Chapter 4

Spectral geometry on quantum SU(2)

In this chapter we provide a detailed treatment of the non-commutative geometry of quantum SU(2). As alluded to in the introduction, it has turned out remarkably difficult to properly unify the theory of quantum groups with Connes' non-commutative geometry, and the general consensus seems to be that one needs to relax Connes' axioms by allowing for certain twists; see [19]. There are by now a number of candidates for Dirac operators on $\mathrm{SU}_q(2)$ with various advantages and disadvantages [8, 9, 13, 21, 38, 41, 42, 62], and here we wish to give a detailed analysis of the Dirac operators proposed in [38,41] from the quantum metric point of view. In order to treat both Dirac operators simultaneously, it will be an advantage to allow for an additional parameter t which, for fixed q, interpolates between the Dirac operator from [38] and that from [41] on $\mathrm{SU}_q(2)$. We emphasise that for $t \neq q$ we do not work with a single Dirac operator but rather with a pair of Dirac operators, aligning with the terminology from classical fiber bundles, we refer to them as the vertical and horizontal Dirac operator, respectively. The vertical and horizontal Dirac operators are in fact incompatible in the sense that their interactions with the coordinate algebra require the use of two different twists.

4.1 The horizontal and vertical Dirac operators

Let us fix two parameters $t, q \in (0,1]$. We define two unbounded operators $\mathcal{D}_q^H$ and $\mathcal{D}_t^V \colon \mathcal{O}(\mathrm{SU}_q(2))^{\oplus 2} \to L^2(\mathrm{SU}_q(2))^{\oplus 2}$. The first of these unbounded operators is referred to as the *horizontal Dirac operator* and is given by the matrix

$$\mathcal{D}_q^H := \begin{pmatrix} 0 & -q^{-\frac{1}{2}} \partial_{fk^{-1}} \\ -q^{\frac{1}{2}} \partial_{ek^{-1}} & 0 \end{pmatrix}. \tag{4.1}$$

We remark that $\mathcal{D}_q^H$ is independent of the parameter $t \in (0,1]$. The second unbounded operator is referred to as the *vertical Dirac operator* and given by the assignment

$$\mathcal{D}_t^V \begin{pmatrix} \xi \\ \eta \end{pmatrix} := \begin{pmatrix} t^{\frac{-n+1}{2}} \left[\frac{n-1}{2}\right]_t & 0 \\ 0 & -t^{\frac{-m-1}{2}} \left[\frac{m+1}{2}\right]_t \end{pmatrix} \cdot \begin{pmatrix} \xi \\ \eta \end{pmatrix} \tag{4.2}$$

for all $\xi \in \mathcal{A}_q^n$ and $\eta \in \mathcal{A}_q^m$. A direct computation verifies that both $\mathcal{D}_t^V$ and $\mathcal{D}_q^H$ are symmetric and for both operators there exists a family of orthogonal finite dimensional invariant subspaces which span a dense subspace in $L^2(\mathrm{SU}_q(2))^{\oplus 2}$; it may even be deduced from (3.9) that we can obtain a joint invariant family of finite dimensional subspaces, by setting

$$V_{ij}^n := \left\{ \begin{pmatrix} \lambda \cdot u_{ij}^n \\ \mu \cdot u_{i,j-1}^n \end{pmatrix} \,\middle|\, \lambda, \mu \in \mathbb{C} \right\} \subseteq \mathcal{O}(\mathrm{SU}_q(2))^{\oplus 2}, \quad n \in \mathbb{N}_0, i, j \in \{0, \dots, n\}.$$

It therefore follows that $\mathcal{D}_q^H$, $\mathcal{D}_t^V$ and $\mathcal{D}_t^V + \mathcal{D}_q^H$ are essentially selfadjoint, and we denote the selfadjoint closures of the horizontal and vertical Dirac operators by D_q^H and D_t^V, respectively. Moreover, we have the following convenient description of the closure of $\mathcal{D}_t^V + \mathcal{D}_q^H : \mathcal{O}(\mathrm{SU}_q(2))^{\oplus 2} \to L^2(\mathrm{SU}_q(2))^{\oplus 2}$:

Lemma 4.1.1. *The unbounded operator $\mathcal{D}_t^V + \mathcal{D}_q^H$ is essentially selfadjoint. Moreover, it holds that* $\mathrm{Dom}(\overline{\mathcal{D}_t^V + \mathcal{D}_q^H}) = \mathrm{Dom}(D_t^V) \cap \mathrm{Dom}(D_q^H)$ *and* $\overline{\mathcal{D}_t^V + \mathcal{D}_q^H} = D_t^V + D_q^H$.

Proof. We already argued that $\mathcal{D}_t^V + \mathcal{D}_q^H$ is essentially selfadjoint. Let $n, m \in \mathbb{Z}$. Using that $\partial_e(\mathcal{A}_q^n) \subseteq \mathcal{A}_q^{n-2}$ and $\partial_f(\mathcal{A}_q^m) \subseteq \mathcal{A}_q^{m+2}$, we obtain for $\xi \in \mathcal{A}_q^n$ and $\eta \in \mathcal{A}_q^m$ that

$$\begin{aligned}
&\mathcal{D}_q^H \mathcal{D}_t^V \begin{pmatrix} \xi \\ \eta \end{pmatrix} \\
&= \mathcal{D}_q^H \begin{pmatrix} t^{-\frac{n-1}{2}} \left[\frac{n-1}{2}\right]_t \xi \\ -t^{-\frac{m+1}{2}} \left[\frac{m+1}{2}\right]_t \eta \end{pmatrix} = \begin{pmatrix} q^{-\frac{1}{2}} t^{-\frac{m+1}{2}} \left[\frac{m+1}{2}\right]_t \partial_{fk^{-1}} \eta \\ -q^{\frac{1}{2}} t^{-\frac{(n-1)}{2}} \left[\frac{n-1}{2}\right]_t \partial_{ek^{-1}} \xi \end{pmatrix} \\
&= - \begin{pmatrix} t^{-\frac{(m+2)-1}{2}} \left[\frac{(m+2)-1}{2}\right]_t & 0 \\ 0 & -t^{-\frac{(n-2)+1}{2}} \left[\frac{(n-2)+1}{2}\right]_t \end{pmatrix} \begin{pmatrix} -q^{-\frac{1}{2}} \partial_{fk^{-1}} \eta \\ -q^{\frac{1}{2}} \partial_{ek^{-1}} \xi \end{pmatrix} \\
&= -\mathcal{D}_t^V \mathcal{D}_q^H \begin{pmatrix} \xi \\ \eta \end{pmatrix}.
\end{aligned}$$

Hence, $\mathcal{D}_t^V$ and $\mathcal{D}_q^H$ anti-commute on the core $\mathcal{O}(\mathrm{SU}_q(2))^{\oplus 2}$ and from [51, Proposition 2.3] it therefore follows that D_t^V and D_q^H weakly anti-commute in the sense of [51, Definition 2.1]. An application of [51, Theorem 2.6] thus gives that $D_t^V + D_q^H$ is selfadjoint on $\mathrm{Dom}(D_t^V) \cap \mathrm{Dom}(D_q^H)$; see also [36, 58]. Hence $\overline{\mathcal{D}_t^V + \mathcal{D}_q^H} \subseteq D_t^V + D_q^H$ and since both operators are selfadjoint the opposite inclusion follows trivially. ∎

4.2 The origin of the Dirac operators

We now describe the precise relationship between the Dirac operators constructed above and those introduced in [38, 41]. Setting $t = q$, a direct computation verifies that

$$\mathcal{D}_q := \mathcal{D}_q^V + \mathcal{D}_q^H = \begin{cases} \begin{pmatrix} \frac{1-q\partial_{k^{-2}}}{q-q^{-1}} & -q^{-\frac{1}{2}}\partial_{fk^{-1}} \\ -q^{\frac{1}{2}}\partial_{ek^{-1}} & \frac{q^{-1}\partial_{k^{-2}}-1}{q-q^{-1}} \end{pmatrix} & \text{for } q \in (0,1) \\ \begin{pmatrix} \frac{1}{2}(\partial_h - 1) & -\partial_f \\ -\partial_e & -\frac{1}{2}(\partial_h + 1) \end{pmatrix} & \text{for } q = 1. \end{cases}$$

Comparing with the Dirac operator $\mathcal{D}_q^{\mathrm{KS}}$ introduced in [38] we then have the identity

$$\mathcal{D}_q = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathcal{D}_q^{\mathrm{KS}} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

In [41], Krähmer, Rennie and Senior proposed another candidate for a Dirac operator, $\mathcal{D}_q^{\mathrm{KRS}}$, which they apply to construct a non-trivial twisted Hochschild 3-cocycle; see [41, Theorem 3.5]. This provides one way of formalising the intuition that $\mathrm{SU}_q(2)$ ought to have dimension 3 as a non-commutative manifold, avoiding the typical dimension drop phenomenon. In our notation, their Dirac operator is given by

$$\mathcal{D}_q^{\mathrm{KRS}} := \mathcal{D}_1^V + \underbrace{\begin{pmatrix} 0 & q^{-\frac{1}{2}}\partial_{kf} \\ q^{\frac{1}{2}}\partial_{ke} & 0 \end{pmatrix}}_{=:\mathcal{D}_{\mathrm{KRS}}^H}.$$

The relationship between our horizontal Dirac operator and the horizontal Dirac operator introduced by Krähmer, Rennie and Senior is governed by the unbounded strictly positive operator $\Gamma_{q,0}$ via the relation

$$\Gamma_{q,0}\mathcal{D}_{\mathrm{KRS}}^H\Gamma_{q,0} = -\mathcal{D}_q^H. \tag{4.3}$$

The vertical and horizontal Dirac operators D_1^V and D_q^H are also compatible with the unbounded Kasparov product in a way which we will now explain; see [37, 58, 59]. It is however important to realise that the triple

$$\big(C(\mathrm{SU}_q(2)), L^2(\mathrm{SU}_q(2))^{\oplus 2}, D_1^V + D_q^H\big)$$

is *not* a spectral triple unless $q = 1$, so that we are formally beyond the scope of the current state of the art in unbounded KK-theory. We let D_q^0 denote the Dirac operator associated with the Dąbrowski–Sitarz spectral triple $(C(S_q^2), H_q^1 \oplus H_q^{-1}, D_q^0)$; see [22]. We are going to discuss this even spectral triple in more details in Section 5.1, but record for the moment that D_q^0 agrees with the closure of the unbounded symmetric operator

$$\mathcal{D}_q^0 := \begin{pmatrix} 0 & -\partial_f \\ -\partial_e & 0 \end{pmatrix} : \mathcal{A}_q^1 \oplus \mathcal{A}_q^{-1} \to H_q^1 \oplus H_q^{-1}.$$

The grading operator on $H_q^1 \oplus H_q^{-1}$ is denoted by $\gamma := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, and the derivation on $\mathcal{O}(S_q^2)$ coming from D_q^0 by taking commutators is denoted by $\partial^0 \colon \mathcal{O}(S_q^2) \to \mathbb{B}(H_q^1 \oplus H_q^{-1})$.

Let E denote the Hilbert C^*-module obtained by completing $\mathcal{O}(\mathrm{SU}_q(2))$ with respect to the $C(S_q^2)$-valued inner product given by $\langle x, y \rangle := \Pi_0^L(x^* y)$. We may turn E into a C^*-correspondence from $C(\mathrm{SU}_q(2))$ to $C(S_q^2)$ where the left action of the unital C^*-algebra $C(\mathrm{SU}_q(2))$ is induced by the product structure in $\mathcal{O}(\mathrm{SU}_q(2))$. The C^*-correspondence E can moreover be equipped with the unbounded selfadjoint and regular operator $N \colon \mathrm{Dom}(N) \to E$ defined on the core $\mathcal{O}(\mathrm{SU}_q(2)) \subseteq E$ by putting $N(x) = n \cdot x$ whenever $x \in \mathcal{A}_q^n$. The pair $(C(\mathrm{SU}_q(2)), E, N)$ is then an odd unbounded Kasparov module from $C(\mathrm{SU}_q(2))$ to $C(S_q^2)$; see [12] for more details.

