Chapter 6

The quantum Berezin transform

We now introduce the second key ingredient in the analysis of the quantum metric structure of $\mathrm{SU}_q(2)$, namely an analogue of the classical Berezin transform (see e.g. [73] and references therein) in this context. The Berezin transform was already essential in Rieffel's seminal results in [71], where he proves that the 2-sphere can be approximated by matrices. The Berezin transform also played a pivotal role in the analysis of the quantum metric structure on the Podleś spheres S_q^2, $q \in (0,1]$, in [3,4]. In the present context it will serve to firstly establish the fact that the maximal and minimal Lip-norm, $L_{t,q}^{\max}$ and $L_{t,q}$ introduced in Definition 4.3.5, actually give rise to the same quantum metric structure (see Corollary 6.4.2 below). Secondly, the Berezin transform provides us with finite dimensional quantum metric spaces which we will show approximate quantum SU(2) in a suitably uniform manner. This, in turn, will be the key to our main continuity result, Theorem D.

6.1 Definition of the Berezin transform

Throughout this section, we fix the deformation parameter $q \in (0,1]$. The other parameter $t \in (0,1]$ is irrelevant in this section, since we are currently only concerned with the C^*-algebras and not the Lip-norms. For each $N, M \in \mathbb{N}_0$ we then define the element

$$\xi_N^M := \frac{1}{\sqrt{M+1}} \sum_{r=N}^{N+M} a^r \cdot \sqrt{\langle r+1 \rangle_q} \in \mathcal{O}(\mathrm{SU}_q(2)), \tag{6.1}$$

and consider the state $\chi_N^M \colon C(\mathrm{SU}_q(2)) \to \mathbb{C}$ given by $\chi_N^M(x) := h((\xi_N^M)^* x \xi_N^M)$. That χ_N^M is indeed a state follows from the formulae in (3.15) since $u_{00}^n = (a^*)^n$ for all $n \in \mathbb{N}_0$. In order to analyse these states in more detail, it is convenient to first introduce a new circle action. Consider again the left and right circle actions σ_L and σ_R on $C(\mathrm{SU}_q(2))$ defined on generators by

$$\sigma_L(z,a) := za, \quad \sigma_L(z,b) := zb \quad \text{and} \quad \sigma_R(z,a) := za, \quad \sigma_R(z,b) = z^{-1}b. \tag{6.2}$$

A direct computation shows that

$$\sigma(z,x) := \sigma_R\big(z, \sigma_L^{-1}(z,x)\big) \quad z \in S^1, \ x \in C(\mathrm{SU}_q(2))$$

defines a strongly continuous circle action on $C(\mathrm{SU}_q(2))$ which preserves $\mathcal{O}(\mathrm{SU}_q(2))$, and we let Π_m^σ, $m \in \mathbb{Z}$, denote the spectral projections associated with σ (cf. Section 5.2). The circle action σ is relevant in connection with the states χ_N^M and ϵ since, as we will see below, these only detect its fixed point algebra. We first determine the fixed point algebra in terms of the standard linear basis of $\mathcal{O}(\mathrm{SU}_q(2))$.

Lemma 6.1.1. *The fixed point algebra of the circle action σ on $C(\mathrm{SU}_q(2))$ agrees with the norm closure of the linear span*

$$\operatorname{span}_{\mathbb{C}}\big\{(b^*b)^m(a^*)^k, a^k(b^*b)^m \,\big|\, k,m \in \mathbb{N}_0\big\}.$$

Proof. Since σ fixes a, a^* and b^*b, it is clear that the span in the statement of the lemma is contained in the fixed point algebra. For the opposite inclusion, one may use the standard linear basis (3.1) together with the spectral projection $\Pi_0^\sigma \colon C(\mathrm{SU}_q(2)) \to C(\mathrm{SU}_q(2))$. Indeed, it holds that

$$\Pi_0^\sigma(\xi^{klm}) = \begin{cases} \xi^{klm} & \text{for } m = l \\ 0 & \text{for } m \neq l \end{cases} \qquad \blacksquare$$

Lemma 6.1.2. *Let $N, M \in \mathbb{N}_0$. We have that $\epsilon = \epsilon \circ \Pi_0^\sigma$ and $\chi_N^M = \chi_N^M \circ \Pi_0^\sigma$.*

Proof. This follows immediately since $\epsilon(\sigma(z,x)) = \epsilon(x)$ and $\chi_N^M(\sigma(z,x)) = \chi_N^M(x)$ for all $z \in S^1$ and $x \in C(\mathrm{SU}_q(2))$. In the case of ϵ it suffices to check the relevant identity on the generators a and b and in the case of χ_N^M the relevant identity follows since $\sigma(z,\xi_N^M) = \xi_N^M$ and $h(\sigma(z,x)) = h(x)$. $\blacksquare$

Lemma 6.1.3. *We have the convergence result $\lim_{N,M\to\infty} \chi_N^M = \epsilon$ with respect to the weak* topology on $\mathcal{S}(C(\mathrm{SU}_q(2)))$.*

Proof. By Lemmas 6.1.1 and 6.1.2, we only need to treat elements of the form

$$(b^*b)^m(a^*)^k \quad \text{and} \quad a^k(b^*b)^m, \quad \text{for } k,m \in \mathbb{N}_0.$$

Since states preserve the involution and

$$(b^*b)^m(a^*)^k = q^{-2km}(a^*)^k(b^*b)^m$$

it is enough to check the claim on elements of the form $(a^*)^k(b^*b)^m$. But since

$$\operatorname{span}_{\mathbb{C}}\big\{(b^*b)^m \,\big|\, m \in \mathbb{N}_0\big\} = \operatorname{span}_{\mathbb{C}}\big\{(a^*)^n a^n \,\big|\, n \in \mathbb{N}_0\big\},$$

we may, equivalently, verify the convergence on elements of the form $(a^*)^{k+n}a^n$. Let now $k,n \in \mathbb{N}_0$ be given. We are left with the task of showing that

$$\lim_{N,M\to\infty} \chi_N^M\big((a^*)^{k+n}a^n\big) = \epsilon\big((a^*)^{k+n}a^n\big) = 1.$$

Let us recall the inner product formulae from (3.15) as well as the fact that $(a^*)^m = u_{00}^m$ for all $m \in \mathbb{N}_0$. For each $N, M \in \mathbb{N}_0$ with $M \geqslant k$ we may thus compute as follows:

$$\begin{aligned}\chi_N^M\big((a^*)^{k+n}a^n\big) &= \frac{1}{M+1}\sum_{i,j=N}^{N+M} \langle i+1\rangle_q^{1/2}\langle j+1\rangle_q^{1/2} h\big((a^*)^{i+k+n}a^{n+j}\big)\\ &= \frac{1}{M+1}\sum_{i=N}^{N+M-k} \langle i+1\rangle_q^{1/2}\langle i+k+1\rangle_q^{1/2} h\big((a^*)^{i+k+n}a^{i+k+n}\big)\\ &= \frac{1}{M+1}\sum_{i=N}^{N+M-k} \frac{\langle i+1\rangle_q^{1/2}\langle i+k+1\rangle_q^{1/2}}{\langle i+k+n+1\rangle_q}.\end{aligned}$$

