, \omega)\colon [0, \infty) \to \mathcal{E}_{Y,k-\frac{3}{2},k-1} \oplus \mathcal{W}_{Y,k-\frac{3}{2},k-1}$$

such that after passing to a subsequence, γ_n converges to γ uniformly in $L^2_{k-\frac{3}{2}}$ on each compact set in $[0, \infty)$. Also, there is a solution $(\hat{\phi}, \widehat{\omega})$ to the Seiberg–Witten equations on X such that after passing to a subsequence, $(\hat{\phi}_n, \widehat{\omega}_n)$ converges to $(\hat{\phi}, \widehat{\omega})$ in L^2_{k-1} on each compact set in the interior of X. We have

$$r_Y(\hat{\phi}, \widehat{\omega}) = (\phi(0), \omega(0)).$$

Since $y_n \in A_n$, we have

$$\|y_n^-\|_k = \|(\phi_n^-(0), \omega_n^-(0))\|_k \leq R.$$

Hence, after passing to subsequence, $(\phi_n^-(0), \omega_n^-(0))$ converges to $(\phi^-(0), \omega^-(0))$ in $L^2_{k-\frac{1}{2}}(Y)$ strongly. By the standard elliptic estimate, we have

$$\begin{aligned}&\|\hat{\phi}_n - \hat{\phi}\|_{L^2_k(X)} \\ &\quad \leq C\big(\|\hat{\phi}_n - \hat{\phi}\|_{L^2(X)} + \|D_X(\hat{\phi}_n - \hat{\phi})\|_{L^2_{k-1}(X)} + \|\phi_n^-(0) - \phi^-(0)\|_{L^2_{k-\frac{1}{2}}(Y)}\big).\end{aligned}$$

From the condition that

$$\|SW_{X,n,\mathfrak{p}}(\hat{\phi}_n, \widehat{\omega}_n)\|_{k-1} + \|v_n\| \leq \varepsilon_n,$$

we have

$$\|D_X(\hat{\phi}_n - \hat{\phi})\|_{k-1} \leq C(\|c_X(\hat{\phi}_n, \widehat{\omega}_n) - c_X(\hat{\phi}, \widehat{\omega})\|_{k-1} + \varepsilon_n).$$

Since $c_X(\hat{\phi}_n, \widehat{\omega}_n)$ converges to $c_X(\hat{\phi}, \widehat{\omega})$ in L^2_{k-1} strongly, $\hat{\phi}_n$ converges to $\hat{\phi}$ in L^2_k strongly.

Similarly, $\widehat{\omega}_n$ converges to $\widehat{\omega}$ in L^2_k strongly. Hence,

$$\|(\hat{\phi}, \widehat{\omega})\|_k = R'.$$

This contradicts Proposition 5.1.1, so case (a) cannot happen.

Next we consider case (b). Let

$$y_n = (\phi_n, \omega_n).$$

After passing to a subsequence, $\phi_n^- \in S_k(F_n^-; R)$ for all n, or $\omega_n^- \in S_k(W_n^-; R)$ for all n. Note that the cases $\phi_n^+ \in S_{k-\frac{1}{2}}(F_n^+; R)$, $\omega_n^+ \in S_{k-\frac{1}{2}}(W_n^+; R)$ do not happen because of (5.1.5).

We consider the case $\phi_n^- \in S_k(F_n^-; R)$. Put

$$\gamma_n(t) = (\phi_n(t), \omega_n(t)) = \varphi_n(y_n, t)$$

for $t \geq 0$. As in the proof of Lemma 2.7.3,

$$\begin{aligned} 0 &\geq \frac{d}{dt}\Big|_{t=0} \|\phi_n^-(t)\|_k^2 \\ &\geq \|\phi_n^-\|_{k+\frac{1}{2}}^2 - CR^2\|\phi_n^-\|_{k+\frac{1}{2}} - CR^2. \end{aligned}$$

Therefore the sequence $\|\phi_n^-\|_{k+\frac{1}{2}}$ is bounded. By the Rellich lemma, ϕ_n^- converges to ϕ^- in L_k^2 strongly and hence

$$\|\phi^-\|_k = R,$$

which contradicts Proposition 5.1.1. Similarly, if $\omega_n^- \in S_k(W_n^-; R)$ for all n, we obtain a contradiction. We have proved that (ii) holds for n large and ε small. ■

Remark 5.1.4. To get (5.1.5), we used the $L_{k-\frac{1}{2}}^2$-norm on the positive component. On the other hand, in the case (ii)-(a), we used the condition that $\|\phi_n^-(0)\|_k$ is bounded (rather than $\|\phi_n^-(0)\|_{k-\frac{1}{2}}$) to have that $\phi_n^-(0)$ converges to $\phi^-(0)$ in $L_{k-\frac{1}{2}}^2$. This is why we used the L_k^2-norm on the negative component to define $K_{n,1}(\varepsilon)$, $K_{n,2}(\varepsilon)$.

