Chapter 5

Quantum metrics on quantum SU(2)

We now return to the general setting, and consider again two parameters $t, q \in (0, 1]$ which will be fixed throughout this section. The aim of this section is to show that $(C(\mathrm{SU}_q(2)), L_{t,q}^{\max})$ is a compact quantum metric space. The proof consists of several steps and we therefore first explain the general strategy. For each $M \in \mathbb{N}_0$, we recall from Section 3.5 that the algebraic spectral M-band is defined as the subspace $\mathcal{B}_q^M := \sum_{m=-M}^{M} \mathcal{A}_q^m \subseteq \mathcal{O}(\mathrm{SU}_q(2))$ and that the spectral M-band B_q^M agrees with the norm closure of $\mathcal{B}_q^M$ with respect to the C^*-norm on $C(\mathrm{SU}_q(2))$. The Lipschitz seminorm

$$L_{t,q}^{\max} : C(\mathrm{SU}_q(2)) \to [0, \infty]$$

restricts to a Lipschitz seminorm $L_{t,q}^{\max} : B_q^M \to [0, \infty]$ with domain given by the intersection $B_q^M \cap \mathrm{Lip}_t(\mathrm{SU}_q(2))$. We start by proving that the pair $(B_q^M, L_{t,q}^{\max})$ is a compact quantum metric space for all $M \in \mathbb{N}_0$. Knowing this, the next step is to construct a Lip-norm contraction $C(\mathrm{SU}_q(2)) \to B_q^M$ for each $M \in \mathbb{N}_0$. This sets the stage for an application of Corollary 2.1.10, from which we will finally deduce that $(C(\mathrm{SU}_q(2)), L_{t,q}^{\max})$ is a compact quantum metric space; see Theorem 5.6.1 below for details. In the following section we first treat the case where $M = 0$, which plays a special role, since this provides the connection with the Podleś sphere, whose quantum metric structure was investigated in [2]; see also [3, 4].

5.1 The Podleś sphere revisited

Notice first of all that the spectral 0-band B_q^0 agrees with the Podleś sphere $C(S_q^2)$. For each $m \in \mathbb{Z}$, we recall from Section 3.5 that $H_q^m \subseteq L^2(\mathrm{SU}_q(2))$ denotes the Hilbert space completion of the algebraic spectral subspace $\mathcal{A}_q^m$ with respect to the inner product coming from the Haar state $h : C(\mathrm{SU}_q(2)) \to \mathbb{C}$. The GNS Hilbert space $L^2(\mathrm{SU}_q(2))$ is then isomorphic to the Hilbert space direct sum

$$L^2(\mathrm{SU}_q(2)) \cong \bigoplus_{m=-\infty}^{\infty} H_q^m .$$

The horizontal Dirac operator $\mathcal{D}_q^H : \mathcal{O}(\mathrm{SU}_q(2))^{\oplus 2} \to L^2(\mathrm{SU}_q(2))^{\oplus 2}$ restricts to the unbounded operator

$$\mathcal{D}_q^0 = \begin{pmatrix} 0 & -\partial_f \\ -\partial_e & 0 \end{pmatrix} : \mathcal{A}_q^1 \oplus \mathcal{A}_q^{-1} \to H_q^1 \oplus H_q^{-1}$$

and we denote the closure by $D_q^0\colon \mathrm{Dom}(D_q^0) \to H_q^1 \oplus H_q^{-1}$. The vertical Dirac operator $\mathcal{D}_t^V\colon \mathcal{O}(\mathrm{SU}_q(2))^{\oplus 2} \to L^2(\mathrm{SU}_q(2))^{\oplus 2}$ restricts to the trivial operator zero on the direct sum $\mathcal{A}_q^1 \oplus \mathcal{A}_q^{-1}$, and the diagonal representation

$$\pi\colon C(\mathrm{SU}_q(2)) \to \mathbb{B}(L^2(\mathrm{SU}_q(2))^{\oplus 2})$$

restricts to a representation

$$\pi^0\colon C(S_q^2) \to \mathbb{B}\big(H_q^1 \oplus H_q^{-1}\big).$$

We equip the Hilbert space $H_q^1 \oplus H_q^{-1}$ with the $\mathbb{Z}/2\mathbb{Z}$-grading operator $\gamma = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ and record that the triple $(C(S_q^2), H_q^1 \oplus H_q^{-1}, D_q^0)$ agrees with the Dąbrowski–Sitarz spectral triple (up to conjugation with the grading operator γ); see [22,61]. We remark that we are now within the standard realm of non-commutative geometry, in so far that $(C(S_q^2), H_q^1 \oplus H_q^{-1}, D_q^0)$ is a genuine (even) spectral triple on $C(S_q^2)$. This is in contrast to the situation for $C(\mathrm{SU}_q(2))$, where we are just relying on the spectral data given by the horizontal and vertical Dirac operators. The Dąbrowski–Sitarz spectral triple therefore has its own Lipschitz algebra $\mathrm{Lip}(S_q^2)$ defined as

$$\big\{x \in C(S_q^2) \mid \pi^0(x)(\mathrm{Dom}(D_q^0)) \subseteq \mathrm{Dom}(D_q^0) \text{ and } \overline{[D_q^0, \pi^0(x)]} \text{ is bounded}\big\}.$$

For each $x \in \mathrm{Lip}(S_q^2)$ we apply the notation $\partial^0(x) := \overline{[D_q^0, \pi^0(x)]}$. The main result in [2] is that the Lipschitz seminorm $L_q^{0,\max}\colon C(S_q^2) \to [0,\infty]$ defined by

$$L_q^{0,\max}(x) := \begin{cases} \|\partial^0(x)\| & x \in \mathrm{Lip}(S_q^2) \\ \infty & x \in C(S_q^2) \setminus \mathrm{Lip}(S_q^2) \end{cases}$$

turns $C(S_q^2)$ into a compact quantum metric space. We shall now prove that the two settings are compatible, in the sense that the restriction of $L_{t,q}^{\max}$ to $C(S_q^2)$ agrees with $L_q^{0,\max}$.

Recall that $\nu^{\frac{1}{2}}$ denotes the algebra automorphism $\partial_{k^{-1}} \circ \delta_{k^{-1}}\colon \mathcal{O}(\mathrm{SU}_q(2)) \to \mathcal{O}(\mathrm{SU}_q(2))$ while $\mathcal{J}\colon \mathcal{O}(\mathrm{SU}_q(2)) \to \mathcal{O}(\mathrm{SU}_q(2))$ denotes the antilinear antihomomorphism $x \mapsto (\delta_k \partial_k)(x^*)$ which extends to the antilinear unitary operator J on $L^2(\mathrm{SU}_q(2))$; see Section 4.5 for more details.

Lemma 5.1.1. *For each $x, y \in \mathcal{O}(\mathrm{SU}_q(2))$ it holds that $J\nu^{\frac{1}{2}}(y)^* J\Lambda(x) = \Lambda(xy)$.*

Proof. This follows from a straightforward computation, using that $\delta_k(x)^* = \delta_{k^{-1}}(x^*)$ and $\partial_k(x^*) = \partial_{k^{-1}}(x)^*$ for all $x \in \mathcal{O}(\mathrm{SU}_q(2))$. ∎

Proposition 5.1.2. *The inclusion $C(S_q^2) \subseteq C(\mathrm{SU}_q(2))$ in an isometry with respect to the seminorms $L_q^{0,\max}\colon C(S_q^2) \to [0,\infty]$ and $L_{t,q}^{\max}\colon C(\mathrm{SU}_q(2)) \to [0,\infty]$. In particular, it holds that $\mathrm{Lip}(S_q^2) \subseteq \mathrm{Lip}_t(\mathrm{SU}_q(2))$.*

Proof. By restricting the Haar state, we obtain a GNS representation $\rho^0 \colon C(S_q^2) \to \mathbb{B}(H_q^0)$, and we denote by $L^\infty(S_q^2) \subseteq \mathbb{B}(H_q^0)$ the enveloping von Neumann algebra $\rho^0(C(S_q^2))''$. By standard von Neumann algebraic techniques, the inclusion $C(S_q^2) \subseteq C(\mathrm{SU}_q(2))$ extends to a normal inclusion $\iota \colon L^\infty(S_q^2) \to L^\infty(\mathrm{SU}_q(2))$ with the property that $\iota(x) \cdot \xi = x \cdot \xi$ for $\xi \in L^2(S_q^2) \subseteq L^2(\mathrm{SU}_q(2))$.

