Chapter 3

Preliminaries on quantum SU(2)

The main object of study in the present text is the unital C^*-algebra $C(\mathrm{SU}_q(2))$, known as *quantum* SU(2), introduced by Woronowicz in [78]. There are numerous good sources describing this object, and in addition to the original texts by Woronowicz we refer the reader to the monographs [40, 75] for general background information. Let $q \in (0, 1]$. Aligning our notation with the papers [2–4, 22, 25], we define the C^*-algebraic version of quantum SU(2) as the universal unital C^*-algebra $C(\mathrm{SU}_q(2))$ with two generators a and b subject to the relations

$$\begin{gathered} ba = qab \quad b^*a = qab^* \quad bb^* = b^*b \\ 1 = a^*a + q^2bb^* \quad aa^* + bb^* = 1. \end{gathered}$$

These relations are best justified by noting that they are equivalent to the requirement that

$$u := \begin{pmatrix} a^* & -qb \\ b^* & a \end{pmatrix} \in \mathbb{M}_2\big(C(\mathrm{SU}_q(2))\big)$$

is a unitary matrix, in the following referred to as the *fundamental unitary*. Inside the unital C^*-algebra $C(\mathrm{SU}_q(2))$ we have the *coordinate algebra* $\mathcal{O}(\mathrm{SU}_q(2))$ defined as the unital $*$-subalgebra generated by a and b. The set $\{\xi^{klm} \mid k \in \mathbb{Z}, l, m \in \mathbb{N}_0\}$ with elements given by

$$\xi^{klm} := \begin{cases} a^k b^l (b^*)^m & k, l, m \geqslant 0 \\ b^l (b^*)^m (a^*)^{-k} & k < 0, l, m \geqslant 0 \end{cases} \tag{3.1}$$

constitutes a linear basis for $\mathcal{O}(\mathrm{SU}_q(2))$; see [75, Proposition 6.2.5]. The coordinate algebra $\mathcal{O}(\mathrm{SU}_q(2))$ is in fact a Hopf $*$-algebra and the coproduct Δ, the antipode S and the counit ϵ are best described in terms of the fundamental unitary by means of the formulae $\Delta(u) = u \otimes u$, $S(u) = u^*$ and $\epsilon(u) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. The coproduct Δ extends to a unital $*$-homomorphism $\Delta\colon C(\mathrm{SU}_q(2)) \to C(\mathrm{SU}_q(2)) \otimes_{\min} C(\mathrm{SU}_q(2))$, which turns $C(\mathrm{SU}_q(2))$ into a C^*-algebraic compact quantum group in the sense of Woronowicz; see [79]. For general C^*-algebraic compact quantum groups, it is not true that one can find a bounded counit, but since $C(\mathrm{SU}_q(2))$ is known to be *coamenable*, the counit $\epsilon\colon \mathcal{O}(\mathrm{SU}_q(2)) \to \mathbb{C}$ actually does extend to a unital $*$-homomorphism $\epsilon\colon C(\mathrm{SU}_q(2)) \to \mathbb{C}$; see [7].

3.1 The quantum enveloping algebra

We are also interested in the *quantum enveloping algebra* $\mathcal{U}_q(\mathfrak{su}(2))$. For $q \in (0,1)$, this is defined (see [40, Chapter 4]) as the universal unital $\mathbb{C}$-algebra with generators e, f, k, k^{-1} subject to the relations

$$kk^{-1} = 1 = k^{-1}k, \quad ek = qke, \quad kf = qfk \quad \text{and} \quad fe - ef = \frac{k^2 - k^{-2}}{q - q^{-1}}. \tag{3.2}$$

The quantum enveloping algebra becomes a unital $*$-algebra for the adjoint operation determined by the formulae $k^* = k$ and $e^* = f$. For $q = 1$, the (quantum) enveloping algebra is defined as the universal unital algebra with generators e, f, h satisfying the relations

$$[h, e] = -2e, \quad [h, f] = 2f \quad \text{and} \quad [f, e] = h,$$

with involution given by $h^* = h$ and $e^* = f$; i.e., it agrees with the enveloping algebra of the Lie algebra $\mathfrak{su}(2)$ as one would expect. Note that we have chosen to follow the notation from [22], and that the quantum enveloping algebra just defined is the one denoted $\breve{U}_q(\mathrm{sl}_2)$ in [40]. The quantum enveloping algebra $\mathcal{U}_q(\mathfrak{su}(2))$ is also a Hopf $*$-algebra. For $q \neq 1$, the comultiplication, antipode and counit are determined by the formulae

$$\begin{array}{lll} \Delta(e) = e \otimes k + k^{-1} \otimes e & S(e) = -q^{-1}e & \epsilon(e) = 0 \\ \Delta(f) = f \otimes k + k^{-1} \otimes f & S(f) = -qf & \epsilon(f) = 0 \\ \Delta(k) = k \otimes k & S(k) = k^{-1} & \epsilon(k) = 1 \end{array}$$

and for $q = 1$ by

$$\begin{array}{lll} \Delta(e) = e \otimes 1 + 1 \otimes e & S(e) = -e & \epsilon(e) = 0 \\ \Delta(f) = f \otimes 1 + 1 \otimes f & S(f) = -f & \epsilon(f) = 0 \\ \Delta(h) = h \otimes 1 + 1 \otimes h & S(h) = -h & \epsilon(h) = 0. \end{array}$$

In order to unify our notation, it is convenient to put $k = 1$ in the case where $q = 1$.