Following the scheme of unbounded KK-theory, we should in principle be able to form the unbounded Kasparov product of the odd unbounded Kasparov module $(C(\mathrm{SU}_q(2)), E, N)$ and the even spectral triple $(C(S_q^2), H_q^1 \oplus H_q^{-1}, D_q^0)$. The result of this operation is in general not a spectral triple on $C(\mathrm{SU}_q(2))$, but we still investigate the involved unbounded operators on the Hilbert space $E \widehat{\otimes}_{C(S_q^2)} (H_q^1 \oplus H_q^{-1})$, which arises as the interior tensor product between the C^*-correspondence E and the C^*-correspondence $H_q^1 \oplus H_q^{-1}$. The interior tensor product $E \widehat{\otimes}_{C(S_q^2)} (H_q^1 \oplus H_q^{-1})$ is isomorphic to $L^2(\mathrm{SU}_q(2))^{\oplus 2}$ and the isomorphism is induced by the product structure in $\mathcal{O}(\mathrm{SU}_q(2))$. The unbounded selfadjoint and regular operator $N \colon \mathrm{Dom}(N) \to E$ gives rise to the unbounded selfadjoint and regular operator $N \widehat{\otimes} \gamma \colon \mathrm{Dom}(N \widehat{\otimes} 1) \to E \widehat{\otimes}_{C(S_q^2)} (H_q^1 \oplus H_q^{-1})$ which is given by $N \otimes \gamma$ on the core

$$\mathrm{Dom}(N) \otimes_{C(S_q^2)} (H_q^1 \oplus H_q^{-1}) \subseteq E \underset{C(S_q^2)}{\widehat{\otimes}} (H_q^1 \oplus H_q^{-1}).$$

Under the isomorphism between $E \widehat{\otimes}_{C(S_q{}^2)} (H_q^1 \oplus H_q^{-1})$ and $L^2(\mathrm{SU}_q(2))^{\oplus 2}$ it can be verified that $N \widehat{\otimes} \gamma$ agrees with D_1^V. This explains the relationship between the vertical Dirac operator D_1^V and the expected formula from unbounded KK-theory.

In order to explain the relationship between the horizontal Dirac operator and constructions appearing in unbounded KK-theory, we define the *Graßmann connection*

$$\nabla \colon \mathcal{O}(\mathrm{SU}_q(2)) \to E \underset{C(S_q^2)}{\widehat{\otimes}} \mathbb{B}(H_q^1 \oplus H_q^{-1})$$

by putting $\nabla(x) := \sum_{i=0}^n (u_{i0}^n)^* \otimes \partial^0(u_{i0}^n \cdot x)$ whenever x belongs to the algebraic spectral subspace $\mathcal{A}_q^n \subseteq \mathcal{O}(\mathrm{SU}_q(2))$. Combining this Graßmann connection with the Dirac operator from the Dąbrowski–Sitarz spectral triple we obtain the linear map

$$1 \otimes_\nabla \mathcal{D}_q^0 \colon \mathcal{O}(\mathrm{SU}_q(2)) \underset{\mathcal{O}(S_q^2)}{\otimes} (\mathcal{A}_q^1 \oplus \mathcal{A}_q^{-1}) \to E \underset{C(S_q^2)}{\widehat{\otimes}} (H_q^1 \oplus H_q^{-1})$$

given by $(1 \otimes_\nabla \mathcal{D}_q^0)(x \otimes y) := \nabla(x)(y) + x \otimes \mathcal{D}_q^0(y)$, where the domain agrees with the balanced tensor product $\mathcal{O}(\mathrm{SU}_q(2)) \otimes_{\mathcal{O}(S_q^2)} (\mathcal{A}_q^1 \oplus \mathcal{A}_q^{-1})$. It can then be verified that $1 \otimes_\nabla \mathcal{D}_q^0$ induces an unbounded symmetric operator on the Hilbert space $E\widehat{\otimes}_{C(S_q^2)}(H_q^1 \oplus H_q^{-1})$. Moreover, this unbounded symmetric operator is unitarily equivalent to $\mathcal{D}_H^{\mathrm{KRS}}\colon \mathcal{O}(\mathrm{SU}_q(2))^{\oplus 2} \to L^2(\mathrm{SU}_q(2))^{\oplus 2}$. The dampening procedure applied in (4.3) in order to pass from the horizontal Dirac operator $\mathcal{D}_H^{\mathrm{KRS}}$ to the horizontal Dirac operator $\mathcal{D}_q^H$ appears in many places and is systematically investigated in [32, 33] from the point of view of unbounded KK-theory. We record however that the modular operators applied in [32, 33] are all assumed to be bounded (even though inverses are allowed to be unbounded).

4.3 Bounded twisted commutators

Recall that $\pi\colon C(\mathrm{SU}_q(2)) \to \mathbb{B}(L^2(\mathrm{SU}_q(2))^{\oplus 2})$ denotes the injective $*$-homomorphism obtained by letting the GNS representation ρ act diagonally. We now wish to describe the interaction between the coordinate algebra $\mathcal{O}(\mathrm{SU}_q(2))$ and the horizontal and vertical Dirac operators. To this end, it is convenient to introduce the linear maps ∂^1 and $\partial^2\colon \mathcal{O}(\mathrm{SU}_q(2)) \to \mathcal{O}(\mathrm{SU}_q(2))$ given by the formulae

$$\partial^1 := q^{\frac{1}{2}}\partial_e, \quad \partial^2 := q^{-\frac{1}{2}}\partial_f, \tag{4.4}$$

as well as the linear map $\partial_t^3\colon \mathcal{O}(\mathrm{SU}_q(2)) \to \mathcal{O}(\mathrm{SU}_q(2))$ given by

$$\partial_t^3(x) := [n/2]_t \cdot x \quad \text{for all } x \in \mathcal{A}_q^n. \tag{4.5}$$

The following lemma shows that suitably twisted commutators with the horizontal and vertical Dirac do indeed give rise to bounded operators, which may be explicitly described via the maps just introduced. Note that for $t = q$ the twist is the same and in this case the lemma below becomes the statement from [38, Lemma 3.2]; cf. Section 4.2.

Lemma 4.3.1. *For each $x \in \mathcal{O}(\mathrm{SU}_q(2))$, it holds that the twisted commutators*

$$\mathcal{D}_q^H \cdot \sigma_L(q^{\frac{1}{2}}, x) - \sigma_L(q^{-\frac{1}{2}}, x) \cdot \mathcal{D}_q^H\colon \mathcal{O}(\mathrm{SU}_q(2))^{\oplus 2} \to L^2(\mathrm{SU}_q(2))^{\oplus 2} \quad \textit{and}$$
$$\mathcal{D}_t^V \cdot \sigma_L(t^{\frac{1}{2}}, x) - \sigma_L(t^{-\frac{1}{2}}, x) \cdot \mathcal{D}_t^V\colon \mathcal{O}(\mathrm{SU}_q(2))^{\oplus 2} \to L^2(\mathrm{SU}_q(2))^{\oplus 2}$$

extend to bounded operators on $L^2(\mathrm{SU}_q(2))^{\oplus 2}$ given, respectively, by

$$\partial_q^H(x) := \begin{pmatrix} 0 & -\partial^2(x) \\ -\partial^1(x) & 0 \end{pmatrix} \quad \textit{and} \quad \partial_t^V(x) := \begin{pmatrix} \partial_t^3(x) & 0 \\ 0 & -\partial_t^3(x) \end{pmatrix}.$$

Proof. Note first that $\sigma_L(s^{\pm\frac{1}{2}}, -)$ preserves $\mathcal{O}(\mathrm{SU}_q(2))$ for all $s \in (0, 1]$ by Lemma 3.6.3, so that the compositions in the lemma are indeed well defined. By linearity, it suffices to fix an $n \in \mathbb{Z}$ and prove the statements for $x \in \mathcal{A}_q^n$. It then holds that

$\sigma_L(q^{\pm\frac{1}{2}}, x) = q^{\pm\frac{n}{2}} x = \partial_{k\pm 1}(x)$. Using the twisted Leibniz rule from (3.7), the first formula may now be verified by a direct computation. For the second equality, one computes the twisted commutator on an arbitrary vector in $\mathcal{A}_q^k \oplus \mathcal{A}_q^m$, and again a direct computation yields the desired formula. ∎

In classical Riemannian spin geometry, it is well known (see e.g. [16, Chapter 6, Lemma 1]) that a continuous function has bounded commutator with the Dirac operator exactly if the function in question is Lipschitz with respect to the Riemannian metric. Our next aim is to provide a suitable counterpart for the algebra of Lipschitz functions in the q-deformed setting. We recall that both of the parameters t and q in $(0, 1]$ are currently fixed.

Definition 4.3.2. Let $x \in C(\mathrm{SU}_q(2))$. We say that x is *horizontally Lipschitz* when

(1) x is analytic of order $-\log(q)/2$;

(2) the bounded operator $\sigma_L(q^{\frac{1}{2}}, x)$ preserves the domain of D_q^H;

(3) the twisted commutator

$$D_q^H \cdot \sigma_L(q^{\frac{1}{2}}, x) - \sigma_L(q^{-\frac{1}{2}}, x) \cdot D_q^H : \mathrm{Dom}(D_q^H) \to L^2(\mathrm{SU}_q(2))^{\oplus 2}$$

extends to a bounded operator $\partial_q^H(x)$ on $L^2(\mathrm{SU}_q(2))^{\oplus 2}$. The set of horizontally Lipschitz elements is denoted $\mathrm{Lip}^H(\mathrm{SU}_q(2))$.

We say that x is *vertically Lipschitz* when

(1) x is analytic of order $-\log(t)/2$;

(2) the bounded operator $\sigma_L(t^{\frac{1}{2}}, x)$ preserves the domain of D_t^V;

(3) the twisted commutator

$$D_t^V \cdot \sigma_L(t^{\frac{1}{2}}, x) - \sigma_L(t^{-\frac{1}{2}}, x) \cdot D_t^V : \mathrm{Dom}(D_t^V) \to L^2(\mathrm{SU}_q(2))^{\oplus 2}$$

extends to a bounded operator $\partial_t^V(x)$ on $L^2(\mathrm{SU}_q(2))^{\oplus 2}$. The set of vertically Lipschitz elements is denoted $\mathrm{Lip}_t^V(\mathrm{SU}_q(2))$

We apply the notation $\mathrm{Lip}_t(\mathrm{SU}_q(2))$ for the subset of $C(\mathrm{SU}_q(2))$ consisting of elements which are both horizontally and vertically Lipschitz.

A few remarks are in place. The subset $\mathrm{Lip}_t(\mathrm{SU}_q(2)) \subseteq C(\mathrm{SU}_q(2))$ is in fact a unital $*$-subalgebra which we refer to as the *Lipschitz algebra*. Moreover, we obtain from Lemma 4.3.1 that $\mathcal{O}(\mathrm{SU}_q(2)) \subseteq \mathrm{Lip}_t(\mathrm{SU}_q(2))$ and hence that $\mathrm{Lip}_t(\mathrm{SU}_q(2))$ is norm-dense in $C(\mathrm{SU}_q(2))$. The basic algebraic properties of the linear maps

$$\partial_q^H \text{ and } \partial_t^V : \mathrm{Lip}_t(\mathrm{SU}_q(2)) \to \mathbb{B}\big(L^2(\mathrm{SU}_q(2))^{\oplus 2}\big)$$

can be summarised as follows.

Lemma 4.3.3. *The linear maps* $\partial_q^H, \partial_t^V : \mathrm{Lip}_t(\mathrm{SU}_q(2)) \to \mathbb{B}(L^2(\mathrm{SU}_q(2))^{\oplus 2})$ *are twisted* $*$*-derivations, in the sense that the formulae*

$$\partial_q^H(x^*) = -\partial_q^H(x)^*, \quad \partial_q^H(x \cdot y) = \partial_q^H(x)\sigma_L(q^{\frac{1}{2}}, y) + \sigma_L(q^{-\frac{1}{2}}, x)\partial_q^H(y) \quad \text{and}$$
$$\partial_t^V(x^*) = -\partial_t^V(x)^*, \quad \partial_t^V(x \cdot y) = \partial_t^V(x)\sigma_L(t^{\frac{1}{2}}, y) + \sigma_L(t^{-\frac{1}{2}}, x)\partial_t^V(y)$$

hold for all $x, y \in \mathrm{Lip}_t(\mathrm{SU}_q(2))$.

Proof. The twisted Leibniz rules are verified through a direct computation, and the $*$-compatibility follows from the selfadjointness of the involved unbounded operators and the formula $\sigma_L(s^{\frac{1}{2}}, x)^* = \sigma_L(s^{-\frac{1}{2}}, x^*)$, which can be derived from (3.20). ■

We are interested in the linear map

$$\partial_{t,q} := \partial_t^V + \partial_q^H : \mathrm{Lip}_t(\mathrm{SU}_q(2)) \to \mathbb{B}\big(L^2(\mathrm{SU}_q(2))^{\oplus 2}\big).$$

It is important to clarify that $\partial_{t,q}$ is *not* a twisted derivation unless $t = q$. It does however hold that $\partial_{t,q}(x^*) = -\partial_{t,q}(x)^*$ for all $x \in \mathrm{Lip}_t(\mathrm{SU}_q(2))$. Later on, in Proposition 5.2.4, we shall moreover see that $\partial_{t,q}$ is closable for the norm topology.