Let $x \in \mathrm{Lip}(S_q^2)$ be given. We recall from [4, Lemma 3.7] that the operator $\delta^0(x) := u \partial^0(x) u^*$ belongs to $\mathbb{M}_2(L^\infty(S_q^2))$ and we may thus define the element

$$\iota(\partial^0(x)) := u^* \iota(\delta^0(x)) u \in \mathbb{M}_2(L^\infty(\mathrm{SU}_q(2))).$$

It clearly holds that $\|\iota(\partial^0(x))\| = \|\partial^0(x)\|$. Moreover, we remark that whenever $\zeta \in H_q^1 \oplus H_q^{-1} \subseteq L^2(\mathrm{SU}_q(2)) \oplus L^2(\mathrm{SU}_q(2))$ it holds that $u \cdot \zeta \in H_q^0 \oplus H_q^0$ and hence that

$$\iota(\partial^0(x)) I y I \zeta = I y I \iota(\partial^0(x)) \zeta = I y I \partial^0(x) \zeta \quad \text{for all } y \in L^\infty(\mathrm{SU}_q(2)). \tag{5.1}$$

Let now $\xi \in \mathcal{O}(\mathrm{SU}_q(2))^{\oplus 2}$. We aim to show that $x \cdot \xi \in \mathrm{Dom}(D_q^H) \cap \mathrm{Dom}(D_t^V)$ and that we have the identities

$$[D_q^H, x]\xi = \iota(\partial^0(x))\xi \quad \text{and} \quad [D_t^V, x]\xi = 0. \tag{5.2}$$

This suffices to prove the present theorem: indeed, since $x \in \mathrm{Lip}(S_q^2)$ both twists involved in the definitions of $\partial_t^V(x)$ and $\partial_q^H(x)$ are trivial. Moreover, if one proves the relations in (5.2) for ξ in the core $\mathcal{O}(\mathrm{SU}_q(2))^{\oplus 2}$, an approximation argument shows that $x(\mathrm{Dom}(D_q^H)) \subseteq \mathrm{Dom}(D_q^H)$, $x(\mathrm{Dom}(D_t^V)) \subseteq \mathrm{Dom}(D_t^V)$ and that the relations in (5.2) hold on the two domains.

Let us start out by proving the claims relating to the vertical Dirac operator. Without loss of generality, we may assume that $\xi \in \mathcal{A}_q^n \oplus \mathcal{A}_q^m$ for some $n, m \in \mathbb{Z}$. Since $x \in C(S_q^2)$, it follows that $x\xi \in H_q^n \oplus H_q^m$. But $H_q^n \oplus H_q^m \subseteq \mathrm{Dom}(D_t^V)$ and the relevant commutator $[D_t^V, x]\xi$ is trivial since the restriction of D_t^V to $H_q^n \oplus H_q^m$ is given by multiplication with the diagonal matrix

$$\begin{pmatrix} t^{\frac{-n+1}{2}} \left[\frac{n-1}{2}\right]_t & 0 \\ 0 & -t^{\frac{-m-1}{2}} \left[\frac{m+1}{2}\right]_t \end{pmatrix}.$$

Next we focus on the claims relating to the horizontal Dirac operator. Let first $\eta \in \mathrm{Dom}(D_q^0)$ and $z \in \mathcal{O}(\mathrm{SU}_q(2))$ be given. We begin by showing that

$$IzI\eta \in \mathrm{Dom}(D_q^H) \quad \text{and} \quad [D_q^H, IzI]\eta = I \partial_q^H(\partial_k(z)) I \eta. \tag{5.3}$$

Since $\mathcal{A}_q^1 \oplus \mathcal{A}_q^{-1}$ is a core for $\mathrm{Dom}(D_q^0)$ we may, without loss of generality, assume that $\eta \in \mathcal{A}_q^1 \oplus \mathcal{A}_q^{-1}$. We then remark that $IzI\eta = \mathcal{I}z\mathcal{I}\eta \in \mathcal{O}(\mathrm{SU}_q(2)) \oplus \mathcal{O}(\mathrm{SU}_q(2))$ and that Lemma 4.5.3 therefore implies that

$$[D_q^H, IzI]\eta = \mathcal{I} \partial_q^H(\partial_k(z)) \mathcal{I} \Gamma_{q,0}^2 \eta.$$

The desired formula for the commutator $[D_q^H, IzI]\eta$ then follows by noting that $\Gamma_{q,0}$ restricts to the identity operator on $\mathcal{A}_q^1 \oplus \mathcal{A}_q^{-1}$. To proceed, we denote the two columns in u^* by $v_1 \in \mathcal{A}_q^1 \oplus \mathcal{A}_q^{-1}$ and $v_2 \in \mathcal{A}_q^1 \oplus \mathcal{A}_q^{-1}$, thus

$$v_1 = \begin{pmatrix} a \\ -qb^* \end{pmatrix} \quad \text{and} \quad v_2 = \begin{pmatrix} b \\ a^* \end{pmatrix}.$$

For our fixed element $\xi \in \mathcal{O}(\mathrm{SU}_q(2))^{\oplus 2}$, we therefore choose $y_1, y_2 \in \mathcal{O}(\mathrm{SU}_q(2)) \subseteq L^2(\mathrm{SU}_q(2))^{\oplus 2}$ such that

$$\xi = u^* u \xi = v_1 \cdot y_1 + v_2 \cdot y_2 = -I\nu^{\frac{1}{2}}(y_1)^* I \cdot v_1 - I\nu^{\frac{1}{2}}(y_2)^* I \cdot v_2,$$

where the last equality follows from Lemma 5.1.1 (suppressing the embedding Λ for notational convenience). To ease the notation, put $z_1 := -\nu^{\frac{1}{2}}(y_1)^*$ and $z_2 := -\nu^{\frac{1}{2}}(y_2)^*$. We then have that

$$\xi = Iz_1I \cdot v_1 + Iz_2I \cdot v_2 \quad \text{and} \quad x\xi = Iz_1I \cdot xv_1 + Iz_2I \cdot xv_2.$$

Since v_1 and v_2 belong to $\mathcal{A}_q^1 \oplus \mathcal{A}_q^{-1} \subseteq \mathrm{Dom}(D_q^0)$ and $x \in \mathrm{Lip}(S_q^2)$ we know that xv_1 and $xv_2 \in \mathrm{Dom}(D_q^0)$. We thus obtain from (5.3) that $x\xi \in \mathrm{Dom}(D_q^H)$ and moreover that

$$\begin{aligned} D_q^H x\xi &= Iz_1I \cdot D_q^H xv_1 + Iz_2I \cdot D_q^H xv_2 + I\partial_q^H(\partial_k(z_1))Ixv_1 \\ &\quad + I\partial_q^H(\partial_k(z_2))Ixv_2 \\ &= Iz_1I \cdot \partial^0(x)v_1 + Iz_2I \cdot \partial^0(x)v_2 + Iz_1IxD_q^H v_1 + Iz_2IxD_q^H v_2 \\ &\quad + xI\partial_q^H(\partial_k(z_1))Iv_1 + xI\partial_q^H(\partial_k(z_2))Iv_2 \\ &= Iz_1I \cdot \partial^0(x)v_1 + Iz_2I \cdot \partial^0(x)v_2 + xD_q^H Iz_1Iv_1 + xD_q^H Iz_2Iv_2 \\ &= \iota(\partial^0(x))\xi + xD_q^H\xi, \end{aligned}$$

where the last equality follows from (5.1). This ends the proof of the present proposition. ∎

Corollary 5.1.3. *We have the identity* $(B_q^0, L_{t,q}^{\max}) = (C(S_q^2), L_q^{0,\max})$. *In particular, it holds that* $(B_q^0, L_{t,q}^{\max})$ *is a compact quantum metric space.*

Proof. It suffices to establish the identity $(B_q^0, L_{t,q}^{\max}) = (C(S_q^2), L_q^{0,\max})$ since [2, Theorem 8.3] already shows that $(C(S_q^2), L_q^{0,\max})$ is a compact quantum metric space. We have $B_q^0 = C(S_q^2)$ and, by Proposition 5.1.2, $\mathrm{Lip}(S_q^2) \subseteq B_q^0 \cap \mathrm{Lip}_t(\mathrm{SU}_q(2))$ with $L_q^{0,\max}(x) = L_{t,q}^{\max}(x)$ for all $x \in \mathrm{Lip}(S_q^2)$. We therefore only need to show that $B_q^0 \cap \mathrm{Lip}_t(\mathrm{SU}_q(2)) \subseteq \mathrm{Lip}(S_q^2)$. Denote by $\iota\colon H_q^1 \oplus H_q^{-1} \to L^2(\mathrm{SU}_q(2)) \oplus L^2(\mathrm{SU}_q(2))$ the inclusion of Hilbert spaces. It can then be verified that $\iota^* \mathcal{D}_q^H \subseteq \mathcal{D}_q^0 \iota^*$ and from this inclusion it follows that $\iota^* D_q^H \subseteq D_q^0 \iota^*$. Similarly, we have the inclusion $\iota D_q^0 \subseteq D_q^H \iota$ of unbounded operators.

The above inclusions can now be applied to see that

$$\pi^0(x)\xi = \iota^*\pi(x)\iota\xi \in \mathrm{Dom}(D_q^0)$$

for each x in $B_q^0 \cap \mathrm{Lip}_t(\mathrm{SU}_q(2))$ and each ξ in $\mathrm{Dom}(D_q^0)$. Moreover, it holds that

$$D_q^0\pi^0(x)\xi = D_q^0\iota^*\pi(x)\iota\xi = \iota^*\partial_q^H(x)\iota\xi + \iota^*\pi(x)\iota D_q^0\xi = \iota^*\partial_q^H(x)\iota\xi + \pi^0(x)D_q^0\xi.$$

This shows that $x \in \mathrm{Lip}(S_q^2)$ and the corollary is proved. ∎

Remark 5.1.4. The algebraic counterpart to Proposition 5.1.2 is basically a triviality. Indeed, for $x \in \mathcal{O}(S_q^2)$ we have that $\partial_{t,q}(x) = \begin{pmatrix} 0 & -q^{-1/2}\partial_f(x) \\ -q^{1/2}\partial_e(x) & 0 \end{pmatrix} = \partial^0(x)$, and it moreover holds that

$$\begin{aligned} &\begin{pmatrix} 0 & -q^{-\frac{1}{2}}\partial_f(x) \\ -q^{\frac{1}{2}}\partial_e(x) & 0 \end{pmatrix}^* \begin{pmatrix} 0 & -q^{-\frac{1}{2}}\partial_f(x) \\ -q^{\frac{1}{2}}\partial_e(x) & 0 \end{pmatrix} \\ &\quad = \partial^0(x)^*\partial^0(x) \in \mathbb{M}_2(\mathcal{O}(S_q^2)). \end{aligned}$$

We thereby obtain that

$$L_{t,q}(x)^2 = \|\partial_{t,q}(x)^*\partial_{t,q}(x)\|_{C(\mathrm{SU}_q(2))} = \|\partial^0(x)^*\partial^0(x)\|_{C(S_q^2)} = L_q^0(x)^2,$$

where L_q^0 denotes the variation of $L_q^{0,\max}$ whose domain is $\mathcal{O}(S_q^2)$.