The coordinate algebra $\mathcal{O}(\mathrm{SU}_q(2))$ and the quantum enveloping algebra $\mathcal{U}_q(\mathfrak{su}(2))$ are related to one another by means of a non-degenerate dual pairing of Hopf $*$-algebras [40, Chapter 4, Theorem 21]. For $q \neq 1$, this pairing can be described as follows:

$$\langle k, u \rangle = \begin{pmatrix} q^{-\frac{1}{2}} & 0 \\ 0 & q^{\frac{1}{2}} \end{pmatrix}, \quad \langle e, u \rangle = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \langle f, u \rangle = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{3.3}$$

and for $q = 1$ the same formulae apply together with the additional identity

$$\langle h, u \rangle = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The dual pairing yields a left action and a right action of $\mathcal{U}_q(\mathfrak{su}(2))$ on $\mathcal{O}(\mathrm{SU}_q(2))$. These actions play a central role in the present text and for $\eta \in \mathcal{U}_q(\mathfrak{su}(2))$ they are defined by the linear endomorphisms

$$\partial_\eta := (1 \otimes \langle \eta, \cdot \rangle)\Delta \quad \text{and} \quad \delta_\eta := (\langle \eta, \cdot \rangle \otimes 1)\Delta$$

of $\mathcal{O}(\mathrm{SU}_q(2))$. Thus, ∂_η denotes the left action associated to η whereas δ_η denotes the corresponding right action. Pairing the generators of $\mathcal{O}(\mathrm{SU}_q(2))$ and $\mathcal{U}_q(\mathfrak{su}(2))$ one obtains the following explicit formulae for the endomorphisms coming from e and f (we are here only listing the non-zero values):

$$\begin{array}{llll} \partial_e(a) = b^* & \partial_f(a^*) = -qb & \delta_e(a^*) = b^* & \delta_f(a) = -qb \\ \partial_e(b) = -q^{-1}a^* & \partial_f(b^*) = a & \delta_e(b) = -q^{-1}a & \delta_f(b^*) = a^*. \end{array} \tag{3.4}$$

The endomorphisms coming from e and f in $\mathcal{U}_q(\mathfrak{su}(2))$ are related to one another via the adjoint operation, meaning that

$$\partial_e(x^*) = -q^{-1}\partial_f(x)^* \quad \text{and} \quad \delta_e(x^*) = -q^{-1}\delta_f(x)^* \tag{3.5}$$

for all $x \in \mathcal{O}(\mathrm{SU}_q(2))$. We furthermore record that ∂_k and δ_k are algebra automorphisms of $\mathcal{O}(\mathrm{SU}_q(2))$. The relationship between these automorphisms and the adjoint operation is given by $\partial_k(x^*) = \partial_k^{-1}(x)^*$ and $\delta_k(x^*) = \delta_k^{-1}(x)^*$ for all $x \in \mathcal{O}(\mathrm{SU}_q(2))$. The relevant formulae on generators are listed here:

$$\partial_k(a) = q^{\frac{1}{2}}a \quad \partial_k(b) = q^{\frac{1}{2}}b \quad \delta_k(a) = q^{\frac{1}{2}}a \quad \delta_k(b^*) = q^{\frac{1}{2}}b^*. \tag{3.6}$$

All these formulae may be derived directly from the defining relations for $\mathcal{O}(\mathrm{SU}_q(2))$ and $\mathcal{U}_q(\mathfrak{su}(2))$ and the definition of a dual pairing of Hopf $*$-algebras [40, Chapter 1, Definition 5 & (41)]. In the same way one sees that both ∂_e and ∂_f are twisted derivations, in the sense that

$$\begin{aligned} \partial_e(xy) &= \partial_e(x)\partial_k(y) + \partial_{k^{-1}}(x)\partial_e(y) \\ \partial_f(xy) &= \partial_f(x)\partial_k(y) + \partial_{k^{-1}}(x)\partial_f(y) \end{aligned} \tag{3.7}$$

for all $x, y \in \mathcal{O}(\mathrm{SU}_q(2))$.

We shall encounter such twisted derivations numerous times in the sections to follow and we therefore formalise this notion in the following short section.

