Chapter 1

Introduction

Over the past century, the theory of operator algebras has given rise to an abundance of non-commutative analogues of classical mathematical theories, many of which have grown into successful independent research areas with exciting applications in theoretical physics. Prominent examples of this phenomenon include the theory of quantum groups [43, 79], non-commutative geometry à la Connes [14, 16–18], and Rieffel's theory of quantum metric spaces [68, 70], which constitute far-reaching non-commutative generalisations of classical topological groups, Riemannian (spin) manifolds and compact metric spaces, respectively. Despite a continuous effort, it has proven very challenging to reconcile the theory of non-commutative geometry with the theory of quantum groups [19, 57], and even for the most fundamental q-deformation, Woronowicz' $\mathrm{SU}_q(2)$, it is not clear how one should modify Connes' axioms to obtain a non-commutative geometry which adequately reflects the underlying q-geometry. Numerous candidates for Dirac operators on $\mathrm{SU}_q(2)$ have been proposed [9, 13, 21, 38, 41, 42, 62], each having their advantages and disadvantages, but it seems unclear which (if any) of these provide $\mathrm{SU}_q(2)$ with the right kind of non-commutative geometry. At the time of writing, it is not even known if any of these Dirac operators give rise to a compact quantum metric structure, so also the connection between the metric geometry and the differential geometry on $\mathrm{SU}_q(2)$ is open[1].

The first aim of the present memoir is to remedy the latter problem by introducing a family of Dirac operators on $\mathrm{SU}_q(2)$ and showing that these give rise to compact quantum metric structures. More precisely, we investigate a new 2-parameter family of Dirac operators $D_{t,q}$, indexed over $(0, 1] \times (0, 1]$, which connect some of the existing constructions in that $D_{q,q}$ agrees with the Dirac operator suggested in [38] and $D_{1,q}$ is closely related to the one studied in [41]. It is important to stress that $D_{t,q}$ does not have bounded commutators with the coordinate algebra $\mathcal{O}(\mathrm{SU}_q(2))$ in general, and we therefore cannot obtain a genuine spectral triple. It does, however, decompose naturally into a "horizontal" part D_q^H and "vertical" part D_t^V, each of which admits bounded *twisted* commutators (though for different twists) with elements from $\mathcal{O}(\mathrm{SU}_q(2))$. The appearance of such twists seems inevitable when non-commutative

[1]Note that one may construct a Dirac operator from a length function on the dual $\widehat{\mathrm{SU}_q(2)}$ whose iterated commutators do give rise to a compact quantum metric space [8, Proposition 4.3 and Theorem 7.4], but this construction seems to be less related to the spin geometry of $\mathrm{SU}_q(2)$. Indeed, for $q = 1$ the resulting Dirac operator is very different from the classical Dirac operator since the K-homology class of the Dirac operator coming from a length function is trivial.

geometry is applied in the context of q-deformed spaces [19], and even though our constructions do not fit exactly with Connes' original axioms for a spectral triple, many of the properties of classical spectral triples admit suitable analogues in our twisted setting, as witnessed by the following result:

Theorem A (See Lemmas 3.6.4, 4.1.1, 4.3.1, 4.5.2, 4.6.1, Proposition 4.5.4 and Section 4.4). *The Dirac operators $D_{t,q} = D_t^V + D_q^H$ are selfadjoint and the following hold:*

(1) *There exists a one-parameter family $(\sigma_r)_{r\in(0,\infty)}$ of algebra automorphisms of $\mathcal{O}(\mathrm{SU}_q(2))$ such that the twisted commutators*

$$D_t^V \sigma_t(x) - \sigma_t^{-1}(x) D_t^V \quad \text{and} \quad D_q^H \sigma_q(x) - \sigma_q^{-1}(x) D_q^H,$$

extend to bounded operators $\partial_t^V(x)$ and $\partial_q^H(x)$ for all $x \in \mathcal{O}(\mathrm{SU}_q(2))$. Moreover, there exists a one-parameter family of unbounded, strictly positive operators $(\Gamma_r)_{r\in(0,\infty)}$ satisfying that $\sigma_r(x) = \overline{\Gamma_r^{-1} x \Gamma_r}$ for all $r \in (0,\infty)$ and all $x \in \mathcal{O}(\mathrm{SU}_q(2))$.