Let now $\varepsilon > 0$ be given. Since $\lim_{s\to\infty} \frac{\langle s\rangle_q}{\langle l+s\rangle_q} = 1$ for all $l \in \mathbb{N}_0$ we may choose $N_0 \in \mathbb{N}_0$ such that

$$\left|\frac{\langle i+1\rangle_q^{1/2}\langle k+i+1\rangle_q^{1/2}}{\langle n+k+i+1\rangle_q} - 1\right| < \varepsilon/2$$

for all $i \geqslant N_0$. Furthermore, we may choose $M_0 \geqslant k$ such that $\frac{k}{M+1} < \varepsilon/2$ for all $M \geqslant M_0$. For all $M \geqslant M_0$ and $N \geqslant N_0$ we then estimate that

$$\begin{aligned}\left|\chi_N^M\big((a^*)^{k+n}a^n\big) - 1\right| &\leqslant \frac{1}{M+1}\sum_{i=N}^{N+M-k}\left|\frac{\langle i+1\rangle_q^{1/2}\langle i+k+1\rangle_q^{1/2}}{\langle i+k+n+1\rangle_q} - 1\right|\\ &\quad + \left|\frac{M-k+1}{M+1} - 1\right|\\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon.\end{aligned}$$

This proves the proposition. ∎

We are now ready to introduce the analogue of the Berezin transform in our q-deformed setting:

Definition 6.1.4. The *quantum Berezin transform* in degree $N, M \in \mathbb{N}_0$ is the completely positive unital map $\beta_N^M : C(\mathrm{SU}_q(2)) \to C(\mathrm{SU}_q(2))$ given by

$$\beta_N^M(x) := \big(1 \otimes \chi_N^M\big)\Delta(x).$$

Remark 6.1.5. In [3], a quantum Berezin transform was introduced for the standard Podleś sphere S_q^2 in a manner very similar to the one above; see also [31] for an alternative and much more general construction of a Berezin transform on quantum homogeneous spaces. In [3], the states defining the Berezin transform were denoted

h_N, $N \in \mathbb{N}_0$, and given by $h_N(x) := \langle N+1\rangle_q h((a^*)^N x a^N)$ for all $x \in C(S_q^2) \subseteq C(\mathrm{SU}_q(2))$. We therefore have that $h_N = \chi_N^0|_{C(S_q^2)}$. In particular, the restriction of β_N^0 to $C(S_q^2)$ agrees with the Berezin transform β_N introduced in [3]. When $q = 1$, we recovered the usual Berezin transform on the classical 2-sphere; see [3, Section 3.2] for details on this. Note also that a Berezin transform for quantum homogeneous spaces was introduced in [72] in the setting of Kac-type quantum groups. Since $\mathrm{SU}_q(2)$ is only of Kac-type when $q = 1$ the constructions in [72] unfortunately do not apply directly in our context. However, as we shall see below, the more ad hoc definition above shares a number of properties with the construction in [72].

6.2 The image of the Berezin transform

In connection with our investigation of the quantum Gromov–Hausdorff continuity of the family $(C(\mathrm{SU}_q(2)), L_{t,q}^{\max})_{t,q\in(0,1]}$, a detailed understanding of the image of the Berezin transform β_N^M turns out to be imperative. In this section we therefore describe this image explicitly in terms of polynomial expressions in the generators a, b, a^*, b^* for $\mathcal{O}(\mathrm{SU}_q(2))$.

For each $r, s \in \mathbb{N}_0$, we introduce the linear functional $\varphi_{r,s} \colon C(\mathrm{SU}_q(2)) \to \mathbb{C}$ given by

$$\varphi_{r,s}(x) := h\big((a^*)^s x a^r\big).$$

These linear functionals are then related to our states χ_N^M (see (6.1)) by the formula

$$\chi_N^M = \frac{1}{M+1} \sum_{r,s=N}^{N+M} \sqrt{\langle r+1\rangle_q \langle s+1\rangle_q} \cdot \varphi_{r,s}, \quad N, M \in \mathbb{N}_0. \tag{6.3}$$

We now wish to determine the image of the Berezin transform β_N^M. To this end we first analyse the linear functionals $\varphi_{r,s}$ in more details.

Lemma 6.2.1. *Let $n, r, s \in \mathbb{N}_0$ and $0 \leqslant i, j \leqslant n$. It holds that $\varphi_{r,s}(u_{ij}^n) \geqslant 0$ and that*

$$\varphi_{r,s}(u_{ij}^n) \neq 0 \Leftrightarrow \big(n - 2j = r - s \text{ and } i = j \text{ and } j \leqslant s\big).$$

Proof. First note that by (3.14) we have the identities

$$\varphi_{r,s}(u_{ij}^n) = h\big((a^*)^s u_{ij}^n a^r\big) = h\big(\nu(a^r)\cdot(a^*)^s u_{ij}^n\big) = q^{-2r} h\big(a^r (a^*)^s u_{ij}^n\big).$$

Applying the formulae (3.11) we obtain that

$$\begin{aligned} a^r \cdot u_{ij}^n &= \sum_{k=0}^{r} \lambda_{n,i,j}(k) \cdot u_{i+k,j+k}^{n+2k-r} \quad \text{and} \\ (a^*)^s \cdot u_{ij}^n &= \sum_{k=0}^{\min\{i,j,s\}} \mu_{n,i,j}(k) \cdot u_{i-k,j-k}^{n-2k+s}, \end{aligned} \tag{6.4}$$

where all the coefficients appearing are strictly positive. Now note that $h(u^m_{kl}) = 0$ for all $m > 0$ and $h(u^0_{00}) = 1$. We see from the formulae in (6.4) that if the matrix coefficient u^0_{00} appears in the double sum expressing $a^r(a^*)^s \cdot u^n_{ij}$, then there are terms of the form u^m_{00} in the sum expressing $(a^*)^s \cdot u^n_{ij}$. This in turn implies that $s \geqslant j$ and $i = j$. We thus arrive at the following expressions:

$$h\big(a^r(a^*)^s \cdot u^n_{ij}\big) = \begin{cases} \lambda_{n-2j+s,0,0}(0) \cdot \mu_{n,j,j}(j) \cdot h\big(u^{n-2j+s-r}_{00}\big) & i = j,\ j \leqslant s \\ 0 & \text{elsewhere} \end{cases}$$
$$= \begin{cases} \lambda_{r,0,0}(0) \cdot \mu_{n,j,j}(j) & i = j,\ j \leqslant s,\ n = r - s + 2j \\ 0 & \text{elsewhere.} \end{cases}$$

This proves the lemma. ∎

Lemma 6.2.2. *Let $N, M \in \mathbb{N}_0$ and $m \in \mathbb{Z}$ with $|m| > M$. It holds that $\beta^M_N(x) = 0$ for all $x \in A^m_q$.*

Proof. Since β^M_N preserves the involution and $A^m_q = (A^{-m}_q)^*$ we may suppose that $m < -M$. Furthermore, we may assume that $x = u^{2j-m}_{ij}$ for some $j \in \mathbb{N}_0$ and $i \in \{0, 1, \dots, 2j - m\}$ since $\mathcal{A}^m_q$ is spanned by such matrix coefficients by (3.16). It then follows from Lemma 6.2.1 that

$$\beta^M_N(x) = \sum_{k=0}^{2j-m} u^{2j-m}_{ik} \cdot \chi^M_N\big(u^{2j-m}_{kj}\big) = u^{2j-m}_{ij} \cdot \chi^M_N\big(u^{2j-m}_{jj}\big) = 0.$$

Indeed, for all $r, s \in \{N, \dots, N + M\}$ we have $2j - m - 2j = -m > M \geqslant r - s$. ∎

Lemma 6.2.3. *Let $N, M \in \mathbb{N}_0$ and let $m \in \{0, \dots, M\}$. Let moreover $j \in \mathbb{N}_0$ and $i \in \{0, 1, \dots, 2j + m\}$. It holds that*

$$\beta^M_N\big(u^{2j+m}_{ij}\big) \neq 0 \Leftrightarrow j \in \{0, \dots, N + M - m\}.$$

