Appendix A

The Conley index and parameterized stable homotopy

In this appendix we define the category in which the Seiberg–Witten stable homotopy type lives, and variations thereon, as well as some background on the Conley index. Let G be a compact Lie group for this section. In Section A.1 we define parameterized homotopy categories we will be interested in. In Section A.2 we give basic definitions for the Conley index. In Section A.3 we give a definition of spectra suitable for the construction. The main point is Theorem A.2.1, which states that the parameterized homotopy class of the (parameterized) Conley index is well defined as a parameterized equivariant homotopy class in $\mathcal{K}_{G,Z}$.

A.1 The unstable parameterized homotopy category

This section is intended both to introduce some notation and to point out that the notions introduced in [43] are compatible with parameterized, equivariant homotopy theory, as considered in [16, 39].[1] In the first part, we follow the discussion of Costenoble–Waner [16, Chapter II] and Mrozek–Reineck–Srzednicki [43, Section 3]. In particular, we will occasionally use the notation of model categories, but the reader unfamiliar with this language may safely ignore these aspects. The main points are Lemma A.1.4, which lets us translate properties from the language of [43] to that of [39], and Proposition A.1.6, which is used in describing the change of the Conley index of approximate Seiberg–Witten flows upon changing the finite-dimensional approximation.

Definition A.1.1. Fix a compactly generated space Z with a continuous G-action. A triple $\mathbf{U} = (U, r, s)$ consisting of a G-space U and G-equivariant continuous maps $r: U \to Z$ and $s: Z \to U$ such that $r \circ s = \mathrm{id}_Z$ is called an (equivariant) *ex-space* over Z.[2] Let $\mathcal{K}_{G,Z}$ be the category of ex-spaces, where morphisms $(U, r, s) \to (U', r', s')$ are given by maps $f: U \to U'$ so that $r' f = r$ and $f s = s'$.

In comparison to the ordinary homotopy category, passing to the parameterized homotopy category results in many more maps (for a highbrow definition of the parameterized homotopy category, refer to Remark A.1.3).

[1] Establishing that [43] and [16, 39] are compatible is, in fact, straightforward. However, at the time that [43] appeared, the May–Sigurdsson parameterized homotopy category had not yet appeared.

[2] In [43], ex-spaces are called *fiberwise-deforming spaces*.

Definition A.1.2. A *fiberwise-deforming map* $f: \mathbf{U} \to \mathbf{U}'$ is an equivariant continuous map $f: (U, s(Z)) \to (U', s'(Z))$ so that $r' \circ f$ is (equivariantly) homotopic to r, relative to $s(Z)$. We say that fiberwise-pointed spaces $\mathbf{U}$ and $\mathbf{U}'$ are *fiberwise-deforming homotopy equivalent* if there exist continuous G-equivariant maps $f: \mathbf{U} \to \mathbf{U}'$, $g: \mathbf{U}' \to \mathbf{U}$ so that

$$\begin{aligned} f \circ s &= s', & g \circ s' &= s, \\ r' \circ f &\simeq r \operatorname{rel} s(Z), & r \circ g &\simeq r' \operatorname{rel} s'(Z), \\ g \circ f &\simeq \mathrm{id}_U \operatorname{rel} s(Z), & f \circ g &\simeq \mathrm{id}_{U'} \operatorname{rel} s'(Z). \end{aligned}$$

We write $[\mathbf{U}]$ for the fiberwise homotopy type of $\mathbf{U}$. We will call a fiberwise-deforming map, along with the choice of a homotopy h between $r' \circ f$ and r, a *lax map*, following [16].

We can also consider homotopies of fiberwise-deforming maps. A *homotopy* of fiberwise-deforming maps will mean a collection of fiberwise-deforming maps $F_t: \mathbf{U} \to \mathbf{U}'$, so that $F: U \times I \to U'$ is continuous. Homotopy of lax maps is similar, but requiring that the homotopy involved in the definition of a lax map is compatible, as we will define below.