(2) *The Dirac operators $D_{t,q}$ are $\mathrm{SU}_q(2)$-equivariant, in the sense that acting on $(L^2(\mathrm{SU}_q(2)) \widehat{\otimes} L^2(\mathrm{SU}_q(2)))^{\oplus 2}$, the selfadjoint unbounded operators $1 \widehat{\otimes} D_t^V$ and $1 \widehat{\otimes} D_q^H$ commute with $W \oplus W$ where $W \in \mathbb{B}(L^2(\mathrm{SU}_q(2)) \widehat{\otimes} L^2(\mathrm{SU}_q(2)))$ denotes the multiplicative unitary for $\mathrm{SU}_q(2)$.*

(3) *There exists an antilinear unitary I with $I^2 = -1$ such that $D_{t,q}$ satisfies the first order condition $[\partial_t^V(x), IyI] = 0 = [\partial_q^H(x), IyI]$ for all $x, y \in \mathcal{O}(\mathrm{SU}_q(2))$. Moreover, I commutes with $D_{t,q}$ up to modular operators in the sense that the relations*

$$D_t^V \Gamma_t^{-1} \cdot I = I \cdot D_t^V \Gamma_t^{-1} \quad \text{and} \quad D_q^H \Gamma_q^{-1} \cdot I = I \cdot D_q^H \Gamma_q^{-1}$$

hold on the dense subspace $\mathcal{O}(\mathrm{SU}_q(2))^{\oplus 2} \subseteq L^2(\mathrm{SU}_q(2))^{\oplus 2}$.

(4) *When $t = q = 1$, the unbounded selfadjoint operator $D_{1,1}$ satisfies that $2 \cdot D_{1,1} + 1 = D_{S^3}$, where D_{S^3} is the classical Dirac operator on $\mathrm{SU}(2) \cong S^3$.*

As already mentioned, one of the main goals of the memoir is to investigate the metric geometry governed by the Dirac operators $D_{t,q}$, by connecting our construction to Rieffel's theory of compact quantum metric spaces [68, 70]. The data defining a compact quantum metric space consists of a unital C^*-algebra A (or, more generally, an operator system) endowed with a densely defined seminorm L, and the central requirement is that the Monge–Kantorovič extended[2] metric d_L on the state

[2] The adjective *extended* here means that d_L is a priory allowed to take the value infinity. Note, however, that by compactness this cannot be the case if d_L metrises the weak* topology.

space $\mathcal{S}(A)$, defined as

$$d_L(\mu,\nu) := \sup\{|\mu(a) - \nu(a)| \mid L(a) \leqslant 1\}, \tag{1.1}$$

metrises the weak* topology. The motivating example of course comes from taking a classical compact metric space (M, d) and associating to it the seminorm

$$L_{\mathrm{Lip}}(f) := \sup\left\{ \frac{|f(x) - f(y)|}{d(x,y)} \,\middle|\, x, y \in M, x \neq y \right\}$$

defined on the Lipschitz functions $\mathrm{Lip}(M)$. Rieffel's definition is also very much inspired by constructions appearing in non-commutative geometry; see [15]. Indeed, one can associate a natural seminorm L_D to a unital spectral triple (A, H, D) by setting

$$L_D(a) := \left\| \overline{[D,a]} \right\|, \tag{1.2}$$

whenever the element a belongs to a specified dense unital $*$-subalgebra $\mathcal{A} \subseteq A$ of "differentiable" operators. Note, however, that this construction does not always yield a compact quantum metric space, and even in the cases where this happens, the argument is often far from trivial; see, e.g., [2, 63, 69].

