Chapter 2

Compact quantum metric spaces

In this chapter, we present the relevant preliminaries on compact quantum metric spaces. For our purposes, the theory of (concrete) operator systems provides the most convenient framework for studying compact quantum metric spaces, and we are thus in line with the recent developments in [20, 77], as well as the C^*-algebra based approaches in [52, 53, 68].

The theory discussed here is also closely related to Rieffel's original theory of *order unit compact quantum metric spaces* [70], via the passage from an operator system to its selfadjoint part (the real subspace of selfadjoint elements). The selfadjoint part of an operator system is indeed an order unit space and the two state spaces can be identified via restriction.

2.1 Definitions and basic properties

Throughout this section, X will be a *complete operator system*; i.e., X will be a norm-closed subspace of a specified unital C^*-algebra A_X such that X is invariant under the adjoint operation and contains the unit from A_X. A *state* on X is a positive linear functional $\mu: X \to \mathbb{C}$ which sends the unit 1_X in X to the unit 1 in $\mathbb{C}$. A state on X automatically has norm 1 [64], and the state space $\mathcal{S}(X)$ therefore becomes a compact Hausdorff space for the weak* topology. Although X is not an algebra, any selfadjoint $x \in X$ may still be written as a difference of positive elements from X as

$$x = \frac{1}{2}(\|x\| \cdot 1_X + x) - \frac{1}{2}(\|x\| \cdot 1_X - x),$$

and from this it follows that any positive map $\Phi: X \to Y$ into another operator system Y satisfies $\Phi(x^*) = \Phi(x)^*$. Lastly, we note the slight subtlety that Φ need not be a contraction, but that it is bounded with $\|\Phi\| \leqslant 2\|\Phi(1_X)\|$; see [64, Proposition 2.1]. If, however, Φ is *completely* positive then $\|\Phi\| = \|\Phi\|_{\text{cb}} = \|\Phi(1_X)\|$. Note also, that if Φ is unital and positive and x is selfadjoint then $-\|x\| \cdot 1_X \leqslant x \leqslant \|x\| \cdot 1_X$ so that $\|\Phi(x)\| \leqslant \|x\|$. As a final observation, we note that if $\Phi: X \to Y$ is instead assumed to be unital and contractive, then Φ is automatically positive; see, e.g., [64, Proposition 2.11]. We will apply these observations without further mentioning in the sections to follow.

The complete operator system X gives rise to a complete order unit space $X_{\text{sa}} := \{x \in X \mid x = x^*\}$, where the order and the unit are inherited from the surrounding unital C^*-algebra A_X. The order unit space X_{sa} also has an associated state space

$\mathcal{S}(X_{\mathrm{sa}})$ and we record that the restriction of states yields an affine homeomorphism $\mathcal{S}(X) \to \mathcal{S}(X_{\mathrm{sa}})$. For an arbitrary element $x \in X$ we let $\mathrm{Re}(x)$ and $\mathrm{Im}(x)$ in X_{sa} denote the real and the imaginary part of x. We are interested in metrics on the state space $\mathcal{S}(X)$ and in particular those metrics which metrise the weak* topology. As realised by Rieffel, these may be constructed from certain seminorms on the operator system X and we now recall the key notions in this connection.

Definition 2.1.1. A seminorm $L\colon X \to [0, \infty]$ is called a *Lipschitz seminorm* when the following hold:

(1) L is *densely defined*, meaning that the domain $\mathrm{Dom}(L) := \{x \in X : L(x) < \infty\}$ is a norm-dense subspace of X;

(2) the kernel of L contains the scalars $\mathbb{C} := \mathbb{C} \cdot 1_X$, thus $L(1_X) = 0$;

(3) L is invariant under the adjoint operation, i.e., $L(x^*) = L(x)$ for all $x \in X$.

It is common to require that the kernel of a Lipschitz seminorm agrees with the scalars $\mathbb{C} = \mathbb{C} \cdot 1_X$, but we find it convenient to work with the above more flexible notion.

Definition 2.1.2. Let $L\colon X \to [0, \infty]$ be a Lipschitz seminorm on the complete operator system X. The *Monge–Kantorovič metric* $d_L\colon \mathcal{S}(X) \times \mathcal{S}(X) \to [0, \infty]$ is defined by

$$d_L(\mu, \nu) := \sup\big\{|\mu(x) - \nu(x)| \;\big|\; L(x) \leqslant 1\big\}, \quad \text{for } \mu, \nu \in \mathcal{S}(X).$$

We remark that the Monge–Kantorovič metric d_L is not, strictly speaking, a metric since it can, a priori, take the value infinity. In fact, it can be proved that if $\ker(L)$ contains non-scalar elements, then there exist states μ_0 and ν_0 on X such that $d_L(\mu_0, \nu_0) = \infty$; see for example [35, Lemma 2.2]. This possibility is excluded when (X, L) is a compact quantum metric space in the following sense:

Definition 2.1.3. Let $L\colon X \to [0, \infty]$ be a Lipschitz seminorm. We say that (X, L) is a *compact quantum metric space* when the Monge–Kantorovič metric d_L metrises the weak* topology on the state space $\mathcal{S}(X)$. In this case, L is referred to as a *Lip-norm*.