Let us denote the standard matrix units in $\mathbb{M}_2(\mathbb{C})$ by e_{ij}, $i, j \in \{0, 1\}$, and introduce the twisted derivations $\partial^1, \partial^2, \partial_t^3 : \mathrm{Lip}_t(\mathrm{SU}_q(2)) \to \mathbb{B}(L^2(\mathrm{SU}_q(2)))$ by putting

$$\begin{aligned}
\partial^1(x) &:= -e_{11} \cdot \partial_{t,q}(x) \cdot e_{00} \\
\partial^2(x) &:= -e_{00} \cdot \partial_{t,q}(x) \cdot e_{11} \\
\partial_t^3(x) &:= e_{00} \cdot \partial_{t,q}(x) \cdot e_{00}
\end{aligned}$$

for all $x \in \mathrm{Lip}_t(\mathrm{SU}_q(2))$. By Lemma 4.3.1, this notation is compatible with the notation introduced in (4.4) and (4.5). The adjective *twisted* above is here to be understood in the sense of Definition 3.2.1 where the twists are given by $\sigma(q^{\frac{1}{2}}, \cdot)$ and $\sigma(q^{-\frac{1}{2}}, \cdot)$ for ∂^1 and ∂^2, and by $\sigma(t^{\frac{1}{2}}, \cdot)$ and $\sigma(t^{-\frac{1}{2}}, \cdot)$ for ∂_t^3.

Remark 4.3.4. Let $x \in \mathrm{Lip}_t(\mathrm{SU}_q(2))$ be given. A direct computation shows that

$$\langle \zeta', e_{00}\partial_q^H(x)e_{00} \cdot \zeta \rangle = \langle \zeta', e_{11}\partial_q^H(x)e_{11} \cdot \zeta \rangle = 0$$

for all $\zeta, \zeta' \in \mathcal{O}(\mathrm{SU}_q(2))^{\oplus 2}$. We thereby obtain $\partial_q^H(x) = \begin{pmatrix} 0 & -\partial^2(x) \\ -\partial^1(x) & 0 \end{pmatrix}$. Similarly, one sees that $\partial_t^V(x) = \begin{pmatrix} \partial_t^3(x) & 0 \\ 0 & \partial_t^4(x) \end{pmatrix}$ for some twisted derivation $\partial_t^4 : \mathrm{Lip}_t(\mathrm{SU}_q(2)) \to \mathbb{B}(L^2(\mathrm{SU}_q(2)))$. As a consequence, the following inequality holds:

$$\max\big\{\|\partial_t^V(x)\|, \|\partial_q^H(x)\|\big\} \leqslant \|\partial_{t,q}(x)\|. \tag{4.6}$$

In analogy with the algebraic case described in Lemma 4.3.1, we shall later show (see Remark 5.3.3) that $\partial_t^4(x) = -\partial_t^3(x)$, implying that

$$\partial_{t,q}(x) = \begin{pmatrix} \partial_t^3(x) & -\partial^2(x) \\ -\partial^1(x) & -\partial_t^3(x) \end{pmatrix} \quad \text{for all } x \in \mathrm{Lip}_t(\mathrm{SU}_q(2)).$$

Definition 4.3.5. We define two seminorms, $L_{t,q}$ and $L_{t,q}^{\max}$, on $C(\mathrm{SU}_q(2))$ by setting

$$L_{t,q}(x) := \begin{cases} \|\partial_{t,q}(x)\| & \text{for } x \in \mathcal{O}(\mathrm{SU}_q(2)) \\ \infty & \text{for } x \in C(\mathrm{SU}_q(2)) \setminus \mathcal{O}(\mathrm{SU}_q(2)) \end{cases}$$

$$L_{t,q}^{\max}(x) := \begin{cases} \|\partial_{t,q}(x)\| & \text{for } x \in \mathrm{Lip}_t(\mathrm{SU}_q(2)) \\ \infty & \text{for } x \in C(\mathrm{SU}_q(2)) \setminus \mathrm{Lip}_t(\mathrm{SU}_q(2)) \end{cases}$$

The (extended) metrics on $\mathcal{S}(\mathrm{SU}_q(2))$ induced by the two seminorms $L_{t,q}$ and $L_{t,q}^{\max}$ through the formula (1.1) will be denoted $d_{t,q}$ and $d_{t,q}^{\max}$, respectively.

Remark 4.3.6. It follows from Lemmas 4.3.1 and 4.3.3 that $L_{t,q}^{\max}$ and $L_{t,q}$ are both Lipschitz seminorms in the sense of Definition 2.1.1.

In Latrémolière's approach to the quantised Gromov–Hausdorff distance [46,47], a central role is played by an axiom demanding that the seminorm in question satisfies a certain Leibniz inequality [47, (1.1)]. Since $\partial_{t,q}$ is not a derivation, we only get a twisted version of the Leibniz inequality, where the operator norm appearing in [47, (1.1)] is replaced by the norm $\|\cdot\|_{t,q}$ introduced in Section 3.6.

Lemma 4.3.7. *Let $x, y \in \mathrm{Lip}_t(\mathrm{SU}_q(2))$. Then we have the estimate*

$$L_{t,q}^{\max}(x \cdot y) \leqslant \|x\|_{t,q} \cdot L_{t,q}^{\max}(y) + L_{t,q}^{\max}(x) \cdot \|y\|_{t,q}.$$

Proof. Let $x, y \in \mathrm{Lip}_t(\mathrm{SU}_q(2))$. We first notice that the following inequalities hold:

$$\begin{aligned} \|\partial_q^H(x \cdot y)\| &\leqslant \|\partial_q^H(x)\| \cdot \|\sigma_L(q^{\frac{1}{2}}, y)\| + \|\sigma_L(q^{-\frac{1}{2}}, x)\| \cdot \|\partial_q^H(y)\| \\ &\leqslant L_{t,q}^{\max}(x) \cdot \|\sigma_L(q^{\frac{1}{2}}, y)\| + \|\sigma_L(q^{-\frac{1}{2}}, x)\| \cdot L_{t,q}^{\max}(y). \end{aligned}$$

Since a similar computation shows that

$$\|\partial_t^V(x \cdot y)\| \leqslant L_{t,q}^{\max}(x) \cdot \|\sigma_L(t^{\frac{1}{2}}, y)\| + \|\sigma_L(t^{-\frac{1}{2}}, x)\| \cdot L_{t,q}^{\max}(y),$$

we obtain the result of the present lemma. ∎

One of the main results of the present memoir is Theorem B, which shows that $L_{t,q}^{\max}$ turns $C(\mathrm{SU}_q(2))$ into a compact quantum metric space. Knowing this, it then follows (cf. Theorem 2.1.5) that $L_{t,q}$ also has this property. The proof of Theorem B is contained in Chapter 5 below, but before proceeding to this, we will need to carry out a rather detailed analysis of the spectral geometry on $\mathrm{SU}_q(2)$ arising from the horizontal and vertical Dirac operators introduced above. We first show how one recovers the classical spin geometry on SU(2) when $t = q = 1$.

4.4 Comparison with the classical Dirac operator

In this section, we analyse the classical case where both of the parameters t and q are equal to one. Consider therefore the compact Lie group $\mathrm{SU}(2)$ of special unitary 2×2-matrices. The unital C^*-algebra of continuous functions on $\mathrm{SU}(2)$ agrees with $C(\mathrm{SU}_1(2))$ and the fundamental representation $U\colon \mathrm{SU}(2) \to U(\mathbb{C}^2)$ identifies with the fundamental unitary $u \in \mathbb{M}_2(C(\mathrm{SU}_1(2)))$. We equip $\mathrm{SU}(2)$ with the Haar measure μ and record that the corresponding state on $C(\mathrm{SU}(2))$ agrees with the Haar state $h\colon C(\mathrm{SU}_1(2)) \to \mathbb{C}$. In particular, the Hilbert space of (equivalence classes) of square integrable functions $L^2(\mathrm{SU}(2))$ coincides with $L^2(\mathrm{SU}_1(2))$. We are now going to explain how the classical Dirac operator on $\mathrm{SU}(2)$ identifies with the sum of the vertical and horizontal Dirac operators, $\mathcal{D}_1^V$ and $\mathcal{D}_1^H$, from (4.2) and (4.1) up to rescaling and addition of a constant.

The Lie algebra of $\mathrm{SU}(2)$ is denoted by $\mathfrak{su}(2)$ and is explicitly given by the space of skew-hermitian (2×2)-matrices of trace zero. We equip the Lie algebra $\mathfrak{su}(2)$ with the inner product defined by

$$\langle X, Y \rangle := \mathrm{TR}(X^* Y) \quad \text{for all } X, Y \in \mathfrak{su}(2),$$

where $\mathrm{TR}\colon \mathbb{M}_2(\mathbb{C}) \to \mathbb{C}$ denotes the normalised trace satisfying that $\mathrm{TR}(1) = 1$. We single out the orthonormal basis for $\mathfrak{su}(2)$ consisting of the matrices

$$X_1 := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad X_2 := \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad X_3 := \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

The elements in $\mathfrak{su}(2)$ can be identified with left-invariant vector fields on $\mathrm{SU}(2)$. Indeed, for each element $X \in \mathfrak{su}(2)$ one obtains a derivation $X\colon C^\infty(\mathrm{SU}(2)) \to C^\infty(\mathrm{SU}(2))$ by the formula

$$X(f)(g) := \frac{d}{dt}\big(f(g \cdot e^{tX})\big)\big|_{t=0} \quad \text{for all } f \in C^\infty(\mathrm{SU}(2)),\ g \in \mathrm{SU}(2). \tag{4.7}$$

In this way, the inner product on the Lie algebra $\mathfrak{su}(2)$ yields a Riemannian metric on $\mathrm{SU}(2)$ and therefore in particular a metric on $\mathrm{SU}(2)$. Upon identifying $\mathrm{SU}(2)$ with the 3-sphere S^3 via the map

$$\begin{pmatrix} z_1 & -\overline{z_2} \\ z_2 & \overline{z_1} \end{pmatrix} \mapsto (z_1, z_2)$$

it can be verified that the corresponding metric on S^3 agrees with the classical round metric. This means that S^3 sits inside $\mathbb{R}^4$ as a sphere of radius one, or more precisely that the standard inclusion $S^3 \to \mathbb{R}^4$ becomes a Riemannian immersion.

The spinor bundle for $\mathrm{SU}(2)$ is the trivial complex hermitian vector bundle of rank 2. The fundamental representation of the Lie algebra $\mathfrak{su}(2)$ on $\mathbb{C}^2$ induces a representation of the Clifford algebra associated to $\mathfrak{su}(2)$ on $\mathbb{C}^2$. The classical Dirac operator $\mathcal{D}_{S^3}: C^\infty(\mathrm{SU}(2))^{\oplus 2} \to L^2(\mathrm{SU}(2))^{\oplus 2}$ on SU(2) is then given by the expression

$$\mathcal{D}_{S^3}(\xi) := \sum_{i=1}^{3} X_i \cdot X_i(\xi) = \begin{pmatrix} iX_3(\xi) & -X_1(\xi) + iX_2(\xi) \\ X_1(\xi) + iX_2(\xi) & -iX_3(\xi) \end{pmatrix};$$

see for example [24, Section 3.5]. Notice that we are here considering $\mathcal{D}_{S^3}$ as an unbounded operator on the Hilbert space of L^2-sections of the spinor bundle. We denote the closure of $\mathcal{D}_{S^3}$ by D_{S^3} and record that D_{S^3} is a selfadjoint unbounded operator.

At the level of the coordinate algebra $\mathcal{O}(\mathrm{SU}_1(2))$, which we tacitly identify with a unital $*$-subalgebra of $C^\infty(\mathrm{SU}(2))$, we now single out the correspondence between the derivations associated to $X_1, X_2, X_3 \in \mathfrak{su}(2)$ and the derivations $\partial_e, \partial_f, \partial_h$ defined in Chapter 3. Using the formula (4.7) on may verify the relations

$$\partial_e = -\frac{1}{2}(X_1 + iX_2) \quad \partial_f = \frac{1}{2}(X_1 - iX_2) \quad \partial_h = iX_3,$$

directly on the generators a, b, a^*, b^*, and since all maps are derivations the same relations hold on all of $\mathcal{O}(\mathrm{SU}_1(2))$. We may thus rewrite the unbounded operator $\mathcal{D}_1^V + \mathcal{D}_1^H : \mathcal{O}(\mathrm{SU}_1(2))^{\oplus 2} \to L^2(\mathrm{SU}_1(2))^{\oplus 2}$ as follows:

$$\mathcal{D}_1^V + \mathcal{D}_1^H = \frac{1}{2} \cdot \begin{pmatrix} iX_3 & -X_1 + iX_2 \\ X_1 + iX_2 & -iX_3 \end{pmatrix} - \frac{1}{2}.$$

At the level of unbounded operators on $L^2(\mathrm{SU}(2))^{\oplus 2}$ we therefore obtain that $2 \cdot (\mathcal{D}_1^V + \mathcal{D}_1^H) + 1 \subseteq \mathcal{D}_{S^3}$. Since both of the unbounded operators $2 \cdot (\mathcal{D}_1^V + \mathcal{D}_1^H) + 1$ and $\mathcal{D}_{S^3}$ are essentially selfadjoint we conclude that their closures agree, resulting in the identity

$$2 \cdot \overline{\mathcal{D}_1^V + \mathcal{D}_1^H} + 1 = D_{S^3}.$$

We moreover recall from Lemma 4.1.1 that $\overline{\mathcal{D}_1^V + \mathcal{D}_1^H} = D_1^V + D_1^H$. Lastly, we spell out some consequences of the above identity of Dirac operators from the point of view of quantum metric spaces. Let us denote the classical round metric by $d_{S^3}: S^3 \times S^3 \to [0, \infty)$ and the corresponding Lipschitz algebra by $\mathrm{Lip}(S^3)$. The Lipschitz constant associated to a Lipschitz function $f: S^3 \to \mathbb{C}$ is denoted by $L_{\mathrm{Lip}}(f)$. For each point $p \in S^3$ we apply the notation $\mathrm{ev}_p: C(\mathrm{SU}_1(2)) \to \mathbb{C}$ for the pure state given by evaluation in the point p. We are here suppressing the $*$-isomorphisms $C(\mathrm{SU}_1(2)) \cong C(\mathrm{SU}(2)) \cong C(S^3)$.

