Chapter 6

Frøyshov-type invariants

In this chapter we will generalize the Frøyshov-type invariants [20, 37] defined for rational homology 3-spheres to 3-manifolds with $b_1 > 0$, making use of the Seiberg–Witten Floer stable homotopy type constructed in this memoir. As applications, we will prove restrictions on the intersection forms of smooth 4-manifolds with boundary.

It may be of interest to compare the material of this section with work of Levine–Ruberman, where similar invariants are defined in the Heegaard Floer setting [31]; also see [9] for further work in the Heegaard Floer setting.

6.1 Equivariant cohomology

We will recall a basic fact about the S^1-equivariant Borel cohomology. For a pointed S^1-CW complex W, we let $\widetilde{H}^*_{S^1}(W;\mathbb{R})$ be the reduced S^1-equivariant Borel cohomology:

$$\widetilde{H}^*_{S^1}(W;\mathbb{R}) = \widetilde{H}^*(W \wedge_{S^1} ES^1_+;\mathbb{R}),$$

where ES^1_+ is a union of ES^1 and a disjoint base point. Note that $\widetilde{H}^*_{S^1}(S^0;\mathbb{R})$ is isomorphic to $\mathbb{R}[T]$ and that $\widetilde{H}^*_{S^1}(W;\mathbb{R})$ is an $\mathbb{R}[T]$-module. We have the following (see [16, Proposition 1.18.2] and [38, Proposition 2.2]).

Proposition 6.1.1. *Let V be an S^1-representation space and $\mathcal{V}$ be the vector bundle*

$$\mathcal{V} = (W \times ES^1) \times_{S^1} V \to W \times_{S^1} ES^1$$

over $W \times_{S^1} ES^1$. The Thom isomorphism for $\mathcal{V}$ induces an $\mathbb{R}[T]$-module isomorphism

$$\widetilde{H}^{*+\dim_{\mathbb{R}} V}_{S^1}(\Sigma^V W;\mathbb{R}) \cong \widetilde{H}^*_{S^1}(W;\mathbb{R}).$$

6.2 Frøyshov-type invariant

Let B be a compact CW-complex and choose a base point $b_0 \in B$. We view B as an S^1-CW-complex, with the trivial action of S^1. The following definition is an S^1-ex-space version of [38, Definition 2.7].

Definition 6.2.1. Let $\mathbf{U} = (W, r, s)$ be a well-pointed S^1-ex-space over B such that W is S^1-homotopy equivalent to an S^1-CW complex. We say that **U** is of SWF type at level t if there is an equivalence, as ex-spaces, from $W^{S^1} \to S_B^{\mathbb{R}^t}$, and so that the S^1-action on $W \setminus W^{S^1}$ is free.

Note that in the situation above, W^{S^1} inherits the structure of an ex-space, as a subspace of W, naturally. Spaces of SWF type are meant to be the class of spaces that are produced by the Seiberg–Witten Floer homotopy-type construction. Indeed, note that in the case that B is a point, spaces of SWF type over B are exactly spaces of SWF type as in [38]. For us, B will always be a Picard torus.

Moreover, for $\mathbf{U} = \mathcal{SWF}(Y)$ for some 3-manifold Y admitting a spectral section (with torsion spin^c structure and spectral section suppressed from the notation), more is true, in that the fixed point set W^{S^1} is actually fiber-preserving homotopy-equivalent, relative to $s(B)$, to $S_B^{\mathbb{R}^t}$, although for the definition of the Frøyshov invariant, this is not strictly needed.

Definition 6.2.2. Let $\mathbf{U} = (W, r, s)$ be a well-pointed S^1-ex-space of SWF type at level t over B. We denote by $\mathcal{I}_\Lambda(\mathbf{U})$ the submodule in $\widetilde{H}^*(B_+;\mathbb{R}) \otimes \mathbb{R}[\![T]\!]$, viewed as a module over the formal power series ring $\mathbb{R}[\![T]\!]$, generated by the image of the homomorphism induced by the inclusion $\iota\colon W^{S^1} \hookrightarrow W$:

$$\begin{aligned}\widetilde{H}^{*+t}_{S^1}(W/s(B);\mathbb{R}) &\xrightarrow{\iota^*} \widetilde{H}^{*+t}_{S^1}(W^{S^1}/s(B);\mathbb{R}) \cong \widetilde{H}^{*+t}_{S^1}(S^{\mathbb{R}^t} \wedge B_+;\mathbb{R}) \\ &= H^*(B;\mathbb{R}) \otimes \mathbb{R}[T] \hookrightarrow H^*(B;\mathbb{R}) \otimes \mathbb{R}[\![T]\!].\end{aligned}$$

We obtain a more specific invariant by considering only $H^0(B;\mathbb{R})$, in the case that B is connected; we impose this condition on B from now on. Let $\mathcal{I}(\mathbf{U})$ denote the ideal in $\mathbb{R}[\![T]\!]$ which is the image of

$$\begin{aligned}&\widetilde{H}^{*+t}_{S^1}(W/s(B);\mathbb{R}) \xrightarrow{\iota^*} \widetilde{H}^{*+t}_{S^1}(W^{S^1}/s(B);\mathbb{R}) \\ &\quad\cong \widetilde{H}^{*+t}_{S^1}(S^{\mathbb{R}^t} \wedge B_+;\mathbb{R}) \to \widetilde{H}^{*+t}_{S^1}(S^{\mathbb{R}^t};\mathbb{R}) = \mathbb{R}[T] \hookrightarrow \mathbb{R}[\![T]\!]\end{aligned}$$

obtained using the inclusion of a fiber $S^{\mathbb{R}^t} \to S^{\mathbb{R}^t} \wedge B_+$.

Then there is a nonnegative integer h such that $\mathcal{I}(\mathbf{U}) = (T^h)$. Here, (T^h) is the ideal generated by T^h. We denote this integer by $h(\mathbf{U})$.

The invariant $h(\mathbf{U})$ defined above is most similar to d_{bot} as in [31], while $\mathcal{I}_\Lambda(\mathbf{U})$ is, roughly, in line with the collection of their "intermediate invariants".

Remark 6.2.3. We also note that the cohomology group $\widetilde{H}^*_{S^1}(W/s(B);\mathbb{R})$ admits an action by $H^*(B)$ as follows. Using the projection map $r\colon W \to B$, we have an algebra morphism $r^*\colon H^*(B;\mathbb{R}) \to H^*(W;\mathbb{R})$. The Mayer–Vietoris sequence for (B, W) splits because of the map $s\colon B \to W$, and we obtain

$$H^*(W;\mathbb{R}) = H^*(W/s(B);\mathbb{R}) \oplus H^*(B;\mathbb{R}),$$

and in fact this splitting is at the level of $H^*(B;\mathbb{R})$-modules, so that the cohomology group $H^*(W/s(B);\mathbb{R})$ inherits an $H^*(B;\mathbb{R})$-action. This is not strictly necessary in the definition of invariants from $\mathcal{I}_\Lambda(\mathbf{U})$ above, but is indicative of the structure of $\mathcal{I}_\Lambda(\mathbf{U})$.

From Proposition 6.1.1, we can see the following.