Definition 2.1.4. For a compact quantum metric space (X, L), the *diameter* is defined as

$$\mathrm{diam}(X, L) := \mathrm{diam}(\mathcal{S}(X), d_L) := \sup\{d_L(\mu, \nu) \mid \mu, \nu \in \mathcal{S}(X)\}.$$

For any norm or seminorm $|||\cdot|||$ on X, $x \in X$ and $r \geqslant 0$ we denote the corresponding open and closed balls as follows:

$$\mathbb{B}_r^{|||\cdot|||}(x) := \{y \in X \mid |||x - y||| < r\} \quad \text{and} \quad \overline{\mathbb{B}}_r^{|||\cdot|||}(x) := \{y \in X \mid |||x - y||| \leqslant r\}.$$

The following convenient characterisation of compact quantum metric spaces can be found in [67, Theorem 1.8]; here we let $[\,\cdot\,]\colon X \to X/\mathbb{C}$ denote the quotient map and $\|\cdot\|_{X/\mathbb{C}}$ denote the quotient norm on $X/\mathbb{C}$.

Theorem 2.1.5 (Rieffel). *Let $L\colon X \to [0,\infty]$ be a Lipschitz seminorm. It holds that (X, L) is a compact quantum metric space if and only if the subset $[\overline{\mathbb{B}}_1^L(0)] \subseteq X/\mathbb{C}$ is totally bounded with respect to the quotient norm $\|\cdot\|_{X/\mathbb{C}}$ on $X/\mathbb{C}$.*

Remark 2.1.6. We recall that a subset of a metric space is said to be *totally bounded* if it can be covered by a finite number of ε-balls for any $\varepsilon > 0$. Moreover, if the ambient metric space is complete (as it is the case for $X/\mathbb{C}$), then a subset is totally bounded if and only if it has compact closure. We moreover notice that if (X, L) is a compact quantum metric space, then the intersection $\overline{\mathbb{B}}_1^{\|\cdot\|}(0) \cap \overline{\mathbb{B}}_1^L$ is totally bounded as a subset of X. This follows by applying the isomorphism of Banach spaces $X \to X/\mathbb{C} \oplus \mathbb{C}$ given by $x \mapsto ([x], \mu(x))$, where $\mu\colon X \to \mathbb{C}$ is a fixed state.

Let us now explain the relationship between the above operator system approach to compact quantum metric spaces and Rieffel's approach developed in the context of order unit spaces. Consider a norm-dense real subspace $V \subseteq X_{\mathrm{sa}}$ satisfying that $1_X \in V$ and let $L^0\colon V \to [0,\infty)$ be a seminorm with $L^0(1_X) = 0$. We call such a seminorm L^0 for an *order unit Lipschitz seminorm*. This data also gives rise to a Monge–Kantorovič metric on the state space $\mathcal{S}(X)$ by putting

$$d_{L^0}(\mu,\nu) := \sup\big\{|\mu(x) - \nu(x)| \;\big|\; x \in V,\ L^0(x) \leqslant 1\big\}.$$

Definition 2.1.7 (Rieffel). The pair (V, L^0) is an *order unit compact quantum metric space* when the Monge–Kantorovič metric d_{L^0} metrises the weak* topology on the state space $\mathcal{S}(X)$.

We now wish to relate the two concepts of compact quantum metric spaces given in Definitions 2.1.3 and 2.1.7.

To every Lipschitz seminorm $L\colon X \to [0,\infty]$ on the operator system X we associate an order unit Lipschitz seminorm $L_{\mathrm{sa}}\colon \mathrm{Dom}(L)_{\mathrm{sa}} \to [0,\infty)$ by restricting L to the selfadjoint part of the domain $\mathrm{Dom}(L)_{\mathrm{sa}} := X_{\mathrm{sa}} \cap \mathrm{Dom}(L)$. Conversely, to every order unit Lipschitz seminorm $L^0\colon V \to [0,\infty)$, we associate a Lipschitz seminorm $L^0_{\mathrm{os}}\colon X \to [0,\infty]$ by defining

$$L^0_{\mathrm{os}}(x) := \sup_{\theta\in[0,2\pi]} L^0\big(\cos(\theta)\,\mathrm{Re}(x) + \sin(\theta)\,\mathrm{Im}(x)\big)$$

for $\mathrm{Re}(x), \mathrm{Im}(x) \in \mathrm{Dom}(L^0)$ and $L^0_{\mathrm{os}}(x) := \infty$, otherwise. We record the formula $(L^0_{\mathrm{os}})_{\mathrm{sa}} = L^0$. The relationship between the two notions of compact quantum metric spaces can now be made precise.

Proposition 2.1.8. *If $L\colon X \to [0,\infty]$ is a Lipschitz seminorm, then we have the identity $d_L = d_{L_{\mathrm{sa}}}$ for the associated Monge–Kantorovič metrics on $\mathcal{S}(X)$. Hence, if (X, L) is a compact quantum metric space, then* $(\mathrm{Dom}(L)_{\mathrm{sa}}, L_{\mathrm{sa}})$ *is an order unit compact quantum metric space.*

Conversely, if $L^0\colon V \to [0,\infty)$ *is an order unit Lipschitz seminorm, then we have the identity* $d_{L^0} = d_{L^0_{\mathrm{os}}}$. *Hence if* (V, L^0) *is an order unit compact quantum metric space, then* (X, L^0_{os}) *is a compact quantum metric space.*

Proof. Let $L\colon X \to [0,\infty]$ be a Lipschitz seminorm. It clearly holds that $d_{L_{\mathrm{sa}}} \leqslant d_L$. Let now $\mu, \nu \in \mathcal{S}(X)$ and consider an element $\xi \in X$ with $L(\xi) \leqslant 1$. Choose a $\lambda \in S^1$ such that $\lambda \cdot (\mu(\xi) - \nu(\xi)) \in \mathbb{R}$. Since $L(\mathrm{Re}(\lambda \cdot \xi)) \leqslant L(\xi)$, we obtain that $\mathrm{Re}(\lambda \cdot \xi) \in \mathrm{Dom}(L)_{\mathrm{sa}}$ and $L_{\mathrm{sa}}(\mathrm{Re}(\lambda \cdot \xi)) \leqslant 1$. We may thus estimate as follows:

$$|\mu(\xi) - \nu(\xi)| = |\mu(\lambda \cdot \xi) - \nu(\lambda \cdot \xi)| = |\mu(\mathrm{Re}(\lambda \cdot \xi)) - \nu(\mathrm{Re}(\lambda \cdot \xi))| \leqslant d_{L_{\mathrm{sa}}}(\mu, \nu).$$