Remark A.1.3. There is a model structure (what May–Sigurdsson call the *q-model structure*) on $\mathcal{K}_{G,Z}$ given by declaring a map in $\mathcal{K}_{G,Z}$ to be a weak equivalence, fibration, or cofibration, if it is such after forgetting the base Z, but May–Sigurdsson point out technical difficulties with this model structure. They define a variant, the *qf-model structure* on $\mathcal{K}_{G,Z}$, whose weak equivalences are those of the q-model structure, but with a smaller class of cofibrations. Let $\mathrm{Ho}\,\mathcal{K}_{G,Z}$ denote the homotopy category of the qf-model structure; we call this the *parameterized homotopy category* and write $[X, Y]_{G,Z}$ for the morphism sets of $\mathrm{Ho}\,\mathcal{K}_{G,Z}$ – these turn out to be the same as the lax maps X to Y up to homotopy, as in [16, Section 2.1].

Let ΛZ denote the set of *Moore paths* of Z:

$$\Lambda Z = \left\{(\lambda, \ell) \in Z^{[0,\infty]} \times [0, \infty) : \lambda(r) = \lambda(\ell) \text{ for } r \geq \ell\right\}.$$

Recall that Moore paths have a strictly associative composition:

$$(\lambda\mu)(t) = \begin{cases} \lambda(t) & \text{if } t \leq \ell_\lambda, \\ \mu(t - \ell_\lambda) & \text{if } t \geq \ell_\lambda. \end{cases}$$

Given $r: X \to Z$, the *Moore path fibration* $LX = L(X, r)$ is defined by

$$LX = X \times_Z \Lambda Z,$$

and there is an inherited projection map $Lr: LX \to Z$ by $Lr((x, \lambda)) = \lambda(\infty)$, as well as an inherited section map $Ls: Z \to LX$ given by $Ls(b) = (s(b), b)$, the path with length zero at $s(b)$. Finally, there is a natural inclusion $\iota\colon X \to LX$, which is a weak-equivalence on total spaces, and hence a weak equivalence in the qf-model structure.

Note that a lax map $X \to Y$ is equivalent to the data of a genuine map $X \to LY$ in $\mathcal{K}_{G,Z}$ (using that Y and LY are weakly equivalent, and basic properties of model categories). In particular, any lax map defines an element of $[X, Y]_{G,Z}$, which may or may not be represented by a map $X \to Y$ in $\mathcal{K}_{G,Z}$. The following lemma is then immediate from the definitions.

Lemma A.1.4. *Fiberwise-deforming homotopy-equivalent spaces are weakly equivalent in $\mathcal{K}_{G,Z}$.*

A *homotopy* between lax maps $f_0\colon X \to Y$ and $f_1\colon X \to Y$ is a lax map $X \wedge_Z [0,1]_+ \to Y$ so that $f|_{X\wedge i} = f_i$ for $i = 0, 1$. By [16, Section 2.1] the homotopy classes of lax maps are in agreement with $[X, Y]_{G,Z}$.

We will encounter collections of fiberwise-deforming spaces related by suspensions. We have the following definition.

Definition A.1.5 ([43, Section 3.10]). Let $\mathbf{U} = (U, r, s)$ and $\mathbf{U}' = (U', r', s')$ be ex-spaces over Z, Z', where U, Z are G-spaces and U', Z' are G'-spaces, for G, G' compact Lie groups. Define an equivalence relation $\sim_\wedge$ on $U \times U'$ by $(u, u') \sim_\wedge (v, v')$ if $(u, u') = (v, v')$ or $u = v \in s(Z)$, $r'(u') = r'(v')$ or $r(u) = r(v)$, $u' = v' \in s'(Z')$. Define the *fiberwise smash product* by

$$U \wedge U' := U \times U' / \sim_\wedge .$$

We call an ex-space $\mathbf{U}$ *well pointed* if the inclusion $s(Z) \to U$ is a cofibration in the category of G-spaces. That is, we require that $s(Z) \subset U$ admits a G-equivariant Strøm structure (for a definition see [43, Section 3]). We record the following result from [43] (the proof in the equivariant case is identical to that for the nonequivariant case).

Proposition A.1.6 ([43, Proposition 3.10]). *Assume that* $\mathbf{U}$, $\mathbf{U}'$, $\mathbf{V}$, $\mathbf{V}'$ *are fiberwise well-pointed spaces, with* $[\mathbf{U}] = [\mathbf{U}']$ *and* $[\mathbf{V}] = [\mathbf{V}']$. *Then* $[\mathbf{U} \wedge \mathbf{V}] = [\mathbf{U}' \wedge \mathbf{V}']$.