Lemma 6.2.4. *Let $\mathbf{U} = (W, r, s)$ be a well-pointed S^1-ex-space of SWF type over B. If V is a real vector space, we have*

$$h(\Sigma_B^V \mathbf{U}) = h(\mathbf{U}).$$

If V is a complex vector space, we have

$$h(\Sigma_B^V \mathbf{U}) = h(\mathbf{U}) + \dim_{\mathbb{C}} V.$$

Proposition 6.2.5. *Let $\mathbf{U}_0 = (W_0, r_0, s_0)$, $\mathbf{U}_1 = (W_1, r_1, s_1)$ be well-pointed S^1-ex-spaces of SWF type at level t over B_0 and B_1, and assume we are given a map $\rho\colon B_0 \to B_1$. Let $\rho_!\mathbf{U}_0$ denote the pushforward of $\mathbf{U}_0$, as an ex-space over B_1. Assume that there is a fiberwise-deforming S^1-map*

$$f\colon \rho_!\mathbf{U}_0 \to \mathbf{U}_1$$

such that the restriction to

$$f^{S^1}\colon \rho_! W_0^{S^1} \to W_1^{S^1},$$

as a fiberwise-deforming morphism over B_1, is homotopy equivalent to

$$\mathrm{id} \wedge \rho\colon (\mathbb{R}^t)^+ \times B_0 \cup_{B_0} B_1 \to (\mathbb{R}^t)^+ \times B_1.$$

Then

$$h(\mathbf{U}_0) \le h(\mathbf{U}_1).$$

As a special case, if B_0 is a point, the hypothesis is that the map f, restricted to fixed point sets, $f^{S^1}\colon W_0^{S^1} \to W_1^{S^1}/s(W_1)$, be homotopic to the inclusion of a fiber.

Proof of Proposition 6.2.5. We have the following diagram:

$$\begin{array}{ccc}
\widetilde{H}^{*+t}(W_0/s(B_0);\mathbb{R}) & \xleftarrow{\quad f^* \quad} & \widetilde{H}^{*+t}(W_1/s(B_1);\mathbb{R}) \\
\downarrow & & \downarrow \\
\widetilde{H}^{*+t}((\mathbb{R}^t)^+ \times B_0/s(B_0);\mathbb{R}) & \xleftarrow{\quad \rho^* \quad} & \widetilde{H}^{*+t}((\mathbb{R}^t)^+ \times B_1/s(B_1);\mathbb{R}) \\
\downarrow & & \downarrow \\
\widetilde{H}^{*+t}((\mathbb{R}^t)^+;\mathbb{R}) = \mathbb{R}[T] & \xleftarrow[\cong]{\quad f^* \quad} & \widetilde{H}^{*+t}((\mathbb{R}^t)^+;\mathbb{R}) = \mathbb{R}[T] \\
& \searrow \quad \mathbb{R}[\![T]\!]. \quad \swarrow &
\end{array}$$

From this diagram, we obtain

$$(T^{h(\mathbf{U}_0)}) \supset (T^{h(\mathbf{U}_1)}),$$

which implies that $h(\mathbf{U}_0) \leq h(\mathbf{U}_1)$. ∎

Definition 6.2.6. For $m, n \in \mathbb{Z}$ and S^1-ex-space $\mathbf{U}$ of SWF type over B, we define

$$h(\Sigma_B^{\mathbb{R}^m \oplus \mathbb{C}^n} \mathbf{U}) = h(\mathbf{U}) + n.$$

Note that this definition is compatible with Lemma 6.2.4.

Definition 6.2.7. For $m_0, n_0, m_0, n_1 \in \mathbb{Z}$ and S^1-ex-spaces $\mathbf{U}_0$, $\mathbf{U}_1$ of SWF type over B, we say that $\Sigma_B^{\mathbb{R}^{m_0} \oplus \mathbb{C}^{n_0}} \mathbf{U}_0$ and $\Sigma_B^{\mathbb{R}^{m_1} \oplus \mathbb{C}^{n_1}} \mathbf{U}_1$ are locally equivalent if there is $N \in \mathbb{Z}_{\geq 0}$ with $N + m_0, N + n_0, N + m_1, N + n_1 \geq 0$ and fiberwise-deforming maps

$$f \colon \Sigma_B^{\mathbb{R}^{N+m_0} \oplus \mathbb{C}^{N+n_0}} \mathbf{U}_0 \to \Sigma_B^{\mathbb{R}^{N+m_1} \oplus \mathbb{C}^{N+n_1}} \mathbf{U}_1,$$
$$g \colon \Sigma_B^{\mathbb{R}^{N+m_1} \oplus \mathbb{C}^{N+n_1}} \mathbf{U}_1 \to \Sigma_B^{\mathbb{R}^{N+m_0} \oplus \mathbb{C}^{N+n_0}} \mathbf{U}_0$$

such that the restrictions

$$f^{S^1} \colon \Sigma_B^{\mathbb{R}^{N+m_0}} (\mathbf{U}_0)^{S^1} \to \Sigma_B^{\mathbb{R}^{N+m_1}} (\mathbf{U}_1)^{S^1},$$
$$g^{S^1} \colon \Sigma_B^{\mathbb{R}^{N+m_1}} (\mathbf{U}_1)^{S^1} \to \Sigma_B^{\mathbb{R}^{N+m_0}} (\mathbf{U}_0)^{S^1}$$

are homotopy equivalent to

$$\mathrm{Id} \colon B \times (\mathbb{R}^t) \to B \times (\mathbb{R}^t)^+$$

as fiberwise-deforming morphisms over B.

It is easy to see that the local equivalence is an equivalence relation.

Corollary 6.2.8. *If* $\Sigma_B^{\mathbb{R}^{m_0} \oplus \mathbb{C}^{n_0}} \mathbf{U}_0$ *and* $\Sigma_B^{\mathbb{R}^{m_1} \oplus \mathbb{C}^{n_1}} \mathbf{U}_1$ *are locally equivalent,*

$$h(\Sigma_B^{\mathbb{R}^{m_0} \oplus \mathbb{C}^{n_0}} \mathbf{U}_0) = h(\Sigma_B^{\mathbb{R}^{m_1} \oplus \mathbb{C}^{n_1}} \mathbf{U}_1).$$

Proof. This is a direct consequence of Proposition 6.2.5. ∎

Let Y be a closed 3-manifold, g be a Riemannian metric, $\mathfrak{s}$ be a torsion spin^c structure on Y. Let B_Y be the Picard torus $\mathrm{Pic}(Y)$ of Y. Assume that $\operatorname{ind} D_Y = 0$ in $K^1(B_Y)$. We take a spectral system

$$\mathfrak{S} = (\mathbf{P}, \mathbf{Q}, \mathbf{W}_P, \mathbf{W}_Q, \{\eta_n^P\}_n, \{\eta_n^Q\}, \{\eta_n^{W_P}\}_n, \{\eta_n^{W_Q}\}_n)$$

for Y. See Definition 3.5.1. Put

$$F_n = P_n \cap Q_n, \quad W_n = W_{P,n} \cap W_{Q,n}$$

as before. Take half-integers k_+, k_- with $k_+, k_- > 5$ and with $|k_+ - k_-| \leq \frac{1}{2}$. We have the approximate Seiberg–Witten flow

$$\varphi_n = \varphi_{n,k_+,k_-} : (F_n \oplus W_n) \times \mathbb{R} \to F_n \oplus W_n.$$

Put

$$A_n = (B_{k_+}(F_n^+; R) \times_{B_Y} B_{k_-}(F_n^-; R)) \times_{B_Y} (B_{k_+}(W_n^+; R) \times_{B_Y} B_{k_-}(W_n^-; R))$$

for $R \gg 0$. Recall that A_n is an isolating neighborhood for $n \gg 0$ (Theorem 2.3.3).

Lemma 6.2.9. *Let* $\mathbf{U}_n = (I_n, r_n, s_n)$ *be the* S^1*-equivariant Conley index for the isolated invariant set* $\mathrm{inv}(A_n, \varphi_n)$ *for* $n \gg 0$*. Then* $\mathbf{U}_n$ *is of SWF type at level* $\mathrm{rank}_{\mathbb{R}} W_n^-$.