Having the Dirac operators $D_{t,q}$ at our disposal we obtain a family of seminorms $L_{t,q} \colon \mathcal{O}(\mathrm{SU}_q(2)) \to [0,\infty)$ by setting

$$L_{t,q}(x) := \left\| \partial_t^V(x) + \partial_q^H(x) \right\|$$

and we may ask whether they yield compact quantum metric structures on $C(\mathrm{SU}_q(2))$. Since the seminorm $L_{t,q}$ comes from two unbounded selfadjoint operators via a twisted commutator construction we may enlarge the domain considerably and replace the coordinate algebra $\mathcal{O}(\mathrm{SU}_q(2))$ with a much larger algebra of Lipschitz elements $\mathrm{Lip}_t(\mathrm{SU}_q(2))$. We are considering the resulting seminorm $L_{t,q}^{\max} \colon \mathrm{Lip}_t(\mathrm{SU}_q(2)) \to [0,\infty)$ as the "maximal" seminorm associated with our spectral data whereas $L_{t,q}$ is regarded as the corresponding "minimal" seminorm. In analogy with the classical case, there is a wide gap between the Hopf algebra $\mathcal{O}(\mathrm{SU}_q(2))$ of polynomial expressions in the generators and the Lipschitz algebra $\mathrm{Lip}_t(\mathrm{SU}_q(2))$, in so far that the intersection of the domain of the closures of the twisted derivations ∂_t^V and ∂_q^H on the coordinate algebra does not agree with the Lipschitz algebra $\mathrm{Lip}_t(\mathrm{SU}_q(2))$; see for example [25, Theorem 3.1]. One may compare the gap between the minimal and maximal seminorms to the gap between the unital C^*-algebra $C(\mathrm{SU}_q(2))$ and its weak closure, the von Neumann algebra $L^\infty(\mathrm{SU}_q(2))$. We are in this text presenting a thorough treatment of the maximal seminorms $L_{t,q}^{\max}$ and this is partly the reason for the appearance of a number of analytic challenges.

Theorem B (see Theorem 5.6.1). *The pair $(C(\mathrm{SU}_q(2)), L_{t,q}^{\max})$ is a compact quantum metric space for all $t, q \in (0,1]$.*

Enlarging the domain of a seminorm increases the difficulty of proving that it defines a compact quantum metric structure, so Theorem B immediately implies the corresponding statement for the minimal seminorm $L_{t,q}$.

Corollary C (see Corollary 5.6.2). *The pair* $(C(\mathrm{SU}_q(2)), L_{t,q})$ *is a compact quantum metric space for all* $t, q \in (0, 1]$.

To prove Theorem B, we develop a set of new general tools which are likely to have applications elsewhere, and we therefore briefly outline the main ideas involved. The central ingredient is the Podleś sphere $C(S_q^2)$ [66], which arises as the fixed point algebra of a certain circle action on $C(\mathrm{SU}_q(2))$ (providing a quantised analogue of the Hopf fibration). In contrast to $\mathrm{SU}_q(2)$, the non-commutative metric geometry of S_q^2 is reasonably well understood. The work of Dąbrowski and Sitarz [22] provides the Podleś sphere with a unital spectral triple, and it was furthermore proven in [2] that $C(S_q^2)$ becomes a compact quantum metric space when equipped with the corresponding seminorm from (1.2). The circle action defining the Podleś sphere also gives rise to an increasing sequence of finitely generated projective modules which suitably exhaust $C(\mathrm{SU}_q(2))$. These finitely generated projective modules are direct sums of spectral subspaces for the circle action and are referred to as spectral bands. The first step in proving Theorem B is to lift the compact quantum metric structure from $C(S_q^2)$ to the spectral bands and we develop the general theory to achieve this in Section 2.3. The second step is then to lift the compact quantum metric structure from the spectral bands all the way up to $C(\mathrm{SU}_q(2))$. Perhaps a bit surprisingly, the main aid here comes from the theory of Schur multipliers, and we unfold this aspect in Section 5.4.