In the case where $b_1(Y) = 0$, we can use the $L_{k-\frac{1}{2}}^2$-norm on both of the positive and negative component. See the proofs of [35, Proposition 6] and [23, Lemma 4.4]. In those proofs, to get the $L_{k-\frac{1}{2}}^2$-convergence of $\phi_n^-(0)$, the following identity was used:

$$e^D\phi_n^-(1) - \phi_n^-(0) = \int_0^1 \frac{d}{dt}(e^{tD}\pi^-\phi_n(t))\,dt. \tag{5.1.6}$$

In the case where $b_1(Y) > 0$, we have

$$\begin{aligned} \frac{d}{dt}(e^{tD}\pi^-\phi_n(t)) &= e^{tD}(D + \nabla_{X_H} D)\pi^-\phi_n(t) + e^{tD}(\nabla_{X_H}\pi^-)\phi_n(t) \\ &\quad - e^{tD}\pi^-\{(\pi_n D + \nabla_{X_H}\pi_{F_n})\phi_n(t) + q(\phi_n(t))\}. \end{aligned}$$

Since $(\nabla_{X_H}\pi_{F_n})\phi_n(t)$ does not converge in $L_{k-\frac{1}{2}}^2$, we cannot deduce that $\phi_n^-(0)$ converges in $L_{k-\frac{1}{2}}^2$ from (5.1.6).

For n large and ε small, let (N_n, L_n) be a regular index pair of $\mathrm{inv}(\varphi_n, A_n)$ with

$$K_{1,n}(\varepsilon) \subset N_n, \quad K_{2,n}(\varepsilon) \subset L_n.$$

Put

$$
\begin{aligned}
S_{B_X}^{U_n \oplus V_n} &:= \bigcup_{a \in B_X} B((U_n \oplus V_n)_a; R)/S((U_n \oplus V_n)_a; R), \\
S_{B_X}^{U_n' \oplus V_n' \oplus \underline{\mathbb{C}}^m} &:= \bigcup_{a \in B_X} B((U_n' \oplus V_n' \oplus \underline{\mathbb{C}}^m)_a; \varepsilon)/S((U_n' \oplus V_n' \oplus \underline{\mathbb{C}}^m)_a; \varepsilon),
\end{aligned}
$$

which are sphere bundles over B_X, and let I_n be the Conley index:

$$
I_n := N_n \cup_{p_{B_Y}|_{L_n}} B_Y.
$$

Here, $p_{B_Y}\colon N_n \to B_Y$ is the projection. We obtain a map

$$
\mathcal{BF}_{[n]}(X, \mathfrak{t})\colon S_{B_X}^{U_n \oplus V_n} \to S_{B_X}^{U_n' \oplus V_n' \oplus \underline{\mathbb{C}}^m} \wedge_{B_X} r_Y^* I_n \tag{5.1.7}
$$

defined by

$$
\begin{aligned}
&\mathcal{BF}_{[n]}(X, \mathfrak{t})([\hat{\phi}, v, \hat{\omega}]) \\
&\quad = \begin{cases} [SW_{X,n,\mathfrak{p}}(\hat{\phi}, v, \hat{\omega}), v] \wedge [\pi_{P_n \oplus W_{P,n}} r_Y(\hat{\phi}, \hat{\omega})] & \text{if (5.1.8) holds,} \\ *_a & \text{otherwise.} \end{cases}
\end{aligned}
$$

Here, $a = p_{B_X}(\hat{\phi}, \hat{\omega})$, $*_a$ denotes the base point of the sphere $S^{(U_n' \oplus V_n' \oplus \underline{\mathbb{C}}^m)_a}$ and we have the following condition:

$$
\begin{gathered}
\|SW_{X,n,\mathfrak{p}}(\hat{\phi}, v, \hat{\omega})\|_{k-1}^2 + \|v\|^2 \leq \varepsilon, \\
\pi_{P_n \oplus W_{P,n}} r_Y(\hat{\phi}, \hat{\omega}) \in K_{n,1}(\varepsilon).
\end{gathered} \tag{5.1.8}
$$

We refer to the map $\mathcal{BF}_n(X, \mathfrak{t})$ as the (relative, nth) *pre-Bauer–Furuta invariant* of $(X, \mathfrak{t})$, to emphasize that it is not yet an invariant of the construction (rather, its stable homotopy equivalence class will turn out to be an invariant).

An alternative version of this relative Bauer–Furuta invariant is obtained instead by considering the map of B_Y spaces:

$$
\mathcal{BF}_{[n]}(X, \mathfrak{t})\colon S_{B_X}^{U_n \oplus V_n} \to S_{B_X}^{U_n' \oplus V_n' \oplus \underline{\mathbb{C}}^m} \wedge_{B_Y} N_n/_{B_Y} L_n,
$$

where $S_{B_X}^{U_n \oplus V_n}$ is a B_Y space using r_Y, and where $N_n/_B L_n$ is the fiberwise quotient.