Proof. We first note that I_n is of the homotopy type of an S^1-CW complex by Proposition 3.6.1. The S^1-fixed point set $(I_n, r_n, s_n)^{S^1}$ is the Conley index for

$$\mathrm{inv}(\varphi_n|_{W_n}, B_{k_+}(W_n^+; R) \times_{B_Y} B_{k_-}(W_n^-; R)).$$

Note that if $\phi = 0$, the quadratic terms $c_1(\gamma)$, $c_2(\gamma)$, $X_H(\phi)$ are all zero. See (2.3.3). Hence the restriction of the flow φ_n to W_n is the flow induced by the linear map $-{*d}|_{W_n}$. In particular, the flow $\varphi_n|_{W_n}$ preserves each fiber of the trivial bundle $W_n = B_Y \times L_k^2(\mathrm{im}\, d^*)_{\lambda_n}^{\mu_n}$ over B_Y. Hence there is an equivalence, as ex-spaces, $(I_n)^{S^1} \cong S_B^{W_n^-}$. (In fact, more is true: there is a fiber-preserving homotopy equivalence $(I_n)^{S^1} \cong S_B^{W_n^-}$.) ∎

Let $\mathcal{SWF}(Y, \mathfrak{s}, [\mathfrak{S}])$ be the Seiberg–Witten Floer parameterized homotopy type (Definition 3.5.8).

Recall that η_n^P, η_n^Q, $\eta_n^{W_P}$, $\eta_n^{W_Q}$ are isomorphisms

$$\begin{aligned} P_{n+1} &\xrightarrow{\cong} P_n \oplus \mathbb{C}^{k_{P,n}}, \\ Q_{n+1} &\xrightarrow{\cong} Q_n \oplus \mathbb{C}^{k_{Q,n}}, \\ W_{n+1}^P &\xrightarrow{\cong} W_n^+ \oplus \mathbb{R}^{k_{W,+,n}}, \\ W_{n+1}^Q &\xrightarrow{\cong} W_n^- \oplus \mathbb{R}^{k_{W,-,n}}. \end{aligned}$$

These induce an S^1-equivariant homotopy equivalence

$$I(\varphi_{n+1}) \cong \Sigma_B^{\mathbb{C}^{k_{Q,n}} \oplus \mathbb{R}^{k_{W,-,n}}} I(\varphi_n)$$

for $n \gg 0$, whose restriction to the S^1-fixed point set is a fiber-preserving homotopy equivalence. See Theorem 3.1.1. This implies that the number

$$h(\mathcal{SWF}(Y,\mathfrak{s},[\mathfrak{S}])) = h(I(\varphi_n)) - D_n^2$$

is independent of the choice of $n \gg 0$ by Lemma 6.2.4 and Corollary 6.2.8. Here, $D_n^2 = \dim(Q_n - Q_0)$.

Also, it follows from Proposition 3.4.1 that $h(\mathcal{SWF}(Y,\mathfrak{s},[\mathfrak{S}]))$ is independent of $k_\pm$. Hence $h(\mathcal{SWF}(Y,\mathfrak{s},[\mathfrak{S}]))$ is well defined.

We will introduce another number. We can take a spin^c 4-manifold $(X,\mathfrak{t})$ with boundary $(Y,\mathfrak{s})$. Since $c_1(\mathfrak{t})|_Y$ is torsion in $H^2(Y;\mathbb{Z})$, there is a positive integer m such that

$$mc_1(\mathfrak{t}) \in H^2(X,Y;\mathbb{Z}).$$

Put

$$c_1(\mathfrak{t})^2 := \frac{1}{m}\langle (mc_1(\mathfrak{t})) \cup c_1(\mathfrak{t}), [X]\rangle \in \mathbb{Q},$$

where $\langle\cdot,\cdot\rangle$ is the pairing

$$H^4(X,Y;\mathbb{Z}) \otimes H_4(X;\mathbb{Z}) \to \mathbb{Z}.$$

We define

$$\begin{aligned} n(Y,g,\mathfrak{s},P_0) &:= \dim \operatorname{ind}(D_X,P_0) - \frac{c_1(\mathfrak{t})^2 - \sigma(X)}{8} \in \mathbb{Q} \\ &= \frac{1}{2}\eta_{D,P_0} - \frac{1}{8}\eta_{Y,\mathrm{sign}}. \end{aligned} \tag{6.2.1}$$

Here, D_X is the Dirac operator on X, $\operatorname{ind}(D,P_0)$ is the index defined in Proposition 2.1.3 and η_{D,P_0}, $\eta_{Y,\mathrm{sign}}$ are the η-invariants of the Dirac operator and signature operator. We have used the index formula [5, 40]. See also [35, Section 6].

Definition 6.2.10. We define $h(Y,\mathfrak{s}) \in \mathbb{Q}$ by

$$h(Y,\mathfrak{s}) := h(\mathcal{SWF}(Y,\mathfrak{s},[\mathfrak{S}])) - n(Y,g,\mathfrak{s},P_0).$$

A priori, the expression in Definition 6.2.10 may depend on both the metric and the spectral system. However, for two spectral systems $\mathfrak{S}_0, \mathfrak{S}_1$ with $\dim \operatorname{ind}(D_X, P_0^0) = \dim \operatorname{ind}(D_X, P_0^1)$, we see that the h-invariants agree, since $\mathcal{SWF}(Y,\mathfrak{s},[\mathfrak{S}_0])$ differs from $\mathcal{SWF}(Y,\mathfrak{s},[\mathfrak{S}_1])$ by suspension by a virtual complex vector bundle of formal dimension zero. In order to see this, we first note that S^1-equivariant Borel cohomology is an S^1-equivariant complex orientable cohomology theory by [12], so that for an S^1-equivariant complex vector bundle V over B and an S^1-ex-space (X,r,s) over B, there is a canonical isomorphism

$$\widetilde{H}^{*+2\operatorname{rank}_{\mathbb{C}} V}_{S^1}(\nu_! \Sigma_B^V X) \cong \widetilde{H}^{*+2\operatorname{rank}_{\mathbb{C}} V}_{S^1}(\mathrm{Th}(r^* V)) \cong \widetilde{H}^*_{S^1}(X).$$

Here, $\nu\colon B \to *$ and we have used (3.6.3). This implies that

$$h(\Sigma_B^V X) = h(X) + 2\,\mathrm{rank}_{\mathbb{C}} V.$$

It follows in particular that

$$h(\mathcal{SWF}(Y, \mathfrak{s}, [\mathfrak{S}_0])) = h(\mathcal{SWF}(Y, \mathfrak{s}, [\mathfrak{S}_1])).$$

Changes in the metric and changes in $\dim \mathrm{ind}(D_X, P_0)$ are treated in a similar way, so we only address the latter. Indeed, if we replace $\mathfrak{S}_0$ with a spectral system $\mathfrak{S}_1$ so that the K-theory class is

$$[\mathfrak{S}_1 - \mathfrak{S}_0] = \underline{\mathbb{C}} \in K(B_Y),$$

then

$$h(\mathcal{SWF}(Y, \mathfrak{s}, [\mathfrak{S}_1])) = h(\mathcal{SWF}(Y, \mathfrak{s}, [\mathfrak{S}_0])) - 1,$$

but $n(Y, g, \mathfrak{s}, P_0^1) = n(Y, g, \mathfrak{s}, P_0^0) - 1$, as needed.

Finally, in the case that $b_1(Y) = 0$, this agrees (by definition) with the δ-invariant defined in [38].

In particular, it is natural to consider the parameterized equivariant homotopy type of the formal desuspension:

$$\Sigma_{B_Y}^{-n(Y,g,\mathfrak{s},P_0)\mathbb{C}} \mathcal{SWF}(Y, \mathfrak{s}, [\mathfrak{S}]),$$

which one can think of as a desuspension so that the grading of a reducible element of $\mathcal{SWF}(Y, \mathfrak{s}, [\mathfrak{S}])$ has been specified. We note that $n(S^1 \times S^2, g, \mathfrak{s}, P_0) = 0$, where g is the product metric on $S^1 \times S^2$, $\mathfrak{s}$ is the torsion spin^c structure and P_0 is the standard spectral section (since the Dirac operator has trivial kernel for each flat connection, this is specified). That is, with our conventions, the grading of each reducible in

$$\mathrm{Pic}(S^1 \times S^2) \simeq \mathcal{SWF}(Y, \mathfrak{s}, [\mathfrak{S}])$$

is zero. This differs from the convention in Heegaard–Floer homology, for which each reducible should be $-\frac{1}{2}$-graded, as in [45].