Theorem 4.4.1. *The pair* $(C(\mathrm{SU}_1(2)), L_{1,1}^{\max})$ *is a compact quantum metric space. The Lipschitz algebra* $\mathrm{Lip}_1(\mathrm{SU}_1(2))$ *identifies with the Lipschitz algebra* $\mathrm{Lip}(S^3)$

and for every $f \in \mathrm{Lip}(S^3)$ *it holds that*

$$L_{1,1}^{\max}(f) = \frac{1}{2} L_{\mathrm{Lip}}(f).$$

In particular, for every pair of points $p_0, p_1 \in S^3$ *we obtain the formula*

$$2 \cdot d_{S^3}(p_0, p_1) = d_{1,1}^{\max}(\mathrm{ev}_{p_0}, \mathrm{ev}_{p_1}),$$

where the metric on the right-hand side denotes the Monge–Kantorovič metric associated with the Lip-norm $L_{1,1}^{\max}$.

Proof. A continuous function $f : S^3 \to \mathbb{C}$ has bounded commutator with $D_{S^3} = 2 \cdot (D_1^V + D_1^H) + 1$ if and only if f is Lipschitz with respect to d_{S^3} [16, Chapter 6, Lemma 1], and by the paragraph following [16, Chapter 6, Lemma 1] one has that $\|\overline{[D_{S^3}, f]}\|$ equals the Lipschitz constant $L_{\mathrm{Lip}}(f)$. Since $t = q = 1$, all twists appearing in the definition of the Lipschitz algebra $\mathrm{Lip}_1(\mathrm{SU}_1(2))$ are trivial. Using that $D_{S^3} = 2 \cdot (D_1^V + D_1^H) + 1$ and, in particular, that the domain of D_{S^3} is the intersection of the domains of D_1^V and D_1^H, it can then be verified that a continuous function $f : S^3 \to \mathbb{C}$ has bounded commutator with D_{S^3} if and only if f is both vertically and horizontally Lipschitz (meaning that f has bounded commutators with D_1^V and with D_1^H). The Lipschitz algebra $\mathrm{Lip}_1(\mathrm{SU}_1(2))$ therefore agrees with the Lipschitz algebra $\mathrm{Lip}(S^3)$ and the formula $L_{1,1}^{\max}(f) = \frac{1}{2} L_{\mathrm{Lip}}(f)$ now follows. The comparison formula for the two metrics d_{S^3} and $d_{1,1}^{\max}$ is now a consequence of [16, Chapter 6, Formula 1]; see also [15, Proposition 1]. ∎

4.5 The real structure

In Connes' non-commutative geometry, one encounters the notion of a *real structure* for a spectral triple (A, H, D); see [17]. A real structure captures the dimension (modulo 8) of the non-commutative spin manifold in question and is encoded by an antilinear unitary $J : H \to H$ (subject to a couple of conditions). Even though we are working on the borderline of non-commutative geometry we shall nevertheless show that one may define an analogue of a real structure in our setting. As one would expect, this real structure gives $\mathrm{SU}_q(2)$ real dimension 3; see Remark 4.5.6 below for more details.

Let us fix the parameters $t, q \in (0, 1]$. Define the antilinear map $\mathcal{J} : \mathcal{O}(\mathrm{SU}_q(2)) \to \mathcal{O}(\mathrm{SU}_q(2))$ by setting $\mathcal{J}(x) = (\partial_k \delta_k)(x^*)$. Using that the modular automorphism ν is given by $\delta_{k^{-2}} \partial_{k^{-2}} : \mathcal{O}(\mathrm{SU}_q(2)) \to \mathcal{O}(\mathrm{SU}_q(2))$ a direct computation shows that $\mathcal{J}$ extends to an antilinear unitary J on $L^2(\mathrm{SU}_q(2))$. In fact, J is the modular conjugation arising when applying Tomita–Takesaki theory (see e.g. [74, Chapter VI]) to the left Hilbert algebra $\mathcal{O}(\mathrm{SU}_q(2))$ equipped with the inner product $\langle x, y \rangle := h(x^* y)$.

In particular, it therefore holds that $JL^\infty(\mathrm{SU}_q(2))J = L^\infty(\mathrm{SU}_q(2))'$; see [74, Chapter VI, Theorem 1.19].

We now define the antilinear map

$$\mathcal{I} := \begin{pmatrix} 0 & \mathcal{J} \\ -\mathcal{J} & 0 \end{pmatrix} : \mathcal{O}(\mathrm{SU}_q(2))^{\oplus 2} \to \mathcal{O}(\mathrm{SU}_q(2))^{\oplus 2}$$

together with the associated antilinear unitary operator

$$I := \begin{pmatrix} 0 & J \\ -J & 0 \end{pmatrix} : L^2(\mathrm{SU}_q(2))^{\oplus 2} \to L^2(\mathrm{SU}_q(2))^{\oplus 2}.$$

We record that $I^2 = -1$. This is the map that will be our substitute for a real structure, and our next aim is therefore to prove a version of the first order condition, which in our setting amounts to a relation of the form $[\partial_{t,q}(x), IyI] = 0$; see Proposition 4.5.4. To achieve this, the unbounded operator $\Gamma_{s,0}$ defined in (3.21) turns out to be essential, and we analyse its interaction with $\mathcal{I}$, $\mathcal{D}_t^V$ and $\mathcal{D}_q^H$ in the following series of lemmas.

Lemma 4.5.1. *The horizontal Dirac operator $\mathcal{D}_q^H$ commutes with $\Gamma_{q,0}$ and the vertical Dirac operator $\mathcal{D}_t^V$ commutes with $\Gamma_{s,0}$ for all $s \in (0,1]$.*

Proof. Let $n, m \in \mathbb{Z}$. By linearity, it suffices prove the two commutation relations on vectors of the form $\binom{\xi}{\eta} \in \mathcal{A}_q^n \oplus \mathcal{A}_q^m$. Since $\Gamma_{q,0}$ preserves the algebraic spectral subspaces and $\partial_e(\mathcal{A}_q^n) \subseteq \mathcal{A}_q^{n-2}$ and $\partial_f(\mathcal{A}_q^m) \subseteq \mathcal{A}_q^{m+2}$ we obtain that:

$$\begin{aligned}
\mathcal{D}_q^H \Gamma_{q,0} \begin{pmatrix} \xi \\ \eta \end{pmatrix} &= \begin{pmatrix} -q^{-\frac{1}{2}} q^{\frac{-1-m}{2}} \partial_{fk^{-1}}(\eta) \\ -q^{\frac{1}{2}} q^{\frac{1-n}{2}} \partial_{ek^{-1}}(\xi) \end{pmatrix} \\
&= \begin{pmatrix} q^{\frac{1-(m+2)}{2}} & 0 \\ 0 & q^{\frac{-1-(n-2)}{2}} \end{pmatrix} \mathcal{D}_q^H \begin{pmatrix} \xi \\ \eta \end{pmatrix} \\
&= \Gamma_{q,0} \mathcal{D}_q^H \begin{pmatrix} \xi \\ \eta \end{pmatrix},
\end{aligned}$$

thus proving the first commutation relation. Since both $\mathcal{D}_t^V$ and $\Gamma_{s,0}$ are diagonal on $\mathcal{A}_q^n \oplus \mathcal{A}_q^m$ they clearly commute here. ■

Lemma 4.5.2. *It holds that $\mathcal{I} \cdot \Gamma_{s,0}^{-1} = \Gamma_{s,0} \cdot \mathcal{I}$ for all $s \in (0,1]$. Moreover, we have the commutation relations*

$$(\mathcal{D}_q^H \Gamma_{q,0}^{-1}) \cdot \mathcal{I} = \mathcal{I} \cdot (\mathcal{D}_q^H \Gamma_{q,0}^{-1}) \quad \text{and} \quad (\mathcal{D}_t^V \Gamma_{t,0}^{-1}) \cdot \mathcal{I} = \mathcal{I} \cdot (\mathcal{D}_t^V \Gamma_{t,0}^{-1}).$$

Proof. By linearity, it suffices to check the three commutation relations on subspaces of the form $\mathcal{A}_q^n \oplus \mathcal{A}_q^m$ for arbitrary $n, m \in \mathbb{Z}$. The first commutation relation $\mathcal{I} \cdot \Gamma_{s,0}^{-1} = \Gamma_{s,0} \cdot \mathcal{I}$ follows on $\mathcal{A}_q^n \oplus \mathcal{A}_q^m$ by noting that $\mathcal{J}(\mathcal{A}_q^k) = \mathcal{A}_q^{-k}$ for all $k \in \mathbb{Z}$. For the second commutation relation, we first remark that $\partial_f \mathcal{J}(\xi) = -\mathcal{J}\partial_e(\xi)$ for all vectors $\xi \in \mathcal{O}(\mathrm{SU}_q(2))$. Indeed, using the defining relations for $\mathcal{U}_q(\mathfrak{su}(2))$ from (3.2) and the $*$-relations from (3.5) we may compute as follows:

$$\partial_f \mathcal{J}(\xi) = \partial_f \partial_k \delta_k(\xi^*) = \partial_k \delta_k \partial_f(\xi^*) \cdot q^{-1} = -\partial_k \delta_k \partial_e(\xi)^* = -\mathcal{J}\partial_e(\xi).$$

Similarly, one sees that $\partial_e \mathcal{J} = -\mathcal{J}\partial_f$, and the second commutation relation then follows by noting that

$$\mathcal{D}_q^H \Gamma_{q,0}^{-1} = \begin{pmatrix} 0 & -q^{-\frac{1}{2}}\partial_{fk^{-1}} \\ -q^{\frac{1}{2}}\partial_{ek^{-1}} & 0 \end{pmatrix} \begin{pmatrix} q^{-\frac{1}{2}}\partial_k & 0 \\ 0 & q^{\frac{1}{2}}\partial_k \end{pmatrix} = \begin{pmatrix} 0 & -\partial_f \\ -\partial_e & 0 \end{pmatrix}.$$

To prove the last commutation relation, observe that the restriction of the unbounded operator $\mathcal{D}_t^V \Gamma_{t,0}^{-1}$ to the subspace $\mathcal{A}_q^n \oplus \mathcal{A}_q^m$ is represented by $\begin{pmatrix} [\frac{n-1}{2}]_t & 0 \\ 0 & -[\frac{m+1}{2}]_t \end{pmatrix}$. Using one more time that $\mathcal{J}(\mathcal{A}_q^k) = \mathcal{A}_q^{-k}$ for all $k \in \mathbb{Z}$, we now obtain the identity $(\mathcal{D}_t^V \Gamma_{t,0}^{-1}) \cdot \mathcal{I} = \mathcal{I} \cdot (\mathcal{D}_t^V \Gamma_{t,0}^{-1})$ on $\mathcal{A}_q^n \oplus \mathcal{A}_q^m$ from a direct computation. ∎

Lemma 4.5.3. *For each $y \in \mathcal{O}(\mathrm{SU}_q(2))$ we have the identities*

$$[\mathcal{D}_q^H, \mathcal{I} y \mathcal{I}] = \Gamma_{q,0} \cdot \mathcal{I}\partial_q^H(y)\mathcal{I} \cdot \Gamma_{q,0} = \mathcal{I}\partial_q^H(\partial_k(y))\mathcal{I} \cdot \Gamma_{q,0}^2$$
$$[\mathcal{D}_t^V, \mathcal{I} y \mathcal{I}] = \Gamma_{t,0} \cdot \mathcal{I}\partial_t^V(y)\mathcal{I} \cdot \Gamma_{t,0} = \mathcal{I}\partial_t^V(\sigma_L(t^{\frac{1}{2}}, y))\mathcal{I} \cdot \Gamma_{t,0}^2$$

on the subspace $\mathcal{O}(\mathrm{SU}_q(2))^{\oplus 2} \subseteq L^2(\mathrm{SU}_q(2))^{\oplus 2}$.