This shows that $d_L \leqslant d_{L_{\mathrm{sa}}}$ and we may conclude that $d_L = d_{L_{\mathrm{sa}}}$. Conversely, suppose that $L^0\colon V \to [0,\infty)$ is an order unit Lipschitz seminorm. Recall that $(L^0_{\mathrm{os}})_{\mathrm{sa}} = L^0$ and hence $d_{L^0} = d_{L^0_{\mathrm{os}}}$ by the first part of the proposition. ■

The following result provides a technical condition for verifying when a pair (X, L) is a compact quantum metric space. The essence of the result is that if (X, L) can be suitably approximated by compact quantum metric spaces, then (X, L) must also be a compact quantum metric space; see Corollary 2.1.10 for the precise statement. In the present text we shall apply this theorem to provide quantum SU(2) with the structure of a compact quantum metric space.

Theorem 2.1.9. *Let* $L\colon X \to [0,\infty]$ *be a Lipschitz seminorm. Suppose that for every* $\varepsilon > 0$ *there exist an operator system* X_ε *equipped with a seminorm* $L_\varepsilon\colon X_\varepsilon \to [0,\infty]$ *and linear maps* $\Phi_\varepsilon\colon X \to X_\varepsilon$ *and* $\Psi_\varepsilon\colon X_\varepsilon \to X$ *such that*

(1) *The kernel of* L_ε *is closed in operator norm and the subset*

$$\big[\overline{\mathbb{B}}_1^{L_\varepsilon}(0)\big] \subseteq X_\varepsilon / \ker(L_\varepsilon)$$

is totally bounded with respect to the quotient operator norm on $X_\varepsilon / \ker(L_\varepsilon)$*;*

(2) *We have the inclusion* $\Psi_\varepsilon(\ker(L_\varepsilon)) \subseteq \mathbb{C}$*;*

(3) Φ_ε *is bounded for the seminorms and* Ψ_ε *is bounded for the operator norms;*

(4) *The inequality* $\|\Psi_\varepsilon \Phi_\varepsilon(x) - x\| \leqslant \varepsilon \cdot L(x)$ *holds for all* $x \in X$.

Then (X, L) *is a compact quantum metric space.*

Before embarking on the proof, it is worth emphasising that the maps Φ_ε and Ψ_ε are not required to be unital, and indeed this additional flexibility will be of importance when applying the criterion to prove Theorem 2.3.3 below.

Proof. By Theorem 2.1.5, it suffices to show that the subset $[\overline{\mathbb{B}}_1^L(0)] \subseteq X/\mathbb{C}$ is totally bounded. Let $\varepsilon > 0$ be given. Put $\varepsilon' := \varepsilon/2$ and choose a constant $C > 0$ such that $L_{\varepsilon'}(\Phi_{\varepsilon'}(x)) \leqslant C \cdot L(x)$ and $\|\Psi_{\varepsilon'}(y)\| \leqslant C \cdot \|y\|$ for all $x \in X$ and all $y \in X_{\varepsilon'}$. Since $\mathbb{C} \subseteq \ker(L)$ the first inequality implies that $\Phi_{\varepsilon'}(\mathbb{C}) \subseteq \ker(L_{\varepsilon'})$ and we therefore have well-defined linear maps $[\Phi_{\varepsilon'}]\colon X/\mathbb{C} \to X_{\varepsilon'}/\ker(L_{\varepsilon'})$, $[\Psi_{\varepsilon'}]\colon X_{\varepsilon'}/\ker(L_{\varepsilon'}) \to X/\mathbb{C}$ at the level of quotient spaces. We record that $[\Phi_{\varepsilon'}][\overline{\mathbb{B}}_1^L(0)] \subseteq [\overline{\mathbb{B}}_C^{L_{\varepsilon'}}(0)]$. Using that the subset $[\overline{\mathbb{B}}_C^{L_{\varepsilon'}}(0)] \subseteq X_{\varepsilon'}/\ker(L_{\varepsilon'})$ is totally bounded, we may put $\delta := \varepsilon'/C = \varepsilon/(2C)$ and choose finitely many elements $y_1, y_2, \ldots, y_n \in X_{\varepsilon'}$ such that

$$[\Phi_{\varepsilon'}]\big[\overline{\mathbb{B}}_1^L(0)\big] \subseteq \bigcup_{j=1}^n \mathbb{B}_\delta^{\|\cdot\|_{X_{\varepsilon'}/\ker(L_{\varepsilon'})}}\big([y_j]\big).$$