5.2 Well-definedness of the relative Bauer–Furuta invariant

We next consider how the construction of the relative Bauer–Furuta invariant in (5.1.7) depends on the choices involved. This is very similar to Chapter 3, so we will abbreviate many of the arguments.

First, we address the perturbation $\mathfrak{p}$.

Lemma 5.2.1. *Let $\mathfrak{p}_1$ be a perturbation for which (5.1.2) is surjective. Let $\mathfrak{q}$ be a linear operator $\mathbb{C}^{m_2} \to \mathcal{E}^-_{X,k-1} \oplus r_Y^*(P_0 \cap L^2_{k-\frac{1}{2}})$. Let $U_n(\mathfrak{p})$, respectively $U_n(\mathfrak{p}+\mathfrak{q})$ be the bundles defined as in (5.1.3) with respect to the perturbations $\mathfrak{p}$, respectively $\mathfrak{p}+\mathfrak{q}$. Let $\mathcal{BF}_{[n],\mathfrak{p}}(X,\mathfrak{t})$, respectively $\mathcal{BF}_{[n],\mathfrak{p}+\mathfrak{q}}(X,\mathfrak{t})$, be the maps defined in (5.1.7) with respect to the perturbations $\mathfrak{p}$ and $\mathfrak{p}+\mathfrak{q}$. Then there is the following commutative diagram:*

$$\begin{tikzcd}
\Sigma^{\mathbb{C}^{m_2}} S_{B_X}^{U_n(\mathfrak{p})\oplus V_n} \arrow[r, "\Sigma^{\mathbb{C}^{m_2}} \mathcal{BF}_{[n],\mathfrak{p}}"] \arrow[d] & S_{B_X}^{U_n'\oplus V_n'\oplus\mathbb{C}^m\oplus\mathbb{C}^{m_2}} \wedge_{B_Y} I_n \arrow[d] \\
S_{B_X}^{U_n(\mathfrak{p}+\mathfrak{q})\oplus V_n} \arrow[r, "\mathcal{BF}_{[n],\mathfrak{p}+\mathfrak{q}}"'] & S_{B_X}^{U_n'\oplus V_n'\oplus\mathbb{C}^m\oplus\mathbb{C}^{m_2}} \wedge_{B_Y} I_n.
\end{tikzcd}$$

Moreover, a choice of map $L\colon \mathbb{C}^{m_2} \to \mathcal{E}^+_{X,k} \oplus \mathbb{C}^m$ so that $((D_X, \pi_{P_0} r_Y) + \mathfrak{p}) \circ L = \mathfrak{q}$ determines the vertical arrows in the diagram.

Proof. Such a choice of L as at the end of the statement exists for any such $\mathfrak{p}$, $\mathfrak{q}$, by surjectivity of (5.1.2). We show how to define maps as in the commutative diagram in terms of such L. Of course, if $\mathfrak{q} = 0$, this is obvious, with $L = 0$.

More generally, we have the following commutative diagram:

$$\begin{tikzcd}
\mathcal{E}^+_{X,k} \oplus \mathbb{C}^m \oplus \mathbb{C}^{m_2} \arrow[r] \arrow[d, "\tilde{L}"'] & \mathcal{E}^-_{X,k-1} \oplus r_Y^*(P_0 \cap L^2_{k-\frac{1}{2}}) \arrow[d, "\mathrm{id}"] \\
\mathcal{E}^+_{X,k} \oplus \mathbb{C}^m \oplus \mathbb{C}^{m_2} \arrow[r] & \mathcal{E}^-_{X,k-1} \oplus r_Y^*(P_0 \cap L^2_{k-\frac{1}{2}}),
\end{tikzcd} \tag{5.2.1}$$

where $\tilde{L}$ is the identity on $\mathcal{E}^+_{X,k} \oplus \mathbb{C}^m$, and $L \oplus \mathrm{id}_{\mathbb{C}^{m_2}}$ on $\mathbb{C}^{m_2}$. The horizontal arrows are $(D_X, \pi_{P_0} r_Y) \oplus \mathfrak{p} \oplus 0$ and $(D_X, \pi_{P_0} r_Y) \oplus \mathfrak{p} \oplus \mathfrak{q}$, respectively.

Comparing with the definition of the Seiberg–Witten map (5.1.1), we see that there is a commutative diagram analogous to (5.2.1), but with the maps $\widetilde{SW}_{X,n,\mathfrak{p}}$ (and similarly for $\mathfrak{q}$) from (5.1.4) along the horizontal arrows.

The definition of $\mathcal{BF}_{[n]}(X,\mathfrak{t})$ then gives the commutative diagram in the lemma statement. ∎

As in Chapter 3, the proof of well-definedness is related to the definition of a families invariant. Let $\mathcal{F}$ be a family of (metrized, spin^c) 4-manifolds with boundary, over a base B, with fiber $(X,\mathfrak{t})$, and let $\mathcal{G}$ be the boundary family (naturally over the base B), where we write $\partial(X,\mathfrak{t}) = (Y,\mathfrak{s})$. See Section 3.2 for family of spin^c manifolds. Assume that we have fixed a sequence of good spectral sections P_n, Q_n on the boundary family.