There is also a pushforward for ex-spaces defined in [39]. Fix an ex-object $\mathbf{U}$ given by $Z \to^s U \to^r Z$ and a map $f\colon Z \to Y$. Define $f_!\mathbf{U} = (f_!U, t, q)$ by the retract diagram

$$\begin{array}{ccc}
Z & \longrightarrow & Y \\
\downarrow & & \downarrow \\
\mathbf{U} & \longrightarrow & f_!U \\
\downarrow & & \downarrow \\
Z & \longrightarrow & Y,
\end{array}$$

where the top square is a pushout, and the bottom is defined by the universal property of pushouts, along with the requirement that $q \circ t = \mathrm{id}$.

Proposition A.1.7 ([39, Proposition 7.3.4]). *Say that* $\mathbf{U}$ *and* $\mathbf{U}'$ *are weakly equivalent* G*-ex-spaces. Then* $f_!\mathbf{U} \simeq f_!\mathbf{U}'$.

Note the simple example that for $\mathbf{U}$ a sectioned spherical fibration over Z, and $f\colon Z \to *$ the collapse, $f_!\mathbf{U}$ is the Thom complex.

For W a real G-vector space and $\mathbf{U} \in \mathcal{K}_{G,Z}$, we define $\Sigma^W \mathbf{U} = \mathbf{U} \wedge W^+$, where W^+ is considered as a parameterized space over a point (we consider $\mathbf{U} \wedge W^+$ as a G-fiberwise deforming space by pulling back along the diagonal map $G \to G \times G$). By Proposition A.1.7, this is well defined on the level of homotopy categories.

Remark A.1.8. For two ex-spaces $\mathbf{U}, \mathbf{U}'$, there is a fiberwise product $\mathbf{U} \times_Z \mathbf{U}'$, which is naturally an ex-space (whose structure maps are inherited from the universal properties of pullbacks), and similarly we obtain a fiberwise smash product $\mathbf{U} \wedge_Z \mathbf{U}'$. That is, we have a functor $\wedge_Z\colon \mathrm{Ho}\,\mathcal{K}_{G,Z} \times \mathrm{Ho}\,\mathcal{K}_{G,Z} \to \mathrm{Ho}\,\mathcal{K}_{G,Z}$. By [39, Proposition 7.3.1], $\wedge_Z$ descends to homotopy categories. The main implication of this from our perspective is that it is legitimate to suspend Conley indices by nontrivial sphere bundles over the base Z.

Definition A.1.9. Fix B a finite G-CW complex. The G-equivariant *parameterized Spanier–Whitehead category* PSW_B is defined as follows. The objects are pairs $(\mathbf{U}, R)$, also denoted by $\Sigma_B^R \mathbf{U}$, for $\mathbf{U}$ an element of $\mathcal{K}_{G,Z}$ (with total space U a finite G-CW complex) and R a virtual real finite-dimensional G-vector space (in a fixed universe). Morphisms are given by

$$\hom((\mathbf{U}, R), (\mathbf{U}', R')) = \mathrm{colim}_W [\Sigma^{W+R}\mathbf{U}, \Sigma^{W+R'}\mathbf{U}']_{G,B},$$

where the colimit is over sufficiently large W. A *stable homotopy equivalence* in $PSW_{G,B}$ will be a stable map that admits some representative which is a weak equivalence. We write $(\mathbf{U}, R) \simeq_{PSW} (\mathbf{U}', R')$ to denote stable homotopy equivalence, omitting the subscript if clear from the context. A *parameterized* G*-equivariant stable homotopy type* is an equivalence class of objects in $PSW_{G,B}$ up to stable homotopy equivalence.

In Definition A.1.9, the colimit may be taken over any sequence of representations which is cofinal in the universe. In particular, in the case of S^1 and Pin(2)-spaces, we will fix the following definitions.