We now claim that $[\overline{\mathbb{B}}_1^L(0)] \subseteq \bigcup_{j=1}^n \mathbb{B}_\varepsilon^{\|\cdot\|_{X/\mathbb{C}}}([\Psi_{\varepsilon'}(y_j)])$. Indeed, for every $x \in \overline{\mathbb{B}}_1^L(0)$ we may choose $j_0 \in \{1, 2, \ldots, n\}$ such that $\|[\Phi_{\varepsilon'}(x)] - [y_{j_0}]\|_{X_{\varepsilon'}/\ker(L_{\varepsilon'})} < \delta$. Recalling that $C \cdot \delta = \varepsilon' = \varepsilon/2$ we then obtain the following inequalities:

$$\begin{aligned}
\big\|[x] - [\Psi_{\varepsilon'}(y_{j_0})]\big\|_{X/\mathbb{C}} &\leqslant \big\|[x - \Psi_{\varepsilon'}\Phi_{\varepsilon'}(x)]\big\|_{X/\mathbb{C}} + \big\|[\Psi_{\varepsilon'}\Phi_{\varepsilon'}(x) - \Psi_{\varepsilon'}(y_{j_0})]\big\|_{X/\mathbb{C}} \\
&\leqslant \varepsilon' \cdot L(x) + C \cdot \big\|[\Phi_{\varepsilon'}(x) - y_{j_0}]\big\|_{X_{\varepsilon'}/\ker(L_{\varepsilon'})} \\
&< \varepsilon' + C \cdot \delta = \varepsilon.
\end{aligned}$$

This shows that $[x] \in \mathbb{B}_\varepsilon^{\|\cdot\|_{X/\mathbb{C}}}([\Psi_{\varepsilon'}(y_{j_0})])$ and the theorem is therefore proved. ■

It is useful to spell out the following particular case of the above theorem.

Corollary 2.1.10. *Let $L\colon X \to [0, \infty]$ be a Lipschitz seminorm. Suppose that for every $\varepsilon > 0$ there exist a compact quantum metric space $(X_\varepsilon, L_\varepsilon)$ and unital linear maps $\Phi_\varepsilon\colon X \to X_\varepsilon$ and $\Psi_\varepsilon\colon X_\varepsilon \to X$ such that*

(1) *Φ_ε is bounded for the Lipschitz seminorms and Ψ_ε is bounded for the operator norms;*

(2) *The inequality $\|\Psi_\varepsilon\Phi_\varepsilon(x) - x\| \leqslant \varepsilon \cdot L(x)$ holds for all $x \in X$.*

Then (X, L) is a compact quantum metric space.

2.2 Quantum Gromov–Hausdorff distance

We now review the notion of quantum Gromov–Hausdorff distance between two compact quantum metric spaces (X, L) and (Y, K). We are in this text applying Rieffel's original notion of quantum Gromov–Hausdorff distance as introduced in [70], although we are paraphrasing the main definitions in order to deal with operator systems instead of order unit spaces. We would, however, like to emphasise the large body of work due to Latrémolière regarding quantised distance concepts in a C^*-algebraic context; see [44–47]. It could, in particular, be interesting to investigate whether our main continuity result for quantum SU(2) (Theorem D) remains valid for Latrémolière's notion of quantum Gromov–Hausdorff propinquity as well.

Definition 2.2.1. A Lipschitz seminorm $M\colon X \oplus Y \to [0,\infty]$ is said to be *admissible* when the pair $(X \oplus Y, M)$ is a compact quantum metric space, $\mathrm{Dom}(M) = \mathrm{Dom}(L) \oplus \mathrm{Dom}(K)$ and the quotient seminorms induced by M_{sa} via the coordinate projections

$$\mathrm{Dom}(M)_{\mathrm{sa}} \to \mathrm{Dom}(L)_{\mathrm{sa}} \quad \text{and} \quad \mathrm{Dom}(M)_{\mathrm{sa}} \to \mathrm{Dom}(K)_{\mathrm{sa}}$$

agree with L_{sa} and K_{sa}, respectively.

Whenever $M\colon X \oplus Y \to [0,\infty]$ is an admissible Lipschitz seminorm it follows that the coordinate projections $X \oplus Y \to X$ and $X \oplus Y \to Y$ induce isometries $\mathcal{S}(X) \to \mathcal{S}(X \oplus Y)$ and $\mathcal{S}(Y) \to \mathcal{S}(X \oplus Y)$ where the state spaces involved are equipped with the Monge–Kantorovič metrics coming from the relevant Lip-norms. In particular, we may measure the Hausdorff distance between the state spaces $\mathcal{S}(X)$ and $\mathcal{S}(Y)$ with respect to the Monge–Kantorovič metric d_M on the state space $\mathcal{S}(X \oplus Y)$. Denoting this quantity by

$$\mathrm{dist}_{\mathrm{H}}^{d_M}(\mathcal{S}(X), \mathcal{S}(Y)) \in [0,\infty)$$

the *quantum Gromov–Hausdorff distance* between (X, L) and (Y, K) is defined as the infimum over all these Hausdorff distances:

$$\mathrm{dist}_{\mathrm{Q}}((X,L);(Y,K)) := \inf\big\{\mathrm{dist}_{\mathrm{H}}^{d_M}(\mathcal{S}(X), \mathcal{S}(Y)) \,\big|\, M\colon X \oplus Y \to [0,\infty] \text{ admissible}\big\}.$$

In the following lemma, we apply the notation

$$\mathrm{dist}_{\mathrm{Q}}\big((\mathrm{Dom}(L)_{\mathrm{sa}}, L_{\mathrm{sa}}); (\mathrm{Dom}(K)_{\mathrm{sa}}, K_{\mathrm{sa}})\big)$$

for the quantum Gromov–Hausdorff distance between the order unit compact quantum metric spaces $(\mathrm{Dom}(L)_{\mathrm{sa}}, L_{\mathrm{sa}})$, $(\mathrm{Dom}(K)_{\mathrm{sa}}, K_{\mathrm{sa}})$. This notion of order unit quantum Gromov–Hausdorff distance was introduced by Rieffel in [70], and is defined via the obvious order unit space analogue of admissible seminorms; see [70, Definition 4.2].