3.2 Twisted derivations

Definition 3.2.1. Let A and B be C^*-algebras and let σ and $\theta\colon \mathcal{A} \to B$ be algebra homomorphisms defined on a dense $*$-subalgebra $\mathcal{A} \subseteq A$. We say that a linear map

$d\colon \mathcal{A} \to B$ is a *twisted derivation* when $d(x \cdot y) = d(x) \cdot \theta(y) + \sigma(x) \cdot d(y)$ for all $x, y \in \mathcal{A}$. A twisted derivation is called a *twisted $*$-derivation* when $d(x^*) = -d(x)^*$ and $\sigma(x^*)^* = \theta(x)$ for all $x \in \mathcal{A}$.

We remark that a twisted derivation $d\colon \mathcal{A} \to B$ is the same thing as a derivation $d\colon \mathcal{A} \to B$ when B is given the bimodule structure determined by the algebra homomorphisms σ and $\theta\colon \mathcal{A} \to B$.

3.2.0.1 q-numbers. We are going to need two versions of q-numbers. For $q \in (0, 1]$ and $n \in \mathbb{N}$ we define the quantity

$$\langle n \rangle_q := 1 + q^2 + \cdots + q^{2(n-1)}. \tag{3.8}$$

Furthermore, the classical *q-number* makes sense for every $a \in \mathbb{R}$ and is defined by

$$[a]_q := \begin{cases} \frac{q^a - q^{-a}}{q - q^{-1}} & q \in (0, 1) \\ a & q = 1. \end{cases}$$

Whenever no confusion can arise, we omit the subscript q from the notation.

3.3 Corepresentation theory

The (co-)representation theory of $\mathrm{SU}_q(2)$ is well understood, and turns out to be equivalent with that of SU(2); see [78, Section 5]. We may therefore choose a complete set of irreducible corepresentation unitaries $u^n \in \mathbb{M}_{n+1}(\mathcal{O}(\mathrm{SU}_q(2)))$, $n \in \mathbb{N}_0$, where the matrix entries u^n_{ij} are labelled by indices $i, j \in \{0, 1, \dots, n\}$. For $q \neq 1$, we fix this choice of irreducible corepresentation unitaries such that

$$\begin{aligned} \langle k, u^n_{ij} \rangle &= \delta_{ij} \cdot q^{j - \frac{n}{2}} \\ \langle e, u^n_{ij} \rangle &= \delta_{i,j-1} \cdot q^{\frac{1-n}{2}} \sqrt{\langle n - j + 1 \rangle_q \langle j \rangle_q} \\ \langle f, u^n_{ij} \rangle &= \delta_{i,j+1} \cdot q^{\frac{1-n}{2}} \sqrt{\langle n - j \rangle_q \langle j + 1 \rangle_q}, \end{aligned} \tag{3.9}$$

and for $q = 1$ we fix the same formulae together with the additional identity

$$\langle h, u^n_{ij} \rangle = \delta_{ij} \cdot (2j - n).$$

We record that the fundamental unitary u agrees with the irreducible corepresentation unitary u^1 and that $u^0 = 1$. We shall often refer to the entries $u^n_{ij} \in \mathcal{O}(\mathrm{SU}_q(2))$ as the *matrix coefficients* and we apply the convention that $u^n_{ij} := 0$ whenever one of the parameters n, i, j is outside of its natural range; i.e., when $n < 0$ or $(i, j) \notin \{0, \dots, n\}^2$. The adjoint operation can be described at the level of the matrix coefficients via the formula

$$(u^n_{ij})^* = (-q)^{j-i} u^n_{n-i,n-j}; \tag{3.10}$$

see for instance [21, Section 2]. For more details on the corepresentation theory for quantum SU(2), we refer the reader to [40, Chapter 3, Theorem 13 & Chapter 4, Propositions 16 and 19]. Using the q-Clebsch–Gordan coefficients (see [21, Section 3] and [40, Chapter 3.4]) one may explicitly describe the products between the generators and the matrix coefficients:

$$\begin{aligned} a^* \cdot u^n_{ij} &= q^{i+j} \frac{\sqrt{\langle n-i+1\rangle\langle n-j+1\rangle}}{\langle n+1\rangle} \cdot u^{n+1}_{ij} + \frac{\sqrt{\langle i\rangle\langle j\rangle}}{\langle n+1\rangle} \cdot u^{n-1}_{i-1,j-1} \\ b^* \cdot u^n_{ij} &= q^{j} \frac{\sqrt{\langle i+1\rangle\langle n-j+1\rangle}}{\langle n+1\rangle} \cdot u^{n+1}_{i+1,j} - q^{i+1} \frac{\sqrt{\langle n-i\rangle\langle j\rangle}}{\langle n+1\rangle} \cdot u^{n-1}_{i,j-1} \\ a \cdot u^n_{ij} &= \frac{\sqrt{\langle i+1\rangle\langle j+1\rangle}}{\langle n+1\rangle} \cdot u^{n+1}_{i+1,j+1} + q^{i+j+2} \frac{\sqrt{\langle n-i\rangle\langle n-j\rangle}}{\langle n+1\rangle} \cdot u^{n-1}_{ij} \\ b \cdot u^n_{ij} &= -q^{i-1} \frac{\sqrt{\langle j+1\rangle\langle n-i+1\rangle}}{\langle n+1\rangle} \cdot u^{n+1}_{i,j+1} + q^{j} \frac{\sqrt{\langle n-j\rangle\langle i\rangle}}{\langle n+1\rangle} \cdot u^{n-1}_{i-1,j}. \end{aligned} \tag{3.11}$$