Let $\mathcal{U}_{S^1} = \underline{\mathbb{C}}^{\oplus\infty} \oplus \underline{\mathbb{R}}^{\oplus\infty}$, where $\mathbb{C}$ is the standard representation of $U(1)$, and $\mathbb{R}$ is the trivial representation. Let $\mathcal{U}_{\mathrm{Pin}(2)} = \underline{\mathbb{H}}^{\oplus\infty} \oplus \underline{\widetilde{\mathbb{R}}}^{\oplus\infty}$, where $\mathbb{H}$ is the quaternion representation of Pin(2) and $\widetilde{\mathbb{R}}$ is the sign representation. There is a full subcategory $\mathfrak{C}_{S^1}$ of $PSW_{S^1,B}$ obtained by considering only those spaces $(\mathbf{U}, R)$ with $R = \underline{\mathbb{C}}^{\oplus n} \oplus \underline{\mathbb{R}}^{\oplus m}$, with $m, n \in \mathbb{Z}$; we use the shorthand $(\mathbf{U}, -2n, -m)$ to denote $(\mathbf{U}, R)$ in $\mathfrak{C}_{S^1}$.

Note that every element of $PSW_{S^1,B}$ on $\mathcal{U}_{S^1}$ is stable homotopy equivalent to an element of $\mathfrak{C}_{S^1}$. Similarly, we write $\mathfrak{C}_{\mathrm{Pin}(2)}$ for the subcategory whose objects are tuples $(\mathbf{U}, R)$ in $PSW_{\mathrm{Pin}(2),B}$ with

$$R = \underline{\mathbb{H}}^{\oplus n} \oplus \underline{\widetilde{\mathbb{R}}}^{\oplus m}.$$

We write $(\mathbf{U}, -4n, -m)$ for the resulting element (so that the notation is consistent with the forgetful functor from Pin(2)-spaces to S^1-spaces).

We note that PSW_*, the parameterized Spanier–Whitehead category over a point, is exactly the ordinary Spanier–Whitehead category. The next lemma follows from the definitions.

Lemma A.1.10. *Let $f: B \to *$. There is an induced functor $f_!: PSW_B \to PSW_*$ defined by $f_!(\mathbf{U}, R) = (f_!\mathbf{U}, R)$ so that $(\mathbf{U}, R) \simeq_{PSW_B} (\mathbf{U}', R')$ implies $f_!(\mathbf{U}, R) \simeq_{PSW_*} f_!(\mathbf{U}', R')$.*

We have the following corollary.

Corollary A.1.11. *Let $f: B \to *$. Then stable-homotopy equivalence classes in PSW_B give well-defined stable-homotopy classes in PSW_*.*

Finally, we remark that May–Sigurdsson [39, Chapters 20–22] define many parameterized homology theories, suitably generalizing the usual definition of a (usual) homology theory, and giving convenient invariants from objects of PSW_*.

A.2 The parameterized Conley index

In this subsection we review the *parameterized Conley index* from [43] (see also Bartsch [7]); we note that we work in considerably less generality than they present. We start by giving the basic definitions in Conley index theory, following [35, Section 5]. Note that the authors of [43] work nonequivariantly; the proofs in the equivariant case are similar.

Let M be a finite-dimensional manifold and φ a flow on M; for a subset $N \subset M$, we define the following sets:

$$\begin{aligned} N^+ &= \{x \in N : \forall t > 0,\ \varphi_t(x) \in N\}, \\ N^- &= \{x \in N : \forall t < 0,\ \varphi_t(x) \in N\}, \\ \operatorname{inv} N &= N^+ \cap N^-. \end{aligned}$$

A compact subset $S \subset M$ is called an *isolated invariant set* if there exists a compact neighborhood $S \subset N$ so that $S = \operatorname{inv}(N) \subset \operatorname{int}(N)$. Such a set N is called an *isolating neighborhood* of S.

A pair (N, L) of compact subsets $L \subset N \subset M$ is an *index pair* for S if the following hold:

(1) $\mathrm{inv}(N \setminus L) = S \subset \mathrm{int}(N \setminus L)$.

(2) L is an exit set for N, that is, for any $x \in N$ and $t > 0$ so that $\varphi_t(x) \notin N$, there exists $\tau \in [0,t)$ with $\varphi_\tau(x) \in L$.

(3) L is *positively invariant* in N. That is, for $x \in L$ and $t > 0$, if $\varphi_{[0,t]}(x) \subset N$, then $\varphi_{[0,t]}(x) \subset L$.