Lemma 2.2.2. *We have the identity*

$$\mathrm{dist}_{\mathrm{Q}}((X,L);(Y,K)) = \mathrm{dist}_{\mathrm{Q}}\big((\mathrm{Dom}(L)_{\mathrm{sa}}, L_{\mathrm{sa}}); (\mathrm{Dom}(K)_{\mathrm{sa}}, K_{\mathrm{sa}})\big).$$

Proof. Suppose that $M\colon X \oplus Y \to [0,\infty]$ is admissible. By Proposition 2.1.8 we then know that $(\mathrm{Dom}(L)_{\mathrm{sa}} \oplus \mathrm{Dom}(K)_{\mathrm{sa}}, M_{\mathrm{sa}})$ is an order unit compact quantum metric space. It moreover follows immediately from Definition 2.2.1 that M_{sa} is admissible in the order unit sense of Rieffel; see [70, Section 4]. Conversely, suppose that $M^0\colon \mathrm{Dom}(L)_{\mathrm{sa}} \oplus \mathrm{Dom}(K)_{\mathrm{sa}} \to [0,\infty)$ is admissible in the order unit sense. By Proposition 2.1.8, we then know that $(X \oplus Y, M^0_{\mathrm{os}})$ is a compact quantum metric space. We record that $\mathrm{Dom}(M^0_{\mathrm{os}}) = \mathrm{Dom}(L) \oplus \mathrm{Dom}(K)$ and since $(M^0_{\mathrm{os}})_{\mathrm{sa}} = M^0$ we obtain that M^0_{os} is admissible in the sense of Definition 2.2.1. The claimed identity between quantum Gromov–Hausdorff distances now follows from Proposition 2.1.8. ∎

Since the quantum Gromov–Hausdorff distance $\mathrm{dist}_{\mathrm{Q}}((X,L);(Y,K))$ is nothing but Rieffel's original definition from [70] applied to the associated order unit compact quantum metric spaces, all the main results from [70] may be imported verbatim. For the readers' convenience, we summarise the key features of $\mathrm{dist}_{\mathrm{Q}}$ in the theorem below. However, before doing so we need to clarify the slightly subtle notion of isometry in the setting of compact quantum metric spaces. Fix a compact quantum metric space (X,L) and consider the associated order unit compact quantum metric space (A,L_A) where $A := \mathrm{Dom}(L)_{\mathrm{sa}}$ and $L_A := L|_A$. We let (A^c, L_A^c) denote the *closed* compact quantum metric space associated to (A,L_A); for more details on this construction see [70, Section 6] and [68, Section 4]. If (Y,K) is another compact quantum metric space with associated order unit compact quantum metric space (B,K_B), then an *isometry* between X and Y is an order unit isomorphism $\varphi\colon A^c \to B^c$ satisfying that $L_B^c \circ \varphi = L_A^c$. The state spaces of (A^c, L_A^c) and (X,L) are naturally identified, and by [70, Corollary 6.4] one has that the isometries from (X,L) to (Y,K) are in bijective correspondence with the affine isometric isomorphisms from $(\mathcal{S}(Y), d_K)$ to $(\mathcal{S}(X), d_L)$.

Theorem 2.2.3 (Rieffel). *The following hold:*

(1) *The quantum Gromov–Hausdorff distance is symmetric and satisfies the triangle inequality.*

(2) *The quantum Gromov–Hausdorff distance between two compact quantum metric spaces is zero if and only if there exists an isometry between them.*

(3) *The set of isometry classes of compact quantum metric spaces is complete for the metric induced by* $\mathrm{dist}_{\mathrm{Q}}$.

The following result provides a convenient way to estimate the distance between two compact quantum metric spaces:

Proposition 2.2.4. *Let (X,L) and (X',L') be compact quantum metric spaces and suppose that $\Phi\colon X \to X'$ and $\Psi\colon X' \to X$ are two unital positive maps satisfying that*

(1) *there exist $C, C' > 0$ such that*

$$L'(\Phi(x)) \leqslant C \cdot L(x) \quad \textit{and} \quad L(\Psi(y)) \leqslant C' \cdot L'(y)$$

for all $x \in X$ and $y \in X'$;

(2) *there exist $\varepsilon, \varepsilon' > 0$ such that*

$$\|\Psi\Phi(x) - x\| \leqslant \varepsilon \cdot L(x) \quad \textit{and} \quad \|\Phi\Psi(y) - y\| \leqslant \varepsilon' \cdot L'(y)$$

for all $x \in X$ and $y \in X'$.

Then the quantum Gromov–Hausdorff distance $\mathrm{dist}_{\mathrm{Q}}((X,L);(X',L'))$ *is dominated by*

$$\max\bigl\{\mathrm{diam}(X,L)\cdot|1-1/C|+\varepsilon/C, \mathrm{diam}(X',L')\cdot|1-1/C'|+\varepsilon'/C'\bigr\}.$$

Proof. To ease the notation, we put

$$r := \max\bigl\{\mathrm{diam}(X,L)\cdot|1-1/C|+\varepsilon/C, \mathrm{diam}(X',L')\cdot|1-1/C'|+\varepsilon'/C'\bigr\},$$

and define a Lipschitz seminorm $K\colon X\oplus X'\to[0,\infty]$ by

$$K(x,y) := \max\left\{L(x), L'(y), \frac{1}{r}\|y-\Phi(x)\|, \frac{1}{r}\|x-\Psi(y)\|\right\}.$$