In this case $\chi^M_N(u^{2j+m}_{jj}) > 0$ and we have the formula

$$\beta^M_N\big(u^{2j+m}_{ij}\big) = u^{2j+m}_{ij} \cdot \chi^M_N\big(u^{2j+m}_{jj}\big).$$

Similarly, we have that

$$\beta^M_N\big(u^{2j+m}_{i,j+m}\big) \neq 0 \Leftrightarrow j \in \{0, \dots, N + M - m\}.$$

In this case $\chi^M_N(u^{2j+m}_{j+m,j+m}) > 0$ and we have the formula

$$\beta^M_N\big(u^{2j+m}_{i,j+m}\big) = u^{2j+m}_{i,j+m} \cdot \chi^M_N\big(u^{2j+m}_{j+m,j+m}\big).$$

Proof. Let $k \in \{0, \dots, 2j+m\}$ be given. Using Lemma 6.2.1 together with (6.3) we obtain that $\chi_N^M(u_{kj}^{2j+m}) \neq 0$ if and only if $k = j$ and there exist $r, s \in \{N, \dots, N+M\}$ with $r - s = m$ and $j \leqslant s$. Since we have assumed that $M \geqslant m \geqslant 0$ we then see that $\chi_N^M(u_{kj}^{2j+m}) \neq 0$ if and only if $k = j$ and $j \in \{0, 1, \dots, N+M-m\}$. In this case, we moreover have that $\chi_N^M(u_{kj}^{2j+m}) > 0$. The first claim of the lemma (regarding u_{ij}^{2j+m}) therefore follows since

$$\beta_N^M\big(u_{ij}^{2j+m}\big) = \sum_{k=0}^{2j+m} u_{ik}^{2j+m} \cdot \chi_N^M\big(u_{kj}^{2j+m}\big) = u_{ij}^{2j+m} \cdot \chi_N^M\big(u_{jj}^{2j+m}\big).$$

The remaining claim is now a consequence of the positivity of the linear maps $\beta_N^M : C(\mathrm{SU}_q(2)) \to C(\mathrm{SU}_q(2))$ and $\chi_N^M : C(\mathrm{SU}_q(2)) \to \mathbb{C}$. Indeed, we know from (3.10) that $(u_{i,j}^{2j+m})^* = (-q)^{j-i} u_{2j+m-i,j+m}^{2j+m}$. ∎

Lemma 6.2.4. *Let $N, M \in \mathbb{N}_0$ and let $m \in \{0, 1, \dots, M\}$. It holds that*

$$\beta_N^M(A_q^{-m}) = \mathrm{span}_{\mathbb{C}}\big\{u_{ij}^{2j+m} \,\big|\, 0 \leqslant j \leqslant N+M-m,\ 0 \leqslant i \leqslant 2j+m\big\} \quad \textit{and}$$
$$\beta_N^M(A_q^{m}) = \mathrm{span}_{\mathbb{C}}\big\{u_{i,j+m}^{2j+m} \,\big|\, 0 \leqslant j \leqslant N+M-m,\ 0 \leqslant i \leqslant 2j+m\big\}.$$

The vector space dimensions are given by

$$\dim_{\mathbb{C}}\big(\beta_N^M(A_q^{-m})\big) = (N+M+1)(N+M+1-m) = \dim_{\mathbb{C}}\big(\beta_N^M(A_q^{m})\big).$$

In particular, we have that $\beta_N^M(A_q^k) \subseteq \mathcal{A}_q^k$ for all $k \in \{-M, -M+1, \dots, M\}$.

Proof. We first remark that the algebraic spectral subspace $\mathcal{A}_q^{-m}$ is spanned by matrix coefficients of the form u_{ij}^{2j+m} with $j \in \mathbb{N}_0$ and $i \in \{0, 1, \dots, 2j+m\}$ by (3.16). Similarly, since $\mathcal{A}_q^m = (\mathcal{A}_q^{-m})^*$ it follows from (3.10) that $\mathcal{A}_q^m$ is spanned by matrix coefficients of the form $u_{i,j+m}^{2j+m}$ with $j \in \mathbb{N}_0$ and $i \in \{0, 1, \dots, 2j+m\}$. The first claim regarding the images is then a consequence of Lemma 6.2.3. The relevant formula for the dimension of the subspaces $\beta_N^M(A_q^{-m})$ and $\beta_N^M(A_q^m)$ now follows from the computation

$$\begin{aligned}\dim_{\mathbb{C}}\big(\beta_N^M(A_q^m)\big) = \dim_{\mathbb{C}}\big(\beta_N^M(A_q^{-m})\big) &= \sum_{j=0}^{N+M-m} (2j+m+1) \\ &= (N+M-m+1)(N+M+1).\end{aligned}$$ ∎

The images of the spectral bands under the Berezin transforms will serve as our finite dimensional (also known as “fuzzy”) approximations, analogous to the fuzzy spheres from [55, 71] and their q-deformed counterparts in [3]. It will, however, also be convenient to have a description available in terms of the generators of $\mathrm{SU}_q(2)$

and we therefore opt to use this as the formal definition. To this end, recall from [3, Definition 3.5] that the quantum fuzzy sphere in degree $N \in \mathbb{N}_0$ is defined as

$$\begin{aligned}&\mathrm{Fuzz}_N(S_q^2)\\&\quad := \mathrm{span}_{\mathbb{C}}\left\{(bb^*)^i(ab^*)^j, (bb^*)^i(ba^*)^j \,\middle|\, i,j \in \mathbb{N}_0,\ i+j \leqslant N\right\} \subseteq \mathcal{O}(S_q^2).\end{aligned} \tag{6.5}$$

We now make the following definition:

Definition 6.2.5. Let $N, m \in \mathbb{N}_0$. We define the *fuzzy spectral subspaces* as the finite dimensional vector spaces

$$\mathrm{Fuzz}_N(A_q^m) := \sum_{k=0}^{m} a^k b^{m-k} \cdot \mathrm{Fuzz}_N(S_q^2) \subseteq \mathcal{A}_q^m \quad \text{and}$$
$$\mathrm{Fuzz}_N(A_q^{-m}) := \sum_{k=0}^{m} (a^*)^k (b^*)^{m-k} \cdot \mathrm{Fuzz}_N(S_q^2) \subseteq \mathcal{A}_q^{-m}.$$

Moreover, for $K \in \mathbb{N}_0$ we define the *fuzzy spectral K-bands* as

$$\mathrm{Fuzz}_N(B_q^K) := \sum_{m=-K}^{K} \mathrm{Fuzz}_N(A_q^m) \subseteq \mathcal{B}_q^K.$$

Note that since $\mathrm{Fuzz}_N(S_q^2)$ increases with $N \in \mathbb{N}_0$, the same is true for $\mathrm{Fuzz}_N(A_q^m)$ for all $m \in \mathbb{Z}$. As mentioned above, the spaces just defined are intimately related to the quantum Berezin transform as the following result shows:

Proposition 6.2.6. *Let $N, M, K \in \mathbb{N}_0$. It holds that $\beta_N^M(A_q^m) = \mathrm{Fuzz}_{N+M-|m|}(A_q^m)$ for all $m \in \{-M, \ldots, M\}$. In particular $\beta_N^M(B_q^K) \subseteq \mathrm{Fuzz}_{N+M}(B_q^K)$ whenever $M \geqslant K$ and $\mathrm{Fuzz}_N(B_q^K)$ is an operator system (without any constraints on $N, K \in \mathbb{N}_0$).*