In particular, it holds that $u^n_{00} = (a^*)^n$ for all $n \in \mathbb{N}_0$, a fact that will be used several times throughout the memoir.

3.4 The Haar state

Quantum SU(2) comes equipped with its *Haar state* $h\colon C(\mathrm{SU}_q(2)) \to \mathbb{C}$ which can be expressed on the matrix coefficients by the simple relations

$$h(1) = 1 \quad \text{and} \quad h(u^n_{ij}) = 0$$

for all $n \in \mathbb{N}$ and $i, j \in \{0, 1, \dots, n\}$; see e.g. [40, Chapter 4, (50)]. On the elements ξ^{klm} of the linear basis (3.1), the Haar state vanishes if $k \neq 0$, and for $k = 0$ it furthermore vanishes when $l \neq m$. Finally, when $k = 0$ and $l = m$ it holds that

$$h(b^m b^{*m}) = \frac{1}{\langle m+1\rangle_q}; \tag{3.12}$$

see e.g. [75, Theorem 6.2.17]. As the name suggests, the Haar state is bi-invariant with respect to the comultiplication in the sense that

$$(h \otimes 1)\Delta(x) = (1 \otimes h)\Delta(x) = h(x) \cdot 1 \quad \text{for all } x \in C(\mathrm{SU}_q(2)).$$

For $q \neq 1$, the Haar state is not a trace, but it is a *twisted trace* with respect to the algebra automorphism $\nu := \delta_{k-2} \circ \partial_{k-2}$, in the sense that

$$h(xy) = h(\nu(y)x) \quad \text{for all } x, y \in \mathcal{O}(\mathrm{SU}_q(2)); \tag{3.13}$$

see [40, Chapter 4, Proposition 15]. Using the formulae in (3.9), one sees that the modular automorphism ν is given by the following formula on the matrix coefficients:

$$\nu(u^n_{ij}) = q^{2(n-i-j)} \cdot u^n_{ij}. \tag{3.14}$$

The algebra automorphisms $\delta_{k-1} \circ \partial_{k-1}$ and $\delta_k \circ \partial_k$ will be denoted $\nu^{\frac{1}{2}}$ and $\nu^{-\frac{1}{2}}$, respectively.

The Haar state is faithful and we denote the corresponding GNS Hilbert space by $L^2(\mathrm{SU}_q(2))$ and the natural embedding $C(\mathrm{SU}_q(2)) \subseteq L^2(\mathrm{SU}_q(2))$ by Λ. Furthermore, we denote the associated injective $*$-homomorphism by $\rho\colon C(\mathrm{SU}_q(2)) \to \mathbb{B}(L^2(\mathrm{SU}_q(2)))$ and the notation $L^\infty(\mathrm{SU}_q(2))$ refers to the enveloping von Neumann algebra so that $L^\infty(\mathrm{SU}_q(2))$ agrees with the double commutant $\rho(C(\mathrm{SU}_q(2)))'' \subseteq \mathbb{B}(L^2(\mathrm{SU}_q(2)))$. Lastly, the diagonal representation of $C(\mathrm{SU}_q(2))$ on two copies of $L^2(\mathrm{SU}_q(2))$ plays a prominent role in the sections to follow and will be denoted by $\pi\colon C(\mathrm{SU}_q(2)) \to \mathbb{B}(L^2(\mathrm{SU}_q(2))^{\oplus 2})$. Whenever convenient, we apply the notation $H_q := L^2(\mathrm{SU}_q(2))$. The matrix units u^n_{ij} constitute an orthogonal basis in $L^2(\mathrm{SU}_q(2))$ and the 2-norms of u^n_{ij} and $(u^n_{ij})^*$ are given by

$$\begin{aligned} \langle u^n_{ij}, u^n_{ij} \rangle &= h\big((u^n_{ij})^* u^n_{ij}\big) = \frac{q^{2(n-i)}}{\langle n+1 \rangle_q} \\ \big\langle (u^n_{ij})^*, (u^n_{ij})^* \big\rangle &= h\big(u^n_{ij} (u^n_{ij})^*\big) = \frac{q^{2j}}{\langle n+1 \rangle_q}; \end{aligned} \tag{3.15}$$

whenever $n \in \mathbb{N}_0$ and $i, j \in \{0, \ldots, n\}$; see [40, Chapter 4, Theorem 17].