We will prove a generalization of [20, Theorem 4].

Theorem 6.2.11. *Let Y_0 be a rational homology 3-sphere and Y_1 be a closed, oriented 3-manifold such that the triple-cup product*

$$\Lambda^3 H^1(Y_1; \mathbb{Z}) \to \mathbb{Z},$$
$$\alpha_1 \wedge \alpha_2 \wedge \alpha_3 \mapsto \langle \alpha_1 \cup \alpha_2 \cup \alpha_3, [Y_1] \rangle$$

is zero. Let $(X, \mathfrak{t})$ be a compact, spin^c negative semidefinite 4-manifold with boundary $-Y_0 \coprod Y_1$ such that $c_1(\mathfrak{t})|_{\partial X}$ is torsion. Then we have

$$\frac{c_1(\mathfrak{t})^2 + b_2^-(X)}{8} + h(Y_0, \mathfrak{t}|_{Y_0}) \le h(Y_1, \mathfrak{t}|_{Y_1}).$$

Proof. Since the triple-cup product is zero, we have $\operatorname{ind} D_{Y_1} = 0$ in $K^1(B_{Y_1})$ by the index formula. (See [30, Proposition 6].) Note that the map $\mathcal{BF}_{[n]}(X, \mathfrak{t})$ constructed in Chapter 5 is a fiber-preserving map. We consider the restriction of $\mathcal{BF}_{[n]}(X, \mathfrak{t})$ to the fiber over a point $[0] \in B_X$. The restriction $\mathcal{BF}_{[n]}(X, \mathfrak{t})$ to the fiber and the duality map

$$I_n(Y_0) \wedge I_n(-Y_0) \to S^{F_n(Y_0) \oplus W_n(Y_0)},$$

defined in [36, Section 2.5], induce an S^1-map

$$f_n \colon \Sigma^{\mathbb{R}^{m_0} \oplus \mathbb{C}^{n_0 + a}} I_n(Y_0) \to \Sigma^{\mathbb{R}^{m_1} \oplus \mathbb{C}^{n_1}} (I_n(Y_1)/s_n(B_{Y_1}))$$

for $n \gg 0$, where

$$\begin{aligned} m_0 - m_1 &= \operatorname{rank}_{\mathbb{R}} W_n(Y_1)^- - \dim_{\mathbb{R}} W_n(Y_0)^-, \\ n_0 - n_1 &= \operatorname{rank}_{\mathbb{C}} F_n(Y_1)^- - \dim_{\mathbb{C}} F_n(Y_0)^-, \\ a &= \dim \operatorname{ind} D_{X,P_0} \\ &= \frac{c_1(\mathfrak{t})^2 + b_2^-(X)}{8} + n(Y_1, g|_{Y_1}, \mathfrak{t}|_{Y_1}, P_0) - n(Y_0, g|_{Y_0}, \mathfrak{t}|_{Y_0}). \end{aligned}$$

The restriction of f_n to the S^1-fixed point set $\Sigma^{\mathbb{R}^{m_0}}(I_n(Y_0))^{S^1}$ is induced by the operator

$$D' = (d^+, \pi^0_{-\infty} r_{-Y_0}, \pi^0_{-\infty} r_{Y_1}) \colon \Omega^1_{CC}(X) \to \Omega^+(X) \oplus (\mathcal{W}_{-Y_0})^0_{-\infty} \oplus (\mathcal{W}_{Y_1})^0_{-\infty}.$$

The operator D' is an isomorphism. Therefore the restriction

$$f_n^{S^1} \colon \Sigma^{\mathbb{R}^{m_0}}(I_n(Y_0))^{S^1} \to \Sigma^{\mathbb{R}^{m_1}}(I_n(Y_1))^{S^1}_{[0]}$$

is a homotopy equivalence. Here, $[0] \in B_{Y_1}$ is the restriction of $[0] \in B_X$ to Y and $(I_n(Y_1))^{S^1}_{[0]}$ is the fiber over $[0]$.

By Lemma 6.2.4 and Proposition 6.2.5, we have

$$\frac{c_1(\mathfrak{t})^2 + b_2^-(X)}{8} + h(Y_0, \mathfrak{t}|_{Y_0}) \leq h(Y_1, \mathfrak{t}|_{Y_1}). \qquad \blacksquare$$

Remark 6.2.12. There is an apparent discrepancy with the statement of [31, Theorem 4.7]. We note that in the translation between these statements, we expect $h(Y, \mathfrak{s})$ to correspond to $\frac{d_{\mathrm{bot}}(Y,\mathfrak{s})}{2} + \frac{b_1(Y)}{4}$, due to the difference in the grading conventions on the reducible; with this observation, the statements are consistent.

Remark 6.2.13. In order to generalize Theorem 6.2.11 to the case $b_1(Y_0) > 0$, we need to establish the duality for the Seiberg–Witten Floer parameterized homotopy types $\mathcal{SWF}(Y_0, \mathfrak{t}|_{Y_0}, [\mathfrak{S}])$ and $\mathcal{SWF}(-Y_0, \mathfrak{t}|_{Y_0}, [\mathfrak{S}_0^\vee])$ to get the parameterized Bauer–Furuta map

$$\mathcal{SWF}(Y_0, \mathfrak{t}|_{Y_0}, [\mathfrak{S}_0]) \to \mathcal{SWF}(Y_1, \mathfrak{t}|_{Y_1}, [\mathfrak{S}_1]).$$

We do not discuss it in this memoir. See Proposition 3.6.2.

Corollary 6.2.14. *Let Y be a closed, connected, oriented 3-manifold such that the triple-cup product is zero. Let $(X, \mathfrak{t})$ be a compact, negative semidefinite,* spin^c *4-manifold with $\partial X = Y$ such that $c_1(\mathfrak{t})|_Y$ is torsion. Then we have*

$$\frac{c_1(\mathfrak{t})^2 + b_2^-(X)}{8} \leq h(Y, \mathfrak{t}|_Y).$$

Proof. Removing a small ball from X, we get a compact spin^c 4-manifold X' with boundary $S^3 \coprod Y$. Applying Theorem 6.2.11 to X', we get the inequality. ∎

Example 6.2.15. Let T^2 be a torus $(\mathbb{R}/\mathbb{Z}) \times (\mathbb{R}/\mathbb{Z})$. Put

$$Y := \mathbb{R} \times T^2/(x, \theta_1, \theta_2) \sim (x+1, -\theta_1, -\theta_2).$$

Then Y is a flat T^2 bundle over S^1, which has a flat metric and $b_1(Y) = 1$. We have

$$H^2(Y; \mathbb{Z}) \cong H_1(Y; \mathbb{Z}) \cong \mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z}).$$

There are four spin^c structures $\mathfrak{s}_0, \dots, \mathfrak{s}_3$. Let $\mathfrak{s}_0$ be the spin^c structure corresponding to the 2-plane field tangent to the fibers. As stated in Example 4.2.2, for $j = 1, 2, 3$, $(Y, \mathfrak{s}_j)$ satisfies the conditions of Theorem 4.1.2. We have

$$\mathcal{SWF}(Y, \mathfrak{s}, [\mathfrak{S}]) \cong S^0_{B_Y}.$$

Here, $\mathfrak{S}$ is a spectral system with $P_0 = \mathcal{E}_0(D)^0_{-\infty}$. As stated in [24, p. 2112],

$$n(Y, \mathfrak{s}_j, g, P_0) = 0$$

for $j = 1, 2, 3$. Therefore we obtain

$$h(Y, \mathfrak{s}_j) = h(\mathcal{SWF}(Y, \mathfrak{s}, [\mathfrak{S}])) - n(Y, \mathfrak{s}_j, g, P_0) = 0.$$