Assume also that we have fixed a sequence of good spectral sections $W_{P,n}$, $W_{Q,n}$ of $*d$ of the boundary family, and assume $W_{P,0}$ is the orthogonal complement of $W_{Q,0}$.

As at the beginning of the section, we now have bundles $\mathcal{E}^{\pm}_{\mathcal{F},k}$ and $\mathcal{W}_{\mathcal{F},k}$, where the fibers over $b \in B$ (with associated 4-manifold $(X, \mathfrak{t})$) are

$$\begin{aligned}\mathcal{E}^{\pm}_{\mathcal{F},k,b} &:= \mathcal{H}^1(\mathcal{F}_b) \times_{H^1(X;\mathbb{Z})} L^2_k(\Gamma(\mathbb{S}^{\pm}_b)),\\ \mathcal{W}_{\mathcal{F},k,b} &:= \mathrm{Pic}(\mathcal{F}_b) \times L^2_k(\Omega^1_{CC}(\mathcal{F}_b)).\end{aligned}$$

Furthermore, the space of sections $L^2_{k-1}(\Omega^+(\mathcal{F}))$ now defines a bundle over B as well, with fiber $L^2_{k-1}(\Omega^+(\mathcal{F}_b))$, the L^2_{k-1}-self-dual 2-forms on the fiber.

The 4-dimensional Seiberg–Witten equations (5.1.1) now define a fiberwise map:

$$SW_{\mathcal{F},n} : \mathcal{E}^+_{\mathcal{F},k} \oplus \mathcal{W}_{\mathcal{F},k} \to \big(\mathcal{E}^-_{\mathcal{F},k-1} \oplus L^2_{k-1}(\Omega^+(\mathcal{F}))\big) \oplus r^*_{\mathcal{G}}(P_n \oplus W_{P,n}).$$

Define U_n as in (5.1.3), and V_n similarly. Exactly as before, define A_n; note that A_n is now a fiber bundle over the total space of the fibration $\mathrm{Pic}(\mathcal{F}) \to B$, a fiber of this latter fibration is $\mathrm{Pic}(\mathcal{F}_b)$. Define subspaces (themselves spaces over the total space of $\mathrm{Pic}(\mathcal{G}) \to B$) $K_{n,1}(\varepsilon)$ and $K_{n,2}(\varepsilon)$ with fibers $K_{n,1,b}(\varepsilon)$ and $K_{n,2,b}(\varepsilon)$ according to

$$\begin{aligned}K_{n,1,b}(\varepsilon) := \big\{y \in A_n :\ &\exists(\hat{\phi}, v, \hat{\omega}) \in B_k(U_n \oplus V_n; R'),\ (\hat{\phi}, v) \in U_n \subset \mathcal{E}^+_{X,k} \oplus \underline{\mathbb{C}}^m,\\ &\hat{\omega} \in V_n,\ \|(SW_{X,n,\mathfrak{p},b}, \mathrm{id}_{\mathbb{C}^m})(\hat{\phi}, v, \hat{\omega})\|_{k-1} \le \varepsilon,\\ &y = \pi_{P_n \oplus W_{P,n}} r_{\mathcal{G}_b}(\hat{\phi}, \hat{\omega})\big\}\end{aligned}$$

and

$$\begin{aligned}K_{n,2,b}(\varepsilon) := \big\{y \in A_n :\ &\exists(\hat{\phi}, v, \hat{\omega}) \in \partial B_k(U_n \oplus V_n; R'),\\ &\|(SW_{X,n,\mathfrak{p},b}, \mathrm{id}_{\underline{\mathbb{C}}^m})(\hat{x}, v, \hat{\omega})\|_{k-1} \le \varepsilon,\\ &y = \pi_{P_n \oplus W_{P,n}} r_{\mathcal{G}_b}(\hat{\phi}, \hat{\omega})\big\}\\ \cup(\partial A_n \cap K_{n,1,b}(\varepsilon)).&\end{aligned}$$

The proof of Proposition 5.1.3 is only changed in this setting according to the procedure in Chapter 3. In particular, the following proposition also relies on a families version of [35, Theorem 4]; the proof thereof is only notationally different from that appearing in [35]. A families version of Proposition 5.1.1 is also used; its proof is a modification of that in [23, Section 4]. We obtain the following proposition.