5.2 Spectral projections and twisted derivations

Let $\theta\colon S^1 \times C(\mathrm{SU}_q(2)) \to C(\mathrm{SU}_q(2))$ be a strongly continuous action of the circle on quantum SU(2). In this section we are investigating the relationship between the spectral projections coming from θ and the twisted $*$-derivations

$$\partial_q^H \text{ and } \partial_t^V\colon \mathrm{Lip}_t(\mathrm{SU}_q(2)) \to \mathbb{B}\big(L^2(\mathrm{SU}_q(2))^{\oplus 2}\big)$$

introduced in Chapter 4. As a first consequence of these efforts, we shall establish, in Proposition 5.2.4 below, that the sum of twisted $*$-derivations $\partial_{t,q} = \partial_q^H + \partial_t^V$ is closable.

We suppose that there exists a 2π-periodic, strongly continuous one-parameter unitary group $(U_r)_{r\in\mathbb{R}}$ acting on the Hilbert space $L^2(\mathrm{SU}_q(2))^{\oplus 2}$ such that

(1) $U_r\pi(x)U_{-r} = \pi(\theta(e^{ir}, x))$ for all $r \in \mathbb{R}$ and $x \in C(\mathrm{SU}_q(2))$, where π is the diagonal unital $*$-homomorphism introduced in Section 3.4.

(2) $[D_q^H, U_r] = 0 = [D_t^V, U_r]$ for all $r \in \mathbb{R}$;

(3) $\theta(z, \sigma_L(w, x)) = \sigma_L(w, \theta(z, x))$ for all $z, w \in S^1$, $x \in C(\mathrm{SU}_q(2))$.

Since the map $r \mapsto U_r$ is strongly continuous, we obtain that the map $r \mapsto U_r T U_{-r}$ is weakly continuous for every $T \in \mathbb{B}(L^2(\mathrm{SU}_q(2))^{\oplus 2})$. For each $n \in \mathbb{Z}$ we may therefore define the *n-th spectral projection* $\Pi_n^\theta \colon \mathbb{B}(L^2(\mathrm{SU}_q(2))^{\oplus 2}) \to \mathbb{B}(L^2(\mathrm{SU}_q(2))^{\oplus 2})$, implicitly, by the formula

$$\langle \xi, \Pi_n^\theta(T)\eta \rangle = \frac{1}{2\pi} \int_0^{2\pi} \langle \xi, U_r T U_{-r} \eta \rangle e^{-irn} \, dr, \quad \xi, \eta \in L^2(\mathrm{SU}_q(2))^{\oplus 2}. \tag{5.4}$$

We remark that the spectral projections separate points; i.e. that for an operator T in $\mathbb{B}(L^2(\mathrm{SU}_q(2))^{\oplus 2})$, it holds that $T = 0$ if and only if $\Pi_n^\theta(T) = 0$ for all $n \in \mathbb{Z}$. It follows from our conditions that the spectral projection

$$\Pi_n^\theta \colon \mathbb{B}(L^2(\mathrm{SU}_q(2))^{\oplus 2}) \to \mathbb{B}(L^2(\mathrm{SU}_q(2))^{\oplus 2})$$

induces a spectral projection $\Pi_n^\theta \colon C(\mathrm{SU}_q(2)) \to C(\mathrm{SU}_q(2))$ satisfying that

$$\Pi_n^\theta(\pi(x)) = \pi(\Pi_n^\theta(x)) \quad \text{for all } x \in C(\mathrm{SU}_q(2)).$$

The spectral projection on $C(\mathrm{SU}_q(2))$ is given by the norm-convergent Riemann integral

$$\Pi_n^\theta(x) = \frac{1}{2\pi} \int_0^{2\pi} \theta(e^{ir}, x) \cdot e^{-irn} \, dr. \tag{5.5}$$

Lemma 5.2.1. *For each $n \in \mathbb{Z}$ and $x \in \mathrm{Lip}_t(\mathrm{SU}_q(2))$, it holds that*

$$\Pi_n^\theta(x) \in \mathrm{Lip}_t(\mathrm{SU}_q(2))$$

and we have the identities

$$\Pi_n^\theta(\partial_q^H(x)) = \partial_q^H(\Pi_n^\theta(x)) \quad \text{and} \quad \Pi_n^\theta(\partial_t^V(x)) = \partial_t^V(\Pi_n^\theta(x)). \tag{5.6}$$

In particular, it holds that the spectral projection $\Pi_n^\theta \colon C(\mathrm{SU}_q(2)) \to C(\mathrm{SU}_q(2))$ is a contraction for our Lipschitz seminorm, meaning that

$$L_{t,q}^{\max}\big(\Pi_n^\theta(x)\big) \leqslant L_{t,q}^{\max}(x) \quad \text{for all } x \in C(\mathrm{SU}_q(2)).$$

Proof. Let $n \in \mathbb{Z}$. The fact that Π_n^θ becomes a contraction for the seminorm $L_{t,q}^{\max}$ is going to follow from the identities in (5.6) together with the fact that the spectral projection Π_n^θ is a norm-contraction. Let $x \in \mathrm{Lip}_t(\mathrm{SU}_q(2))$ be given. We focus on showing that $\Pi_n^\theta(x)$ is horizontally Lipschitz and that the identity $\Pi_n^\theta(\partial_q^H(x)) = \partial_q^H(\Pi_n^\theta(x))$ is satisfied. The vertical case follows by a similar argument.

We first record that the assumption (3) on θ implies that

$$\sigma_L(e^{ir}, \Pi_n^\theta(x)) = \Pi_n^\theta(\sigma_L(e^{ir}, x)) \quad \text{for all } r \in \mathbb{R}.$$

Hence, Lemma 3.6.2 shows that $\Pi_n^\theta(x)$ is analytic of order $-\log(q)/2$ and that the identities

$$\Pi_n^\theta\big(\sigma_L(q^{\frac{1}{2}}, x)\big) = \sigma_L\big(q^{\frac{1}{2}}, \Pi_n^\theta(x)\big) \quad \text{and} \quad \Pi_n^\theta\big(\sigma_L(q^{-\frac{1}{2}}, x)\big) = \sigma_L\big(q^{-\frac{1}{2}}, \Pi_n^\theta(x)\big)$$

are satisfied. Let now $\xi, \eta \in \mathrm{Dom}(D_q^H)$ be given. We may then compute as follows:

$$\begin{aligned}
&\big\langle D_q^H \xi, \sigma_L\big(q^{\frac{1}{2}}, \Pi_n^\theta(x)\big)\eta\big\rangle \\
&= \frac{1}{2\pi}\int_0^{2\pi} \big\langle D_q^H \xi, U_r \sigma_L(q^{\frac{1}{2}}, x) U_{-r}\eta\big\rangle \cdot e^{-irn}\, dr \\
&= \frac{1}{2\pi}\int_0^{2\pi} \big\langle \xi, U_r D_q^H \sigma_L(q^{\frac{1}{2}}, x) U_{-r}\eta\big\rangle \cdot e^{-irn}\, dr \\
&= \frac{1}{2\pi}\int_0^{2\pi} \big\langle \xi, U_r \partial_q^H(x) U_{-r}\eta\big\rangle \cdot e^{-irn}\, dr \\
&\quad + \frac{1}{2\pi}\int_0^{2\pi} \big\langle \xi, U_r \sigma_L(q^{-\frac{1}{2}}, x) U_{-r} D_q^H \eta\big\rangle \cdot e^{-irn}\, dr \\
&= \big\langle \xi, \Pi_n^\theta\big(\partial_q^H(x)\big)\eta\big\rangle + \big\langle \xi, \sigma_L\big(q^{-\frac{1}{2}}, \Pi_n^\theta(x)\big) D_q^H \eta\big\rangle.
\end{aligned}$$

This shows that $\sigma_L(q^{\frac{1}{2}}, \Pi_n^\theta(x))\eta \in \mathrm{Dom}((D_q^H)^*) = \mathrm{Dom}(D_q^H)$, and moreover that $\Pi_n^\theta(x)$ is horizontally Lipschitz with $\partial_q^H(\Pi_n^\theta(x)) = \Pi_n^\theta(\partial_q^H(x))$. ∎

Our prime example, where the above lemma applies, is given by the 2π-periodic strongly continuous one-parameter unitary group $(U_r^L)_{r\in\mathbb{R}}$ defined by

$$U_r^L \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} e^{ir(k-1)} \cdot \xi \\ e^{ir(m+1)} \cdot \eta \end{pmatrix} \qquad \xi \in H_q^k,\ \eta \in H_q^m, \tag{5.7}$$

This unitary group induces the left circle action on $C(\mathrm{SU}_q(2))$ in the sense that

$$U_r^L \pi(x) U_{-r}^L = \pi\big(\sigma_L(e^{ir}, x)\big) \quad \text{for all } r \in \mathbb{R} \text{ and } x \in C(\mathrm{SU}_q(2)). \tag{5.8}$$

For each $n \in \mathbb{Z}$ we denote the corresponding spectral projection by Π_n^L. The following lemma now verifies that the last assumption (2) is indeed satisfied.