Proof. Let $m \in \mathbb{Z}$ with $|m| \leqslant M$ be given. We focus on the case where $m \in \mathbb{N}_0$ since the case where $m < 0$ follows from similar arguments. We begin by recalling from [3, Lemmas 3.4 and 3.7] that

$$\mathrm{Fuzz}_{N+M-m}(S_q^2) = \mathrm{span}_{\mathbb{C}}\left\{u_{ij}^{2j} \,\middle|\, 0 \leqslant j \leqslant N+M-m,\ 0 \leqslant i \leqslant 2j\right\}. \tag{6.6}$$

Similarly, we recall from Lemma 6.2.4 that

$$\beta_N^M(A_q^m) = \mathrm{span}_{\mathbb{C}}\left\{u_{i,j+m}^{2j+m} \,\middle|\, 0 \leqslant j \leqslant N+M-m,\ 0 \leqslant i \leqslant 2j+m\right\}.$$

For $m = 0$, the identity $\mathrm{Fuzz}_{N+M-m}(A_q^m) = \beta_N^M(A_q^m)$ therefore follows immediately. We may thus suppose that $m > 0$. Let us start out by proving the inclusion

$$\mathrm{Fuzz}_{N+M-m}(A_q^m) \subseteq \beta_N^M(A_q^m). \tag{6.7}$$

For each $l, i, j \in \mathbb{N}_0$ with $i \leqslant 2j + l$ it follows from (3.11) that

$$a \cdot u^{2j+l}_{i,j+l} \in \operatorname{span}_{\mathbb{C}}\left\{u^{2j+(l+1)}_{i+1,j+(l+1)},\ u^{2(j-1)+(l+1)}_{i,j-1+(l+1)}\right\} \quad \text{and}$$
$$b \cdot u^{2j+l}_{i,j+l} \in \operatorname{span}_{\mathbb{C}}\left\{u^{2j+(l+1)}_{i,j+(l+1)},\ u^{2(j-1)+(l+1)}_{i-1,j-1+(l+1)}\right\}.$$

Hence, for all $k \in \{0, \ldots, m\}$, $j \in \{0, \ldots, N + M - m\}$ and $i \in \{0, \ldots, 2j\}$ it holds that

$$a^k b^{m-k} u^{2j}_{ij} \in \operatorname{span}_{\mathbb{C}}\left\{u^{2j+m}_{i,j+m} \,\middle|\, 0 \leqslant j \leqslant N + M - m,\ 0 \leqslant i \leqslant 2j + m\right\} = \beta^M_N(A^m_q).$$

By definition of $\operatorname{Fuzz}_{N+M-m}(A^m_q)$, the inclusion in (6.7) therefore follows. In order to show that $\operatorname{Fuzz}_{N+M-m}(A^m_q) = \beta^M_N(A^m_q)$, it now suffices to establish that

$$\dim_{\mathbb{C}}\left(\operatorname{Fuzz}_{N+M-m}(A^m_q)\right) \geqslant \dim_{\mathbb{C}}\left(\beta^M_N(A^m_q)\right).$$

Rewriting the definition of the quantum fuzzy sphere from (6.5) slightly we obtain

$$\begin{aligned} &\operatorname{Fuzz}_{N+M-m}(S^2_q) \\ &\quad = \operatorname{span}_{\mathbb{C}}\left\{a^j b^i (b^*)^{i+j},\ (a^*)^j b^{i+j} (b^*)^i \,\middle|\, i, j \in \mathbb{N}_0,\ i + j \leqslant N + M - m\right\}. \end{aligned} \tag{6.8}$$

From the two extremes, $k = 0$ and $k = m$, in Definition 6.2.5 we obtain that

$$\begin{aligned} M_1 &:= \left\{a^{m+j} b^i (b^*)^{i+j},\ (a^*)^j b^{m+i+j} (b^*)^i \,\middle|\, i, j \in \mathbb{N}_0,\ i + j \leqslant N + M - m\right\} \\ &\subseteq \operatorname{Fuzz}_{N+M-m}(A^m_q). \end{aligned}$$

Similarly, fixing $j = 0$ in (6.8) and letting k vary in $\{1, \ldots, m - 1\}$ we obtain that

$$\begin{aligned} M_2 &:= \left\{a^k b^{m-k+i} (b^*)^i \,\middle|\, 1 \leqslant k \leqslant m - 1,\ 0 \leqslant i \leqslant N + M - m\right\} \\ &\subseteq \operatorname{Fuzz}_{N+M-m}(A^m_q). \end{aligned}$$

Since $m \geqslant 1$, we see from (3.1) that the set $M_1 \cup M_2$ consists of linearly independent vectors and its cardinality is given by

$$\begin{aligned} &(m - 1) \cdot (N + M - m + 1) + 2 \cdot \left(\sum_{i=1}^{N+M-m+1} i\right) \\ &\quad = (N + M - m + 1) \cdot (N + M + 1), \end{aligned}$$

which is exactly $\dim_{\mathbb{C}}(\beta^M_N(A^m_q))$ by Lemma 6.2.4. This completes the proof of the first part of the lemma.

The last two statements of the lemma follow from the first part. Firstly, for $M \geqslant K$ we have that

$$\beta^M_N(B^K_q) = \sum_{m=-K}^{K} \operatorname{Fuzz}_{N+M-|m|}(A^m_q) \subseteq \sum_{m=-K}^{K} \operatorname{Fuzz}_{N+M}(A^m_q) = \operatorname{Fuzz}_{N+M}(B^K_q).$$

Secondly, for arbitrary $N, K \in \mathbb{N}_0$, $\mathrm{Fuzz}_N(B_q^K)$ is an operator system since the Berezin transforms are $*$-preserving:

$$\mathrm{Fuzz}_N(A_q^m)^* = \beta_N^{|m|}(A_q^m)^* = \beta_N^{|m|}(A_q^{-m}) = \mathrm{Fuzz}_N(A_q^{-m}). \qquad \blacksquare$$

Corollary 6.2.7. *Let* $m \in \mathbb{Z}$ *and* $N, K \in \mathbb{N}_0$. *It holds that* $\dim_{\mathbb{C}}(\mathrm{Fuzz}_N(A_q^m)) = (N + |m| + 1)(N + 1)$. *In particular, both* $\dim_{\mathbb{C}}(\mathrm{Fuzz}_N(A_q^m))$ *and* $\dim_{\mathbb{C}}(\mathrm{Fuzz}_N(B_q^K))$ *are independent of* $q \in (0, 1]$.