Proposition 5.2.2. *There is an $\varepsilon_0 > 0$ such that if $0 < \varepsilon < \varepsilon_0$, for n large, we can find a regular fiberwise index pair (N_n, L_n) of* $\mathrm{inv}(A_n; \varphi_{n,k,k-\frac{1}{2}})$ *with*

$$K_{n,1}(\varepsilon) \subset N_n \subset A_n, \quad K_{n,2}(\varepsilon) \subset L_n.$$

Put

$$S^{U_n\oplus V_n}_{\mathrm{Pic}(\mathcal{F})} := \bigcup_{a\in\mathrm{Pic}(\mathcal{F})} B((U_n\oplus V_n)_a; R)/S((U_n\oplus V_n)_a; R),$$
$$S^{U_n'\oplus V_n'\oplus\underline{\mathbb{C}}^m}_{\mathrm{Pic}(\mathcal{F})} := \bigcup_{a\in\mathrm{Pic}(\mathcal{F})} B((U_n'\oplus V_n'\oplus\underline{\mathbb{C}}^m)_a; \varepsilon)/S((U_n'\oplus V_n'\oplus\underline{\mathbb{C}}^m)_a; \varepsilon).$$

Let

$$I_n(\mathcal{G}) = N_n \cup_{p_{\mathrm{Pic}(\mathcal{G})}|_{L_n}} \mathrm{Pic}(\mathcal{G}),$$

where $p_{\mathrm{Pic}(\mathcal{G})}$ is the projection to $\mathrm{Pic}(\mathcal{G})$ of $F\times W$.

We obtain a fiber-preserving map over $\mathrm{Pic}(\mathcal{G})$:

$$\mathcal{BF}_{[n]}(\mathcal{F})\colon S^{U_n\oplus V_n}_{\mathrm{Pic}(\mathcal{G})} \to S^{U_n'\oplus V_n'\oplus\underline{\mathbb{C}}^m}_{\mathrm{Pic}(\mathcal{G})} \wedge_{\mathrm{Pic}(\mathcal{G})} I_n(\mathcal{G}).$$

Here, $S^{U_n\oplus V_n}_{\mathrm{Pic}(\mathcal{G})}$ and $S^{U_n'\oplus V_n'\oplus\underline{\mathbb{C}}^m}_{\mathrm{Pic}(\mathcal{G})}$ are spaces over $\mathrm{Pic}(\mathcal{G})$ by pushing forward $S^{U_n\oplus V_n}_{\mathrm{Pic}(\mathcal{F})}$ and $S^{U_n'\oplus V_n'\oplus\underline{\mathbb{C}}^m}_{\mathrm{Pic}(\mathcal{F})}$ along the restriction map $\mathrm{Pic}(\mathcal{F})\to\mathrm{Pic}(\mathcal{G})$ (see Appendix A.1).

In particular, we obtain that the homotopy class of the map $\mathcal{BF}_{[n]}(X,\mathfrak{t})$ in (5.1.7) is independent of the metric on X used in its construction. To be more precise, we have the following lemma.

Lemma 5.2.3. *Let $(X,\mathfrak{t})$ be a compact* spin^c 4*-manifold with boundary (admitting a Floer framing) $(Y,\mathfrak{s})$. Let g_t for $t\in[0,1]$ be a path of metrics on X, along with a path of perturbations $\mathfrak{p}_t$ with surjectivity in (5.1.2) for all t. There exist good spectral sections $P_{n,t}$, $Q_{n,t}$, $W_{P,n,t}$, $W_{Q,n,t}$ on the boundary Y, say, forming a spectral system $\mathfrak{S}$. Let $I_n = \mathcal{SWF}_{[n]}(Y,\mathfrak{s},\mathfrak{S})$ denote the family Seiberg–Witten invariant of the boundary. Let p denote the projection $p\colon B_Y\times I\to B_Y$, where $I=[0,1]$. Then there exists a map*

$$\mathcal{BF}_{[n],I}(X,\mathfrak{t})\colon S^{U_n\oplus V_n}_{B_X\times I} \to S^{U_n'\oplus V_n'\oplus\underline{\mathbb{C}}^m}_{B_X\times I} \wedge_{B_Y\times I} p^* I_n.$$

The map $\mathcal{BF}_{[n],I}(X,\mathfrak{t})$ is a map respecting the projection on each side to $B_Y\times I$.

In particular, for a fixed trivialization *of the families $U_{n,t}$, $V_{n,t}$, $U'_{n,t}$, $V'_{n,t}$ and I_n over I_+, together with a path of perturbations $\mathfrak{p}_t$, there is an (equivariant) homotopy equivalence from $\mathcal{BF}_{[n],0,\mathfrak{p}_0}$ and $\mathcal{BF}_{[n],1,\mathfrak{p}_1}$ which is well defined up to (equivariant) homotopy.*

Proof. The existence of the spectral sections follows from Chapter 2. Otherwise the lemma is a restatement of the definition of the families relative Bauer–Furuta invariant. There is no issue in choosing a good spectral section for $*d$ of the boundary family in this situation, since on $[0,1]$, each $*d$ may be written as a (small) compact perturbation of $*_g d$, where g is some fixed metric. ∎

Further, the homotopy class of $\mathcal{BF}_{[n]}(X,\mathfrak{t})$ does not depend on the Sobolev norm used in its construction. The proof of the following lemma is analogous to the work in Section 3.4, and is left to the reader. We state the result for the unparameterized case; the parameterized case is not substantially different.