Since both (X,L) and (X',L') are compact quantum metric spaces, we get that K turns $X\oplus X'$ into a compact quantum metric space. Indeed, fix a state $\mu\in\mathcal{S}(X)$ and put $\nu:=\mu\circ\Psi$. The fact that the image of $\overline{\mathbb{B}}_1^K(0)$ becomes totally bounded in $(X\oplus X')/\mathbb{C}$ then follows since the map

$$(X\oplus X')/\mathbb{C}\ni[(x,y)]\mapsto(\mu(x)-\nu(y),[x],[y])\in\mathbb{C}\oplus X/\mathbb{C}\oplus X'/\mathbb{C}$$

is an isomorphism of Banach spaces. We now show that K is admissible. Clearly, $\mathrm{Dom}(K)=\mathrm{Dom}(L)\oplus\mathrm{Dom}(L')$. Let thus $x\in\mathrm{Dom}(L)_{\mathrm{sa}}$ be given and let $\mu\colon X\to\mathbb{C}$ be a state. Put $z=x-\mu(x)1_X$ and define the element $y:=\frac{1}{C}\Phi(z)+\mu(x)1_{X'}\in\mathrm{Dom}(L')_{\mathrm{sa}}$. We then obtain the estimates:

- $L'(y)\leqslant\frac{1}{C}L'(\Phi(z))\leqslant L(x)$;
- $\frac{1}{r}\|y-\Phi(x)\|=\frac{1}{r}\|\frac{1}{C}\Phi(z)-\Phi(z)\|\leqslant\frac{|1-1/C|}{r}\|z\|$
  $\leqslant\frac{|1-1/C|\cdot\mathrm{diam}(X,L)}{r}\cdot L(x)\leqslant L(x)$;
- $\frac{1}{r}\|x-\Psi(y)\|\leqslant\frac{1}{r}\|z-\frac{1}{C}\Psi\Phi(z)\|\leqslant\|z\|\cdot\frac{|1-1/C|}{r}+\frac{1}{r}\cdot\frac{1}{C}\cdot\|z-\Psi\Phi(z)\|$
  $\leqslant\frac{|1-1/C|\cdot\mathrm{diam}(X,L)}{r}\cdot L(x)+\frac{1}{r}\cdot\frac{\varepsilon}{C}\cdot L(x)\leqslant L(x)$.

This shows that $K(x,y)\leqslant L(x)$. Similarly, we obtain that

$$K\left(\frac{1}{C'}\Psi(x'-\nu(x')1_{X'})+\nu(x')1_X, x'\right)\leqslant L'(x')$$

whenever $\nu\colon X'\to\mathbb{C}$ is a state and $x'\in\mathrm{Dom}(L')_{\mathrm{sa}}$. We conclude that K is an admissible seminorm. Finally, given $\mu\in\mathcal{S}(X)$ it holds that $\nu:=\mu\circ\Psi\in\mathcal{S}(X')$ and $d_K(\mu,\nu)\leqslant r$. By symmetry, we obtain from this that

$$\mathrm{dist}_{\mathrm{Q}}\bigl((X,L);(X',L')\bigr)\leqslant\mathrm{dist}_{\mathrm{H}}^{d_K}\bigl(\mathcal{S}(X),\mathcal{S}(X')\bigr)\leqslant r$$

and this ends the proof of the present proposition. ■

We spell out the following useful consequence of the above proposition.

Corollary 2.2.5. *Let (X, L) be a compact quantum metric space and let $Y \subseteq X$ be a sub-operator system such that* $\mathrm{Dom}(L) \cap Y$ *is norm-dense in Y. Suppose there exist a constant $D \geqslant 0$ and an $\varepsilon > 0$ as well as a unital positive map $\Phi\colon X \to Y$ such that $L(\Phi(x)) \leqslant (1 + D) \cdot L(x)$ and $\|x - \Phi(x)\| \leqslant \varepsilon \cdot L(x)$ for all $x \in X$. Then (Y, L) is a compact quantum metric space and we have the estimate*

$$\mathrm{dist}_{\mathrm{Q}}\big((X, L); (Y, L)\big) \leqslant \mathrm{diam}(X, L) \cdot \frac{D}{1 + D} + \varepsilon.$$

In particular, if Φ is a Lip-norm contraction then $\mathrm{dist}_{\mathrm{Q}}((X, L); (Y, L)) \leqslant \varepsilon$.

Proof. That (Y, L) is a compact quantum metric space follows from Rieffel's criterion in Theorem 2.1.5. We apply Proposition 2.2.4 to the unital positive map $\Phi\colon X \to Y$ and the inclusion $\iota\colon Y \to X$. We then obtain that

$$\begin{aligned}\mathrm{dist}_{\mathrm{Q}}\big((X, L); (Y, L)\big) &\leqslant \max\big\{\mathrm{diam}(X, L) \cdot \big|1 - 1/(1 + D)\big| + \varepsilon/(1 + D), \varepsilon\big\} \\ &\leqslant \mathrm{diam}(X, L) \cdot \frac{D}{1 + D} + \varepsilon.\end{aligned}$$

∎

Remark 2.2.6. Under the assumptions in Corollary 2.2.5, if Z is an intermediate operator system (i.e., $Y \subseteq Z \subseteq X$) such that $\mathrm{Dom}(L) \cap Z$ is dense in Z, then (Z, L) is a compact quantum metric space as well, and the same estimate on the quantum Gromov–Hausdorff distance holds with (Z, L) instead of (Y, L). Indeed, one may simply enlarge the codomain of Φ from Y to Z and remark that the assumptions in Corollary 2.2.5 are still satisfied.

Corollary 2.2.7. *Under the assumptions of Corollary 2.2.5 we have the estimate*

$$d_L(\mu, \nu) \leqslant \frac{D}{1 + D} \cdot \mathrm{diam}(X, L) + 2\varepsilon + d_L(\mu|_Y, \nu|_Y)$$

for all $\mu, \nu \in \mathcal{S}(X)$.