Proof. The first identity follows from Lemma 6.2.4 since $\mathrm{Fuzz}_N(A_q^m) = \beta_N^{|m|}(A_q^m)$ by Proposition 6.2.6. Since $\mathrm{Fuzz}_N(A_q^m) \subseteq \mathcal{A}_q^m$ one has that

$$\dim_{\mathbb{C}}(\mathrm{Fuzz}_N(B_q^K)) = \sum_{m=-K}^{K} \dim_{\mathbb{C}}(\mathrm{Fuzz}_N(A_q^m)),$$

and $\dim_{\mathbb{C}}(\mathrm{Fuzz}_N(B_q^K))$ is therefore also independent of $q \in (0, 1]$. $\blacksquare$

Inspecting the proof of Proposition 6.2.6, we obtain an explicit linear basis for the fuzzy spectral subspaces:

Corollary 6.2.8. *For each* $N \in \mathbb{N}$ *and* $m \in \mathbb{Z}$ *the fuzzy spectral subspace* $\mathrm{Fuzz}_N(A_q^m)$ *admits a linear basis consisting of a subset of the standard linear basis* (3.1) *for* $\mathcal{O}(\mathrm{SU}_q(2))$ *which is independent of the value of* q. *Concretely the basis can be chosen as follows:*

- *For* $m > 0$ *it is given by*

$$\big\{a^{j+m} b^i (b^*)^{i+j},\ (a^*)^j b^{i+j+m} (b^*)^i \,\big|\, i, j \in \mathbb{N}_0,\ 0 \le i + j \le N\big\}$$
$$\cup \big\{a^k b^{i+m-k} (b^*)^i \,\big|\, k \in \{1, \dots, m-1\},\ i \in \{0, \dots, N\}\big\}.$$

- *For* $m = 0$ *it is given by*

$$\big\{a^j b^i (b^*)^{i+j},\ (a^*)^j b^{i+j} (b^*)^i \,\big|\, j \in \{1, \dots, N\},\ i \in \{0, \dots, N-j\}\big\}$$
$$\cup \big\{b^i (b^*)^i \,\big|\, i \in \{0, \dots, N\}\big\}.$$

- *For* $m < 0$ *it is given by*

$$\big\{(a^*)^{j-m} b^{i+j} (b^*)^i,\ a^j b^i (b^*)^{i+j-m} \,\big|\, i, j \in \mathbb{N}_0,\ 0 \le i + j \le N\big\}$$
$$\cup \big\{(a^*)^k b^i (b^*)^{i-m-k} (b^*)^i \,\big|\, k \in \{1, \dots, -m-1\},\ i \in \{0, \dots, N\}\big\}.$$

The fuzzy approximations of the 2-sphere originated in physics [29, 55, 56] and have the feature of carrying an action of SU(2). In the q-deformed setting fuzzy approximations of the Podleś sphere have also been studied in the mathematical physics literature; see [5, 27, 28]. In some sense these ideas can be traced back to the work of Podleś [66].

Similarly to the quantum fuzzy spheres, our fuzzy spectral band also carries a coaction of quantum SU(2):

Proposition 6.2.9. *For each $N, K \in \mathbb{N}_0$, the operator system*

$$\mathrm{Fuzz}_N(B_q^K) \subseteq \mathcal{O}(\mathrm{SU}_q(2))$$

is $\mathcal{O}(\mathrm{SU}_q(2))$-coinvariant.

Proof. For every $x \in \mathrm{Fuzz}_N(B_q^K)$ we need to show that

$$\Delta(x) \in \mathcal{O}(\mathrm{SU}_q(2)) \otimes \mathrm{Fuzz}_N(B_q^K).$$

Let $m \in \{-K, -K+1, \dots, K\}$. We shall in fact see that

$$\Delta(x) \in \mathcal{O}(\mathrm{SU}_q(2)) \otimes \mathrm{Fuzz}_N(A_q^m)$$

whenever $x \in \mathrm{Fuzz}_N(A_q^m)$. Indeed, since $\mathrm{Fuzz}_N(A_q^m) = \beta_N^{|m|}(A_q^m)$ by Proposition 6.2.6, the relevant inclusion follows from Lemma 6.2.4 together with the formula for the coproduct on matrix coefficients. ■

Remark that for $q = 1$, the comultiplication on $\mathcal{O}(\mathrm{SU}(2))$ is dual to the group multiplication. Letting λ denote the left regular action of SU(2) on $\mathcal{O}(\mathrm{SU}(2))$ and $\mathrm{ev}_g : \mathcal{O}(\mathrm{SU}(2)) \to \mathbb{C}$ denote the evaluation at a point $g \in \mathrm{SU}(2)$, we have the formula $\lambda_{g^{-1}} f = (\mathrm{ev}_g \otimes 1)\Delta(f)$ for all $f \in \mathcal{O}(\mathrm{SU}(2))$. Thus, in this case the coinvariance in Proposition 6.2.9 does indeed correspond to invariance of $\mathrm{Fuzz}_N(B_1^K)$ under the left regular action of SU(2).

In the section to follow, we need to apply the Berezin transform, which is at the moment only defined on $C(\mathrm{SU}_q(2))$, to elements in the von Neumann algebra $L^\infty(\mathrm{SU}_q(2))$. Since Δ extends to a normal $*$-homomorphism at the von Neumann algebraic level and each χ_N^M is normal (being a vector state in the GNS representation), the slice map formula $(1 \otimes \chi_N^M)\Delta(x)$ also makes sense at the level of $L^\infty(\mathrm{SU}_q(2))$. We could therefore simply extend the Berezin transform β_N^M to $L^\infty(\mathrm{SU}_q(2))$ using the same formula. However, it will be important to view this extension as a composition of a finite-dimensional projection and the original Berezin transform and we therefore take this point of view as our point of departure.

For each finite dimensional subspace $F \subseteq \mathcal{O}(\mathrm{SU}_q(2))$ we let $P_F : L^2(\mathrm{SU}_q(2)) \to L^2(\mathrm{SU}_q(2))$ denote the orthogonal projection with image $\Lambda(F) \subseteq L^2(\mathrm{SU}_q(2))$. We then define the linear map $\Phi_F : \mathbb{B}(L^2(\mathrm{SU}_q(2))) \to \mathcal{O}(\mathrm{SU}_q(2))$ by the formula

$$\Lambda(\Phi_F(T)) := P_F(T \cdot \Lambda(1)) \quad \text{for all } T \in \mathbb{B}(L^2(\mathrm{SU}_q(2))). \tag{6.9}$$

We record that $\mathrm{Im}(\Phi_F) = F$ and that Φ_F is WOT-norm continuous, where we recall that WOT refers to the weak operator topology on $\mathbb{B}(L^2(\mathrm{SU}_q(2)))$. Notice, moreover, that $\Phi_{F_0}\Phi_{F_1} = \Phi_{F_1}\Phi_{F_0} = \Phi_{F_0}$ when $F_0, F_1 \subseteq \mathcal{O}(\mathrm{SU}_q(2))$ are finite dimensional subspaces with $F_0 \subseteq F_1$. For each $N, M \in \mathbb{N}_0$, we have that $\beta_N^M(B_q^M) \subseteq \mathcal{B}_q^M \subseteq \mathcal{O}(\mathrm{SU}_q(2))$ is a finite-dimensional subspace and we apply the notation

$$\Phi_N^M := \Phi_{\beta_N^M(B_q^M)} : \mathbb{B}\big(L^2(\mathrm{SU}_q(2))\big) \to \mathcal{O}(\mathrm{SU}_q(2))$$

for the associated linear map. Recalling from Proposition 6.2.6 that

$$\beta_N^M(B_q^M) \subseteq \mathrm{Fuzz}_{N+M}(B_q^M),$$

we now define the *extended Berezin transform* $\widetilde{\beta}_N^M : L^\infty(\mathrm{SU}_q(2)) \to \mathrm{Fuzz}_{N+M}(B_q^M)$ by setting

$$\widetilde{\beta}_N^M(x) := \beta_N^M(\Phi_N^M(x)) \quad \text{for all } x \in L^\infty(\mathrm{SU}_q(2)). \tag{6.10}$$

Lemma 6.2.10. *Let $N, M \in \mathbb{N}_0$. The extended Berezin transform $\widetilde{\beta}_N^M$ is ucp (unital completely positive) and satisfies that $\widetilde{\beta}_N^M(x) = \beta_N^M(x)$ for all $x \in C(\mathrm{SU}_q(2))$.*

Proof. We will start by showing that the extended Berezin transform does indeed extend the Berezin transform. By norm-density and linearity, it suffices to verify that $\beta_N^M(\Phi_N^M(u_{ij}^n)) = \beta_N^M(u_{ij}^n)$ for all $n \in \mathbb{N}_0$ and $i, j \in \{0, \dots, n\}$. If u_{ij}^n is not one of the matrix coefficients spanning $\beta_N^M(B_q^M) = \sum_{m=-M}^{M} \beta_N^M(A_q^m)$ then we obtain from Lemmas 6.2.2, 6.2.3 and 6.2.4 that both sides of the claimed identity are equal to zero (recall here that the different matrix coefficients are orthogonal to one another when embedded in $L^2(\mathrm{SU}_q(2))$). Conversely, if $u_{ij}^n \in \beta_N^M(B_q^M)$, it holds that $\Phi_N^M(u_{ij}^n) = u_{ij}^n$, and the relevant identity therefore holds trivially.