Lemma 5.2.2. *It holds that* $[D_q^H, U_r^L] = 0 = [D_t^V, U_r^L]$ *for all* $r \in \mathbb{R}$.

Proof. Recall that $\mathcal{O}(\mathrm{SU}_q(2))^{\oplus 2}$ is a core for both of the unbounded selfadjoint operators D_q^H and D_t^V. Moreover, we know that $\mathcal{O}(\mathrm{SU}_q(2))$ agrees with the algebraic linear span of the algebraic spectral subspaces $\mathcal{A}_q^k$, $k \in \mathbb{Z}$. It therefore suffices to prove the relevant commutator identities on vectors of the form $\big(\begin{smallmatrix} \xi \\ \eta \end{smallmatrix}\big)$ with $\xi \in \mathcal{A}_q^n$ and $\eta \in \mathcal{A}_q^m$ for some $n, m \in \mathbb{Z}$. The vanishing result for the commutator with the vertical Dirac operator $\mathcal{D}_t^V$ is then clearly satisfied, so we focus on the horizontal Dirac operator $\mathcal{D}_q^H$. In this case, the vanishing result follows since

$$\mathcal{D}_q^H \begin{pmatrix} \xi \\ \eta \end{pmatrix} = - \begin{pmatrix} q^{-\frac{1}{2}} \partial_{fk^{-1}}(\eta) \\ q^{\frac{1}{2}} \partial_{ek^{-1}}(\xi) \end{pmatrix} \in \mathcal{A}_q^{m+2} \oplus \mathcal{A}_q^{n-2}. \qquad \blacksquare$$

The assumptions in Lemma 5.2.1 are therefore met, and it yields the following:

Corollary 5.2.3. *Let $x \in \mathrm{Lip}_t(\mathrm{SU}_q(2))$, $n \in \mathbb{Z}$. It holds that $\Pi_n^L(x) \in \mathrm{Lip}_t(\mathrm{SU}_q(2))$ and we have the identities*

$$\partial_t^V(\Pi_n^L(x)) = \Pi_n^L(\partial_t^V(x)) \quad \textit{and} \quad \partial_q^H(\Pi_n^L(x)) = \Pi_n^L(\partial_q^H(x)).$$

In particular, it holds that $L_{t,q}^{\max}(\Pi_n^L(x)) \leqslant L_{t,q}^{\max}(x)$.

Proposition 5.2.4. *The sum of twisted $*$-derivations*

$$\partial_{t,q} = \partial_q^H + \partial_t^V : \mathrm{Lip}_t(\mathrm{SU}_q(2)) \to \mathbb{B}\big(L^2(\mathrm{SU}_q(2))^{\oplus 2}\big)$$

is closable.

Proof. We first record that (4.6) yields that

$$\max\big\{\|\partial_q^H(x)\|, \|\partial_t^V(x)\|\big\} \leqslant \|\partial_{t,q}(x)\| \leqslant \|\partial_q^H(x)\| + \|\partial_t^V(x)\|$$

for all $x \in \mathrm{Lip}_t(\mathrm{SU}_q(2))$.

To see that $\partial_{t,q}$ is closable it thus suffices to show that ∂_q^H and ∂_t^V are both closable. We focus on showing that ∂_q^H is closable since the proof is almost the same for ∂_t^V. For $m \in \mathbb{Z}$, we first remark that the restriction $\partial_q^H : \mathrm{Lip}_t(\mathrm{SU}_q(2)) \cap A_q^m \to \mathbb{B}(L^2(\mathrm{SU}_q(2))^{\oplus 2})$ is closable: indeed, for each $x \in \mathrm{Lip}_t(\mathrm{SU}_q(2)) \cap A_q^m$ and each $\xi \in \mathrm{Dom}(D_q^H)$ we obtain from Lemma 3.6.3 that

$$\partial_q^H(x)\xi = D_q^H \sigma_L(q^{\frac{1}{2}}, x)\xi - \sigma_L(q^{-\frac{1}{2}}, x) D_q^H \xi = q^{\frac{m}{2}} D_q^H x\xi - q^{-\frac{m}{2}} x D_q^H \xi.$$

The fact that the relevant restriction is closable then follows from the selfadjointness of D_q^H. An application of Corollary 5.2.3 now shows that $\partial_q^H : \mathrm{Lip}_t(\mathrm{SU}_q(2)) \to \mathbb{B}(L^2(\mathrm{SU}_q(2))^{\oplus 2})$ is closable. Indeed, as already remarked, for a bounded operator $y \in \mathbb{B}(L^2(\mathrm{SU}_q(2))^{\oplus 2})$ it holds that $y = 0$ if and only if $\Pi_n^L(y) = 0$ for all $n \in \mathbb{Z}$. $\blacksquare$

5.3 Spectral bands as compact quantum metric spaces

We fix again our two parameters $t, q \in (0, 1]$ together with an $M \in \mathbb{N}_0$. We are now going to establish that the spectral band $B_q^M = \sum_{m=-M}^{M} A_q^m$ becomes a quantum metric space when equipped with the Lipschitz seminorm $L_{t,q}^{\max} : B_q^M \to [0, \infty]$ introduced in Definition 4.3.5. To this end, we utilise the general theory about finitely generated projective modules developed in Section 2.3. We emphasise that the domain of the restriction $L^{\max}_{t,q}\colon B^M_q \to [0,\infty]$ is substantially larger than the algebraic spectral band $\mathcal{B}^M_q$. We start out by stating (and reproving) a well-known result regarding the spectral subspaces $(A^m_q)_{m\in\mathbb{Z}}$ (see e.g. [30, Proposition 3.5]). Recall that $\Pi^L_m\colon C(\mathrm{SU}_q(2))\to C(\mathrm{SU}_q(2))$ denotes the spectral projection defined in (5.5) associated with the circle action σ_L.

Lemma 5.3.1. *For any $x \in C(\mathrm{SU}_q(2))$ and any $m \in \mathbb{Z}$ it holds that*

$$\Pi^L_m(x) = \begin{cases} \sum_{i=0}^m (u^m_{i0})^* \cdot \Pi^L_0(u^m_{i0}\cdot x) & m \geq 0 \\ \sum_{i=0}^{|m|} \big(u^{|m|}_{i|m|}\big)^* \cdot \Pi^L_0\big(u^{|m|}_{i|m|}\cdot x\big) & m < 0 \end{cases}$$

In particular, we obtain that A^m_q is finitely generated and projective as a right module over $A^0_q = C(S^2_q)$.

Proof. By continuity and density, it suffices to check the identity for $x \in \mathcal{O}(\mathrm{SU}_q(2))$ and by linearity we may furthermore assume that $x \in \mathcal{A}^k_q$ for some $k \in \mathbb{Z}$. If $m \geq 0$, then $u^m_{i0} \in \mathcal{A}^{-m}_q$ (cf. (3.16)) and hence $u^m_{i0}x \in \mathcal{A}^{k-m}_q$. Hence both sides are zero if $m \neq k$ and for $m = k$ the identity follows from the fact that u^m is a unitary matrix. The final statement about projectivity now follows, since the identity just proven shows that the map $A^m_q \to (A^0_q)^{\oplus(m+1)}$ given by $x \mapsto u_{\bullet 0}\cdot x$ provides an embedding of A^m_q as a direct summand in a finitely generated free module. The case $m < 0$ follows analogously. ∎

To show that $(B^M_q, L^{\max}_{t,q})$ is a compact quantum metric space, we wish to apply Theorem 2.3.3, and we therefore need to compare the Lipschitz seminorm $L^{\max}_{t,q}$ with the operator norm on quantum SU(2) (see Assumption 2.3.2). This comparison takes place in the next two lemmas.