Example 6.2.16. Let Σ be a closed, oriented surface with $g(\Sigma) > 0$ and Y be the sphere bundle of the complex line bundle over Σ of degree d. Suppose that $0 < g < d$, where $g := g(\Sigma)$. Let $\mathfrak{s}_q$ be the spin^c structure in Proposition 4.2.3. For $q \in \{g, g+1, \dots, d-1\}$, we have

$$\mathcal{SWF}(Y, \mathfrak{s}_q, [\mathfrak{S}]) \cong S^0_B$$

by Theorem 4.2.5. Here, $\mathfrak{S}$ is a spectral system with $P_0 = \mathcal{E}_0(D_r)^0_{-\infty}$. The value of $n(Y, g_r, \mathfrak{s}_q, P_0)$ was computed in [24, Section 8.2] and we have

$$n(Y, g_r, \mathfrak{s}_q, P_0) = -\frac{d-1}{8} - \frac{(g-1-q)(d+g-1-q)}{2d}. \tag{6.2.2}$$

(Note that the definition of $n(Y, g, \mathfrak{s}_q, P_0)$ of this memoir is -1 times that of [24].) Hence

$$\begin{aligned} h(Y, \mathfrak{s}_q, g) &= h(\mathcal{SWF}(Y, \mathfrak{s}_q, [\mathfrak{S}])) - n(Y, g, \mathfrak{s}_q, P_0) \\ &= \frac{d-1}{8} + \frac{(g-1-q)(d+g-1-q)}{2d}. \end{aligned}$$

6.3 K-theoretic Frøyshov invariant

In analogy to the previous section on the (homological) Frøyshov invariant, we now generalize the invariant $\kappa(Y)$ constructed in [37]. For details on Pin(2)-equivariant complex K-theory, we refer to [37].

Let $\widetilde{\mathbb{R}}$ be the nontrivial real representation of $\mathrm{Pin}(2) = S^1 \coprod jS^1$. Let B be a compact, connected Pin(2)-CW complex with a Pin(2)-fixed marked (though we do not consider B itself to be an object in the category of pointed spaces) point $b_0 \in B^{\mathrm{Pin}(2)}$, such that the S^1-action on B is trivial and the action of j is an involution.

Definition 6.3.1. Let $\mathbf{U} = (W, r, s)$ be a well-pointed Pin(2)-ex-space over B such that W is Pin(2)-homotopy equivalent to a Pin(2)-CW complex. We say that $\mathbf{U}$ is of SWF type at level t if there is an ex-space Pin(2)-homotopy equivalence from W^{S^1} to $S_B^{\widetilde{\mathbb{R}}^t}$ and if the Pin(2)-action on $W \setminus W^{S^1}$ is free.

As before, in fact for us there is the stronger condition that there is a fiber-preserving (equivariant) homotopy equivalence $W^{S^1} \to S_B^{\widetilde{\mathbb{R}}^t}$.

Let $R(\mathrm{Pin}(2))$ be the representation ring of Pin(2). That is,

$$R(\mathrm{Pin}(2)) \cong \mathbb{Z}[z, w]/(w^2 - 2w, zw - 2w),$$

where

$$w = 1 - [\widetilde{\mathbb{C}}], \quad z = 2 - [\mathbb{H}].$$

We will generalize [37, Definition3] to Pin(2)-ex-spaces.

Definition 6.3.2. Let $\mathbf{U} = (W, r, s)$ be a well-pointed Pin(2)-ex-space of SWF type at level $2t$ over B so that W is Pin(2)-homotopy equivalent to a Pin(2)-CW complex. We denote by $\mathcal{I}_\Lambda(\mathbf{U})$ the submodule in $K_{\mathbb{Z}/2}(B)$, viewed as a module over $R(\mathrm{Pin}(2))$, generated by the image of the homomorphism induced by the inclusion $\iota\colon W^{S^1} \hookrightarrow W$:

$$\begin{aligned}\widetilde{K}_{\mathrm{Pin}(2)}(W/s(B)) \xrightarrow{\iota^*} \widetilde{K}_{\mathrm{Pin}(2)}(W^{S^1}/s(B)) &\cong \widetilde{K}_{\mathrm{Pin}(2)}(S^{\widetilde{\mathbb{C}}^t} \wedge B_+) \\ &= K_{\mathbb{Z}/2}(B).\end{aligned}$$

We obtain a more specific invariant by considering only a single fiber. Let $\mathcal{I}(\mathbf{U})$ denote the ideal in $R(\mathrm{Pin}(2))$ which is the image of

$$\begin{aligned}&\widetilde{K}_{\mathrm{Pin}(2)}(W/s(B)) \xrightarrow{\iota^*} \widetilde{K}_{\mathrm{Pin}(2)}(W^{S^1}/s(B)) \\ &\qquad \cong \widetilde{K}_{\mathrm{Pin}(2)}(S^{\widetilde{\mathbb{C}}^t} \wedge B_+) \to \widetilde{K}_{\mathrm{Pin}(2)}(S^{\widetilde{\mathbb{C}}^t}; \mathbb{R}) = R(\mathrm{Pin}(2))\end{aligned}$$

obtained using the inclusion of a fiber $S^{\mathbb{R}^t} \to S^{\mathbb{R}^t} \wedge B_+$, over the marked point $b_0 \in B^{\mathrm{Pin}(2)}$. In particular, the invariant $k(\mathbf{U})$ depends on a choice of the point $b_0 \in B$, which does not appear in the notation.

We define $k(\mathbf{U}) \in \mathbb{Z}_{\geq 0}$ by

$$k(\mathbf{U}) = \min\{k \in \mathbb{Z}_{\geq 0} : \exists x \in \mathcal{I}(\mathbf{U}),\ wx = 2^k w\}.$$

If $\mathcal{I}(\mathbf{U})$ is of the form (z^k) for some nonnegative integer k, we say that $\mathbf{U}$ is $K_{\mathrm{Pin}(2)}$-split.

Lemma 6.3.3.

$$k(\Sigma_B^{\tilde{\mathbb{C}}}\mathbf{U}) = k(\mathbf{U}), \quad k(\Sigma_B^{\mathbb{H}}\mathbf{U}) = k(\mathbf{U}) + 1.$$

Proof. Since

$$(\Sigma_B^{\tilde{\mathbb{C}}} W)/s(B) = \Sigma^{\tilde{\mathbb{C}}}[W/s(B)],$$
$$(\Sigma_B^{\mathbb{H}} W)/s(B) = \Sigma^{\mathbb{H}}[W/s(B)],$$

we can apply [37, Lemma 3.4]. ∎

Proposition 6.3.4. *Let* $\mathbf{U}_0 = (W_0, r_0, s_0)$*,* $\mathbf{U}_1 = (W_1, r_1, s_1)$ *be* Pin(2)*-ex-spaces of SWF type at level* $2t_0$*,* $2t_1$ *over* B_0 *and* B_1*, and assume we are given an inclusion* $\rho\colon B_0 \to B_1$*. Let* $\rho_!\mathbf{U}_0$ *denote the pushforward of* $\mathbf{U}_0$*, as an ex-space over* B_1*. Assume that there is a fiberwise-deforming* S^1*-map*

$$f\colon \rho_!\mathbf{U}_0 \to \mathbf{U}_1$$

such that the restriction to

$$f^{S^1}\colon \rho_! W_0^{S^1} \to W_1^{S^1},$$

as a fiberwise-deforming morphism over B_1*, is homotopy equivalent to*

$$\ell \cup \rho\colon ((\tilde{\mathbb{C}}^{t_0})^+ \times B_0) \cup_{B_0} B_1 \to (\tilde{\mathbb{C}}^{t_1})^+ \times B_1,$$

where ℓ *is the map on one-point compactifications induced by a map of representations* $\tilde{\mathbb{C}}^{t_0} \to \tilde{\mathbb{C}}^{t_1}$*, which is an inclusion if* $t_0 \leq t_1$*. Say that* ρ *sends the marked point* $b_0 \in B_0$ *to* $b_1 \in B_1$*:*