Proof. Using Lemmas 3.6.4, 4.5.1 and 4.5.2, we may compute as follows:

$$\begin{aligned} \mathcal{D}_q^H \cdot \mathcal{I} y \mathcal{I} &= \mathcal{D}_q^H \Gamma_{q,0}^{-1} \cdot \Gamma_{q,0} \mathcal{I} y \mathcal{I} \\ &= \mathcal{D}_q^H \Gamma_{q,0}^{-1} \cdot \mathcal{I}\sigma_L(q^{\frac{1}{2}}, y)\mathcal{I} \cdot \Gamma_{q,0} \\ &= \Gamma_{q,0} \cdot \mathcal{I}\mathcal{D}_q^H \sigma_L(q^{\frac{1}{2}}, y)\mathcal{I} \cdot \Gamma_{q,0} \\ &= \Gamma_{q,0} \cdot \mathcal{I}\partial_q^H(y)\mathcal{I} \cdot \Gamma_{q,0} + \Gamma_{q,0} \cdot \mathcal{I}\sigma_L(q^{-\frac{1}{2}}, y)\mathcal{D}_q^H \mathcal{I} \cdot \Gamma_{q,0} \\ &= \Gamma_{q,0} \cdot \mathcal{I}\partial_q^H(y)\mathcal{I} \cdot \Gamma_{q,0} + \mathcal{I} y \mathcal{I} \cdot \mathcal{D}_q^H. \end{aligned}$$

This proves the first identity regarding the commutator with the horizontal Dirac operator. The second one follows by a similar computation, using the same series of lemmas as above:

$$\begin{aligned} \Gamma_{q,0}\mathcal{I}\partial_q^H(y)\mathcal{I}\Gamma_{q,0} &= \mathcal{I}\Gamma_{q,0}^{-1}\big(\mathcal{D}_q^H \Gamma_{q,0}^{-1} y \Gamma_{q,0} - \Gamma_{q,0} y \Gamma_{q,0}^{-1}\mathcal{D}_q^H\big)\mathcal{I}\Gamma_{q,0} \\ &= \mathcal{I}\big(\mathcal{D}_q^H \Gamma_{q,0}^{-2} y \Gamma_{q,0}^2 - y\mathcal{D}_q^H\big)\Gamma_{q,0}^{-1}\mathcal{I}\Gamma_{q,0} \\ &= \mathcal{I}\partial_q^H(\partial_k(y))\mathcal{I}\Gamma_{q,0}^2. \end{aligned}$$

The remaining identities regarding the commutator with the vertical Dirac operator are proven by completely analogous computations. ∎

With the above lemmas at our disposal, we may now state and prove the analogue of the first order condition.

Proposition 4.5.4. *For each* $y \in L^\infty(\mathrm{SU}_q(2))$ *and* $x \in \mathrm{Lip}_t(\mathrm{SU}_q(2))$ *we have the identities*

$$[IyI, \partial_q^H(x)] = 0 = [IyI, \partial_t^V(x)].$$

Proof. Since the von Neumann algebra $L^\infty(\mathrm{SU}_q(2))$ agrees with the closure of the coordinate algebra $\mathcal{O}(\mathrm{SU}_q(2)) \subseteq \mathbb{B}(L^2(\mathrm{SU}_q(2)))$ with respect to the strong operator topology, it suffices to treat the case where $y \in \mathcal{O}(\mathrm{SU}_q(2))$. Let thus $y \in \mathcal{O}(\mathrm{SU}_q(2))$ be given. We will just focus on proving that IyI commutes with $\partial_t^V(x)$ since the proof of the analogous result for $\partial_q^H(x)$ follows the same pattern. From Lemmas 3.6.4 and 4.5.3 we obtain the identities

$$\begin{aligned} & IyI \cdot \sigma_L(t^{-\frac{1}{2}}, x)\mathcal{D}_t^V \\ &\quad = \sigma_L(t^{-\frac{1}{2}}, x)\mathcal{I}y\mathcal{I} \cdot \mathcal{D}_t^V \\ &\quad = \sigma_L(t^{-\frac{1}{2}}, x)\mathcal{D}_t^V \cdot \mathcal{I}y\mathcal{I} - \sigma_L(t^{-\frac{1}{2}}, x)\Gamma_{t,0} \cdot \mathcal{I}\partial_t^V(y)\mathcal{I} \cdot \Gamma_{t,0} \\ &\quad = \sigma_L(t^{-\frac{1}{2}}, x)\mathcal{D}_t^V \cdot \mathcal{I}y\mathcal{I} - \Gamma_t \cdot I\partial_t^V(y)Ix \cdot \Gamma_{t,0} \end{aligned} \tag{4.8}$$

of unbounded operators defined on the dense subspace $\mathcal{O}(\mathrm{SU}_q(2))^{\oplus 2}$ in the Hilbert space $L^2(\mathrm{SU}_q(2))^{\oplus 2}$.

Similarly, using Lemmas 3.6.4 and 4.5.3 one more time, we obtain that

$$\begin{aligned} & \langle IyI \cdot D_t^V \sigma_L(t^{\frac{1}{2}}, x)\xi, \eta\rangle \\ &\quad = \langle \sigma_L(t^{\frac{1}{2}}, x)\xi, \mathcal{D}_t^V \cdot \mathcal{I}y^*\mathcal{I}\eta\rangle \\ &\quad = \langle \sigma_L(t^{\frac{1}{2}}, x)\xi, \mathcal{I}y^*\mathcal{I} \cdot \mathcal{D}_t^V\eta\rangle + \langle \sigma_L(t^{\frac{1}{2}}, x)\xi, \Gamma_{t,0} \cdot \mathcal{I}\partial_t^V(y^*)\mathcal{I} \cdot \Gamma_{t,0}\eta\rangle \\ &\quad = \langle D_t^V \sigma_L(t^{\frac{1}{2}}, x) \cdot IyI\xi, \eta\rangle - \langle \Gamma_t \cdot I\partial_t^V(y)Ix \cdot \Gamma_{t,0}\xi, \eta\rangle \end{aligned} \tag{4.9}$$

for all $\xi, \eta \in \mathcal{O}(\mathrm{SU}_q(2))^{\oplus 2}$. Combining the identities in (4.8) and (4.9) we see that

$$\begin{aligned} IyI \cdot \partial_t^V(x)(\xi) &= IyI \cdot D_t^V \sigma_L(t^{\frac{1}{2}}, x)(\xi) - IyI \cdot \sigma_L(t^{-\frac{1}{2}}, x)\mathcal{D}_t^V(\xi) \\ &= D_t^V \sigma_L(t^{\frac{1}{2}}, x) \cdot IyI(\xi) - \Gamma_t \cdot I\partial_t^V(y)Ix \cdot \Gamma_{t,0}(\xi) \\ &\quad - \sigma_L(t^{-\frac{1}{2}}, x)\mathcal{D}_t^V \cdot \mathcal{I}y\mathcal{I}(\xi) + \Gamma_t \cdot I\partial_t^V(y)Ix \cdot \Gamma_{t,0}(\xi) \\ &= \partial_t^V(x) \cdot IyI(\xi) \end{aligned}$$

for all $\xi \in \mathcal{O}(\mathrm{SU}_q(2))^{\oplus 2}$. This proves the proposition. ∎

Corollary 4.5.5. *The twisted $*$-derivations*

$$\partial_t^V \text{ and } \partial_q^H \colon \mathrm{Lip}_t(\mathrm{SU}_q(2)) \to \mathbb{B}\big(L^2(\mathrm{SU}_q(2))^{\oplus 2}\big)$$

both take values in $\mathbb{M}_2(L^\infty(\mathrm{SU}_q(2)))$.

Proof. Since J is the modular conjugation for the left Hilbert algebra $\mathcal{O}(\mathrm{SU}_q(2))$ with inner product coming from the Haar state, it holds that

$$L^\infty(\mathrm{SU}_q(2))' = JL^\infty(\mathrm{SU}_q(2))J$$

as an identity between operator algebras in $\mathbb{B}(L^2(\mathrm{SU}_q(2)))$; see [74, Chapter VI, Theorem 1.19]. For $x \in \mathrm{Lip}_t(\mathrm{SU}_q(2))$, it therefore suffices to show that each entry in $\partial_q^H(x), \partial_t^V(x) \in \mathbb{B}(L^2(\mathrm{SU}_q(2))^{\oplus 2}) = \mathbb{M}_2(\mathbb{B}(L^2(\mathrm{SU}_q(2))))$ belongs to the commutant $(JL^\infty(\mathrm{SU}_q(2))J)'$. For $y \in L^\infty(\mathrm{SU}_q(2))$ it holds that

$$IyI = -\begin{pmatrix} JyJ & 0 \\ 0 & JyJ \end{pmatrix},$$

and hence it suffices to show that $[IyI, \partial_t^V(x)] = [IyI, \partial_q^H(x)] = 0$, but this was already proven in Proposition 4.5.4. ■

Remark 4.5.6. In the classical setting of non-commutative geometry, a real 3-dimensional structure for an odd spectral triple (A, H, D) with coordinate algebra $\mathcal{A} \subseteq A$ is given by an antilinear unitary $J \colon H \to H$. This data is then supposed to satisfy the conditions $J^2 = -1$, $DJ = JD$ and for all $a, b \in \mathcal{A}$ one has $[a, JbJ] = 0$ and $[\overline{[D,a]}, JbJ] = 0$; see [17].

In our setting, the antilinear unitary $I \in \mathbb{B}(L^2(\mathrm{SU}_q(2))^{\oplus 2})$ provides the substitute for a real structure. Lemma 4.5.2 may thus be viewed as a twisted analogue of the relation $DJ = JD$, while Proposition 4.5.4 is the analogue of the first order condition $[\overline{[D,a]}, JbJ] = 0$. The relation $[a, IbI] = 0$ also holds by Tomita–Takesaki theory as already remarked in the beginning of the present section.

4.6 The equivariance condition

We are now going to investigate the equivariance properties of the spectral geometric data governed by our pair of Dirac operators. In some of the literature on Dirac operators on q-deformed spaces (see e.g. [21, 22]) the equivariance is to be understood in the sense that the Dirac operator in question commutes with the right action of $\mathcal{U}_q(\mathfrak{su}(2))$; i.e. with the diagonal action of operators of the form δ_η with $\eta \in \mathcal{U}_q(\mathfrak{su}(2))$ on the core $\mathcal{O}(\mathrm{SU}_q(2))^{\oplus 2} \subseteq L^2(\mathrm{SU}_q(2))^{\oplus 2}$. Since $\mathcal{D}_q^H$ is constructed explicitly using the *left* action it clearly commutes with δ_η, and since δ_η preserves the spectral subspaces it also follows easily that $\mathcal{D}_t^V$ commutes with δ_η. Thus, this type of equivariance is basically built into the construction of $D_{t,q}$. In this section we shall show another kind of equivariance, in that we will show that our spectral data is compatible with the coproduct on the C^*-algebraic quantum group $C(\mathrm{SU}_q(2))$. More precisely, we will show in Lemma 4.6.1 below that the vertical and horizontal Dirac operators both commute with the multiplicative unitary for $\mathrm{SU}_q(2)$, which seems to be an equivariance condition which is more closely related with the SU(2)-equivariance of the classical Dirac operator on S^3; see Remark 4.6.2 for more details. Throughout the section, we are still keeping the two parameters t and q in $(0, 1]$ fixed unless explicitly stated otherwise.

Let us consider the Hilbert space tensor product $L^2(\mathrm{SU}_q(2))\widehat{\otimes}L^2(\mathrm{SU}_q(2))$ and introduce the unitary operator

$$W\colon L^2(\mathrm{SU}_q(2))\widehat{\otimes}L^2(\mathrm{SU}_q(2)) \to L^2(\mathrm{SU}_q(2))\widehat{\otimes}L^2(\mathrm{SU}_q(2))$$

given by the formula $W(x \otimes y) := \Delta(y) \cdot (x \otimes 1)$ for all elements $x, y \in \mathcal{O}(\mathrm{SU}_q(2))$. We record that $W(\mathcal{O}(\mathrm{SU}_q(2)) \otimes \mathcal{O}(\mathrm{SU}_q(2))) = \mathcal{O}(\mathrm{SU}_q(2)) \otimes \mathcal{O}(\mathrm{SU}_q(2))$ and hence that $W^*(\mathcal{O}(\mathrm{SU}_q(2)) \otimes \mathcal{O}(\mathrm{SU}_q(2))) = \mathcal{O}(\mathrm{SU}_q(2)) \otimes \mathcal{O}(\mathrm{SU}_q(2))$ as well. The unitary operator W implements the coproduct $\Delta\colon C(\mathrm{SU}_q(2)) \to C(\mathrm{SU}_q(2)) \otimes_{\min} C(\mathrm{SU}_q(2))$ in the sense that

$$\Delta(z) = W(1 \otimes z)W^* \quad \text{for all } z \in C(\mathrm{SU}_q(2)).$$