We now focus on showing that $\widetilde{\beta}_N^M$ is completely positive. We first note that this is indeed the case for the Berezin transform β_N^M, being defined as the composition of the unital $*$-homomorphism Δ with the slice map induced by the state χ_N^M. Let $x \in L^\infty(\mathrm{SU}_q(2)) \otimes \mathbb{M}_d(\mathbb{C})$ be given. Then there exists a net $(x_\alpha)_\alpha$ in $C(\mathrm{SU}_q(2)) \otimes \mathbb{M}_d(\mathbb{C})$ converging in the strong operator topology to x. The net $(x_\alpha^* x_\alpha)_\alpha$ therefore converges in the weak operator topology to x^*x and since Φ_N^M is WOT-norm continuous we obtain the net $((\beta_N^M \otimes 1_d)(x_\alpha^* x_\alpha))_\alpha$ converges in norm to $(\widetilde{\beta}_N^M \otimes 1_d)(x^*x)$. Since each $(\beta_N^M \otimes 1_d)(x_\alpha^* x_\alpha)$ is positive and the positive cone is norm closed we may conclude that $(\widetilde{\beta}_N^M \otimes 1_d)(x^*x)$ is positive. This proves that the extended Berezin transform is completely positive. ∎

6.3 Estimates on the Berezin transform

Our next aim is to analyse the interplay between the Berezin transforms and the twisted derivations defining the Lip-norms $L_{t,q}^{\max}$. At the algebraic level, i.e. with $L_{t,q}$ instead of $L_{t,q}^{\max}$, this analysis is slightly less complicated (see the remarks preceding Proposition 6.3.4), but at the analytic level things are more subtle. In the first series of lemmas below, we show how one may, nevertheless, reduce certain questions to the algebraic setting by means of the projections Φ_F introduced in (6.9).

Throughout this section, we fix the two parameters t and q in $(0,1]$ unless explicitly stated otherwise.

Lemma 6.3.1. *Let $\xi, \eta \in \mathcal{O}(\mathrm{SU}_q(2))^{\oplus 2}$. Then there exists a finite-dimensional subspace $F_0 \subseteq \mathcal{O}(\mathrm{SU}_q(2))$ such that*

$$\langle \xi, \partial_q^H(x)\eta \rangle = \langle \xi, \partial_q^H(\Phi_F(x))\eta \rangle \quad \textit{and} \quad \langle \xi, \partial_t^V(x)\eta \rangle = \langle \xi, \partial_t^V(\Phi_F(x))\eta \rangle$$

whenever $F \subseteq \mathcal{O}(\mathrm{SU}_q(2))$ is a finite-dimensional subspace with $F_0 \subseteq F$ and $x \in \mathrm{Lip}_t(\mathrm{SU}_q(2))$.

Proof. First consider $y, z \in \mathcal{O}(\mathrm{SU}_q(2))$ and let $F \subseteq \mathcal{O}(\mathrm{SU}_q(2))$ be any finite dimensional subspace containing the vector $y \cdot \nu(z)^* \in \mathcal{O}(\mathrm{SU}_q(2))$. Using that the Haar state is a twisted trace (see (3.13)), we then have that

$$\langle y, x \cdot z \rangle = \langle y \cdot \nu(z)^*, x \rangle = \langle y \cdot \nu(z)^*, \Phi_F(x) \rangle = \langle y, \Phi_F(x) \cdot z \rangle \tag{6.11}$$

for all $x \in C(\mathrm{SU}_q(2))$.

Let us now focus on the case of the horizontal Dirac operator. The argument is similar for the vertical Dirac operator. Let $x \in \mathrm{Lip}_t(\mathrm{SU}_q(2))$ and $\xi, \eta \in \mathcal{O}(\mathrm{SU}_q(2))^{\oplus 2}$. By definition of $\partial_q^H(x)$ and by Lemma 3.6.4 we have that

$$\begin{aligned}
\langle \xi, \partial_q^H(x)\eta \rangle &= \langle \xi, D_q^H \sigma_L(q^{\frac{1}{2}}, x)\eta \rangle - \langle \xi, \sigma_L(q^{-\frac{1}{2}}, x)\mathcal{D}_q^H \eta \rangle \\
&= \langle \mathcal{D}_q^H \xi, \Gamma_q^{-1} x \Gamma_q \eta \rangle - \langle \xi, \Gamma_q x \Gamma_q^{-1} \mathcal{D}_q^H \eta \rangle \\
&= \langle \Gamma_q^{-1} \mathcal{D}_q^H \xi, x \Gamma_q \eta \rangle - \langle \Gamma_q \xi, x \Gamma_q^{-1} \mathcal{D}_q^H \eta \rangle.
\end{aligned}$$

Since the unbounded operators $\Gamma_q, \Gamma_q^{-1}\mathcal{D}_q^H$ both preserve the subspace $\mathcal{O}(\mathrm{SU}_q(2))^{\oplus 2}$ we obtain the result of the lemma by applying the observation from (6.11) and running the last computation backwards. ∎

Lemma 6.3.2. *Let $n, i, j \in \mathbb{N}_0$ satisfy that $i, j \leqslant n$. It holds that $u_{ij}^n \in \beta_N^M(B_q^M)$ for all $N, M \in \mathbb{N}_0$ with $N + M \geqslant n$ and $M \geqslant |2j - n|$. In particular, for any finite dimensional subspace $F \subseteq \mathcal{O}(\mathrm{SU}_q(2))$ we may choose a $K_0 \in \mathbb{N}_0$ such that $F \subseteq \beta_0^K(B_q^K)$ for all $K \geqslant K_0$.*

Proof. Let $N, M \in \mathbb{N}_0$ with $N + M \geqslant n$ and $M \geqslant |2j - n|$ be given. Put $m := |2j - n|$ so that $M \geqslant m$. Suppose first that $m = n - 2j$. We then have that $u_{ij}^n = u_{ij}^{2j+m}$ and it follows from Lemma 6.2.4 that $u_{ij}^n \in \beta_N^M(A_q^{-m}) \subseteq \beta_N^M(B_q^M)$, since $j \leqslant 2j = n - m \leqslant N + M - m$. Suppose next that $m = 2j - n$. Put $k := j - m$ and notice that $k \geqslant 0$ since $k = n - j$. We then have that $u_{ij}^n = u_{ij}^{2j-m} = u_{i,k+m}^{2k+m}$ and it again follows from Lemma 6.2.4 that $u_{ij}^n \in \beta_N^M(A_q^m)$ since $k = j - m \leqslant n - m \leqslant N + M - m$. ∎

We define the linear map $\delta\colon \mathrm{Lip}_q(\mathrm{SU}_q(2)) \to \mathbb{M}_2(L^\infty(\mathrm{SU}_q(2)))$ by putting

$$\delta(x) := u \cdot \partial_{q,q}(x) \cdot u^* \quad \text{for all } x \in \mathrm{Lip}_q(\mathrm{SU}_q(2)).$$

Note that δ does indeed take values in the von Neumann algebra $\mathbb{M}_2(L^\infty(\mathrm{SU}_q(2)))$ since $\partial_{q,q} = \partial_q^V + \partial_q^H$ takes values here by Corollary 4.5.5. We moreover remark that δ extends the twisted $*$-derivation

$$\delta = \begin{pmatrix} \delta^3 & -\delta^2 \\ -\delta^1 & -\delta^3 \end{pmatrix} : \mathcal{O}(\mathrm{SU}_q(2)) \to \mathbb{M}_2\big(\mathcal{O}(\mathrm{SU}_q(2))\big)$$

as can be seen by an application of Proposition 4.7.4.