Lemma 5.2.4. *Let $(X,\mathfrak{t})$ be a compact* spin^c 4*-manifold with boundary (admitting a Floer framing) $(Y,\mathfrak{s})$. Let U'_n be a sequence of finite-dimensional subbundles of $\mathcal{E}^-_{X,k}$ for $k > 11/2$, and $V'_n = B_X \times V'_{n,0}$ be a sequence of finite-dimensional subbundles of $B_X \times L^2_k(\Omega^+(X))$, where $V'_{n,0} \subset L^2_k(\Omega^+(X))$, with $\pi_{U'_n} \to \mathrm{id}_{\mathcal{E}^-_{X,k}}$ and $\pi_{V'_n} \to \mathrm{id}_{B_X\times L^2_k(\Omega^+(X))}$ strongly. Let $\mathcal{BF}_{[n],k+1}(X)$ and $\mathcal{BF}_{[n],k}(X)$ be the pre-Bauer–Furuta invariants defined with respect to the L^2_{k+1} and L^2_k-norms respectively. Write I for the interval* $[0,1]$. *Then there is a family of maps over the interval,*

$$\mathcal{BF}_{[n],I}(X,\mathfrak{t})\colon S^{U_n\oplus V_n}_{B_X\times I} \to S^{U'_n\oplus V'_n\oplus\underline{\mathbb{C}}^m}_{B_X\times I} \wedge_{B_Y\times I} \mathcal{SWF}_{[n]}(Y)_I,$$

where $\mathcal{SWF}_{[n]}(Y)_I$ is the parameterized Conley index coming from the I-family of flows used in the proof of Proposition 3.4.1. In particular, for the given homotopy equivalence in Proposition 3.4.1, the maps $\mathcal{BF}_{[n],k}(X,\mathfrak{t})$ and $\mathcal{BF}_{[n],k+1}(X,\mathfrak{t})$ are homotopic by a homotopy well defined up to homotopy.

We next consider the effect of stabilization on $\mathcal{BF}_{[n]}$. There are two separate stabilizations: increasing U'_n, V'_n, or increasing P_n, Q_n, $W^{\pm}_n$. Fix trivializations of $U'_{n+1}/U'_n = \underline{\mathbb{C}}^{c_n}$ and $V'_{n+1}/V'_n = \underline{\mathbb{R}}^{d_n}$. Recall the definition of a *spectral system* from Definition 3.5.1. By construction, U_{n+1} is naturally identified with $U_n \oplus \underline{\mathbb{C}}^{k_{Q_n}+c_n}$ for $k_{P,n}$, $k_{Q,n}$ as in Theorem 3.1.1, using the isomorphism $\eta\colon P_{n+1} \to P_n \oplus \underline{\mathbb{C}}^{k_{P,n}}$, and similarly for $k_{Q,n}$. Analogously, V_{n+1} is identified with $V_n \oplus \underline{\mathbb{R}}^{k_{W,-,n}+d_n}$. Let $\varphi_{n+1,t}$ denote the family of flows as in Theorem 3.1.1, with n chosen large enough. Recall that there is an induced homotopy equivalence

$$\Sigma^{\underline{\mathbb{C}}^{k_{Q,n}}\oplus\underline{\mathbb{R}}^{k_{W,-,n}}}_{B_Y} \mathcal{SWF}_{[n]}(Y) \to \mathcal{SWF}_{[n+1]}(Y)$$

as in Theorem 3.1.1.

Stabilization of the Bauer–Furuta invariant is as follows. Let $c'_n = c_n + k_{Q,n}$ and $d'_n = d_n + k_{W,-,n}$.

Proposition 5.2.5. *For appropriate choices of index pairs, there is a homotopy-commuting square of parameterized spaces, defined by Conley index continuation maps:*

$$\begin{tikzcd}
S^{\underline{\mathbb{C}}^{c'_n}\oplus\underline{\mathbb{R}}^{d'_n}}_{B_X} \wedge_{B_X} S^{T_n}_{B_X} \arrow[rr] \arrow[d, "\mathrm{id}\wedge_{B_X}\mathcal{BF}_{[n]}"'] & & S^{T_{n+1}}_{B_X} \arrow[d, "\mathcal{BF}_{[n+1]}"] \\
S^{\underline{\mathbb{C}}^{c'_n}\oplus\underline{\mathbb{R}}^{d'_n}\oplus T'_n\oplus\underline{\mathbb{C}}^m}_{B_X} \mathcal{SWF}_{[n]}(Y) \arrow[r] & S^{T'_{n+1}\oplus\underline{\mathbb{C}}^m}_{B_X} & \wedge_{B_Y} \mathcal{SWF}_{[n+1]}(Y),
\end{tikzcd} \tag{5.2.2}$$

$T_n = U_n \oplus V_n$, $T_n' = U_n' \oplus V_n'$. *In particular,* (5.2.2) *is a homotopy-commuting square of (unparameterized) connected simple systems.*

Proof. The proof is similar to the proof of Theorem 3.1.1, and we will only roughly sketch the details. Indeed, the bottom arrow of (5.2.2) is exactly the map defined in that theorem.