3.5 Circle actions

The unital C^*-algebra $C(\mathrm{SU}_q(2))$ carries two distinguished circle actions

$$\sigma_L \quad \text{and} \quad \sigma_R\colon S^1 \times C(\mathrm{SU}_q(2)) \to C(\mathrm{SU}_q(2))$$

referred to as the *left circle action* and the *right circle action*, respectively. These two circle actions are given on the matrix coefficients by the formulae

$$\sigma_L(z, u^n_{ij}) = z^{2j-n} u^n_{ij} \quad \text{and} \quad \sigma_R(z, u^n_{ij}) = z^{2i-n} u^n_{ij} \tag{3.16}$$

for all $z \in S^1$, $n \in \mathbb{N}_0$ and $i, j \in \{0, 1, \ldots, n\}$; see for example [41, Section 2.2]. The spectral subspaces for the left circle action play a special role in the present text and they are denoted by

$$A^m_q := \big\{ x \in C(\mathrm{SU}_q(2)) \,\big|\, \sigma_L(z, x) = z^m \cdot x \text{ for all } z \in S^1 \big\}, \quad m \in \mathbb{Z}.$$

For each $m \in \mathbb{Z}$ we define the algebraic spectral subspace $\mathcal{A}_q^m := A_q^m \cap \mathcal{O}(\mathrm{SU}_q(2))$. Note that the *Podleś sphere* (see [66]) agrees with the fixed point algebra so that $C(S_q^2) = A_q^0$, and the coordinate algebra $\mathcal{O}(S_q^2)$ agrees with the algebraic fixed point algebra $\mathcal{A}_q^0$. The algebraic spectral subspaces are left comodules over $\mathcal{O}(\mathrm{SU}_q(2))$ in the sense that the coproduct restricts to a coaction $\Delta\colon \mathcal{A}_q^m \to \mathcal{O}(\mathrm{SU}_q(2)) \otimes \mathcal{A}_q^m$ for each $m \in \mathbb{Z}$. The spectral subspace A_q^m comes with an associated *spectral projection* $\Pi_m^L\colon C(\mathrm{SU}_q(2)) \to A_q^m$ defined by the norm-convergent Riemann integral

$$\Pi_m^L(x) = \frac{1}{2\pi}\int_0^{2\pi} \sigma_L(e^{ir}, x) \cdot e^{-irm} dr. \tag{3.17}$$

Note that Π_m^L is a contraction and that $\Pi_m^L(\mathcal{O}(\mathrm{SU}_q(2))) \subseteq \mathcal{A}_q^m$. We apply the notation $H_q^m \subseteq H_q$ for the Hilbert space closure of $\Lambda(\mathcal{A}_q^m) \subseteq H_q$. For each $M \in \mathbb{N}_0$, we introduce the *spectral band*

$$B_q^M := \sum_{m=-M}^{M} A_q^m. \tag{3.18}$$

The spectral band also exists in an algebraic version, namely $\mathcal{B}_q^M := \sum_{m=-M}^{M} \mathcal{A}_q^m$. We note that B_q^M agrees with the norm-closure of the algebraic spectral band, where the non-trivial inclusion follows by using the spectral projections.

3.6 Analytic elements

For each $s \in (0, 1]$, we define the closed strip

$$I_s := \left\{ z \in \mathbb{C} \;\middle|\; \mathrm{Im}(z) \in \left[\frac{\log(s)}{2}, -\frac{\log(s)}{2}\right]\right\} \subseteq \mathbb{C}. \tag{3.19}$$

Definition 3.6.1. Let $s \in (0, 1]$. We say that an element $x \in C(\mathrm{SU}_q(2))$ is *analytic of order* $-\log(s)/2$ when the continuous map $\mathbb{R} \to C(\mathrm{SU}_q(2))$ given by $r \mapsto \sigma_L(e^{ir}, x)$ extends to a continuous map $I_s \to C(\mathrm{SU}_q(2))$ which is analytic on the interior $I_s^\circ \subseteq I_s$. If so, we denote this (unique) continuous extension by $z \mapsto \sigma_L(e^{iz}, x)$.

Let $x, y \in C(\mathrm{SU}_q(2))$ be analytic of order $-\log(s)/2$. Applying the basic properties of operator valued analytic maps we obtain that $x \cdot y$ and x^* are analytic of order $-\log(s)/2$ and that we have the relations

$$\sigma_L(e^{iz}, x \cdot y) = \sigma_L(e^{iz}, x) \cdot \sigma_L(e^{iz}, y) \quad \text{and} \quad \sigma_L(e^{iz}, x^*) = \sigma_L(e^{i\cdot\bar{z}}, x)^* \tag{3.20}$$

for all $z \in I_s$. The set of elements that are analytic of order $-\log(s)/2$ thus constitutes a unital $*$-subalgebra.