(1) *If* $t_0 \leq t_1$*, we have*

$$k(\mathbf{U}_0) + t_0 \leq k(\mathbf{U}_1) + t_1.$$

(2) *If* $t_0 < t_1$ *and* $\mathbf{U}_0$ *is* $K_{\mathrm{Pin}(2)}$*-split, we have*

$$k(\mathbf{U}_0) + t_0 + 1 \leq k(\mathbf{U}_1) + t_1.$$

Proof. We have the following commutative diagram:

$$\begin{array}{ccc}
\widetilde{K}_{\mathrm{Pin}(2)}(W_0/s_0(B_0)) & \xleftarrow{\quad f^* \quad} & \widetilde{K}_{\mathrm{Pin}(2)}(W_1/s_1(B_1)) \\
{\scriptstyle \iota_0^*}\big\downarrow & & \big\downarrow{\scriptstyle \iota_1^*} \\
\widetilde{K}_{\mathrm{Pin}(2)}(((\widetilde{\mathbb{C}}^{t_0})^+ \times B_0) \cup_{B_0} B_1/s(B_1)) & \xleftarrow{(\ell\cup\rho)^*} & \widetilde{K}_{\mathrm{Pin}(2)}((\widetilde{\mathbb{C}}^{t_1})^+ \times B_1/s(B_1)) \\
{\scriptstyle \iota^*}\big\downarrow & & \big\downarrow{\scriptstyle \iota^*} \\
\widetilde{K}_{\mathrm{Pin}(2)}((\widetilde{\mathbb{C}}^{t_0})^+) & \xleftarrow{\quad \ell^* \quad} & \widetilde{K}_{\mathrm{Pin}(2)}((\widetilde{\mathbb{C}}^{t_1})^+) \\
{\scriptstyle \cdot w^{t_0}}\big\downarrow & & \big\downarrow{\scriptstyle \cdot w^{t_1}} \\
\widetilde{K}_{\mathrm{Pin}(2)}(S^0) & \xleftarrow[\quad \mathrm{id} \quad]{} & \widetilde{K}_{\mathrm{Pin}(2)}(S^0).
\end{array}$$

Here we have used ι to denote various inclusions. Note that f^* in the first row is well defined, because $s_0(B_0) \subset s_0(B_1)$, using the definition of the pushforward $\rho_! \mathbf{U}_0$ (this does not require that ρ be an inclusion). In fact, more is true, in that $\rho_! W_0/s_0(B_1)$ is exactly $W_0/s_0(B_0)$.

We can apply the arguments in the proofs of [37, Lemmas 3.10 and 3.11] so that the result follows. ∎

Definition 6.3.5. For $m, n \in \mathbb{Z}$ and Pin(2)-ex-space $\mathbf{U}$ of SWF type at even level, we define

$$k(\Sigma_B^{\widetilde{\mathbb{R}}^{2m} \oplus \mathbb{H}^n} \mathbf{U}) = k(\mathbf{U}) + n.$$

Note that this definition is compatible with Lemma 6.3.3.

Definition 6.3.6. For $m_0, n_0, m_1, n_1 \in \mathbb{Z}$ and Pin(2)-ex-spaces $\mathbf{U}_0$, $\mathbf{U}_1$ of SWF type at even level over B, we say that $\Sigma_B^{\widetilde{\mathbb{R}}^{2m_0} \oplus \mathbb{H}^{n_0}} \mathbf{U}_0$ and $\Sigma_B^{\widetilde{\mathbb{R}}^{2m_1} \oplus \mathbb{H}^{n_1}} \mathbf{U}_1$ are locally equivalent if there are $N \in \mathbb{Z}$ with $N + m_0, N + n_0, N + m_1, N + n_1 \geq 0$ and Pin(2)-fiberwise deforming maps

$$f \colon \Sigma_B^{\widetilde{\mathbb{R}}^{2(N+m_0)} \oplus \mathbb{H}^{N+n_0}} \mathbf{U}_0 \to \Sigma_B^{\widetilde{\mathbb{R}}^{2(N+m_1)} \oplus \mathbb{H}^{N+n_1}} \mathbf{U}_1,$$
$$g \colon \Sigma_B^{\widetilde{\mathbb{R}}^{2(N+m_1)} \oplus \mathbb{H}^{N+n_1}} \mathbf{U}_1 \to \Sigma_B^{\widetilde{\mathbb{R}}^{2(N+m_0)} \oplus \mathbb{H}^{N+n_0}} \mathbf{U}_0,$$

such that the restrictions

$$f^{S^1} \colon \Sigma_B^{\widetilde{\mathbb{R}}^{2(N+m_0)}} \mathbf{U}_0^{S^1} \to \Sigma_B^{\widetilde{\mathbb{R}}^{2(N+m_1)}} \mathbf{U}_1^{S^1}, \quad g^{S^1} \colon \Sigma_B^{\widetilde{\mathbb{R}}^{2(N+m_1)}} \mathbf{U}_1^{S^1} \to \Sigma_B^{\widetilde{\mathbb{R}}^{2(N+m_0)}} \mathbf{U}_0^{S^1}$$

are homotopy equivalent to

$$\mathrm{Id} \colon B \times (\mathbb{R}^t)^+ \to B \times (\mathbb{R}^t)^+$$

as Pin(2)-fiberwise-deforming morphisms.

Corollary 6.3.7. *If* $\Sigma_B^{\widetilde{\mathbb{R}}^{2m_0}\oplus\mathbb{H}^{n_0}}\mathbf{U}_0$ *and* $\Sigma_B^{\widetilde{\mathbb{R}}^{2m_1}\oplus\mathbb{H}^{n_1}}\mathbf{U}_1$ *are locally equivalent, we have*

$$k(\Sigma_B^{\widetilde{\mathbb{R}}^{2m_0}\oplus\mathbb{H}^{n_0}}\mathbf{U}_0) = k(\Sigma_B^{\widetilde{\mathbb{R}}^{2m_1}\oplus\mathbb{H}^{n_1}}\mathbf{U}_1).$$

Proof. This is a direct consequence of Proposition 6.3.4. ∎

Let $\mathfrak{s}$ be a spin structure (not just a self-conjugate spin^c structure, although we will also write $\mathfrak{s}$ for the induced self-conjugate spin^c structure) of Y. Then the Seiberg–Witten equations (2.3.4) and the finite-dimensional approximations (2.3.10) have Pin(2)-symmetry. Let B_Y be the Picard torus of Y, which is homeomorphic to the torus $\widetilde{\mathbb{R}}^{b_1(Y)}/\mathbb{Z}^{b_1(Y)}$, where we have chosen coordinates so that $0 \in \widetilde{\mathbb{R}}^{b_1(Y)}$ corresponds to the selected spin structure on Y. We choose $[0] \in B_Y$ as base point. Assume that $\operatorname{ind} D_Y = 0$ in $KQ^1(B_Y)$. By Theorem 2.4.8, we can choose a Pin(2)-spectral system

$$\mathfrak{S} = (\mathbf{P}, \mathbf{Q}, \mathbf{W}_P, \mathbf{W}_Q, \{\eta_n^P\}_n, \{\eta_n^Q\}, \{\eta_n^{W_P}\}_n, \{\eta_n^{W_Q}\}_n)$$

for Y. Put

$$F_n = P_n \cap Q_n, \quad W_n = W_{P,n} \cap W_{Q,n}.$$

We have the Pin(2)-equivariant Conley index (I_n, r_n, s_n) for the isolated invariant set $\operatorname{inv}(A_n, \varphi_{k_+,k_-,n})$ for $n \gg 0$.

Lemma 6.3.8. *The* Pin(2)*-equivariant Conley index* (I_n, r_n, s_n) *is of SWF type at level* $\operatorname{rank}_{\mathbb{R}} W_n^-$ *for* $n \gg 0$.