Recall that we have fixed identifications $U_{n+1}/U_n = \underline{\mathbb{C}}^{c_n+k_{Q,n}}$ To obtain that (5.2.2) homotopy-commutes, we deform $\widetilde{SW}_{X,n+1,\mathfrak{p}} = \widetilde{SW}_{X,n+1,\mathfrak{p},0}$ by a family $\widetilde{SW}_{X,n+1,\mathfrak{p},t}$, by removing (linearly in t) the nonlinear terms in $SW_{X,n,\mathfrak{p}}$ on the U_{n+1}/U_n and V_{n+1}/V_n-factors to a map $\widetilde{SW}_{X,n+1,\mathfrak{p},1}$ which is the sum of maps

$$H\colon U_{n+1}/U_n \oplus V_{n+1}/V_n \to U'_{n+1}/U'_n \oplus V'_{n+1}/V'_n \oplus \underline{\mathbb{C}}^{k_{Q,n}} \oplus \underline{\mathbb{R}}^{k_{W,-,n}}$$

and

$$\widetilde{SW}_{X,n,\mathfrak{p}}\colon U_n \oplus V_n \to U_n' \oplus V_n' \oplus r_Y^*(F_n \oplus W_n) \oplus \underline{\mathbb{C}}^m.$$

Here, H is some linear isomorphism (from the linearization of $SW_{X,n}$).

We define A_n as before, and require that A_n is an isolating neighborhood of the flow $\varphi_{n+1,t}$ for all $t \in [0,1]$.

We then define

$$\begin{aligned} K_{n,1}(\varepsilon) := \big\{(y,t) \in A_n \times [0,1] : {}& \exists(\hat{\phi}, v, \hat{\omega}) \in B_k(U_n \oplus V_n; R'),\\ & \|(SW_{X,n,\mathfrak{p},t}, \mathrm{id}_{\mathbb{C}^m})(\hat{\phi}, v, \hat{\omega})\|_{k-1} \le \varepsilon,\\ & y = \pi_{P_n \oplus W_{P,n}} r_Y(\hat{\phi}, \hat{\omega})\big\} \end{aligned}$$

and

$$\begin{aligned} K_{n,2}(\varepsilon) := \big\{(y,t) \in A_n \times [0,1] : {}& \exists(\hat{\phi}, v, \hat{\omega}) \in \partial B_k(U_n \oplus V_n; R'),\\ & \|(SW_{X,n,\mathfrak{p},t}, \mathrm{id}_{\underline{\mathbb{C}}^m})(\hat{x}, v, \hat{\omega})\|_{k-1} \le \varepsilon,\\ & y = \pi_{P_n \oplus W_{P,n}} r_Y(\hat{\phi}, \hat{\omega})\big\}\\ \cup\big(((\partial A_n) \times [0,1]) \cap K_{n,1}(\varepsilon)\big). \end{aligned}$$

One then establishes the analog of Proposition 5.1.3 for the family of flows $\varphi_{n+1,t}$.

Writing $I = [0,1]$, there results a map

$$\begin{aligned} \mathcal{BF}_{[n+1],I}(X,\mathrm{t})\colon S_{B_X \times I}^{\underline{\mathbb{C}}^{c_n+k_{Q,n}} \oplus \underline{\mathbb{R}}^{d_n+k_{W,-,n}}} \wedge_{B_Y \times I} S_{B_X \times I}^{U_n \oplus V_n}\\ \to S_{B_X \times I}^{U'_{n+1} \oplus V'_{n+1} \oplus \underline{\mathbb{C}}^m} \wedge_{B_Y \times I} \mathcal{SWF}_{[n+1]}(Y). \end{aligned}$$

At $t = 1$ this is the composite from first going down in (5.2.2), while for $t = 0$, this restricts to $\mathcal{BF}_{[n+1]}$. The homotopy commutativity of (5.2.2) follows.

The claim on the well-definedness of the maps in (5.2.2) follows from Theorem A.2.3. ∎

Proposition 5.2.6. *The map $\mathcal{BF}_{[n]}$ is independent of the choice of regular index pair (N_n, L_n) with $K_{n,1}(\varepsilon) \subset N_n$, $K_{n,2}(\varepsilon) \subset L_n$ for n large and ε small, up to isomorphisms in $PSW_{S^1,B}$.*

Proof. We will follow the argument in [23, Appendix]. Take another regular index pair (N'_n, L'_n) with $K_{1,n}(\varepsilon) \subset N'_n$, $K_{2,n}(\varepsilon) \subset L'_n$ for n large and ε small. Let I'_n denote the parameterized Conley index associated to (N'_n, L'_n).