Lemma 5.3.2. *For every $m \in \mathbb{Z}$, it holds that $A^m_q \subseteq \mathrm{Lip}^V_t(\mathrm{SU}_q(2))$ and*

$$\partial^V_t(x) = \begin{pmatrix} [m/2]_t x & 0 \\ 0 & -[m/2]_t x \end{pmatrix} \quad \textit{for all } x \in A^m_q.$$

Proof. Let $m \in \mathbb{Z}$ and $x \in A^m_q$ be given. We then know from Lemma 3.6.3 that x is analytic of order $-\log(t)/2$. Let now $n,k \in \mathbb{Z}$ and $y \in \mathcal{A}^n_q \oplus \mathcal{A}^k_q$ be given. We then have that

$$\sigma_L(t^{\frac{1}{2}},x)\cdot y = t^{\frac{m}{2}} x\cdot y \in A^{m+n}_q \oplus A^{m+k}_q \subseteq \mathrm{Dom}(D^V_t).$$

Using the relation $[r+s]_t - t^{-r}[s]_t = t^s[r]_t$, which is valid for all $r,s \in \mathbb{R}$, a direct computation shows that

$$\big(D^V_t\sigma_L(t^{\frac{1}{2}},x) - \sigma_L(t^{-\frac{1}{2}},x)D^V_t\big)y = \begin{pmatrix} [m/2]_t\cdot x & 0 \\ 0 & -[m/2]_t\cdot x \end{pmatrix}\cdot y.$$

This proves that the twisted commutator

$$D_t^V \sigma_L(t^{\frac{1}{2}}, x) - \sigma_L(t^{-\frac{1}{2}}, x) D_t^V$$

is well defined on the core $\mathcal{O}(\mathrm{SU}_q(2))^{\oplus 2}$ for the vertical Dirac operator and that it extends to the bounded operator $[m/2]_t \begin{pmatrix} x & 0 \\ 0 & -x \end{pmatrix}$. From this it follows immediately that $\sigma_L(t^{\frac{1}{2}}, x)$ preserves $\mathrm{Dom}(D_t^V)$ and that $\partial_t^V(x) = \begin{pmatrix} [m/2]_t x & 0 \\ 0 & -[m/2]_t x \end{pmatrix}$ as desired. ∎

Remark 5.3.3. As an aside, we remark that it is now easy to verify that the algebraic formula for ∂_t^V obtained in Lemma 4.3.1 actually extends to the whole Lipschitz algebra, in the sense that

$$\partial_t^V(x) = \begin{pmatrix} \partial_t^3(x) & 0 \\ 0 & -\partial_t^3(x) \end{pmatrix} \quad \text{for all } x \in \mathrm{Lip}_t(\mathrm{SU}_q(2)). \tag{5.9}$$

By Remark 4.3.4, we already know that the off-diagonal elements in $\partial_t^V(x)$ are zero and that the upper left-hand entry is $\partial_t^3(x)$. Conjugating $\partial_t^V(x)$ with the unitary $S := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ interchanges the diagonal entries, so it suffices to show that $S\partial_t^V(x)S = -\partial_t^V(x)$. A direct computation shows that the unitaries $(U_r^L)_{r\in\mathbb{R}}$ defined in (5.7) satisfies

$$\mathrm{SU}_r^L = \begin{pmatrix} e^{2ir} & 0 \\ 0 & e^{-2ir} \end{pmatrix} U_r^L S \quad \text{for all } r \in \mathbb{R}$$

and since $\partial_t^V(x)$ is diagonal it commutes with the unitary $\begin{pmatrix} e^{2ir} & 0 \\ 0 & e^{-2ir} \end{pmatrix}$. Using this, it is not difficult to see that

$$\Pi_n^L(S\partial_t^V(x)S) = S\Pi_n^L(\partial_t^V(x))S \quad \text{for all } n \in \mathbb{Z}.$$

By Lemma 5.3.2 we know that (5.9) is valid whenever x belongs to a spectral subspace, and since Π_n^L commutes with ∂_t^V (see Corollary 5.2.3) we therefore obtain that

$$\Pi_n^L(S\partial_t^V(x)S) = S\Pi_n^L(\partial_t^V(x))S = S\partial_t^V(\Pi_n^L x)S = -\partial_t^V(\Pi_n^L x) = -\Pi_n^L(\partial_t^V(x)).$$

Since the spectral projections separate points, it follows that $S\partial_t^V(x)S = -\partial_t^V(x)$ and hence that (5.9) holds.

Lemma 5.3.4. *For each $m \in \mathbb{Z}$, it holds that $|[m/2]_t| \cdot \|\Pi_m^L(x)\| \leqslant L_{t,q}^{\max}(x)$ for all $x \in \mathrm{Lip}_t(\mathrm{SU}_q(2))$.*

Proof. The result follows from Corollary 5.2.3 and Lemma 5.3.2 via the estimate

$$|[m/2]_t| \cdot \|\Pi_m^L(x)\| = \|\partial_t^V(\Pi_m^L(x))\| \leqslant L_{t,q}^{\max}(\Pi_m^L(x)) \leqslant L_{t,q}^{\max}(x). \qquad ∎$$

Lastly, in order to apply Theorem 2.3.3, we need to verify that Assumption 2.3.2 (5) is satisfied, which is the contents of the following lemma.

Lemma 5.3.5. *Let $v \in \mathrm{Lip}_t(\mathrm{SU}_q(2))$ and let $M \in \mathbb{N}_0$. Then the left-multiplication operator*

$$m(v)\colon B_q^M \cap \ker(\Pi_0^L) \to C(\mathrm{SU}_q(2))$$

is bounded with respect to the seminorm $L_{t,q}^{\max}$.

Proof. We first remark that Lemma 5.3.4 shows that there exists a constant $D_M > 0$ such that $\|x\| \leqslant D_M \cdot L_{t,q}^{\max}(x)$ for all $x \in B_q^M \cap \ker(\Pi_0^L)$. Next, it follows from Lemmas 3.6.5 and 4.3.7 that

$$\begin{aligned} L_{t,q}^{\max}(v \cdot x) &\leqslant \|v\|_{t,q} \cdot L_{t,q}^{\max}(x) + L_{t,q}^{\max}(v) \cdot \|x\|_{t,q} \\ &\leqslant \left(\|v\|_{t,q} + \sum_{m=-M}^{M} (t^{m/2} + q^{m/2}) D_M \right) \cdot L_{t,q}^{\max}(x) \end{aligned}$$

for all $x \in B_q^M \cap \ker(\Pi_0^L)$. This proves the present lemma. ■

We are now in position to state and prove the main result of this section, which shows that the spectral bands are compact quantum metric spaces. Notice that it follows from Lemma 5.3.1 that the spectral bands are finitely generated projective modules. In fact, with a little extra effort it can be proved that they are free (but this does not help to ease the argumentation).

Theorem 5.3.6. *Let $M \in \mathbb{N}_0$. The spectral band $B_q^M \subseteq C(\mathrm{SU}_q(2))$, the conditional expectation $\Pi_0^L\colon C(\mathrm{SU}_q(2)) \to C(S_q^2)$ and the Lipschitz seminorm $L_{t,q}^{\max}\colon C(\mathrm{SU}_q(2)) \to [0,\infty]$ satisfy Assumptions 2.3.1 and 2.3.2. In particular, it holds that the restriction $L_{t,q}^{\max}\colon B_q^M \to [0,\infty]$ provides B_q^M with the structure of a compact quantum metric space.*

Proof. It follows from Lemma 5.3.1 that B_q^M satisfies Assumption 2.3.1: indeed we may apply the elements in $C(\mathrm{SU}_q(2))$, defined, for each $m \in \{-M, -M+1, \ldots, M\}$ and each $i \in \{0, 1, \ldots, |m|\}$, by

$$v_{im} := \begin{cases} u_{i,0}^m & m \geqslant 0 \\ u_{i,-m}^{-m} & m < 0 \end{cases} \quad \text{and} \quad w_{im} := v_{im}^*.$$

Notice in this respect that $1 = v_{00} = w_{00}$ and that $\Pi_0^L(v_{im}) = 0 = \Pi_0^L(w_{im})$ as soon as $(i, m) \neq (0, 0)$ (cf. (3.16)). To see that conditions (1)–(5) in Assumption 2.3.2 are satisfied, notice that (1) follows from Corollary 5.2.3, while (2) follows from Corollary 5.1.3. Condition (3) is a consequence of Lemma 5.3.4 and condition (4) is trivially satisfied since $v_{im}, w_{im} \in \mathcal{O}(\mathrm{SU}_q(2))$ for all $m \in \{-M, -M+1, \ldots, M\}$ and $i \in \{0, 1, \ldots, |m|\}$. Condition (5) is exactly the contents of Lemma 5.3.5 and Theorem 2.3.3 therefore shows that $(B_q^M, L_{t,q}^{\max})$ is a compact quantum metric space. ■

Knowing that the spectral bands are compact quantum metric spaces, our next main goal will be to show that the same is true for quantum SU(2). We wish to do so by an application of Corollary 2.1.10, but verifying that the assumptions there are indeed fulfilled turns out to be a slightly delicate matter. One of our objectives will be to construct an "anti-derivative" of the twisted $*$-derivation $\partial_t^V\colon \mathrm{Lip}_t(\mathrm{SU}_q(2)) \to \mathbb{B}(L^2(\mathrm{SU}_q(2))^{\oplus 2})$. To this end we need the theory of Schur multipliers, and we gather all the results needed within this context in the following section.

