Afterword: Finite-dimensional approximation in other settings

Outside of Seiberg–Witten theory, we expect that the notion of parameterized finite-dimensional approximation may be applicable in some cases in symplectic topology. The methods of this memoir rely, roughly speaking, on a few special features of the Seiberg–Witten equations, relative to other Floer-type problems:

(1) The configuration space is naturally a bundle over a compact, finite-dimensional manifold.

(2) Bubbling phenomena do not occur.

(3) With respect to the bundle structure in (1), the Seiberg–Witten equations are "close to linear" on the fibers.

(4) There is a relatively good understanding of the spectrum of the Dirac operator.

Perhaps the item most likely to elicit worry more generally is (1). However, we note that it is classical that for any compact subset K of a Hilbert manifold, there is an open sub-Hilbert-manifold B containing K which is diffeomorphic to the total space of a Hilbert bundle over a compact finite-dimensional manifold.

Lemma 1. *Let M be a separable Hilbert manifold and $K \subset M$ a compact subset. Then there exists some open $U \supset K$ diffeomorphic to $V \times H$, where H is a separable Hilbert space, and V is a finite-dimensional smooth manifold.*

Proof. By compactness, choose a good open cover $\mathcal{C}'$ of K, with finite subcover $\mathcal{C} = \{U_i\}_i$, which is once again good, with $U = \bigcup_i U_i$. The nerve $N(\mathcal{C})$ is then homotopy equivalent to U. Moreover, $N(\mathcal{C})$ may be embedded in some finite-dimensional Euclidean space and has a regular neighborhood which is a smooth manifold V, with $N(\mathcal{C}) \simeq V$. By [10, 41], separable infinite-dimensional homotopy-equivalent Hilbert manifolds are diffeomorphic. Then $V \times H$ is diffeomorphic to U, as needed. ∎

In particular, (1) holds locally around the moduli space (of gradient flows of the Chern–Simons functional, symplectic action, etc.) in many situations of interest (there is the technical point that a version of Lemma 1 which respected L^2_k-norms for multiple values of k would be more appropriate, but we have not attempted it). Although it is not at all clear how to perform finite-dimensional approximation in the presence of bubbling, nonetheless items (2) and (4) also hold in various geometric situations. The problem then amounts to establishing appropriate versions of (3) in specific situations; this appears challenging except when the configuration space is very special.

We finally note that the finite-dimensional approximation process of this memoir can also be applied locally. In particular, it can be applied in the neighborhood of a broken trajectory. Here, the base space is some smooth trajectory very close to the broken trajectory, so that there is a neighborhood containing the broken trajectory, and on which (1)–(4) hold. Finite-dimensional approximation then produces a sequence of flows, whose finite-energy integral curves converge to solutions of the Seiberg–Witten equations. Assuming nondegeneracy, one may be able to assemble these locally constructed approximating submanifolds into the data of a flow category as in [11]. The hoped-for result of this process would be replacing the need to give a smooth structure to the corners for the moduli spaces of the Seiberg–Witten equations themselves, with the problem of putting a smooth structure on the trajectory spaces of a finite-dimensional approximation. The main obstruction to this approach is likely the need to establish that the approximating submanifolds constructed this way are suitably independent of the choices involved in their construction, which may be difficult.