Lemma 3.6.2. *Let $s \in (0,1]$ and let x be an analytic element of order $-\log(s)/2$. If $T\colon C(\mathrm{SU}_q(2)) \to C(\mathrm{SU}_q(2))$ is a bounded operator which is equivariant with respect to the circle action σ_L, then $T(x)$ is analytic of order $-\log(s)/2$ and it holds that $T(\sigma_L(e^{iz},x)) = \sigma_L(e^{iz},T(x))$ for all $z \in I_s$.*

Proof. Since T is bounded, the map $I_s \ni z \mapsto T(\sigma_L(e^{iz},x)) \in C(\mathrm{SU}_q(2))$ is continuous and analytic on the interior I_s°. Moreover, for $r \in \mathbb{R}$ we have $T(\sigma_L(e^{ir},x)) = \sigma_L(e^{ir},Tx)$, so it follows that $T(x)$ is analytic of order $-\log(s)/2$ and, by the identity theorem for analytic functions, that $T(\sigma_L(e^{iz},x)) = \sigma_L(e^{iz},T(x))$ for all $z \in I_s$. ■

Lemma 3.6.3. *Let $m \in \mathbb{Z}$ and $x \in A_q^m$. It holds that x is analytic of order $-\log(s)/2$ for all $s \in (0,1]$ and that the associated extension is given by*

$$\sigma_L(e^{iz},x) = e^{iz\cdot m}\cdot x \quad \textit{for all } z \in \mathbb{C}.$$

Proof. This follows since $\sigma_L(e^{it},x) = e^{it\cdot m}\cdot x$ and since $z \mapsto e^{iz\cdot m}$ is analytic. ■

It follows from Lemma 3.6.3 that every $x \in \mathcal{O}(\mathrm{SU}_q(2))$ is analytic of order $-\log(s)/2$ for all $s \in (0,1]$ and that we have an algebra automorphism

$$\sigma_L(e^{iz},\cdot)\colon \mathcal{O}(\mathrm{SU}_q(2)) \to \mathcal{O}(\mathrm{SU}_q(2))$$

for all $z \in \mathbb{C}$. Moreover, it holds that $\sigma_L(e^{iz},\sigma_L(e^{iw},x)) = \sigma_L(e^{i(z+w)},x)$ for all $z,w \in \mathbb{C}$ and $x \in \mathcal{O}(\mathrm{SU}_q(2))$. As a consequence Lemma 3.6.3 we also obtain that

$$\sigma_L(q^{\frac{1}{2}},x) = \partial_k(x) \quad \text{for all } x \in \mathcal{O}(\mathrm{SU}_q(2)).$$

For each $s \in (0,\infty)$, we also introduce the unbounded operator $\Gamma_{s,0}\colon \mathcal{O}(\mathrm{SU}_q(2))^{\oplus 2} \to L^2(\mathrm{SU}_q(2))^{\oplus 2}$ given by the formula

$$\Gamma_{s,0}\begin{pmatrix}\xi\\ \eta\end{pmatrix} := \begin{pmatrix} s^{\frac{1-n}{2}} & 0 \\ 0 & s^{\frac{-1-m}{2}}\end{pmatrix}\begin{pmatrix}\xi\\ \eta\end{pmatrix} \tag{3.21}$$

for all $\xi \in \mathcal{A}_q^n$ and $\eta \in \mathcal{A}_q^m$. Since $\Gamma_{s,0}$ admits an orthonormal basis of eigenvectors with strictly positive eigenvalues, we obtain that $\Gamma_{s,0}$ is closable and that the closure is a positive unbounded operator with dense image. We denote this closure by

$$\Gamma_s\colon \mathrm{Dom}(\Gamma_s) \to L^2(\mathrm{SU}_q(2))^{\oplus 2}.$$

The inverse of Γ_s is again a positive unbounded operator with dense image and we have the following identities regarding images and domains:

$$\mathrm{Dom}(\Gamma_s^{-1}) = \mathrm{Im}(\Gamma_s) \quad \text{and} \quad \mathrm{Im}(\Gamma_s^{-1}) = \mathrm{Dom}(\Gamma_s).$$

The inverse Γ_s^{-1} agrees with the closure of the unbounded operator

$$\Gamma_{s^{-1},0} \colon \mathcal{O}(\mathrm{SU}_q(2))^{\oplus 2} \to L^2(\mathrm{SU}_q(2))^{\oplus 2}$$

and we therefore have the identity $\Gamma_s^{-1} = \Gamma_{s^{-1}}$.