Proof. The proof is similar to that of Lemma 6.2.9 and omitted. ∎

Let $\mathcal{SWF}^{\mathrm{Pin}(2)}(Y, \mathfrak{s}, [\mathfrak{S}])$ be the Pin(2)-Seiberg–Witten Floer parameterized homotopy type. As before, the local equivalence class of $\mathcal{SWF}^{\mathrm{Pin}(2)}(Y, \mathfrak{s}, [\mathfrak{S}])$ is independent of $k_\pm$, n. See [49] for the study of the local equivalence class of the Pin(2)-Seiberg–Witten Floer homotopy type in the case $b_1(Y) = 0$. We may assume that $\dim_{\mathbb{R}} W_n^-$ are even for all n. Then we have the well-defined number

$$k(\mathcal{SWF}^{\mathrm{Pin}(2)}(Y, \mathfrak{s}, [\mathfrak{S}])) \in \mathbb{Z}.$$

Definition 6.3.9. Fix $(Y, \mathfrak{s})$ as above. We define $\kappa(Y, \mathfrak{s}) \in \mathbb{Q} \cup \{-\infty\}$ by

$$\kappa(Y, \mathfrak{s}) := \inf_{g,\mathfrak{S}} 2\Big(k(\mathcal{SWF}^{\mathrm{Pin}(2)}(Y, \mathfrak{s}, [\mathfrak{S}])) - \frac{1}{2}n(Y, g, \mathfrak{s}, P_0)\Big).$$

We say that $(Y, \mathfrak{s})$ is Floer $K_{\mathrm{Pin}(2)}$-split if (I_n, r_n, s_n) is $K_{\mathrm{Pin}(2)}$-split for n large, where (I_n, r_n, s_n) realizes equality in the definition of $\kappa(Y, \mathfrak{s})$.

Note that this invariant indeed depends a priori on $\mathfrak{s}$ as a spin structure, in what we have chosen as the marked point in B_Y that is used in the definition of κ.

Unlike the case for homology, we have not shown that the invariant

$$k(\mathcal{SWF}^{\mathrm{Pin}(2)}(Y, \mathfrak{s}, [\mathfrak{S}]))$$

is invariant under changes of spectral section that lie in $\widetilde{KQ}(B)$ (essentially since we do not have access to a notion of Pin(2)-complex orientable cohomology theories). We expect that the quantity appearing in the inf is, in fact, independent of $[\mathfrak{S}]$, however.

We do not know whether a self-conjugate spin^c structure may have different κ-invariants associated to different underlying spin structures. The invariant $\kappa(Y, \mathfrak{s})$, for Y a rational homology 3-sphere, agrees with Manolescu's definition [37], by construction.

Corollary 6.3.10. *The reduction* mod2 *of the* κ *invariant satisfies*

$$\mu(Y, \mathfrak{s}) = \kappa(Y, \mathfrak{s}) \bmod 2,$$

where $\mu(Y, \mathfrak{s})$ *is the Rokhlin invariant of* $(Y, \mathfrak{s})$.

Proof. Indeed, $n(Y, g, \mathfrak{s}, P_0)$ mod 2 is the Rokhlin invariant of $(Y, \mathfrak{s})$ by its construction. The corollary then follows from the definition of κ and the fact that k is an integer. ∎

Corollary 6.3.10 indicates that $\kappa(Y, \mathfrak{s})$ may depend on $\mathfrak{s}$, as a spin structure. Note that if $(Y, \mathfrak{s})$ admits a Pin(2)-equivariant spectral section, for a self-conjugate spin^c structure $\mathfrak{s}$, then $\mu(Y, -)$ is constant on all spin structures underlying $\mathfrak{s}$; by Lin's result [33], this condition, coupled with the triple-cup product vanishing, characterizes 3-manifolds which admit a Pin(2)-equivariant spectral section. However, if the Pin(2)-equivariant K-theory could be extended to 3-manifolds without a Pin(2)-spectral section, so that Corollary 6.3.10 held, it would of course also imply that $\kappa(Y, \mathfrak{s})$ depends on the spin structure and not just the spin^c structure.

Using our invariant $\kappa(Y, \mathfrak{s})$, we can prove a $\frac{10}{8}$-type inequality for smooth 4-manifolds with boundary, which generalizes the results of [21] and [37].

Theorem 6.3.11. *Let* $(Y_0, \mathfrak{s}_0)$ *be a spin, rational homology* 3*-sphere and* $(Y_1, \mathfrak{s}_1)$ *be a closed, spin* 3*-manifold such that the index* ind D_{Y_1} *is zero in* $KQ^1(B_{Y_1})$.

(1) *Let* $(X, \mathfrak{t})$ *be a compact, smooth, spin, negative semidefinite* 4*-manifold with boundary* $-(Y_0, \mathfrak{s}_0) \coprod (Y_1, \mathfrak{s}_1)$. *Then we have*

$$\frac{1}{8} b_2^-(X) + \kappa(Y_0, \mathfrak{s}_0) \leq \kappa(Y_1, \mathfrak{s}_1).$$

(2) *Let* $(X, \mathfrak{t})$ *be a compact, smooth, spin* 4*-manifold with boundary* $-(Y_0, \mathfrak{s}_0) \coprod (Y_1, \mathfrak{s}_1)$. *Then we have*

$$-\frac{\sigma(X)}{8} + \kappa(Y_0, \mathfrak{s}_0) - 1 \leq b^+(X) + \kappa(Y_1, \mathfrak{s}_1).$$

Moreover, if Y_0 is Floer $K_{\mathrm{Pin}(2)}$-split and $b^+(X) > 0$, we have

$$-\frac{\sigma(X)}{8} + \kappa(Y_0, \mathfrak{s}_0) + 1 \leq b^+(X) + \kappa(Y_1, \mathfrak{s}_1).$$

Proof. Let $[0] \in B_X = \mathrm{Pic}(X)$ be the element corresponding to the flat spin connection. Recall that $\mathcal{BF}_{[n]}$ is a fiber-preserving map. The restriction $\mathcal{BF}_{[n]}(X, \mathfrak{t})$ to the fiber over [0] and the duality map

$$I_n(Y_0) \wedge I_n(-Y_0) \to S^{F_n(Y_0) \oplus W_n(Y_0)}$$

defined in [36, Section 2.5], give a Pin(2)-map

$$f_n \colon \Sigma^{\tilde{\mathbb{R}}^{m_0} \oplus \mathbb{H}^{n_0}} I_n(Y_0) \to \Sigma^{\tilde{\mathbb{R}}^{m_1} \oplus \mathbb{H}^{n_1}} (I_n(Y_1)/s_n(B_{Y_1}))$$

such that

$$f_n((\Sigma^{\tilde{\mathbb{R}}^{m_0} \oplus \mathbb{H}^{n_0}} I_n(Y_0))^{S^1}) \subset (\Sigma^{\tilde{\mathbb{R}}^{m_1} \oplus \mathbb{H}^{n_1}} I_n(Y_1)_{[0]})^{S^1},$$
$$f_n((\Sigma^{\tilde{\mathbb{R}}^{m_0} \oplus \mathbb{H}^{n_0}} I_n(Y_0))^{\mathrm{Pin}(2)}) \subset (\Sigma^{\tilde{\mathbb{R}}^{m_1} \oplus \mathbb{H}^{n_1}} I_n(Y_1)_{[0]})^{\mathrm{Pin}(2)}.$$

Here, $[0] \in \mathrm{Pic}(Y_1)$ is the element corresponding to the flat spin connection, and

$$\begin{aligned} m_0 - m_1 &= \mathrm{rank}_{\mathbb{R}} W_n(Y_1)^- - \dim_{\mathbb{R}} W_n(Y_0)^- - b^+(X), \\ n_0 - n_1 &= \mathrm{rank}_{\mathbb{H}} F_n(Y_1)^- - \dim_{\mathbb{H}} F_n(Y_0)^- \\ &\quad + \frac{1}{2} n(Y_1, g|_{Y_1}, \mathfrak{t}|_{Y_1}, P_0) - \frac{1}{2} n(Y_0, g|_{Y_0}, \mathfrak{t}|_{Y_0}) - \frac{\sigma(X)}{16}. \end{aligned}$$