Here the quantity $d_L(\mu|_Y, \nu|_Y)$ is to be understood as the Monge–Kantorovič metric on $\mathcal{S}(Y)$ arising from the restriction of L, which indeed provides Y with a quantum metric structure by Corollary 2.2.5.

Proof. Let $\mu, \nu \in \mathcal{S}(X)$. By Proposition 2.1.8 it suffices to show that

$$|\mu(x) - \nu(x)| \leqslant \frac{D}{1 + D} \cdot \mathrm{diam}(X, L) + 2\varepsilon + d_L(\mu|_Y, \nu|_Y)$$

for all $x \in X_{\mathrm{sa}}$ with $L(x) \leqslant 1$. Let $x \in X_{\mathrm{sa}}$ with $L(x) \leqslant 1$ be given. By [68, Proposition 2.2] it holds that $\inf_{\lambda \in \mathbb{R}} \|x - \lambda \cdot 1_X\| \leqslant \mathrm{diam}(X, L)/2$. Since Φ is unital and positive (and x is selfadjoint), we then have that $\inf_{\lambda \in \mathbb{R}} \|\Phi(x - \lambda \cdot 1_X)\| \leqslant \mathrm{diam}(X, L)/2$.

We moreover notice that the estimate $\frac{1}{1+D}\Phi(x - \lambda \cdot 1_X) \in Y$ and that the estimate $\frac{1}{1+D}L(\Phi(x - \lambda \cdot 1_X)) \leqslant 1$ is satisfied for all $\lambda \in \mathbb{R}$. For every $\lambda \in \mathbb{R}$ we put $x_\lambda := x - \lambda \cdot 1_X$ and compute as follows:

$$\begin{aligned} |\mu(x) - \nu(x)| &= \inf_{\lambda \in \mathbb{R}} \big|\mu(x_\lambda) - \nu(x_\lambda)\big| \\ &\leqslant \inf_{\lambda \in \mathbb{R}} \bigg(\Big|\mu(x_\lambda) - \frac{1}{1+D}\mu(\Phi(x_\lambda))\Big| + \frac{1}{1+D}\big|\mu(\Phi(x_\lambda)) - \nu(\Phi(x_\lambda))\big| \\ &\qquad + \Big|\frac{1}{1+D}\nu(\Phi(x_\lambda)) - \nu(x_\lambda)\Big| \bigg) \\ &\leqslant 2 \cdot \|x - \Phi(x)\| + 2 \cdot \inf_{\lambda \in \mathbb{R}} \Big\| \frac{D}{1+D}\Phi(x_\lambda) \Big\| + d_L(\mu|_Y, \nu|_Y) \\ &\leqslant 2\varepsilon + \frac{D}{1+D}\operatorname{diam}(X, L) + d_L(\mu|_Y, \nu|_Y). \end{aligned}$$ ■

The first step in proving that $C(\mathrm{SU}_q(2))$ is a compact quantum metric space, is to utilise that this is known to be the case for the C^*-subalgebra $C(S_q^2)$ (see [2]), and then bootstrap to certain finitely generated projective modules over $C(S_q^2)$. We therefore need to develop a bit of general theory to ensure that our finitely generated projective modules do indeed become compact quantum metric spaces, and we carry out this part of the program in the following section.

2.3 Finitely generated projective modules

Let A be a unital C^*-algebra, let $B \subseteq A$ be a unital C^*-subalgebra and suppose that $E: A \to B$ is a conditional expectation. Remark that E is automatically unital and completely positive and the operator norm of E is therefore equal to one. We moreover consider a complete operator system $X \subseteq A$ such that $B \subseteq X$ and suppose in addition that the multiplication in A induces a right B-module structure on X. On top of this data we fix a Lipschitz seminorm $L: A \to [0, \infty]$, and suppose that the domain of L is a unital $*$-subalgebra of A. Our aim is now to impose conditions which ensure that (X, L) is a compact quantum metric space. On the algebraic side we make the following:

Assumption 2.3.1. *Let $n \in \mathbb{N}_0$ and assume that there exist elements $v_j \in A$ and $w_j \in X$ for $j = 0, 1, \ldots, n$ with $v_0 = w_0 = 1_A$ such that*

$$\sum_{j=0}^{n} w_j \cdot E(v_j \cdot x) = x \quad \textit{for all } x \in X.$$

Assume, moreover, that $E(v_j) = 0$ for all $j \in \{1, 2, \ldots, n\}$.

We define the B-linear maps

$$\Phi\colon X \to \bigoplus_{j=0}^{n} B \quad \text{by} \quad \Phi(x) = \sum_{j=0}^{n} e_j \cdot E(v_j \cdot x) \quad \text{and}$$

$$\Psi\colon \bigoplus_{j=0}^{n} B \to X \quad \text{by} \quad \Psi\Big(\sum_{j=0}^{n} e_j \cdot b_j\Big) = \sum_{j=0}^{n} w_j \cdot b_j,$$

where $e_0, \ldots, e_n$ denotes the standard basis in the free module $\bigoplus_{j=0}^{n} B$. It then follows from Assumption 2.3.1 that $(\Psi \circ \Phi)(x) = x$ for all $x \in X$. In particular, we obtain that X is finitely generated projective as a right B-module. Since $E(v_j) = 0$ for all $j \in \{1, 2, \ldots, n\}$ we moreover get that

$$\Phi(b) = \sum_{j=0}^{n} e_j \cdot E(v_j \cdot b) = \sum_{j=0}^{n} e_j \cdot E(v_j) \cdot b = e_0 \cdot b \tag{2.1}$$

for all $b \in B \subseteq X$.