The operator W is referred to as the *multiplicative unitary* for quantum SU(2); see [6] for more details on these matters. For each $x \in \mathrm{Lip}_t(\mathrm{SU}_q(2))$ we may use the multiplicative unitary to make sense of the expressions $\Delta(\partial_q^H(x))$ and $\Delta(\partial_t^V(x))$. Indeed, since $\partial_q^H(x)$ and $\partial_t^V(x)$ are bounded operators on $L^2(\mathrm{SU}_q(2))^{\oplus 2}$ we may apply the following definitions:

$$\Delta(\partial_q^H(x)) := (W \oplus W)(1 \otimes \partial_q^H(x))(W \oplus W)^* \quad \text{and}$$
$$\Delta(\partial_t^V(x)) := (W \oplus W)(1 \otimes \partial_t^V(x))(W \oplus W)^*,$$

where both of the right-hand sides are bounded operators on the Hilbert space tensor product $L^2(\mathrm{SU}_q(2))\widehat{\otimes}L^2(\mathrm{SU}_q(2))^{\oplus 2}$. We would like to commute the coproduct past the twisted $*$-derivations ∂_q^H and ∂_t^V obtaining formulae of the form

$$(1 \otimes \partial_q^H)\Delta(x) = \Delta(\partial_q^H(x)) \quad \text{and} \quad (1 \otimes \partial_t^V)\Delta(x) = \Delta(\partial_t^V(x)).$$

In order to make sense of the left-hand sides of these expressions we first investigate the unbounded selfadjoint operators $1\widehat{\otimes}D_q^H$ and $1\widehat{\otimes}D_t^V$, defined, respectively, as the closures of the unbounded symmetric operators

$$1 \otimes \mathcal{D}_q^H\colon \mathcal{O}(\mathrm{SU}_q(2)) \otimes \mathcal{O}(\mathrm{SU}_q(2))^{\oplus 2} \to L^2(\mathrm{SU}_q(2))\widehat{\otimes}L^2(\mathrm{SU}_q(2))^{\oplus 2},$$
$$1 \otimes \mathcal{D}_t^V\colon \mathcal{O}(\mathrm{SU}_q(2)) \otimes \mathcal{O}(\mathrm{SU}_q(2))^{\oplus 2} \to L^2(\mathrm{SU}_q(2))\widehat{\otimes}L^2(\mathrm{SU}_q(2))^{\oplus 2}.$$

Lemma 4.6.1. *The unitary operator* $W \oplus W$ *preserves the subspaces* $\mathrm{Dom}(1 \widehat{\otimes} D_q^H)$ *and* $\mathrm{Dom}(1 \widehat{\otimes} D^V) \subseteq L^2(\mathrm{SU}_q(2)) \widehat{\otimes} L^2(\mathrm{SU}_q(2))^{\oplus 2}$. *Moreover, it holds that*

$$\begin{aligned} &[1 \widehat{\otimes} D_q^H, W \oplus W](\xi) = 0 \quad \textit{for all } \xi \in \mathrm{Dom}(1 \widehat{\otimes} D_q^H) \quad \textit{and} \\ &[1 \widehat{\otimes} D_t^V, W \oplus W](\zeta) = 0 \quad \textit{for all } \zeta \in \mathrm{Dom}(1 \widehat{\otimes} D_t^V). \end{aligned} \tag{4.10}$$

Proof. We first remark that the direct sum $W \oplus W$ preserves the common core $\mathcal{O}(\mathrm{SU}_q(2)) \otimes \mathcal{O}(\mathrm{SU}_q(2))^{\oplus 2}$ for the two selfadjoint unbounded operators $1 \widehat{\otimes} D_q^H$ and $1 \widehat{\otimes} D_t^V$. Using standard results on commutators with selfadjoint unbounded operators, it therefore suffices to verify the identities in (4.10) for elements of the form $\xi = \zeta = x \otimes y$ with $x \in \mathcal{O}(\mathrm{SU}_q(2))$ and $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \mathcal{O}(\mathrm{SU}_q(2))^{\oplus 2}$. Using the coassociativity of Δ, one sees that $\Delta \partial_\eta(w) = (1 \otimes \partial_\eta)\Delta(w)$ for all $\eta \in \mathcal{U}_q(\mathfrak{su}(2))$ and $w \in \mathcal{O}(\mathrm{SU}_q(2))$. It therefore follows that

$$\begin{aligned} &(1 \otimes \mathcal{D}_q^H)(W \oplus W)(\xi) \\ &= - \begin{pmatrix} 0 & 1 \otimes q^{-\frac{1}{2}} \partial_{fk^{-1}} \\ 1 \otimes q^{\frac{1}{2}} \partial_{ek^{-1}} & 0 \end{pmatrix} \begin{pmatrix} \Delta(y_1) \cdot (x \otimes 1) \\ \Delta(y_2) \cdot (x \otimes 1) \end{pmatrix} \\ &= - \begin{pmatrix} q^{-\frac{1}{2}} \Delta(\partial_{fk^{-1}}(y_2)) \cdot (x \otimes 1) \\ q^{\frac{1}{2}} \Delta(\partial_{ek^{-1}}(y_1)) \cdot (x \otimes 1) \end{pmatrix} \\ &= (W \oplus W)(1 \otimes \mathcal{D}_q^H)(\xi). \end{aligned}$$

This proves the relevant identity in the case of the horizontal Dirac operator. To treat the commutator with the vertical Dirac operator, we simply record that W preserves the subspace $\mathcal{O}(\mathrm{SU}_q(2)) \otimes \mathcal{A}_q^n$ for all values of $n \in \mathbb{Z}$. The commutation relation now follows since $\mathcal{D}_t^V$ acts as a diagonal scalar matrix on $\mathcal{A}_q^n \oplus \mathcal{A}_q^m$ for all $n, m \in \mathbb{Z}$. ■

Remark 4.6.2. In the situation where $t = q = 1$, Lemma 4.6.1 together with the formulae $\lambda_g \xi = (\mathrm{ev}_{g^{-1}} \otimes 1)\Delta(\xi) = (\mathrm{ev}_{g^{-1}} \otimes 1)(W \oplus W)(1 \otimes \xi)$ for the left translation operator $\lambda_g \colon \mathcal{O}(\mathrm{SU}(2)) \to \mathcal{O}(\mathrm{SU}(2))$ implies that $\lambda_g \circ \mathcal{D}_{1,1} = \mathcal{D}_{1,1} \circ \lambda_g$ as operators on $\mathcal{O}(\mathrm{SU}(2))^{\oplus 2}$. Lemma 4.6.1 thus recovers the SU(2)-equivariance of the classical Dirac operator in this case (cf. Section 4.4).

Next, we wish to introduce the analogue of the Lipschitz algebra $\mathrm{Lip}_t(\mathrm{SU}_q(2))$ for the minimal tensor product $C(\mathrm{SU}_q(2)) \otimes_{\min} C(\mathrm{SU}_q(2))$, but in this case associated with the unbounded selfadjoint operators $1 \widehat{\otimes} D_q^H$ and $1 \widehat{\otimes} D_t^V$ instead of the unbounded selfadjoint operators D_q^H and D_t^V. For this to make sense, we let the minimal tensor product of C^*-algebras $C(\mathrm{SU}_q(2)) \otimes_{\min} C(\mathrm{SU}_q(2))$ act on

$$L^2(\mathrm{SU}_q(2)) \widehat{\otimes} L^2(\mathrm{SU}_q(2))^{\oplus 2}$$

via the representation $\rho \otimes \pi$, which we will from now on often suppress. Note that in this representation, the coproduct is implemented by $W \oplus W$ in the sense that

$$(\rho \otimes \pi)(\Delta(x)) = (W \oplus W)(1 \otimes \pi(x))(W \oplus W)^* \quad \text{for all } x \in C(\mathrm{SU}_q(2)). \tag{4.11}$$

We start out by expanding our notion of analytic elements. To this end, we record that the left circle action $\sigma_L \colon S^1 \times C(\mathrm{SU}_q(2)) \to C(\mathrm{SU}_q(2))$ induces a left circle action $1 \otimes \sigma_L$ on the minimal tensor product $C(\mathrm{SU}_q(2)) \otimes_{\min} C(\mathrm{SU}_q(2))$ given on simple tensors by

$$(1 \otimes \sigma_L)(z, x \otimes y) := x \otimes \sigma_L(z, y) \quad \text{for all } z \in S^1,\ x, y \in C(\mathrm{SU}_q(2)).$$

We recall that the closed strip $I_s \subseteq \mathbb{C}$ was introduced in (3.19) for all values of $s \in (0, 1]$.

Definition 4.6.3. Let $s \in (0, 1]$. An element $x \in C(\mathrm{SU}_q(2)) \otimes_{\min} C(\mathrm{SU}_q(2))$ is called *analytic of order* $-\log(s)/2$ when the continuous map

$$\mathbb{R} \to C(\mathrm{SU}_q(2)) \otimes_{\min} C(\mathrm{SU}_q(2))$$

given by $r \mapsto (1 \otimes \sigma_L)(e^{ir}, x)$ extends to a continuous function

$$I_s \to C(\mathrm{SU}_q(2)) \otimes_{\min} C(\mathrm{SU}_q(2))$$

which is analytic on the interior $I_s^\circ \subseteq I_s$. We denote this (unique) continuous extension by $z \mapsto (1 \otimes \sigma_L)(e^{iz}, x)$.

We record that the $*$-algebra structure on the minimal tensor product

$$C(\mathrm{SU}_q(2)) \otimes_{\min} C(\mathrm{SU}_q(2))$$

induces a $*$-algebra structure on the subset of elements which are analytic of order $-\log(s)/2$.

Lemma 4.6.4. *Let $s \in (0, 1]$. If $x \in C(\mathrm{SU}_q(2))$ is analytic of order $-\log(s)/2$ in the sense of Definition 3.6.1, then $\Delta(x)$ is analytic of order $-\log(s)/2$ in the sense of Definition 4.6.3. Moreover, we have the formula*

$$(1 \otimes \sigma_L)(e^{iz}, \Delta(x)) = \Delta(\sigma_L(e^{iz}, x)) \quad \textit{for all } z \in I_s.$$

Proof. We first notice that $(1 \otimes \sigma_L)(e^{ir}, \Delta(x)) = \Delta(\sigma_L(e^{ir}, x))$ for all $x \in C(\mathrm{SU}_q(2))$ and $r \in \mathbb{R}$. To verify this identity, it suffices to use the formula in (3.16) for the matrix coefficients and then extend to all of $C(\mathrm{SU}_q(2))$ by continuity and linearity. Suppose next that $x \in C(\mathrm{SU}_q(2))$ is analytic of order $-\log(s)/2$. The map $r \mapsto \Delta(\sigma_L(e^{ir}, x))$ then extends continuously to I_s and the extension is analytic on I_s°. It follows that $\Delta(x)$ is analytic of order $-\log(s)/2$ as well. The desired formula for all $z \in I_s$ is then a consequence of the identity theorem in complex analysis. ∎

We now have the data needed in order to formally introduce the Lipschitz algebras associated to the unbounded selfadjoint operators $1 \widehat{\otimes} D_q^H$ and $1 \widehat{\otimes} D_t^V$. Let $x \in C(\mathrm{SU}_q(2)) \otimes_{\min} C(\mathrm{SU}_q(2))$. We say that x is *horizontally Lipschitz* when

(1) x is analytic of order $-\log(q)/2$ (with respect to $1 \otimes \sigma_L$);

(2) the bounded operator $(1 \otimes \sigma_L)(q^{\frac{1}{2}}, x)$ preserves the domain of $1 \hat{\otimes} D_q^H$;

(3) the twisted commutator

$$(1 \hat{\otimes} D_q^H) \cdot (1 \otimes \sigma_L)(q^{\frac{1}{2}}, x) - (1 \otimes \sigma_L)(q^{-\frac{1}{2}}, x) \cdot (1 \hat{\otimes} D_q^H)$$

extends to a bounded operator, denoted $(1 \otimes \partial_q^H)(x)$, on the Hilbert space $L^2(\mathrm{SU}_q(2)) \hat{\otimes} L^2(\mathrm{SU}_q(2))^{\oplus 2}$.

We say that x is *vertically Lipschitz* when

(1) x is analytic of order $-\log(t)/2$ (with respect to $1 \otimes \sigma_L$);

(2) the bounded operator $(1 \otimes \sigma_L)(t^{\frac{1}{2}}, x)$ preserves the domain of $1 \hat{\otimes} D_t^V$;

(3) the twisted commutator

$$(1 \hat{\otimes} D_t^V) \cdot (1 \otimes \sigma_L)(t^{\frac{1}{2}}, x) - (1 \otimes \sigma_L)(t^{-\frac{1}{2}}, x) \cdot (1 \hat{\otimes} D_t^V)$$

extends to a bounded operator, denoted $(1 \otimes \partial_t^V)(x)$, on the Hilbert space $L^2(\mathrm{SU}_q(2)) \hat{\otimes} L^2(\mathrm{SU}_q(2))^{\oplus 2}$.

The *Lipschitz algebra* $\mathrm{Lip}_t(\mathrm{SU}_q(2) \times \mathrm{SU}_q(2))$, then consists of the elements in $C(\mathrm{SU}_q(2)) \otimes_{\min} C(\mathrm{SU}_q(2))$ which are both horizontally and vertically Lipschitz. We record that the Lipschitz algebra $\mathrm{Lip}_t(\mathrm{SU}_q(2) \times \mathrm{SU}_q(2))$ is a norm-dense $*$-subalgebra of the minimal tensor product $C(\mathrm{SU}_q(2)) \otimes_{\min} C(\mathrm{SU}_q(2))$.