First we consider the case when $(N_n, L_n) \subset (N'_n, L'_n)$. The map

$$\iota_n \colon I_n \to I'_n$$

induced by the inclusion is an isomorphism in $PSW_{S^1,B}$ by [43, Theorem 6.2] and the following diagram is commutative:

$$\begin{tikzcd}
 & S_{B_X}^{U'_n \oplus V'_n \oplus \underline{\mathbb{C}}^m} \wedge_{B_Y} I_n \arrow[dd, "\mathrm{id}\wedge\iota_n"] \\
S_{B_X}^{U_n \oplus V_n} \arrow[ur, "\mathcal{BF}_{[n]}"] \arrow[dr, "\mathcal{BF}'_{[n]}"'] & \\
 & S_{B_X}^{U'_n \oplus V'_n \oplus \underline{\mathbb{C}}^m} \wedge_{B_Y} I'_n.
\end{tikzcd}$$

Next we consider the general case. As shown in [23, p. 1653], we have index pairs $(\widetilde{N}_n, \tilde{L}_n)$, $(N_{n,1}, L_{n,1})$, $(N'_{n,1}, L'_n)$ such that

$$(N_n, L_n) \subset (N_{n,1}, L_{n,1}), \quad (N'_n, L'_n) \subset (N'_{n,1}, L'_{n,1}),$$
$$(K_{n,1}(\varepsilon), K_{2,n}(\varepsilon)) \subset (\widetilde{N}_n, \tilde{L}_n) \subset (N_{n,1}, L_{n,1}) \cap (N'_{n,1}, L'_{n,1}).$$

We can assume that $(\widetilde{N}_n, \tilde{L}_n)$, $(N_{n,1}, L_{n,1})$, $(N'_{n,1}, L'_{n,1})$ are all regular by thickening the exits slightly ([47, Remark 5.4]). The statement follows from the commutative diagram

$$\begin{tikzcd}
(N_{n,1}, L_{n,1}) & & (N'_{n,1}, L'_{n,1}) \\
 & (\widetilde{N}_n, \tilde{L}_n) \arrow[ul, hook] \arrow[ur, hook] & \\
(N_n, L_n) \arrow[uu, hook] & & (N'_n, L'_n). \arrow[uu, hook]
\end{tikzcd}$$

∎

Recall that we have defined the virtual bundle $\mathrm{ind}(D_X, P)$ following equation (5.1.3). For a normal spectral system $\mathfrak{P}$ whose nth section is P_n, we write $\mathrm{ind}(D_X, \mathfrak{P})$, since $\mathrm{ind}(D_X, P_n)$ and $\mathrm{ind}(D_X, P_{n+1})$ are canonically identified for all n. For $V =$

$V_1 \ominus V_2$ a virtual vector bundle over a base B, we define an element S_B^V of the stable-homotopy category PSW_B (see Definition A.1.9) by $(S_B^{V_1}, -V_2)$, where $S_B^{V_1}$ is the sphere bundle associated to V_1; the stable-homotopy type of this space does not depend on a choice of universe.

For V a vector bundle over B, let Th_B^V denote the Thom space of V; we will abuse notation and also write Th_B^V for the suspension spectrum of Th_B^V. Write $\ker(D_X, \mathfrak{P})$ for the kernel of the map in (5.1.2), which depends on the perturbation $\mathfrak{p}$.

For topological spaces W, Z, a *map class* from W to Z will refer to a homotopy class $W \to Z$, up to self-homotopy-equivalence of W, Z. We can now prove Theorem 1.3.1 from the introduction, which we restate as follows.

Corollary 5.2.7. *Fix a Floer framing $\mathfrak{P}$ on Y. There is a well-defined (parameterized, equivariant, stable) map class*

$$\mathcal{BF}(X, \mathfrak{t})\colon S_{\mathrm{Pic}(X)}^{\mathrm{ind}(D_X, \mathfrak{P})} \to \mathcal{SWF}(Y, \mathfrak{P}).$$

For a choice of perturbation $\mathfrak{p}$ as in (5.1.2), there is a well-defined (equivariant, unparameterized) weak map of spectra:

$$\mathbf{BF}_{\mathfrak{p}}(X, \mathfrak{t})\colon \mathrm{Th}_{\mathrm{Pic}(X)}^{\ker(D_X, \mathfrak{P})} \to \Sigma^{\mathbb{C}^m} \mathbf{SWF}^u(Y, \mathfrak{P}).$$

Moreover, if $\mathfrak{p}_0$ and $\mathfrak{p}_1$ are related by a family $\mathfrak{p}_t$ of perturbations satisfying (5.1.2), $\mathbf{BF}_{\mathfrak{p}_0}$ is homotopic to $\mathbf{BF}_{\mathfrak{p}_1}$.

Proof. The class $\mathcal{BF}_{\mathfrak{p}}$ is well defined by Proposition 5.2.5. Independence (as a map class) from $\mathfrak{p}$ follows from Lemma 5.2.1.

The unparameterized case follows from Proposition 5.2.5, and an argument for families as before. ∎

Analogous results hold for the Pin(2)-equivariant versions, mutatis mutandis.