Assumption 2.3.2. *We impose the following extra conditions on our data:*

(1) *The conditional expectation $E\colon A \to B$ is bounded for the seminorm $L\colon A \to [0, \infty]$;*

(2) *The restriction $L\colon B \to [0, \infty]$ gives B the structure of a compact quantum metric space;*

(3) *There exists a constant $C_0 > 0$ such that $\|x - E(x)\| \leqslant C_0 \cdot L(x)$ for all $x \in X$;*

(4) *The elements v_j and w_j belong to* $\mathrm{Dom}(L)$ *for all $j = 0, 1, \ldots, n$;*

(5) *For each $v \in$* Dom(L) *the left-multiplication operator $m(v)\colon X \cap \ker(E) \to A$ is bounded with respect to the seminorm L.*

A few remarks are in place. First of all, since $L\colon A \to [0, \infty]$ is a Lipschitz seminorm it follows from Assumption 2.3.2 (1) that $\mathrm{Dom}(L) \cap B \subseteq B$ is norm-dense and hence that the restriction $L\colon B \to [0, \infty]$ is a Lipschitz seminorm. Next, since $\Psi : \bigoplus_{j=0}^{n} B \to X$ is surjective and $\mathrm{Dom}(L)$ is an algebra, we obtain from Assumption 2.3.2 (4) that $X \cap \mathrm{Dom}(L) \subseteq X$ is norm-dense and hence that the restriction $L\colon X \to [0, \infty]$ is also a Lipschitz seminorm. As the following theorem shows, this restriction is actually a Lip-norm.

Theorem 2.3.3. *Under Assumptions 2.3.1 and 2.3.2, the restriction $L\colon X \to [0, \infty]$ provides X with the structure of a compact quantum metric space.*

Proof. We first record that the direct sum

$$Y := \bigoplus_{j=0}^{n} B \cong C\big(\{0,1,2,\ldots,n\}, B\big)$$

becomes a unital C^*-algebra when equipped with the supremum norm. We are going to apply Theorem 2.1.9 with $\Phi_\varepsilon = \Phi\colon X \to Y$ and $\Psi_\varepsilon = \Psi\colon Y \to X$ for all $\varepsilon > 0$. Indeed, condition (4) in Theorem 2.1.9 is satisfied since $(\Psi \circ \Phi)(x) = x$ for all $x \in X$. Let us define the seminorm $K\colon Y \to [0,\infty]$ by

$$K\Bigg(\sum_{j=0}^{n} e_j \cdot b_j\Bigg) := \max\big\{L(b_0), L(b_1), \ldots, L(b_n), \|b_1\|, \ldots, \|b_n\|\big\}.$$

Assumption 2.3.2 (2) then implies that the kernel of K is given by the closed subspace $\ker(K) = \mathbb{C} \cdot e_0 \subseteq Y$. Moreover, Theorem 2.1.5 and Remark 2.1.6 together with Assumption 2.3.2 (2) shows that the subset $[\overline{\mathbb{B}}_1^K(0)] \subseteq Y/\ker(K)$ is contained in

$$\big[\mathbb{B}_1^L(0)\big] \times \big(\mathbb{B}_1^L(0) \cap \mathbb{B}_1^{\|\cdot\|}(0)\big) \times \cdots \times \big(\mathbb{B}_1^L(0) \cap \mathbb{B}_1^{\|\cdot\|}(0)\big) \subseteq B/\mathbb{C} \oplus B^{\oplus n}$$

and therefore totally bounded with respect to the quotient operator norm. We have thus verified condition (1) in Theorem 2.1.9. Condition (2) in Theorem 2.1.9 follows immediately since $\Psi(e_0) = w_0$ and $w_0 = 1_A = 1_X$. It is moreover clear that $\Psi\colon Y \to X$ is bounded for the operator norms. In order to establish the remaining condition (3) in Theorem 2.1.9 we therefore only need to show that $\Phi\colon X \to Y$ is bounded for the seminorms involved. By Assumption 2.3.2 (1) we may choose a constant $C_1 > 0$ such that $L(E(x)) \leqslant C_1 \cdot L(x)$ for all $x \in A$. Moreover, by Assumption 2.3.2 (5) we may choose constants $D_j > 0$ for $j = 1,2,\ldots,n$ such that

$$L(v_j \cdot x) \leqslant D_j \cdot L(x) \quad \text{for all } x \in \ker(E) \cap X.$$

Using that $\Phi(E(x)) = e_0 \cdot E(x)$ (see (2.1)) and that $v_0 = 1$, we then obtain that

$$\begin{aligned}
K(\Phi(x)) &\leqslant K\big(\Phi(x - E(x))\big) + K(\Phi(E(x))) \\
&= K\Bigg(\sum_{j=1}^{n} e_j \cdot E\big(v_j \cdot (x - E(x))\big)\Bigg) + L(E(x)) \\
&\leqslant K\Bigg(\sum_{j=1}^{n} e_j \cdot E\big(v_j \cdot (x - E(x))\big)\Bigg) + C_1 \cdot L(x)
\end{aligned}$$

for all $x \in X$. Moreover, for each $j \in \{1,2,\ldots,n\}$ and $x \in X$ we obtain the inequality

$$\big\|E\big(v_j \cdot (x - E(x))\big)\big\| \leqslant \|v_j\| \cdot \|x - E(x)\| \leqslant \|v_j\| \cdot C_0 \cdot L(x)$$

together with the inequality

$$L\big(E\big(v_j \cdot (x - E(x))\big)\big) \leqslant C_1 \cdot L\big(v_j \cdot (x - E(x))\big) \leqslant C_1 \cdot D_j \cdot L(x - E(x))$$
$$\leqslant C_1 \cdot D_j \cdot (1 + C_1) \cdot L(x).$$

This shows that $\Phi\colon X \to Y$ is indeed bounded for the seminorms involved and we have proved the theorem. ■