Lemma 4.6.5. *For $x \in \mathrm{Lip}_t(\mathrm{SU}_q(2))$ it holds that $\Delta(x) \in \mathrm{Lip}_t(\mathrm{SU}_q(2) \times \mathrm{SU}_q(2))$ and we have the formulae $(1 \otimes \partial_q^H)\Delta(x) = \Delta(\partial_q^H(x))$ and $(1 \otimes \partial_t^V)\Delta(x) = \Delta(\partial_t^V(x))$.*

Proof. Let $x \in \mathrm{Lip}_t(\mathrm{SU}_q(2))$ be given. We focus on showing that $\Delta(x)$ is horizontally Lipschitz and that $(1 \otimes \partial_q^H)\Delta(x) = \Delta(\partial_q^H(x))$, since the same argument applies to the vertical case as well. First note that $\Delta(x)$ is analytic of order $-\log(q)/2$ by Lemma 4.6.4. Let $\xi \in \mathcal{O}(\mathrm{SU}_q(2)) \otimes \mathcal{O}(\mathrm{SU}_q(2))^{\oplus 2}$ be an element in the core for $1 \hat{\otimes} D_q^H$, and recall that $(W^* \oplus W^*)(\xi) \in \mathcal{O}(\mathrm{SU}_q(2)) \otimes \mathcal{O}(\mathrm{SU}_q(2))^{\oplus 2}$. Using Lemmas 4.6.1, 4.6.4 and (4.11) we then see that

$$\begin{aligned}(1 \otimes \sigma_L)\big(q^{\frac{1}{2}}, \Delta(x)\big)(\xi) &= \Delta\big(\sigma_L(q^{\frac{1}{2}}, x)\big)(\xi) \\ &= (W \oplus W)\big(1 \otimes \sigma_L(q^{\frac{1}{2}}, x)\big)(W^* \oplus W^*)(\xi) \in \mathrm{Dom}(1 \hat{\otimes} D_q^H).\end{aligned}$$

Using this, another application of Lemmas 4.6.1, 4.6.4 and (4.11) shows that the twisted commutator may be computed on ξ as follows:

$$\begin{aligned}&(1 \hat{\otimes} D_q^H)(1 \otimes \sigma_L)\big(q^{\frac{1}{2}}, \Delta(x)\big)(\xi) - (1 \otimes \sigma_L)\big(q^{-\frac{1}{2}}, \Delta(x)\big)(1 \hat{\otimes} D_q^H)(\xi) \\ &\quad = (W \oplus W)\big(1 \otimes D_q^H \sigma_L(q^{\frac{1}{2}}, x) - 1 \otimes \sigma_L(q^{-\frac{1}{2}}, x) D_q^H\big)(W^* \oplus W^*)(\xi) \\ &\quad = (W \oplus W)\big(1 \otimes \partial_q^H(x)\big)(W^* \oplus W^*)(\xi) = \Delta\big(\partial_q^H(x)\big)(\xi).\end{aligned}$$

The result of the lemma now follows since $\Delta(\partial_q^H(x))$ is a bounded operator and since $\mathcal{O}(\mathrm{SU}_q(2)) \otimes \mathcal{O}(\mathrm{SU}_q(2))^{\oplus 2}$ is a core for the selfadjoint unbounded operator $1 \hat{\otimes} D_q^H$. ■

For $\xi, \zeta \in L^2(\mathrm{SU}_q(2))$ we let $\phi_{\xi,\zeta} : C(\mathrm{SU}_q(2)) \to \mathbb{C}$ denote the bounded linear functional $\phi_{\xi,\zeta}(x) := \langle \xi, \rho(x)\zeta \rangle$. Let us moreover introduce the two bounded operators T_ξ and $T_\zeta : L^2(\mathrm{SU}_q(2))^{\oplus 2} \to L^2(\mathrm{SU}_q(2)) \hat{\otimes} L^2(\mathrm{SU}_q(2))^{\oplus 2}$ given by the formulae $T_\xi(\eta) := \xi \otimes \eta$ and $T_\zeta(\eta) := \zeta \otimes \eta$. We define the bounded operator

$$\phi_{\xi,\zeta} \otimes 1 : \mathbb{B}\big(L^2(\mathrm{SU}_q(2)) \hat{\otimes} L^2(\mathrm{SU}_q(2))^{\oplus 2}\big) \to \mathbb{B}\big(L^2(\mathrm{SU}_q(2))^{\oplus 2}\big)$$

given by $(\phi_{\xi,\zeta} \otimes 1)(z) := T_\xi^* z T_\zeta$, and record that we have the estimate $\|\phi_{\xi,\zeta} \otimes 1\| \leqslant \|\xi\| \cdot \|\zeta\|$ on the operator norm.

The last result of the present section shows how the Lipschitz seminorm and the coproduct interact with the slice maps just introduced. This result will be essential in our analysis of the Berezin transform; see Proposition 6.3.6.

Proposition 4.6.6. *For each $\xi, \zeta \in L^2(\mathrm{SU}_q(2))$ and $z \in \mathrm{Lip}_t(\mathrm{SU}_q(2) \times \mathrm{SU}_q(2))$ it holds that $(\phi_{\xi,\zeta} \otimes 1)(z) \in \mathrm{Lip}_t(\mathrm{SU}_q(2))$ and we have the identities*

$$\partial_q^H\big((\phi_{\xi,\zeta} \otimes 1)(z)\big) = (\phi_{\xi,\zeta} \otimes 1)(1 \otimes \partial_q^H)(z) \quad \textit{and}$$
$$\partial_t^V\big((\phi_{\xi,\zeta} \otimes 1)(z)\big) = (\phi_{\xi,\zeta} \otimes 1)(1 \otimes \partial_t^V)(z).$$

In particular, we have the estimate

$$L_{t,q}^{\max}\big((\phi_{\xi,\zeta} \otimes 1)(\Delta(x))\big) \leqslant \|\xi\|\|\zeta\| \cdot L_{t,q}^{\max}(x)$$

for all $x \in \mathrm{Lip}_t(\mathrm{SU}_q(2))$.

Proof. Let $\xi, \zeta \in L^2(\mathrm{SU}_q(2))$ and $z \in \mathrm{Lip}_t(\mathrm{SU}_q(2) \times \mathrm{SU}_q(2))$ be given. We focus on showing that $(\phi_{\xi,\zeta} \otimes 1)(z)$ is vertically Lipschitz and that $\partial_t^V((\phi_{\xi,\zeta} \otimes 1)(z)) = (\phi_{\xi,\zeta} \otimes 1)(1 \otimes \partial_t^V)(z)$. The analogous claim regarding the horizontal Dirac operator follows by a similar argument. Notice first that we have the inclusion $T_\zeta D_t^V \subseteq (1 \hat{\otimes} D_t^V) T_\zeta$ of unbounded operators on $L^2(\mathrm{SU}_q(2))^{\oplus 2}$. Since the same inclusion holds with T_ξ instead of T_ζ we also obtain the inclusion $T_\xi^*(1 \hat{\otimes} D_t^V) \subseteq D_t^V T_\xi^*$ by applying the adjoint operation. Secondly, since $(\phi_{\xi,\zeta} \otimes 1)(y_1 \otimes y_2) = \phi_{\xi,\zeta}(y_1) y_2$ for $y_1, y_2 \in C(\mathrm{SU}_q(2))$ it follows that

$$\sigma_L(e^{ir}, (\phi_{\xi,\zeta} \otimes 1)(y_1 \otimes y_2)) = (\phi_{\xi,\zeta} \otimes 1)(1 \otimes \sigma_L)(e^{ir}, y_1 \otimes y_2) \quad \text{for all } r \in \mathbb{R},$$

and hence the same formula holds globally on $C(\mathrm{SU}_q(2)) \otimes_{\min} C(\mathrm{SU}_q(2))$ by linearity and density. We thereby obtain that $x := (\phi_{\xi,\zeta} \otimes 1)(z) \in C(\mathrm{SU}_q(2))$ is analytic of order $-\log(t)/2$ (in the sense of Definition 3.6.1) and that we have the identity

$$\sigma_L(e^{iw}, x) = (\phi_{\xi,\zeta} \otimes 1)(1 \otimes \sigma_L)(e^{iw}, z) \quad \text{for all } w \in I_t.$$

It follows from the above observations that the bounded operator

$$\sigma_L(t^{\frac{1}{2}}, x) = (\phi_{\xi,\zeta} \otimes 1)(1 \otimes \sigma_L)(t^{\frac{1}{2}}, z) = T_\xi^*(1 \otimes \sigma_L)(t^{\frac{1}{2}}, z)T_\zeta$$

preserves the domain of D_t^V. Moreover, we may compute as follows for any vector $\eta \in \mathrm{Dom}(D_t^V)$:

$$\begin{aligned} D_t^V \cdot \sigma_L(t^{\frac{1}{2}}, x)(\eta) &= D_t^V \cdot T_\xi^*(1 \otimes \sigma_L)(t^{\frac{1}{2}}, z)T_\zeta(\eta) \\ &= T_\xi^*(1 \widehat{\otimes} D_t^V) \cdot (1 \otimes \sigma_L)(t^{\frac{1}{2}}, z)T_\zeta(\eta) \\ &= T_\xi^*(1 \otimes \partial_t^V)(z)T_\zeta(\eta) + T_\xi^*(1 \otimes \sigma_L)(t^{-\frac{1}{2}}, z) \cdot (1 \widehat{\otimes} D_t^V)T_\zeta(\eta) \\ &= (\phi_{\xi,\zeta} \otimes 1)(1 \otimes \partial_t^V)(z)(\eta) + \sigma_L(t^{-\frac{1}{2}}, x) \cdot D_t^V(\eta). \end{aligned}$$

This ends the proof of the first part of the lemma. The second part of the lemma (regarding the estimate relating to the seminorm $L_{t,q}^{\max}$) now follows immediately by an application of Lemma 4.6.5. ■

4.7 Conjugating the Dirac element with the fundamental unitary

The main technical tool for proving quantum Gromov–Hausdorff continuity for the Podleś spheres S_q^2 [3, Theorem A] is a trivialisation of the "spinor bundle" $\mathcal{A}_q^1 \oplus \mathcal{A}_q^{-1}$, implemented by the fundamental corepresentation unitary $u := u^1 \in \mathbb{M}_2(\mathcal{O}(\mathrm{SU}_q(2)))$. Note that this trivialisation is not compatible with the $\mathbb{Z}/2\mathbb{Z}$-grading on the spinor bundle. As in Section 4.2, we let $\partial^0 \colon \mathcal{O}(S_q^2) \to \mathbb{B}(H_q^1 \oplus H_q^{-1})$ denote the derivation arising by taking the commutator with the Dąbrowski–Sitarz Dirac operator; see [22]. In the work [3] we analysed the linear map $\delta^0 := u\partial^0 u^*$, a key feature of which is that it gives rise to the same seminorm as ∂^0 after composition with the operator norm. Moreover, we saw in [3, Proposition 3.12] that δ^0 can be described by means of the right action of the quantum enveloping algebra $\mathcal{U}_q(\mathfrak{su}(2))$ on the coordinate algebra $\mathcal{O}(\mathrm{SU}_q(2))$.

To obtain quantum Gromov–Hausdorff continuity also at the level of quantum SU(2), it is therefore relevant to analyse the analogue of δ^0 in this context as well, and we carry out the relevant details in this section. For the analysis below to work of out we need the vertical and horizontal derivations to obey the same twisted Leibniz rule. We thus focus exclusively on the special case where the two parameters t and $q \in (0, 1]$ agree, and consider the twisted $*$-derivation (see Definition 3.2.1)

$$\partial := \partial_{q,q} = \begin{pmatrix} \partial_q^3 & -\partial^2 \\ -\partial^1 & -\partial_q^3 \end{pmatrix} : \mathcal{O}(\mathrm{SU}_q(2)) \to \mathbb{M}_2\big(\mathcal{O}(\mathrm{SU}_q(2))\big),$$

where the twists are given by ∂_k and $\partial_{k^{-1}}$ so that $\partial(xy) = \partial(x)\partial_k(y) + \partial_{k^{-1}}(x)\partial(y)$ for all $x, y \in \mathcal{O}(\mathrm{SU}_q(2))$.