Lemma 3.6.4. *Let $s \in (0,1]$. If $x \in C(\mathrm{SU}_q(2))$ is analytic of order $-\log(s)/2$, then it holds that $x(\mathrm{Dom}(\Gamma_s)) \subseteq \mathrm{Dom}(\Gamma_s)$ and $x(\mathrm{Im}(\Gamma_s)) \subseteq \mathrm{Im}(\Gamma_s)$ and we have the relations*

$$\Gamma_s x \Gamma_s^{-1}(\xi) = \sigma_L(s^{-\frac{1}{2}}, x)(\xi) \quad \textit{and} \quad \Gamma_s^{-1} x \Gamma_s(\eta) = \sigma_L(s^{\frac{1}{2}}, x)(\eta)$$

for all $\xi \in \mathrm{Im}(\Gamma_s)$ and $\eta \in \mathrm{Dom}(\Gamma_s)$.

Proof. Suppose that $x \in C(\mathrm{SU}_q(2))$ is analytic of order $-\log(s)/2$. We focus on showing that $x(\mathrm{Dom}(\Gamma_s)) \subseteq \mathrm{Dom}(\Gamma_s)$ and that $\Gamma_s x \Gamma_s^{-1}(\xi) = \sigma_L(s^{-\frac{1}{2}}, x)(\xi)$ for all $\xi \in \mathrm{Dom}(\Gamma_s)$, since the remaining identities follow by similar arguments. We apply the notation $\mathcal{E} := \mathcal{O}(\mathrm{SU}_q(2))^{\oplus 2}$ for the defining core for Γ_s. It then suffices to show that

$$\langle \Gamma_s \eta, x \Gamma_s^{-1} \xi \rangle = \big\langle \eta, \sigma_L(s^{-\frac{1}{2}}, x)\xi \big\rangle \tag{3.22}$$

for all $\xi, \eta \in \mathcal{E}$. Let thus $\xi, \eta \in \mathcal{E}$ be given. For each $r \in \mathbb{R}$ we consider the unitary operator $\Gamma_s^{ir} \colon L^2(\mathrm{SU}_q(2))^{\oplus 2} \to L^2(\mathrm{SU}_q(2))^{\oplus 2}$. It can then be verified that these unitary operators implement the left circle action in the sense that the identity

$$\Gamma_s^{ir} x \Gamma_s^{-ir} = \sigma_L\big(e^{-ir\log(s)/2}, x\big) \tag{3.23}$$

holds for all $r \in \mathbb{R}$. Indeed, when x belongs to a spectral subspace the above identity follows from Lemma 3.6.3 and therefore holds in general by density and continuity.

Let us define the closed strip $I := \{z \in \mathbb{C} \mid \mathrm{Im}(z) \in [-1,1]\}$ together with the continuous functions $f, g \colon I \to \mathbb{C}$ given by the formulae

$$f(z) := \langle \Gamma_s^{i\cdot\bar{z}} \eta, x \Gamma_s^{i\cdot z} \xi \rangle \quad \text{and} \quad g(z) := \big\langle \eta, \sigma_L(e^{i\cdot z\log(s)/2}, x)\xi \big\rangle$$

for all $z \in \mathbb{C}$. Notice that the first of these functions makes sense since $\eta, \xi \in \mathcal{E}$ and the second makes sense because x is analytic of order $-\log(s)/2$. Both of these functions are then holomorphic on the interior of the strip I° and they agree on the real line $\mathbb{R} \subseteq I$ by an application of (3.23). This implies that $f(z) = g(z)$ for all $z \in I$ and we obtain the identity in (3.22) by evaluating at $z = i$. ∎

For each $t \in (0,1]$, we apply the notation $\mathrm{Ana}_t(\mathrm{SU}_q(2))$ for the unital $*$-subalgebra of $C(\mathrm{SU}_q(2))$ consisting of elements $x \in C(\mathrm{SU}_q(2))$ which are analytic of order $\max\{-\log(q)/2, -\log(t)/2\}$. We equip $\mathrm{Ana}_t(\mathrm{SU}_q(2))$ with the norm $\|\cdot\|_{t,q}$ defined by

$$\|x\|_{t,q} := \max\big\{\big\|\sigma_L(t^{\frac{1}{2}}, x)\big\| + \big\|\sigma_L(q^{\frac{1}{2}}, x)\big\|, \big\|\sigma_L(t^{-\frac{1}{2}}, x)\big\| + \big\|\sigma_L(q^{-\frac{1}{2}}, x)\big\|\big\}$$

and record that $\mathrm{Ana}_t(\mathrm{SU}_q(2))$ is then a unital Banach $*$-algebra. For more details, see for instance [11, Example 1.5]. We end this section by a small lemma providing an estimate on the norm $\|\cdot\|_{t,q}$ on a fixed spectral band.