5.4 Schur multipliers

Let $\|\cdot\|_2\colon \ell^2(\mathbb{Z}) \to [0,\infty)$ denote the usual Hilbert space norm on the Hilbert space of ℓ^2-sequences indexed by $\mathbb{Z}$. The standard basis vectors in $\ell^2(\mathbb{Z})$ are denoted by e_i, $i \in \mathbb{Z}$. We recall the following essential result due to Grothendieck:

Proposition 5.4.1 (Grothendieck). *Let H and K be Hilbert spaces and assume that they are $\mathbb{Z}$-graded as $H = \bigoplus_{i=-\infty}^{\infty} H_i$ and $K = \bigoplus_{i=-\infty}^{\infty} K_i$ such that each bounded operator $T \in \mathbb{B}(H,K)$ is represented by a matrix $(T_{ij})_{i,j\in\mathbb{Z}}$. Let $\varphi\colon \mathbb{Z}\times\mathbb{Z} \to \mathbb{C}$ be given and assume that there exist $a(i), b(i) \in \ell^2(\mathbb{Z})$ for every $i \in \mathbb{Z}$ such that*

(1) *$c(a) := \sup_{i\in\mathbb{Z}} \|a(i)\|_2 < \infty$ and $c(b) := \sup_{i\in\mathbb{Z}} \|b(i)\|_2 < \infty$;*

(2) *$\varphi(i,j) = \langle a(i), b(j)\rangle$ for all $i, j \in \mathbb{Z}$.*

Then for every $T \in \mathbb{B}(H,K)$ the matrix $(\varphi(i,j)T_{ij})_{i,j\in\mathbb{Z}}$ also defines a bounded operator from H to K and the map $\mathbb{M}(\varphi)\colon \mathbb{B}(H,K) \to \mathbb{B}(H,K)$, which associates to T the bounded operator with matrix $(\varphi(i,j)T_{ij})_{i,j}$, is completely bounded with cb-norm at most $c(a)c(b)$.

Under the hypotheses of the theorem above, the map φ is called a *Schur multiplier*. For a more elaborate treatment of the theory of Schur multipliers the reader is referred to [65], but for the readers' convenience we sketch the proof of Proposition 5.4.1 here.

Proof. Defining $a\colon H \to \ell^2(\mathbb{Z})\hat{\otimes} H = \bigoplus_{i\in\mathbb{Z}} \ell^2(\mathbb{Z})\hat{\otimes} H_i$ by $a((\xi_i)_i) := (a(i)\otimes\xi_i)_i$ and $b\colon K \to \ell^2(\mathbb{Z})\hat{\otimes} K$ by $b((\eta_i)_i) := (b(i)\otimes\eta_i)_i$, one sees that a and b are bounded with $\|a\| \leqslant c(a)$ and $\|b\| \leqslant c(b)$. Moreover, one verifies that $\mathbb{M}(\varphi)(T) = b^*(1\otimes T)a$ and hence we get $\|\mathbb{M}(\varphi)\| \leqslant c(a)c(b)$. The same argument works over matrices, so we indeed obtain that $\|\mathbb{M}(\varphi)\|_{\mathrm{cb}} \leqslant c(a)c(b)$. ∎

For each $t, q \in (0,1]$, we wish to construct an anti-derivative of

$$\partial_t^V\colon \mathrm{Lip}_t(\mathrm{SU}_q(2)) \to \mathbb{B}\big(L^2(\mathrm{SU}_q(2))^{\oplus 2}\big),$$

which will be given in terms of a Schur multiplier $\varphi_t\colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{C}$ defined by the formula

$$\varphi_t(i,j) := \begin{cases} \frac{1}{[(i-j)/2]_t} & i \neq j \\ 0 & i = j. \end{cases} \tag{5.10}$$

In order to show that $\varphi_t\colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{C}$ is indeed a Schur multiplier we start out by recording a well-known lemma on q-numbers (including the proof for lack of a good reference):

Lemma 5.4.2. *It holds that* $[n/2]_q \geq \frac{n}{q^{1/2}+q^{-1/2}}$ *for all* $q \in (0,1]$ *and* $n \in \mathbb{N}$.

Proof. Let $n \in \mathbb{N}$ be given. The inequality clearly holds for $q = 1$, so assume that $q \in (0,1)$. We first notice that $[n]_q \geqslant n$. Indeed, for $n = 2k$ even this inequality follows since

$$\begin{aligned}[2k]_q - 2k &= (q^{-2k+1} + q^{2k-1} - 2) + (q^{-2k+3} + q^{2k-3} - 2) \\ &\quad + \cdots + (q^{-1} + q - 2) \geqslant 0\end{aligned}$$

and for $n = 2k+1$ odd we obtain the inequality since

$$\begin{aligned}&[2k+1]_q - (2k+1) \\ &\quad = (q^{-2k} + q^{2k} - 2) + (q^{-2k+2} + q^{2k-2} - 2) + \cdots + (q^{-2} + q^2 - 2) \\ &\quad \geqslant 0.\end{aligned}$$

We then obtain that

$$[n/2]_q = \frac{q^{n/2} - q^{-n/2}}{(q^{1/2} - q^{-1/2})(q^{1/2} + q^{-1/2})} = \frac{[n]_{q^{1/2}}}{q^{1/2} + q^{-1/2}} \geq \frac{n}{q^{1/2} + q^{-1/2}}. \qquad \blacksquare$$

Lemma 5.4.3. *Let* $t \in (0,1]$. *The function* $\varphi_t\colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{C}$ *is a Schur multiplier and we have the estimate*

$$\|\mathbb{M}(\varphi_t)\|_{\mathrm{cb}} \leqslant \frac{\pi \cdot (t^{1/2} + t^{-1/2})}{\sqrt{3}}$$

on the cb-norm of the associated completely bounded operator.

Proof. In order to apply Proposition 5.4.1, we define the sequences

$$a(i) := \sum_{k=-\infty}^{\infty} \varphi_t(i,k) \cdot e_k \quad \text{and} \quad b(i) := e_i$$

for all $i \in \mathbb{Z}$ and note that $\varphi_t(i,j) = \langle a(i), b(j) \rangle$. For each $i \in \mathbb{Z}$, we then apply Lemma 5.4.2 to obtain the estimate

$$\|a(i)\|_2^2 = \|a(0)\|_2^2 = 2\sum_{k=1}^{\infty} \frac{1}{[k/2]_t^2} \leqslant 2\sum_{k=1}^{\infty} \frac{(t^{1/2} + t^{-1/2})^2}{k^2} = \frac{\pi^2(t^{1/2} + t^{-1/2})^2}{3}$$

on the Hilbert space norm. Since we moreover have that $\|b(i)\|_2^2 = 1$ for all $i \in \mathbb{Z}$, the relevant estimate on the cb-norm now follows from Proposition 5.4.1:

$$\|\mathbb{M}(\varphi_t)\|_{\text{cb}} \leqslant \sup_{i \in \mathbb{Z}} \|a(i)\|_2 \leqslant \frac{\pi \cdot (t^{1/2} + t^{-1/2})}{\sqrt{3}}. \qquad \blacksquare$$

We shall also need a systematic method for approximating elements in $C(\mathrm{SU}_q(2))$ by elements in the spectral bands B_q^M, $M \in \mathbb{N}_0$. This approximation will also take place by means of Schur multipliers. For each $M \in \mathbb{N}_0$, we define the function $\gamma_M : \mathbb{Z} \times \mathbb{Z} \to \mathbb{C}$ by the formula

$$\gamma_M(i,j) := \begin{cases} \frac{M+1-|i-j|}{M+1} & |i-j| \leqslant M \\ 0 & |i-j| > M. \end{cases} \tag{5.11}$$

Lemma 5.4.4. *For each $M \in \mathbb{N}_0$, the function $\gamma_M : \mathbb{Z} \times \mathbb{Z} \to \mathbb{C}$ is a Schur multiplier and we have the estimate*

$$\|\mathbb{M}(\gamma_M)\|_{\text{cb}} \leqslant 1$$

on the cb-norm of the associated completely bounded operator.

Proof. We are going to apply Proposition 5.4.1. Let $M \in \mathbb{N}_0$ be given, and define, for each $i \in \mathbb{Z}$, the sequences

$$a(i) = b(i) := \frac{1}{\sqrt{M+1}} \cdot \sum_{k=i}^{i+M} e_k.$$

We record that $\|a(i)\|_2^2 = 1 = \|b(i)\|_2^2$. Let now $i, j \in \mathbb{Z}$ be given, and assume first that $i \leqslant j$. We then compute that

$$\begin{aligned} \langle a(i), b(j) \rangle &= \frac{1}{M+1} \cdot \left\langle \sum_{k=i}^{i+M} e_k, \sum_{l=j}^{j+M} e_l \right\rangle = \begin{cases} \frac{1}{M+1} \sum_{k=j}^{i+M} 1 & j \leqslant i+M \\ 0 & j > i+M \end{cases} \\ &= \begin{cases} \frac{M+i-j+1}{M+1} & j-i \leqslant M \\ 0 & j-i > M \end{cases} = \gamma_M(i,j). \end{aligned}$$

For $j \leqslant i$ we get from the above identities that $\langle a(i), b(j) \rangle = \langle a(j), b(i) \rangle = \gamma_M(j,i) = \gamma_M(i,j)$. The proof is therefore complete. $\blacksquare$

For each number $\delta \in (0,1)$ we define the null-sequence of positive real numbers $(\varepsilon(\delta, M))_{M=0}^{\infty}$ by putting

$$\varepsilon(\delta, M) := 2^{\frac{1}{2}} \cdot (\delta^{\frac{1}{2}} + \delta^{-\frac{1}{2}}) \cdot \left(\frac{M}{(M+1)^2} + \sum_{k=M+1}^{\infty} \frac{1}{k^2} \right)^{\frac{1}{2}} \quad \text{for all } M \in \mathbb{N}_0. \tag{5.12}$$

In particular, we record that $\varepsilon(\delta, 0) = \frac{\pi \cdot (\delta^{1/2} + \delta^{-1/2})}{\sqrt{3}}$.