The restriction of f_n to $(\Sigma^{\tilde{\mathbb{R}}^{m_0} \oplus \mathbb{H}^{n_0}} I_n(Y_0))^{S^1}$ is induced by the operator

$$(d^+, \pi^0_{-\infty} r_{-Y_0}, \pi^0_{-\infty} r_{Y_1}) \colon \Omega^1_{CC}(X) \to \Omega^+(X) \oplus (\mathcal{W}_{-Y_0})^0_{-\infty} \oplus (\mathcal{W}_{Y_1, [0]})^0_{-\infty}$$

and is a homotopy equivalence

$$(\Sigma^{\tilde{\mathbb{R}}^{m_0} \oplus \mathbb{H}^{n_0}} I_n(Y_0))^{\mathrm{Pin}(2)} \to (\Sigma^{\tilde{\mathbb{R}}^{m_1} \oplus \mathbb{H}^{n_1}} I_n(Y_1)_{[0]})^{\mathrm{Pin}(2)};$$

indeed, both of these are just S^0 consisting of 0 and the base point. Moreover, if $b^+(X) = 0$, the restriction of f_n to $(\Sigma^{\tilde{\mathbb{R}}^{m_0} \oplus \mathbb{H}^{n_0}} I_n(Y_0))^{S^1}$ is a Pin(2)-homotopy equivalence

$$\Sigma^{\tilde{\mathbb{R}}^{m_0}} I_n(Y_0)^{S^1} \to \Sigma^{\tilde{\mathbb{R}}^{m_1}} I_n(Y_1)^{S^1}_{[0]}.$$

We may assume that m_0, m_1 are even and we can use Proposition 6.3.4 (1) to get the first statement.

If $b^+(X)$ is even, $\Sigma^{\tilde{\mathbb{R}}^{m_0} \oplus \mathbb{H}^{n_0}} I_n(Y_0)$ and $\Sigma^{\tilde{\mathbb{R}}^{m_1} \oplus \mathbb{H}^{n_1}} I_n(Y_1)$ are of SWF type at even levels and we can apply Proposition 6.3.4 (1), (2) to f_n to obtain the second statement. If $b^+(X)$ is odd, we take a connected sum $X \# S^2 \times S^2$, and then we can apply Proposition 6.3.4. In this second part, we take advantage of the fact that $\kappa(Y, \mathfrak{s}) \bmod 2$ agrees with the Rokhlin invariant, as is used in [37, Proof of Theorem 1.4]. ∎

Corollary 6.3.12. *Let* $(X, \mathfrak{t})$ *be a compact spin* 4*-manifold with boundary* Y*. Assume that the index bundle* ind D_Y *is zero in* $KQ^1(B_Y)$*. Then we have*

$$-\frac{\sigma(X)}{8} - 1 \leq b^+(X) + \kappa(Y, \mathfrak{t}|_Y).$$

Moreover, if $b^+(X) > 0$ *we have*

$$-\frac{\sigma(X)}{8} + 1 \leq b^+(X) + \kappa(Y, \mathfrak{t}|_Y).$$

Proof. Removing a small disk from X, we get a bordism X' with boundary $S^3 \coprod Y$. Since $\kappa(S^3) = 0$ and S^3 is Floer $K_{\mathrm{Pin}(2)}$-split, applying Theorem 6.3.11 to X', we obtain the inequalities. ∎

Since the spin bordism group Ω_3^{spin} is zero, we obtain the following.

Corollary 6.3.13. $\kappa(Y, \mathfrak{s}) > -\infty$.

Example 6.3.14. Let $\mathfrak{s}$ be a spin structure on $S^1 \times S^2$. Since $S^1 \times S^2$ has a positive scalar curvature metric g, the conditions of Theorem 4.1.2 are satisfied. Hence $\mathcal{SWF}(Y, \mathfrak{s}, \mathfrak{S}) \cong S^0_{B_Y}$. Here, $\mathfrak{S}$ is a spectral system with $P_0 = \mathcal{E}_0(D)^0_{-\infty}$. Also we have $n(S^1 \times S^2, g, \mathfrak{s}, P_0) = 0$, because there is an orientation-reversing diffeomorphism of $S^1 \times S^2$. So we obtain

$$\kappa(S^1 \times S^2, \mathfrak{s}) \leq 0.$$

Note that $\mathfrak{s}$ extends to a spin structure $\mathfrak{t}$ on $S^1 \times D^3$. Applying Theorem 6.3.12 to $(S^1 \times D^3)\#(S^2 \times S^2)$, we get $\kappa(S^1 \times S^2, \mathfrak{s}) \geq 0$. Hence

$$\kappa(S^1 \times S^2, \mathfrak{s}) = 0.$$

If X is a compact, oriented, spin 4-manifold with boundary $S^1 \times S^2$ and with $b^+(X) > 0$, we have

$$-\frac{\sigma(X)}{8} + 1 \leq b^+(X)$$

by Corollary 6.3.12. This inequality can be also obtained from the $\frac{10}{8}$-inequality [21] for the closed 4-manifold $X \cup (S^1 \times D^3)$ and the additivity of the signature.

Example 6.3.15. Let Y be the flat 3-manifold and $\mathfrak{s}_1$, $\mathfrak{s}_2$, $\mathfrak{s}_3$ be the spin^c structures in Example 6.2.15. As in Example 6.2.15, for any underlying spin structure, we have

$$\kappa(Y, \mathfrak{s}_j) \leq 0$$

for $j = 1, 2, 3$.

Example 6.3.16. Let $p\colon Y \to \Sigma$ be the sphere bundle of the complex line bundle N_d on a closed, oriented surface Σ of degree d. Assume that d is even and that $0 < g(\Sigma) < \frac{d}{2} + 1$. Using a connection on N_d, we have an identification

$$TN_d = p^* T\Sigma \oplus p^* N_d.$$

Let $s\colon Y \to p^* N_d|_Y$ be the tautological section. Then we have

$$TY = p^* T\Sigma \oplus i\mathbb{R}s. \tag{6.3.1}$$

Choose spin structures of Σ and N_d. This is equivalent to choosing complex line bundles $K_\Sigma^{\frac{1}{2}}$, $N_d^{\frac{1}{2}}$ and isomorphisms $K_\Sigma^{\frac{1}{2}} \otimes K_\Sigma^{\frac{1}{2}} \cong K_\Sigma$, $N_d^{\frac{1}{2}} \otimes N_d^{\frac{1}{2}} \cong N_d$. Also we consider the natural spin structure of the trivial bundle $i\mathbb{R}s$. The spin structures of Σ, $i\mathbb{R}s$ and (6.3.1) induce a spin structure $\mathfrak{s}'$ on Y. Note that $p^*(N_d^{\frac{1}{2}} \otimes N_d^{\frac{1}{2}}) \cong p^* N_d = \underline{\mathbb{C}}$ and hence the structure group of $p^* N_d^{\frac{1}{2}}$ is $\{\pm 1\}$. Put $\mathfrak{s} := \mathfrak{s}' \otimes p^* N_d^{\frac{1}{2}}$. Then $\mathfrak{s}$ is a spin structure of Y with spinor bundle $\mathbb{S} = p^*((K_\Sigma^{-\frac{1}{2}} \oplus K_\Sigma^{\frac{1}{2}}) \otimes N_d^{\frac{1}{2}})$. The spin^c structure induced by $\mathfrak{s}$ is $\mathfrak{s}_{g-1+\frac{d}{2}}$ of Proposition 4.2.3. Since $g \leq g - 1 + \frac{d}{2} < d$, we can apply Theorem 4.2.5 and we get

$$\mathcal{SWF}^{\mathrm{Pin}(2)}(Y, \mathfrak{s}, [\mathfrak{S}]) \cong S_B^0.$$

Here, $\mathfrak{S}$ is as in Theorem 4.2.5. Taking q to be $g - 1 + \frac{d}{2}$ in (6.2.2), we have

$$n(Y, \mathfrak{s}, g_r, P_0) = \frac{1}{8}.$$

Thus we obtain

$$\kappa(Y, \mathfrak{s}) \leq -\frac{1}{8}.$$