One of the main virtues of Rieffel's theory of compact quantum metric spaces, is that it allows for a natural generalisation of the classical Gromov–Hausdorff distance between compact metric spaces [23, 26], naturally dubbed the quantum Gromov–Hausdorff distance [70]. This concept has been further developed by, among others, Kerr [39], Li [52–54] and Latrémolière [46–49], and by now exists in several different versions which take into account more structure than Rieffel's original definition. The existence of such a distance function allows one to study the class of compact quantum metric spaces from a more analytical point of view, and opens the possibility to investigate a wealth of natural continuity questions. Over the past two decades, many positive answers have been obtained, and examples include Rieffel's fundamental result that the 2-sphere can be approximated by the fuzzy spheres (matrix algebras) [71], as well as the more recent proof [3] that the Podleś spheres S_q^2 vary continuously in the deformation parameter $q \in (0, 1]$; for many more examples see [1, 35, 44, 50, 54, 70].

In light of Theorem B, the next natural question to ask is whether one obtains quantum Gromov–Hausdorff continuity in the deformation parameters t and q, and through a series of approximation arguments we are able to answer this in the affirmative:

Theorem D (See Theorem 7.3.1). *The quantum metric spaces* $(C(\mathrm{SU}_q(2)), L^{\max}_{t,q})$ *vary continuously in the deformation parameter* $(t,q) \in (0,1] \times (0,1]$ *with respect to the quantum Gromov–Hausdorff distance.*

We single out the following special case of Theorem D, which was the original motivation for the study undertaken in the present memoir. Denoting by d_{S^3} the usual round metric on $\mathrm{SU}(2) \cong S^3 \subseteq \mathbb{R}^4$ and by $L_{\mathrm{Lip}}\colon \mathrm{Lip}(\mathrm{SU}(2)) \to [0,\infty)$ the Lipschitz constant seminorm on $C(\mathrm{SU}(2))$ associated with the rescaled metric $2 \cdot d_{S^3}$, combining Theorems D and A yields the following:

Corollary E (see Corollary 7.3.2). *The quantum metric spaces* $(C(\mathrm{SU}_q(2)), L^{\max}_{t,q})$ *converge in quantum Gromov–Hausdorff distance to* $(C(\mathrm{SU}(2)), L_{\mathrm{Lip}})$ *as* (t,q) *tends to* $(1,1)$.

The rescaling of the metric on S^3 may at first sight seem strange, but it is exactly this factor of 2 which makes the Hopf fibration $S^3 \to S^2$ a Riemannian submersion when the 2-sphere is endowed with its round metric arising from the natural embedding into $\mathbb{R}^3$.

The road to Theorem D is quite long, but involves a number of constructions which are of independent interest. As in the case of the Podleś sphere [3], the key to such a continuity result is to construct an $\mathrm{SU}_q(2)$ version of the Berezin transform. By means of the Berezin transform we obtain finite dimensional compact quantum metric spaces $\mathrm{Fuzz}_N(B_q^K) \subseteq \mathcal{O}(\mathrm{SU}_q(2))$ indexed by $N, K \in \mathbb{N}_0$. We think of these compact quantum metric spaces as *fuzzy spectral bands*. These fuzzy spectral bands are $\mathcal{O}(\mathrm{SU}_q(2))$-coinvariant and it is possible to describe them explicitly in terms of the usual generators for $\mathrm{SU}_q(2)$. In Chapter 6 we construct our Berezin transform and prove that the fuzzy spectral bands approximate $\mathrm{SU}_q(2)$:

Theorem F (see Corollary 6.4.4). *The quantum metric spaces* $(\mathrm{Fuzz}_N(B_q^K), L_{t,q})$ *converge in quantum Gromov–Hausdorff distance to* $(C(\mathrm{SU}_q(2)), L^{\max}_{t,q})$ *as* N *and* K *tend to infinity.*