Lemma 5.4.5. *Let* $\delta \in (0,1)$. *It holds that*

$$\left\|\mathbb{M}(\varphi_t)(1-\mathbb{M}(\gamma_M))\right\|_{\mathrm{cb}} \leqslant \varepsilon(\delta, M)$$

for all $M \in \mathbb{N}_0$ *and all* $t \in [\delta, 1]$.

Proof. Let $M \in \mathbb{N}_0$ and $t \in [\delta, 1]$ be given. We are going to apply Proposition 5.4.1 to the function $\rho_{t,M} \colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{C}$ given by the formula

$$\rho_{t,M}(i,j) := \varphi_t(i,j) \cdot \big(1-\gamma_M(i,j)\big) = \begin{cases} \frac{|i-j|}{(M+1)\cdot[(i-j)/2]_t} & 0 < |i-j| \leqslant M \\ 0 & |i-j| = 0 \\ \frac{1}{[(i-j)/2]_t} & |i-j| > M. \end{cases}$$

For each $i, j \in \mathbb{Z}$ we define the sequences

$$a(i) := \sum_{k=-\infty}^{\infty} \rho_{t,M}(i,k) \cdot e_k \quad \text{and} \quad b(j) = e_j.$$

Applying Lemma 5.4.2, we may estimate the Hilbert space norm of $a(i)$ as follows:

$$\begin{aligned} \|a(i)\|_2^2 = \|a(0)\|_2^2 &= \sum_{k=-\infty}^{\infty} |\rho_{t,M}(0,k)|^2 \\ &= 2 \cdot \sum_{k=M+1}^{\infty} \frac{1}{[k/2]_t^2} + \frac{2}{(M+1)^2} \cdot \sum_{k=1}^{M} \frac{k^2}{[k/2]_t^2} \\ &\leqslant 2\big(t^{\frac{1}{2}} + t^{-\frac{1}{2}}\big)^2 \cdot \left(\sum_{k=M+1}^{\infty} \frac{1}{k^2} \right) + \frac{2M}{(M+1)^2} \cdot \big(t^{\frac{1}{2}} + t^{-\frac{1}{2}}\big)^2 \\ &\leqslant 2\big(\delta^{\frac{1}{2}} + \delta^{-\frac{1}{2}}\big)^2 \cdot \left(\frac{M}{(M+1)^2} + \sum_{k=M+1}^{\infty} \frac{1}{k^2} \right) = \varepsilon(\delta, M)^2. \end{aligned}$$

This shows that $\rho_{t,M}$ is a Schur multiplier satisfying the estimate $\|\mathbb{M}(\rho_{t,M})\|_{\mathrm{cb}} \leqslant \varepsilon(\delta, M)$ on the cb-norm of the associated completely bounded operator. The result of the present lemma now follows by noting that $\mathbb{M}(\rho_{t,M}) = \mathbb{M}(\varphi_t) \cdot (1 - \mathbb{M}(\gamma_M))$ by construction. ∎

We end this subsection by re-introducing spectral projections in the context of Schur multipliers. For each $n \in \mathbb{Z}$ we define the Schur multiplier

$$\delta_n \colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{C} \quad \delta_n(i,j) := \delta_{n,i-j}. \tag{5.13}$$

The associated operator $\mathbb{M}(\delta_n)$ is then completely contractive and can be interpreted in terms of spectral projections. To explain this, suppose that $H = \bigoplus_{m=-\infty}^{\infty} H_m$ is a

$\mathbb{Z}$-graded Hilbert space and define the unitary operator $V_r : H \to H$ by

$$V_r\left(\sum_{m=-\infty}^{\infty} e_m \cdot \xi_m\right) := \sum_{m=-\infty}^{\infty} e_m \cdot e^{irm}\xi_m$$

for every $r \in \mathbb{R}$. This yields a 2π-periodic strongly continuous one-parameter unitary group $(V_r)_{r\in\mathbb{R}}$ and it holds that

$$\langle \xi, \mathbb{M}(\delta_n)(T)\eta\rangle = \frac{1}{2\pi}\int_0^{2\pi} \langle \xi, V_r T V_{-r}\eta\rangle e^{-irn} dr$$

for all $T \in \mathbb{B}(H)$ and $\xi, \eta \in H$. Thus, $\mathbb{M}(\delta_n)$ is the n-th spectral projection associated with our 2π-periodic strongly continuous unitary group $(V_r)_{r\in\mathbb{R}}$; cf. (5.4).

5.5 Projecting onto the spectral bands

Throughout this section we again fix the parameters $t, q \in (0, 1]$. We are going to apply the theory of Schur multipliers to the $\mathbb{Z}$-grading

$$L^2(\mathrm{SU}_q(2))^{\oplus 2} = \bigoplus_{m=-\infty}^{\infty} \big(H_q^{m+1} \oplus H_q^{m-1}\big). \tag{5.14}$$

This $\mathbb{Z}$-grading is simply the spectral subspace decomposition associated with the circle action on $L^2(\mathrm{SU}_q(2))^{\oplus 2}$ induced by the 2π-periodic strongly continuous one-parameter unitary group $(U_r^L)_{r\in\mathbb{R}}$ introduced in (5.7). For $M \in \mathbb{N}_0$ and $n \in \mathbb{Z}$, we consider the Schur multipliers $\gamma_M, \delta_n : \mathbb{Z}\times\mathbb{Z} \to \mathbb{C}$ introduced in (5.11) and (5.13) and apply the following notation for the completely bounded operators they induce:

$$E_M^L := \mathbb{M}(\gamma_M), \quad \Pi_n^L := \mathbb{M}(\delta_n) : \mathbb{B}\big(L^2(\mathrm{SU}_q(2))^{\oplus 2}\big) \to \mathbb{B}\big(L^2(\mathrm{SU}_q(2))^{\oplus 2}\big).$$

This notation is compatible with our already existing notation for spectral projections by the remarks at the end of Section 5.4. We emphasise that both Π_n^L and E_M^L induce operators on $C(\mathrm{SU}_q(2))$ via the relations

$$\Pi_n^L(\pi(x)) = \pi(\Pi_n^L(x)) \quad \text{and} \quad E_M^L(\pi(x)) = \pi(E_M^L(x))$$

for all $x \in C(\mathrm{SU}_q(2))$. Notice in this respect that

$$E_M^L = \sum_{m=-M}^{M} \frac{M+1-|m|}{M+1}\Pi_m^L.$$

The aim of this subsection is to prove that E_M^L is an $L_{t,q}^{\max}$-contraction onto the spectral M-band B_q^M and that E_M^L approximates the identity map on the $L_{t,q}^{\max}$-unit ball better and better as M grows, thus setting the stage for an application of Corollary 2.1.10.

We are also interested in the completely bounded operator

$$\int_t^V : \mathbb{B}(L^2(\mathrm{SU}_q(2))^{\oplus 2}) \to \mathbb{B}(L^2(\mathrm{SU}_q(2))^{\oplus 2})$$

defined by the formula

$$\int_t^V T := \mathbb{M}(\varphi_t)(\gamma \cdot T), \tag{5.15}$$

where we recall that $\gamma := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathbb{B}(L^2(\mathrm{SU}_q(2))^{\oplus 2})$ and that $\varphi_t \colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{C}$ was introduced in (5.10). Remark that the Schur multiplier $\mathbb{M}(\varphi_t)$ is also defined relative to the spectral subspace decomposition given in (5.14). We record that $\mathbb{M}(\varphi_t)$ induces a bounded operator on $C(\mathrm{SU}_q(2))$: indeed, for each $m \in \mathbb{Z}$ and $x \in A_q^m$ we have the formula

$$\mathbb{M}(\varphi_t)(x) = \begin{cases} \frac{1}{[m/2]_t} \cdot x & m \neq 0 \\ 0 & m = 0 \end{cases},$$

from which it follows that $\mathbb{M}(\varphi_t)$ preserves $\mathcal{O}(\mathrm{SU}_q(2))$ and hence also $C(\mathrm{SU}_q(2))$ by boundedness. We start out by proving that $\int_t^V$ serves as an anti-derivative with respect to ∂_t^V providing a non-commutative analogue of the fundamental theorem of calculus.

Proposition 5.5.1. *It holds that*

$$\int_t^V \partial_t^V(x) = (1 - \Pi_0^L)(x) \quad \textit{for all } x \in \mathrm{Lip}_t\big(\mathrm{SU}_q(2)\big).$$

Proof. Let $x \in \mathrm{Lip}_t(\mathrm{SU}_q(2))$ be given. First note that if $x \in \mathrm{Lip}_t(\mathrm{SU}_q(2)) \cap A_q^m$ for some $m \in \mathbb{Z}$, then the statement follows from Lemma 5.3.2: indeed, in this case we have that

$$\int_t^V \partial_t^V(x) = \int_t^V \begin{pmatrix} [m/2]_t x & 0 \\ 0 & -[m/2]_t x \end{pmatrix} = [m/2]_t \cdot \mathbb{M}(\varphi_t)(x) = (1 - \Pi_0^L)(x).$$

To prove the general statement, it suffices to show that

$$\Pi_n^L\Big(\int_t^V \partial_t^V(x)\Big) = \Pi_n^L(1 - \Pi_0^L)(x) \quad \text{for all } n \in \mathbb{Z}.$$

Let thus $n \in \mathbb{Z}$ be given. Since both Π_n^L and $\mathbb{M}(\varphi_t)$ are Schur multipliers with respect to the same $\mathbb{Z}$-grading on $L^2(\mathrm{SU}_q(2))^{\oplus 2}$, they commute. Moreover, we notice that the grading operator γ preserves the spectral subspace $H_q^{m+1} \oplus H_q^{m-1} \subseteq L^2(\mathrm{SU}_q(2))^{\oplus 2}$ for all $m \in \mathbb{Z}$ and hence it holds that left multiplication with γ commutes with Π_n^L.