For an index pair $P = (P_1, P_2)$ of an isolated invariant set S, we define $\tau_P \colon P_1 \to [0,\infty]$ by

$$\tau_P(x) = \begin{cases} \sup\{t \geq 0 : \varphi_{[0,t]}(x) \subset P_1 \setminus P_2\} & \text{if } x \in P_1 \setminus P_2, \\ 0 & \text{if } x \in P_2. \end{cases}$$

We say that an index pair P is *regular* if τ_P is continuous.

For Z a Hausdorff space, $\omega\colon M \to Z$ a continuous map, and a regular index pair $P = (P_1, P_2)$, define the *parameterized Conley index* $I_\omega(P)$ as $P_1 \cup_{\omega|_{P_2}} Z$, namely,

$$I_\omega(P) = (Z \times 0) \cup (P_1 \times 1)/\sim,$$

where $(x,1) \sim (\omega(x), 0)$ for all $x \in P_2 \times 1$.

The space $I_\omega(P)$ is naturally an ex-space, with embedding $s_P\colon Z \to I_\omega(P)$ given by $z \to [z,0]$, and projection $r_P\colon I_\omega(P) \to Z$ given by $r_P([x,1]) = \omega(x)$, $r_P([z,0]) = z$. By construction, $r_P \circ s_P = \mathrm{id}_Z$.

For $Z = *$, we sometimes write $I^u(P)$ for $I_\omega(P)$, to specify the "unparameterized" Conley index.

Theorem A.2.1 ([43, Theorem 2.1]). *If P and Q are two regular index pairs for an isolated invariant set S, then $(I_\omega(P), r_P, s_P)$ and $(I_\omega(Q), r_Q, s_Q)$ have the same equivariant homotopy type over Z, and are both fiberwise well pointed.*

Proof. In [43] it is proved that the two indices have the same fiberwise-deforming type; Lemma A.1.4 then implies the statement. The well-pointedness is [43, Proposition 6.1]. ∎

Definition A.2.2 ([13], [47, Definition 2.6]). A *connected simple system* is a collection I_0 of pointed spaces along with a collection of I_h of homotopy classes of maps among them, so that

(1) for each pair $X, X' \in I_0$, there is a unique class $[f] \in I_h$ from $X \to X'$;

(2) for $f, f' \in I_h$ with $f\colon X \to X'$ and $f'\colon X' \to X''$, the composite $f' \circ f$ is in I_h;

(3) for each $X \in I_0$, the morphism $f\colon X \to X$ is [id].

Of course, the notion of a connected simple system has an obvious generalization in any category with an associated homotopy category.

Theorem A.2.3 ([47]). *Fix notation as in Theorem A.2.1. The* unparameterized Conley indices $I^u(P) = I_\omega(P)/Z$*, ranging over regular index pairs for* S*, form a* connected simple system.

We conjecture that in fact the parameterized Conley indices also have this property.

Conjecture A.2.4. Fix notation as in Theorem A.2.1. Then the parameterized Conley indices $(I_\omega(P), r_P, s_P)$, running over all regular index pairs for the isolated invariant set S, form a connected simple system.

In Chapter 3 we encounter the parameterized Conley indices for product flows. We have the following theorem.

Theorem A.2.5 ([43, Theorem 2.4]). *Let* S*,* S' *be isolated invariant sets for* φ*,* φ'*. Then*

$$I_{\omega\times\omega'}(S \times S', \varphi \times \varphi') \simeq I_\omega(S, \varphi) \wedge I_{\omega'}(S', \varphi').$$

Moreover, the usual deformation invariance of the Conley index continues for the parameterized Conley index.

Theorem A.2.6 ([43, Theorem 2.5], [47, Corollary 6.8]). *If* N *is an isolating neighborhood with respect to flows* φ^λ *continuously depending on* $\lambda \in [0, 1]$*, with a continuous family of isolated invariant sets* S^λ *inside of* N*, then the fiberwise-deforming homotopy type of* $I_\omega(S^\lambda, \varphi^\lambda)$ *is independent of* λ*.*

In the case of the unparameterized Conley index, for each $\lambda_1, \lambda_2 \in [0, 1]$*, there is a well-defined, up to homotopy, map of connected simple systems:*

$$F(\lambda_1, \lambda_2)\colon I^u(S^{\lambda_1}, \varphi^{\lambda_1}) \to I^u(S^{\lambda_2}, \varphi^{\lambda_2}).$$