Note that the twisted $*$-derivation ∂_q^3 can be described by the formula

$$\partial^3 := \partial_q^3 = \begin{cases} \frac{\partial_k - \partial_{k^{-1}}}{q - q^{-1}} & \text{for } q \neq 1 \\ \frac{1}{2}\partial_h & \text{for } q = 1, \end{cases}$$

so that ∂ is defined entirely in terms of the *left* action of $\mathcal{U}_q(\mathfrak{su}(2))$ on the coordinate algebra $\mathcal{O}(\mathrm{SU}_q(2))$. The main point is to show that when ∂ is conjugated with the fundamental corepresentation unitary we obtain a twisted $*$-derivation which can be expressed in terms of the *right* action of the quantum enveloping algebra. Recall that for $\eta \in \mathcal{U}_q(\mathfrak{su}(2))$, the right action of η is defined by the linear endomorphism $\delta_\eta \colon \mathcal{O}(\mathrm{SU}_q(2)) \to \mathcal{O}(\mathrm{SU}_q(2))$ given by the formula $\delta_\eta := (\langle \eta, \cdot \rangle \otimes 1)\Delta$, and in this way we obtain three twisted derivations $\delta^1, \delta^2, \delta^3 \colon \mathcal{O}(\mathrm{SU}_q(2)) \to \mathcal{O}(\mathrm{SU}_q(2))$ by setting

$$\delta^1 := q^{\frac{1}{2}}\delta_e \quad \delta^2 := q^{-\frac{1}{2}}\delta_f \quad \text{and} \quad \delta^3 := \begin{cases} \frac{\delta_k - \delta_{k^{-1}}}{q - q^{-1}} & \text{for } q \neq 1 \\ \frac{1}{2}\delta_h & \text{for } q = 1, \end{cases}$$

which are all twisted by the automorphisms δ_k and $\delta_{k^{-1}}$ so that $\delta^i(xy) = \delta^i(x)\delta_k(y) + \delta_{k^{-1}}(x)\delta^i(y)$ for all $x, y \in \mathcal{O}(\mathrm{SU}_q(2))$ and $i \in \{1,2,3\}$. We now assemble this data into a single twisted $*$-derivation

$$\delta := \begin{pmatrix} \delta^3 & -\delta^2 \\ -\delta^1 & -\delta^3 \end{pmatrix} \colon \mathcal{O}(\mathrm{SU}_q(2)) \to \mathbb{M}_2\big(\mathcal{O}(\mathrm{SU}_q(2))\big),$$

where the twists are again given by δ_k and $\delta_{k^{-1}}$. Recalling that $u = u^1$ denotes the fundamental corepresentation unitary, the main result of this section is the identity

$$u\partial(x)u^* = \delta(x) \quad \text{for all } x \in \mathcal{O}(\mathrm{SU}_q(2)), \tag{4.12}$$

which will play crucial role in our further analysis. The strategy for proving (4.12) will be to first show that $u\partial(-)u^*$ satisfies the same twisted Leibniz rule as δ, thus reducing the proof to verifying (4.12) on the generators of $\mathcal{O}(\mathrm{SU}_q(2))$. To this end, we will need the algebra automorphism $\nu^{-\frac{1}{2}} := \delta_k \circ \partial_k \colon \mathcal{O}(\mathrm{SU}_q(2)) \to \mathcal{O}(\mathrm{SU}_q(2))$. Notice that it follows from the defining commutation relations in $\mathcal{O}(\mathrm{SU}_q(2))$ that

$$bx = \nu^{-\frac{1}{2}}(x)b \quad \text{and} \quad b^*x = \nu^{-\frac{1}{2}}(x)b^* \quad \text{for all } x \in \mathcal{O}(\mathrm{SU}_q(2)). \tag{4.13}$$

Before we proceed we will introduce some relevant notation: if $\sigma, \theta \colon \mathcal{O}(\mathrm{SU}_q(2)) \to \mathcal{O}(\mathrm{SU}_q(2))$ are algebra automorphisms, we shall write

$${}_\sigma[y, x]_\theta := y\theta(x) - \sigma(x)y$$

for the twisted commutator between two elements $x, y \in \mathcal{O}(\mathrm{SU}_q(2))$.

Lemma 4.7.1. *We have the identities*

$$
{}_{\delta_k}[a^*, x]_{\partial_k} = (1-q^2) b \partial^1(x) \quad \text{and} \quad {}_{\delta_k}[a, x]_{\partial_k} = (1-q^2) q^{-1} b^* \partial^2(x)
$$
$$
{}_{\delta_k}[b^*, x]_{\partial_k} = b^*(\partial_k - \partial_{k-1})(x) \quad \text{and} \quad {}_{\delta_k}[b, x]_{\partial_k} = b(\partial_k - \partial_{k-1})(x)
$$

for all $x \in \mathcal{O}(\mathrm{SU}_q(2))$.

Proof. A direct computation reveals that the operation $x \mapsto {}_{\delta_k}[a^*, x]_{\partial_k}$ satisfies the following twisted Leibniz rule:

$$
{}_{\delta_k}[a^*, xy]_{\partial_k} = {}_{\delta_k}[a^*, x]_{\partial_k} \cdot \partial_k(y) + \delta_k(x) \cdot {}_{\delta_k}[a^*, y]_{\partial_k} \quad \text{for all } x, y \in \mathcal{O}(\mathrm{SU}_q(2)).
$$

It moreover follows from (4.13) that the operation $x \mapsto b\partial^1(x)$ satisfies the same twisted Leibniz rule so that

$$
b\partial^1(xy) = b\partial^1(x)\partial_k(y) + \delta_k(x) b \partial^1(y) \quad \text{for all } x, y \in \mathcal{O}(\mathrm{SU}_q(2)).
$$

In order to prove the first identity of the lemma, it thus suffices to check that

$$
a^* \partial_k(x) - \delta_k(x) a^* = (1-q^2) b \partial^1(x)
$$

for $x \in \{a, a^*, b, b^*\}$. In these four cases, one may verify the relevant identity by a straightforward computation. The second identity of the lemma can be proved by a similar argument. The two last identities (those involving twisted commutators with b and b^*) follow immediately from (4.13). ∎

Lemma 4.7.2. *We have the identities*

$$
{}_{\delta_k}[u, x]_{\partial_k} = (q^2 - 1) \begin{pmatrix} 0 & b \\ q^{-1} b^* & 0 \end{pmatrix} \partial(x) \quad \text{and}
$$
$$
{}_{\partial_{k-1}}[u^*, x]_{\delta_{k-1}} = (q^2 - 1) \partial(x) \begin{pmatrix} 0 & q^{-1} b \\ b^* & 0 \end{pmatrix}
$$

for all $x \in \mathcal{O}(\mathrm{SU}_q(2))$.

Proof. The relevant identities are trivially satisfied for $q = 1$, so we focus on the case where $q \neq 1$ and let $x \in \mathcal{O}(\mathrm{SU}_q(2))$ be given. Applying the definition of the fundamental corepresentation unitary u together with Lemma 4.7.1 we obtain that

$$
\begin{aligned}
{}_{\delta_k}[u, x]_{\partial_k} &= \begin{pmatrix} {}_{\delta_k}[a^*, x]_{\partial_k} & -q \cdot {}_{\delta_k}[b, x]_{\partial_k} \\ {}_{\delta_k}[b^*, x]_{\partial_k} & {}_{\delta_k}[a, x]_{\partial_k} \end{pmatrix} \\
&= \begin{pmatrix} (1-q^2) b \partial^1(x) & -q b (\partial_k - \partial_{k-1})(x) \\ b^*(\partial_k - \partial_{k-1})(x) & (1-q^2) q^{-1} b^* \partial^2(x) \end{pmatrix} \\
&= (q^2 - 1) \begin{pmatrix} 0 & b \\ q^{-1} b^* & 0 \end{pmatrix} \partial(x).
\end{aligned}
$$

This proves the first identity of the lemma. The remaining identity then follows from the first via the following computation:

$$\begin{aligned}
&(q^2-1)\partial(x)\begin{pmatrix}0 & q^{-1}b\\ b^* & 0\end{pmatrix}\\
&\quad = (1-q^2)\left(\left(\begin{pmatrix}0 & b\\ q^{-1}b^* & 0\end{pmatrix}\partial(x^*)\right)^*\right) = (\delta_k(x^*)u - u\partial_k(x^*))^*\\
&\quad = u^*\delta_{k-1}(x) - \partial_{k-1}(x)u^* = {}_{\partial_{k-1}}[u^*, x]_{\delta_{k-1}}.
\end{aligned}$$

■

We are now ready to show that the operation $x \mapsto u\partial(x)u^*$ is a twisted derivation.

Proposition 4.7.3. *It holds that*

$$u\partial(xy)u^* = u\partial(x)u^*\delta_k(y) + \delta_{k-1}(x)u\partial(y)u^*$$

for all $x, y \in \mathcal{O}(\mathrm{SU}_q(2))$.

Proof. Let $x, y \in \mathcal{O}(\mathrm{SU}_q(2))$ be given. We compute that

$$\begin{aligned}
u\partial(xy)u^* &= u\partial(x)\partial_k(y)u^* + u\partial_{k-1}(x)\partial(y)u^*\\
&= u\partial(x)u^*u\partial_k(y)u^* + u\partial_{k-1}(x)u^*u\partial(y)u^*\\
&= u\partial(x)u^*\delta_k(y) + \delta_{k-1}(x)u\partial(y)u^*\\
&\quad + u\partial(x)u^* \cdot {}_{\delta_k}[u, y]_{\partial_k}u^* - u \cdot {}_{\partial_{k-1}}[u^*, x]_{\delta_{k-1}}u\partial(y)u^*.
\end{aligned}$$

Notice now that

$$\begin{pmatrix}0 & q^{-1}b\\ b^* & 0\end{pmatrix}u = \begin{pmatrix}q^{-1}bb^* & ab\\ q^{-1}a^*b^* & -qb^*b\end{pmatrix} = u^*\begin{pmatrix}0 & b\\ q^{-1}b^* & 0\end{pmatrix}.$$

Thus, applying Lemma 4.7.2 we obtain that

$$\begin{aligned}
u\partial(x)u^* \cdot {}_{\delta_k}[u, y]_{\partial_k}u^* &= (q^2-1)u\partial(x)u^*\begin{pmatrix}0 & b\\ q^{-1}b^* & 0\end{pmatrix}\partial(y)u^*\\
&= (q^2-1)u\partial(x)\begin{pmatrix}0 & q^{-1}b\\ b^* & 0\end{pmatrix}u\partial(y)u^*\\
&= u \cdot {}_{\partial_{k-1}}[u^*, x]_{\delta_{k-1}} \cdot u\partial(y)u^*.
\end{aligned}$$

This proves the proposition. ■

We are now ready to verify that u conjugates ∂ into δ.

Proposition 4.7.4. *It holds that* $u\partial(x)u^* = \delta(x)$ *for all* $x \in \mathcal{O}(\mathrm{SU}_q(2))$.

Proof. Using Proposition 4.7.3 we see that the operations $x \mapsto u\partial(x)u^*$ and $x \mapsto \delta(x)$ satisfy the same twisted Leibniz rule. Since they also behave in the same way with respect to the adjoint operation, it therefore suffices to verify the required identity on the generators $a, b \in \mathcal{O}(\mathrm{SU}_q(2))$. To treat the case $q = 1$ and $q < 1$ on the same footing, we define

$$\mu := [1/2]_q = \frac{1}{q^{1/2} + q^{-1/2}},$$

so that $\partial^3(a) = \mu a$ and $\partial^3(b) = \mu b$. The two relations may now be proven by a straightforward computation, indeed:

$$\begin{aligned}
u\partial(a)u^* &= \begin{pmatrix} a^* & -qb \\ b^* & a \end{pmatrix} \begin{pmatrix} \mu \cdot a & 0 \\ -q^{\frac{1}{2}} b^* & -\mu \cdot a \end{pmatrix} \begin{pmatrix} a & b \\ -qb^* & a^* \end{pmatrix} \\
&= \begin{pmatrix} \mu \cdot a^* a^2 + q^{\frac{3}{2}} bb^* a - q^2 \mu \cdot bab^* & \mu \cdot a^* ab + q^{\frac{3}{2}} bb^* b + q\mu \cdot baa^* \\ \mu \cdot b^* a^2 - q^{\frac{1}{2}} ab^* a + q\mu \cdot a^2 b^* & \mu \cdot b^* ab - q^{\frac{1}{2}} ab^* b - \mu \cdot a^2 a^* \end{pmatrix} \\
&= \begin{pmatrix} \mu \cdot a & q^{\frac{1}{2}} b \\ 0 & -\mu \cdot a \end{pmatrix} = \delta(a), \quad \text{and} \\
u\partial(b)u^* &= \begin{pmatrix} a^* & -qb \\ b^* & a \end{pmatrix} \begin{pmatrix} \mu \cdot b & 0 \\ q^{-\frac{1}{2}} a^* & -\mu \cdot b \end{pmatrix} \begin{pmatrix} a & b \\ -qb^* & a^* \end{pmatrix} \\
&= \begin{pmatrix} \mu \cdot a^* ba - q^{\frac{1}{2}} ba^* a - q^2 \mu \cdot b^2 b^* & \mu \cdot a^* b^2 - q^{\frac{1}{2}} ba^* b + q\mu \cdot b^2 a^* \\ \mu \cdot b^* ba + q^{-\frac{1}{2}} aa^* a + q\mu \cdot abb^* & \mu \cdot b^* b^2 + q^{-\frac{1}{2}} aa^* b - \mu \cdot aba^* \end{pmatrix} \\
&= \begin{pmatrix} -\mu \cdot b & 0 \\ q^{-\frac{1}{2}} a & \mu \cdot b \end{pmatrix} = \delta(b).
\end{aligned}$$

■

We now have the tools needed to properly investigate the quantum metric space structure on $\mathrm{SU}_q(2)$, and we proceed to do so in the following chapter.