The relevant identity therefore becomes a consequence of Corollary 5.2.3 through the following computation:

$$\Pi_n^L\Big(\int_t^V \partial_t^V(x)\Big) = \int_t^V \partial_t^V(\Pi_n^L x) = (1-\Pi_0^L)\Pi_n^L(x) = \Pi_n^L(1-\Pi_0^L)(x). \quad \blacksquare$$

The next step is to prove that E_M^L is a contraction for $L_{t,q}^{\max}$, thus verifying part of the hypotheses in Corollary 2.1.10.

Lemma 5.5.2. *Let $M \in \mathbb{N}_0$ and $x \in \mathrm{Lip}_t(\mathrm{SU}_q(2))$. It holds that $E_M^L(x) \in B_q^M \cap \mathrm{Lip}_t(\mathrm{SU}_q(2))$ and $L_{t,q}^{\max}(E_M^L(x)) \leqslant L_{t,q}^{\max}(x)$.*

Proof. We start out by recalling that $E_M^L = \sum_{m=-M}^{M} \frac{M+1-|m|}{M+1}\Pi_m^L$. It therefore follows from Lemma 5.4.4 and Corollary 5.2.3 that $E_M^L(x) \in \mathrm{Lip}_t(\mathrm{SU}_q(2)) \cap B_q^M$ and that

$$L_{t,q}^{\max}(E_M^L(x)) = \|\partial_{t,q}(E_M^L(x))\| = \|E_M^L(\partial_{t,q}(x))\| \leqslant \|\partial_{t,q}(x)\| = L_{t,q}^{\max}(x). \quad \blacksquare$$

We now show that the sequence of $L_{t,q}^{\max}$-contractions $(E_M^L)_{M=0}^\infty$ approximates the identity map on the $L_{t,q}^{\max}$-unit ball, thus verifying the last hypothesis in Corollary 2.1.10. In fact, this approximation can be obtained uniformly in the deformation parameters $t,q \in (0,1]$. For each $\delta \in (0,1)$ we recall the definition of the null-sequence of positive real numbers $(\varepsilon(\delta,M))_{M=0}^\infty$ from (5.12).

Proposition 5.5.3. *Let $\delta \in (0,1)$. It holds that*

$$\|x - E_M^L(x)\| \leqslant \varepsilon(\delta,M)\cdot L_{t,q}^{\max}(x)$$

for all $M \in \mathbb{N}_0$, $(t,q) \in [\delta,1]\times(0,1]$ and $x \in \mathrm{Lip}_t(\mathrm{SU}_q(2))$.

Proof. We apply Proposition 5.5.1 in combination with Lemma 5.4.5 to obtain that

$$\begin{aligned}\|(1-E_M^L)(x)\| &= \|(1-E_M^L)(1-\Pi_0^L)(x)\| = \Big\|(1-E_M^L)\int_t^V \partial_t^V(x)\Big\| \\ &\leqslant \big\|(1-\mathbb{M}(\gamma_M))\mathbb{M}(\varphi_t)\big\|\cdot\|\gamma\cdot\partial_t^V(x)\| \leqslant \varepsilon(\delta,M)\cdot L_{t,q}^{\max}(x)\end{aligned}$$

for all $M \in \mathbb{N}_0$, $t \in [\delta,1]$ and $x \in \mathrm{Lip}_t(\mathrm{SU}_q(2))$. $\blacksquare$

5.6 Quantum SU(2) as a compact quantum metric space

We are now ready to show that quantum SU(2) becomes a compact quantum metric space when equipped with the Lipschitz seminorm $L_{t,q}^{\max}\colon C(\mathrm{SU}_q(2)) \to [0,\infty]$.

Theorem 5.6.1. *The pair $(C(\mathrm{SU}_q(2)), L_{t,q}^{\max})$ is a compact quantum metric space for all $t,q \in (0,1]$.*

Proof. For each $M \in \mathbb{N}_0$ we know from Theorem 5.3.6 that the spectral band B_q^M becomes a compact quantum metric space when equipped with the restricted Lipschitz seminorm $L_{t,q}^{\max}\colon B_q^M \to [0,\infty]$. We may then apply Corollary 2.1.10 using the compact quantum metric spaces $(B_q^M, L_{t,q}^{\max})$, together with the unital linear maps $E_M^L\colon C(\mathrm{SU}_q(2)) \to B_q^M$ and the inclusions $\iota_M\colon B_q^M \to C(\mathrm{SU}_q(2))$. That the assumptions in Corollary 2.1.10 are indeed met by this data follows from Lemma 5.5.2 and Proposition 5.5.3. ∎

Corollary 5.6.2. *The pair $(C(\mathrm{SU}_q(2)), L_{t,q})$ is a compact quantum metric space for all $t, q \in (0,1]$*

Proof. Since the $L_{t,q}$-unit ball is contained in the $L_{t,q}^{\max}$-unit ball this follows from Theorems 5.6.1 and 2.1.5. ∎

We can also show that the spectral bands converge towards quantum SU(2) in the quantum Gromov–Hausdorff distance. In fact, as the following theorem shows, the convergence can even be obtained in a uniform manner with respect to the deformation parameters $t, q \in (0,1]$. For each $\delta \in (0,1)$ we recall the definition of the null-sequence of positive real numbers $(\varepsilon(\delta, M))_{M=0}^{\infty}$ from (5.12).

Theorem 5.6.3. *Let $\delta \in (0,1)$. It holds that*

$$\mathrm{dist}_{\mathrm{Q}}\big((C(\mathrm{SU}_q(2)), L_{t,q}^{\max}); (B_q^M, L_{t,q}^{\max})\big) \leqslant \varepsilon(\delta, M)$$

for all $M \in \mathbb{N}_0$ and $(t,q) \in [\delta,1] \times (0,1]$. Moreover, for all $\mu, \nu \in \mathcal{S}(C(\mathrm{SU}_q(2)))$ it holds that

$$d_{t,q}^{\max}(\mu,\nu) \leqslant 2 \cdot \varepsilon(\delta, M) + d_{t,q}^{\max}\big(\mu|_{B_q^M}, \nu|_{B_q^M}\big)$$

for all $M \in \mathbb{N}_0$, all $(t,q) \in [\delta,1] \times (0,1]$.

Proof. By Lemma 5.5.2 and Proposition 5.5.3, the unital positive operator

$$E_M^L\colon C(\mathrm{SU}_q(2)) \to B_q^M$$

satisfies the assumptions in Corollary 2.2.5 with $D = 0$ and $\varepsilon = \varepsilon(\delta, M)$. The first statement therefore follows from Corollary 2.2.5 and the second from Corollary 2.2.7. ∎

We may also provide an estimate on the diameter (see Definition 2.1.4) of quantum SU(2) in terms of the diameter of the Podleś sphere.

Proposition 5.6.4. *For all $t, q \in (0,1]$ it holds that*

$$\mathrm{diam}\big(C(\mathrm{SU}_q(2)), L_{t,q}^{\max}\big) \leqslant \frac{2\pi \cdot (t^{1/2} + t^{-1/2})}{\sqrt{3}} + \mathrm{diam}\big(C(S_q^2), L_q^{0,\max}\big).$$

Proof. By [70, Proposition 5.5] we have that

$$\operatorname{diam}\big(C(\mathrm{SU}_q(2)), L_{t,q}^{\max}\big) = 2 \cdot \operatorname{dist}_{\mathrm{Q}}\big((C(\mathrm{SU}_q(2)), L_{t,q}^{\max}); (\mathbb{C}, 0)\big).$$

Using the triangle inequality for the quantum Gromov–Hausdorff distance we then obtain that

$$\begin{aligned} &\operatorname{diam}\big(C(\mathrm{SU}_q(2)), L_{t,q}^{\max}\big) \\ &\quad \leqslant 2 \cdot \operatorname{dist}_{\mathrm{Q}}\big((C(\mathrm{SU}_q(2)), L_{t,q}^{\max}); (C(S_q^2), L_q^{0,\max})\big) + \operatorname{diam}\big(C(S_q^2), L_q^{0,\max}\big). \end{aligned}$$

The result of the proposition now follows from Corollary 5.1.3 and Theorem 5.6.3 in the case where $M = 0$. ■

Remark 5.6.5. In [3, Theorem 4.18] we proved that the family of Podleś spheres $((C(S_q^2), L_q^{0,\max}))_{q \in (0,1]}$ varies continuously in the quantum Gromov–Hausdorff distance, and thus, in particular, that the function

$$(0, 1] \ni q \mapsto \operatorname{diam}\big(C(S_q^2), L_q^{0,\max}\big) = 2 \cdot \operatorname{dist}_{\mathrm{Q}}((C(S_q^2), L_q^{0,\max}); (\mathbb{C}, 0))$$

is continuous. An application of Proposition 5.6.4 therefore shows that the function $(t, q) \mapsto \operatorname{diam}(C(\mathrm{SU}_q(2)), L_{t,q}^{\max})$ is bounded on compact subsets of $(0, 1] \times (0, 1]$.