This theorem should be viewed as an $\mathrm{SU}_q(2)$-analogue of Rieffel's original result [71], showing that the 2-sphere can be approximated in quantum Gromov–Hausdorff distance by the fuzzy 2-spheres (matrix algebras). The concrete techniques used in the construction of the Berezin transform and the fuzzy spectral bands build on the corresponding constructions for the Podleś sphere developed in [3]. An interesting consequence of the above fuzzy approximation is that the maximal and minimal seminorm actually give rise to the same compact quantum metric structure on $\mathrm{SU}_q(2)$.

Theorem G (see Corollary 6.4.2). *The quantum Gromov–Hausdorff distance between* $(C(\mathrm{SU}_q(2)), L_{t,q}^{\max})$ *and* $(C(\mathrm{SU}_q(2)), L_{t,q})$ *is zero. Moreover, the Monge–Kantorovič metrics* $d_{t,q}^{\max}$ *and* $d_{t,q}$ *on* $\mathcal{S}(C(\mathrm{SU}_q(2)))$*, induced by the two seminorms via the formula* (1.1)*, agree.*

In particular, the continuity results in Theorem D and Corollary E, which pertain to the *maximal* seminorm $L_{t,q}^{\max}$ automatically hold true for the *minimal* seminorm $L_{t,q}$:

Corollary H. *The quantum metric spaces* $(C(\mathrm{SU}_q(2)), L_{t,q})$ *vary continuously in the deformation parameters* $(t,q) \in (0,1] \times (0,1]$ *with respect to the quantum Gromov–Hausdorff distance. In particular,* $(C(\mathrm{SU}_q(2)), L_{t,q})$ *converges to* $(C(\mathrm{SU}(2)), L_{\mathrm{Lip}})$ *as* (t,q) *tends to* $(1,1)$*.*

The rest of the memoir is structured as follows: Chapter 2 contains the necessary background on compact quantum metric spaces as well as the new tools needed for the present memoir. Chapter 3 contains a detailed introduction to $\mathrm{SU}_q(2)$. In Chapter 4 we introduce our family of Dirac operators and prove Theorem A. Chapter 5 is devoted to proving Theorem B and in Chapter 6 we construct the Berezin transform and prove Theorems F and G. The final Chapter 7 pieces everything together into a proof of the main continuity result, Theorem D.

1.1 Notation and standing assumptions

Unless otherwise stated, we shall always apply the notation $\|\cdot\|$ for the unique C^*-norm on a C^*-algebra A or, more generally, for its restriction to a complete operator system $X \subseteq A$. Since the Greek letter epsilon is the standard symbol both for the counit in a quantum group and an arbitrarily small positive number, we will use the symbol ϵ for the former and the symbol ε for the latter. As for tensor products, the symbols $\otimes$, $\otimes_{\min}$ and $\hat{\otimes}$ will denote algebraic, minimal C^*-algebraic, and Hilbert space tensor products, respectively. The theory of unbounded operators plays a central role in the memoir, and if T is an unbounded closable operator in a Hilbert space we will denote its closure by $\overline{T}$. Lastly, we will use the abbreviations WOT and SOT for the weak- and strong operator topology, respectively, and ucp for unital completely positive.

1.2 Note added in proof

Since the writing of the present memoir, the research on quantum metrics on q-deformations has progressed further. In [60], it was proven that the D'Andrea–Dąbrowski spectral triples provide all quantum projective spaces $\mathbb{C}P_q^{\ell}$ with compact quantum metric structures (the case $\ell = 1$ corresponding to the Podleś sphere), and in [34] the higher-dimensional Vaksman–Soibelman spheres were treated, thus providing a generalisation of Theorem B. The work [34] also features an updated treatment of finitely generated projective modules in the context of quantum metric spaces.