Lemma 6.3.3. *For each $N, M \in \mathbb{N}_0$ there exists a $K_0 \in \mathbb{N}_0$ such that*

(1) $\Phi_N^M \delta(x) = \Phi_N^M \delta(\Phi_0^K(x))$ *for all $x \in \mathrm{Lip}_q(\mathrm{SU}_q(2))$ and $K \geqslant K_0$.*
(2) $\Phi_N^M \partial_t^V(x) = \Phi_N^M \partial_t^V(\Phi_0^K(x))$ *for all $x \in \mathrm{Lip}_t(\mathrm{SU}_q(2))$ and $K \geqslant K_0$.*

Proof. We will only carry out the argumentation for δ since the remaining case follows by a similar but slightly easier argument.

Consider the finite dimensional subspace $\beta_N^M(B_q^M) \subseteq \mathcal{O}(\mathrm{SU}_q(2))$ and let $d \in \mathbb{N}$ denote its dimension. Let us choose a subset $\{\zeta_k \mid k = 1, 2, \ldots, d\} \subseteq \beta_N^M(B_q^M)$ so that $\{\Lambda(\zeta_k) \mid k = 1, 2, \ldots, d\}$ constitutes an orthonormal basis for the subspace $\Lambda(\beta_N^M(B_q^M)) \subseteq L^2(\mathrm{SU}_q(2))$. The map Φ_N^M is then given by the expression

$$\Phi_N^M(T) = \sum_{k=1}^d \zeta_k \big\langle \Lambda(\zeta_k), T\Lambda(1)\big\rangle \quad T \in \mathbb{B}\big(L^2(\mathrm{SU}_q(2))\big).$$

For every vector $\zeta \in L^2(\mathrm{SU}_q(2))$ we apply the notation

$$\zeta^0 := \begin{pmatrix} \zeta \\ 0 \end{pmatrix} \quad \text{and} \quad \zeta^1 := \begin{pmatrix} 0 \\ \zeta \end{pmatrix} \in L^2(\mathrm{SU}_q(2))^{\oplus 2},$$

and let $e_{ij} \in \mathbb{M}_2(\mathbb{C})$ denote the standard matrix units for $i, j \in \{0, 1\}$. The linear map Φ_N^M can then be described at the level of 2×2-matrices by the expression

$$\Phi_N^M(T) = \sum_{i,j=0}^1 \sum_{k=1}^d e_{ij} \cdot \zeta_k \big\langle \Lambda(\zeta_k^i), T\Lambda(1)^j\big\rangle \quad \text{for all } T \in \mathbb{M}_2\big(\mathbb{B}(L^2(\mathrm{SU}_q(2)))\big).$$

In particular, we have that

$$\begin{aligned} \Phi_N^M(\delta(x)) &= \sum_{i,j=0}^1 \sum_{k=1}^d e_{ij} \cdot \zeta_k \big\langle \Lambda(\zeta_k^i), \delta(x)\Lambda(1)^j\big\rangle \\ &= \sum_{i,j=0}^1 \sum_{k=1}^d e_{ij} \cdot \zeta_k \big\langle u^*\Lambda(\zeta_k^i), \partial_{q,q}(x)u^*\Lambda(1)^j\big\rangle \end{aligned}$$

for all $x \in \mathrm{Lip}_q(\mathrm{SU}_q(2))$. It therefore follows from Lemma 6.3.1 that we may choose a finite-dimensional subspace $F_0 \subseteq \mathcal{O}(\mathrm{SU}_q(2))$ such that

$$\Phi_N^M(\delta(x)) = \Phi_N^M\big(\delta(\Phi_F(x))\big)$$

for all finite dimensional subspaces $F \subseteq \mathcal{O}(\mathrm{SU}_q(2))$ with $F_0 \subseteq F$ and all $x \in \mathrm{Lip}_q(\mathrm{SU}_q(2))$. The result of the present lemma is now a consequence of Lemma 6.3.2.

■

Let $N, M \in \mathbb{N}_0$ be given. Recall that the Berezin transform $\beta_N^M : C(\mathrm{SU}_q(2)) \to C(\mathrm{SU}_q(2))$ is defined by slicing the coproduct Δ on the right tensor-leg with a state, while endomorphisms of the form δ_η with $\eta \in \mathcal{U}_q(\mathfrak{su}(2))$ are defined by slicing the coproduct on the left tensor-leg. An application of the coassociativity of Δ therefore shows that $\beta_N^M(\delta_\eta(x)) = \delta_\eta(\beta_N^M(x))$ for all $x \in \mathcal{O}(\mathrm{SU}_q(2))$ and $\eta \in \mathcal{U}_q(\mathfrak{su}(2))$. In particular, we obtain that

$$\beta_N^M(\delta(x)) = \delta(\beta_N^M(x)) \quad \text{for all } x \in \mathcal{O}(\mathrm{SU}_q(2)). \tag{6.12}$$

Furthermore, for each element x belonging to an algebraic spectral subspace $\mathcal{A}_q^m$ for some $m \in \mathbb{Z}$, we get from Lemmas 4.3.1, 6.2.2 and 6.2.4 that

$$\beta_N^M\big(\partial_t^V(x)\big) = \begin{pmatrix} [m/2]_t \beta_N^M(x) & 0 \\ 0 & -[m/2]_t \beta_N^M(x) \end{pmatrix} = \partial_t^V(\beta_N^M(x)) \quad \text{for all } t \in (0,1].$$

We may thus conclude that

$$\beta_N^M(\partial_t^V(x)) = \partial_t^V(\beta_N^M(x)) \quad \text{for all } x \in \mathcal{O}(\mathrm{SU}_q(2)).$$

As a consequence of the analysis carried out above, we shall now see that these identities remain valid also at the level of the Lipschitz algebra. Recall, in this connection, that $\widetilde{\beta}_N^M$ denotes the extension of β_N^M to $L^\infty(\mathrm{SU}_q(2))$ introduced in (6.10).

Proposition 6.3.4. *For $M, N \in \mathbb{N}_0$, the following identities are valid:*

(1) $\delta(\beta_N^M(x)) = \widetilde{\beta}_N^M \delta(x)$ *for all* $x \in \mathrm{Lip}_q(\mathrm{SU}_q(2))$*;*

(2) $\partial_t^V(\beta_N^M(x)) = \widetilde{\beta}_N^M \partial_t^V(x)$ *for all* $x \in \mathrm{Lip}_t(\mathrm{SU}_q(2))$.