Lemma 3.6.5. *Let $M \in \mathbb{N}_0$ and $x \in B_q^M$. It holds that $x \in \mathrm{Ana}_t(\mathrm{SU}_q(2))$ and we have the estimate*

$$\|x\|_{t,q} \leqslant \sum_{m=-M}^{M} (t^{\frac{m}{2}} + q^{\frac{m}{2}}) \cdot \|x\| \quad \textit{for all } t \in (0,1].$$

Proof. This follows from Lemma 3.6.3. Indeed, for every $s \in (0,1]$ we have the estimate:

$$\|\sigma_L(s^{\pm\frac{1}{2}}, x)\| = \left\| \sum_{m=-M}^{M} s^{\pm\frac{m}{2}} \Pi_m^L(x) \right\| \leqslant \sum_{m=-M}^{M} s^{\pm\frac{m}{2}} \|x\|. \qquad \blacksquare$$

3.7 The continuous field

It is possible to consider the unital C^*-algebras $C(\mathrm{SU}_q(2))$ for different values of $q \in (0,1]$ as fibres in a continuous field of C^*-algebras, as was shown by Blanchard in [10]. For the sake of clarity, we will, for a moment, adorn the elements in $\mathrm{SU}_q(2)$ with an additional q, thus writing a_q and b_q for the generators. Let us fix $\delta \in (0,1)$. We obtain from [10, Théorème 3.3 & Proposition 7.1] that there exists a unital continuous field of C^*-algebras $C(\mathrm{SU}_\bullet(2))$ over $[\delta,1]$ whose fibre at q agrees with $C(\mathrm{SU}_q(2))$. Concretely, the continuous field $C(\mathrm{SU}_\bullet(2))$ is defined as the universal C^*-algebra generated by three elements $a_\bullet$, $b_\bullet$ and f subject to the relations

- f commutes with $a_\bullet$ and $b_\bullet$;
- f is selfadjoint and the spectrum of f agrees with the interval $[\delta,1]$;
- $u_\bullet = \begin{pmatrix} a_\bullet^* & -f b_\bullet \\ b_\bullet^* & a_\bullet \end{pmatrix}$ is a unitary element in $\mathbb{M}_2(C(\mathrm{SU}_\bullet(2)))$.

For each $q \in [\delta,1]$, the evaluation homomorphism $\mathrm{ev}_q \colon C(\mathrm{SU}_\bullet(2)) \to C(\mathrm{SU}_q(2))$ is defined by sending the generators $a_\bullet$ and $b_\bullet$ to the corresponding generators a_q and b_q in $C(\mathrm{SU}_q(2))$ and by sending f to the scalar q. In what follows, we will tacitly identify $C^*(f)$ with $C([\delta,1])$. We denote by $\mathcal{O}(\mathrm{SU}_\bullet(2))$ the unital $*$-subalgebra generated by $C^*(f)$, $a_\bullet$ and $b_\bullet$. Note that it follows from the discussion in the beginning of Chapter 3 that the elements

$$\xi_\bullet^{klm} := \begin{cases} a_\bullet^k b_\bullet^l (b_\bullet^*)^m & k,l,m \geqslant 0 \\ b_\bullet^l (b_\bullet^*)^m (a_\bullet^*)^{-k} & k < 0, l,m \geqslant 0 \end{cases} \tag{3.24}$$

constitute a basis for $\mathcal{O}(\mathrm{SU}_\bullet(2))$ when considered as a $C([\delta,1])$-module. Let $k \in \mathbb{Z}$ and $l,m \in \mathbb{N}_0$. As a consequence of the twisted Leibniz rule from (3.7) there exists a unique element $\partial_{e_\bullet}(\xi_\bullet^{klm}) \in \mathcal{O}(\mathrm{SU}_\bullet(2))$ such that

$$\mathrm{ev}_q\big(\partial_{e_\bullet}(\xi_\bullet^{klm})\big) = \partial_e\big(\mathrm{ev}_q(\xi_\bullet^{klm})\big) \quad \text{for all } q \in [\delta, 1].$$

We may thus define $\partial_{e_\bullet}\colon \mathcal{O}(\mathrm{SU}_\bullet(2)) \to \mathcal{O}(\mathrm{SU}_\bullet(2))$, by mapping each basis element $\xi_\bullet^{klm}$ to $\partial_{e_\bullet}(\xi_\bullet^{klm})$ and extending by $C([\delta, 1])$-linearity. By construction, it holds that

$$\mathrm{ev}_q\big(\partial_{e_\bullet}(x_\bullet)\big) = \partial_e\big(\mathrm{ev}_q(x_\bullet)\big) \quad \text{for all } x_\bullet \in \mathcal{O}(\mathrm{SU}_\bullet(2)) \text{ and } q \in [\delta, 1].$$

In a similar fashion, we define a $C([\delta, 1])$-linear map $\partial_{f_\bullet}\colon \mathcal{O}(\mathrm{SU}_\bullet(2)) \to \mathcal{O}(\mathrm{SU}_\bullet(2))$ satisfying that

$$\mathrm{ev}_q\big(\partial_{f_\bullet}(x_\bullet)\big) = \partial_f\big(\mathrm{ev}_q(x_\bullet)\big) \quad \text{for all } x_\bullet \in \mathcal{O}(\mathrm{SU}_\bullet(2)) \text{ and } q \in [\delta, 1].$$