Furthermore, for all $\lambda_1, \lambda_2, \lambda_3 \in [0, 1]$*,*

$$F(\lambda_2, \lambda_3) \circ F(\lambda_1, \lambda_2) \sim F(\lambda_1, \lambda_3),$$
$$F(\lambda_1, \lambda_1) \sim \mathrm{id}.$$

Lemma A.2.7. *Fix a flow* φ *on a manifold* X*, along with a map* $p\colon X \to B$*, and write* $\pi\colon B \to *$ *as the map collapsing* B *to a point. Then the pushforward of the parameterized Conley index* $I(\varphi)$*, namely* $\pi_! I(\varphi)$*, is the ordinary Conley index* $I^u(\varphi)$*.*

Proof. This is immediate from the definitions. ∎

We also note the behavior under time reversal.

Theorem A.2.8 ([15, Theorem 3.5], [38, Proposition 3.8]). *Let* M *be a stably parallelized* G*-manifold for a compact Lie group* G*. For* φ *a flow on* M*, the (unparameterized) Conley index of an isolated invariant set* S *with respect to the time-reversed flow* $-\varphi$*, denoted* $I^u(S, -\varphi)$*, is equivariantly Spanier–Whitehead dual to* $I^u(S, \varphi)$*.*

A.3 Spectra

For G a compact Lie group, we define a G-*universe* $\mathcal{U}$ to be a countably infinite-dimensional orthogonal representation of G.

Definition A.3.1. Let $\mathcal{U}$ be a universe with a direct sum decomposition $\mathcal{U} = \bigoplus_{i=1}^{n} V_i^{\infty}$, for finite-dimensional G-representations V_i. A sequential G-spectrum X on $\mathcal{U}$ is a collection $X(V)$ of spaces, indexed on the subspaces of $\mathcal{U}$ of the form $V = \bigoplus_{i=1}^{n} V_i^{k_i}$ for some $k_i \geq 0$, along with transition maps, whenever $W \subset V$,

$$\sigma_{V-W} : \Sigma^{V-W} X(W) \to X(V),$$

where $V - W$ is the orthogonal complement of W in V. For $V = W$, the transition map is required to be the identity, and the maps σ are required to be transitive in the usual way. The space $X(V)$ is sometimes referred to as the Vth *level* of the spectrum.

If σ_{V-W} is a homotopy equivalence for V, W sufficiently large, we say that X is a G-*suspension spectrum.*

We will only work with suspension spectra in this memoir.

A *morphism* of spectra $X \to Y$ will be a collection of morphisms

$$\phi_V : X(V) \to Y(V)$$

compatible with the transition maps.

We will also consider a generalization of morphisms, as follows.

Definition A.3.2. A *weak* morphism of spectra $\phi: X \to Y$ is a collection of morphisms

$$\phi_V : X(V) \to Y(V)$$

for V sufficiently large, so that the diagram

$$\begin{array}{ccc} \Sigma^{W-V} X(V) & \xrightarrow{\Sigma^{W-V}\phi_V} & \Sigma^{W-V} Y(V) \\ \downarrow & & \downarrow \\ X(W) & \xrightarrow[\phi_W]{} & Y(W) \end{array}$$

homotopy commutes for W sufficiently large. Weak morphisms ϕ_0, ϕ_1 are said to be *homotopic* if there exists a weak morphism $\phi_{[0,1]}: X \wedge [0,1]_+ \to Y$ restricting to ϕ_j at $X \wedge \{j\}$ for $j = 0, 1$.

We will also need the notion of a *connected simple system of spectra.* Indeed, instead of using the direct generalization for spaces, the Seiberg–Witten Floer spectrum, as currently defined, requires that we work with weak morphisms instead, as follows.

Definition A.3.3. A *connected simple system of G-spectra* is a collection I_0 of G-spectra, along with a collection I_h of weak homotopy classes of maps between them, so that the analogs of (1)–(3) of Definition A.2.2 are satisfied.

Remark A.3.4. In Section 3.5 we could have used nonsequential G-spectra, but we have no need for the added generality in the memoir, and it slightly complicates the notation.

Remark A.3.5. If higher naturality is established for the Conley index, then it would be possible to replace *weak* morphisms in the definition of **SWF**, and Definition A.3.3 could be replaced with ordinary morphisms of spectra.