Proof. We focus on proving the identity regarding the map δ. A similar argumentation applies to the twisted $*$-derivation ∂_t^V. Let $x \in \mathrm{Lip}_q(\mathrm{SU}_q(2))$ be given. By Lemmas 6.3.2 and 6.3.3, we may choose a $K \in \mathbb{N}_0$ such that $\beta_N^M(B_q^M) \subseteq \beta_0^K(B_q^K)$ and such that

$$\Phi_N^M \delta(x) = \Phi_N^M \delta\big(\Phi_0^K(x)\big).$$

We now remark that $\Phi_0^K(x) \in \mathcal{O}(\mathrm{SU}_q(2))$ and that $\Phi_N^M \Phi_0^K = \Phi_N^M$.

Applying these facts together with (6.12) and Lemma 6.2.10 we obtain the desired result:

$$\widetilde{\beta}_N^M \delta(x) = \beta_N^M \Phi_N^M \delta(x) = \beta_N^M \Phi_N^M \delta\big(\Phi_0^K(x)\big) = \delta\big(\beta_N^M \Phi_N^M \Phi_0^K(x)\big) = \delta\big(\beta_N^M(x)\big).$$

■

In the special situation where $t = q$ we have the identity $u \cdot \partial_{q,q} \cdot u^* = \delta$ and, as we saw above, the map δ commutes with the Berezin transform. As the following result shows, this has the effect that the Berezin transform becomes a contraction for the associated Lip-norm $L^{\max}_{q,q}$. There is no reason to expect this to be the case when $t \neq q$, but Proposition 6.3.10 below provides an estimate on how far away the Berezin transform is from being a contraction for the Lip-norm $L^{\max}_{t,q}$.

Corollary 6.3.5. *Let $N, M \in \mathbb{N}_0$. The Berezin transform*

$$\beta_N^M : \mathrm{Lip}_q(\mathrm{SU}_q(2)) \to \mathcal{O}(\mathrm{SU}_q(2))$$

is a Lip-norm contraction for $L^{\max}_{q,q}$; i.e. we have the inequality

$$L^{\max}_{q,q}\big(\beta_N^M(x)\big) \leqslant L^{\max}_{q,q}(x)$$

for all $x \in \mathrm{Lip}_q(\mathrm{SU}_q(2))$.

Proof. Let $x \in \mathrm{Lip}_q(\mathrm{SU}_q(2))$. By Corollary 4.5.5, we have that

$$\delta(x) = u \cdot \partial_{q,q}(x) \cdot u^* \in \mathbb{M}_2\big(L^\infty(\mathrm{SU}_q(2))\big)$$

and by Lemma 6.2.10 the map $\widetilde{\beta}_N^M : L^\infty(\mathrm{SU}_q(2)) \to C(\mathrm{SU}_q(2))$ is ucp, and hence a complete contraction. Using this together with Proposition 6.3.4, we obtain the relevant inequality:

$$L^{\max}_{q,q}\big(\beta_N^M(x)\big) = \big\|\delta(\beta_N^M(x))\big\| = \big\|\widetilde{\beta}_N^M(\delta(x))\big\| \leqslant \|\delta(x)\| = L^{\max}_{q,q}(x). \qquad \blacksquare$$

We now return to the general setting, and will prove that the Berezin transform suitably approximates the identity operator on the Lip-unit ball. Most of the results below will be needed in two versions: one version for all of quantum SU(2) and one version which is fine tuned to hold on the spectral bands. For $K \in \mathbb{N}_0$, we will also use $d_{t,q}$ and $d^{\max}_{t,q}$ to denote the metrics on the state space $\mathcal{S}(B_q^K)$ arising from the restriction of the seminorms $L_{t,q}$ and $L^{\max}_{t,q}$ to the spectral band B_q^K having domains $\mathcal{B}_q^K$ and $B_q^K \cap \mathrm{Lip}_t(\mathrm{SU}_q(2))$, respectively. Hence, for $\mu, \nu \in \mathcal{S}(C(\mathrm{SU}_q(2)))$ we specify that

$$d^{\max}_{t,q}(\mu,\nu) := \sup\big\{|\mu(x) - \nu(x)| \,\big|\, x \in C(\mathrm{SU}_q(2)),\ L^{\max}_{t,q}(x) \leqslant 1\big\}$$
$$d^{\max}_{t,q}(\mu|_{B_q^K}, \nu|_{B_q^K}) := \sup\big\{|\mu(x) - \nu(x)| \,\big|\, x \in B_q^K,\ L^{\max}_{t,q}(x) \leqslant 1\big\},$$

and similarly for $d_{t,q}$. Note that by Lemma 5.3.2 it holds that the domain of the restricted seminorm $L^{\max}_{t,q}|_{B_q^K}$ is independent of t, in that we have

$$\mathrm{Lip}_t(\mathrm{SU}_q(2)) \cap B_q^K = \mathrm{Lip}^H(\mathrm{SU}_q(2)) \cap B_q^K, \tag{6.13}$$

where $\mathrm{Lip}^H(\mathrm{SU}_q(2))$ is the algebra of horizontally Lipschitz elements introduced in Definition 4.3.2.

Proposition 6.3.6. *Let $N, M, K \in \mathbb{N}_0$. It holds that*

$$\|\beta_N^M(x) - x\| \leqslant d_{t,q}^{\max}\big(\chi_N^M, \epsilon\big) \cdot L_{t,q}^{\max}(x) \quad \textit{for all } x \in C(\mathrm{SU}_q(2)) \quad \textit{and}$$
$$\|\beta_N^M(x) - x\| \leqslant d_{t,q}^{\max}\big(\chi_N^M|_{B_q^K}, \epsilon|_{B_q^K}\big) \cdot L_{t,q}^{\max}(x) \quad \textit{for all } x \in B_q^K.$$

Proof. When proving the two statements we may focus on the case where x belongs to $\mathrm{Lip}_t(\mathrm{SU}_q(2))$ or $\mathrm{Lip}_t(\mathrm{SU}_q(2)) \cap B_q^K$ since the seminorms on the right-hand side otherwise take the value infinity. Notice first that for every $y \in C(\mathrm{SU}_q(2))$ it holds that

$$\|y\| = \sup\big\{|\phi_{\xi,\eta}(y)| \,\big|\, \xi, \eta \in L^2(\mathrm{SU}_q(2)),\ \|\xi\|, \|\eta\| = 1\big\}, \tag{6.14}$$

where we recall that $\phi_{\xi,\eta}$ denotes the linear functional $x \mapsto \langle \xi, \rho(x)\eta\rangle$. Let now $x \in \mathrm{Lip}_t(\mathrm{SU}_q(2))$ be given and let $\xi, \eta \in L^2(\mathrm{SU}_q(2))$ be unit vectors. Using the identity (6.14), it suffices to show that

$$\big|\phi_{\xi,\eta}(\beta_N^M(x) - x)\big| \leqslant d_{t,q}^{\max}(\chi_N^M, \epsilon) \cdot L_{t,q}^{\max}(x).$$

This inequality follows from Proposition 4.6.6 and the Fubini Theorem for slice maps [76] via the estimates:

$$\begin{aligned}
\big|\phi_{\xi,\eta}(\beta_N^M(x) - x)\big| &= \big|(\chi_N^M - \epsilon)(\phi_{\xi,\eta} \otimes 1)\Delta(x)\big| \\
&\leqslant d_{t,q}^{\max}(\chi_N^M, \epsilon) \cdot L_{t,q}^{\max}\big((\phi_{\xi,\eta} \otimes 1)\Delta(x)\big) \\
&\leqslant d_{t,q}^{\max}(\chi_N^M, \epsilon) \cdot L_{t,q}^{\max}(x).
\end{aligned}$$

This proves the first part of the statement.

If $x \in B_q^K$ then $\Delta(x) \in C(\mathrm{SU}_q(2)) \otimes_{\min} B_q^K$ since each of the algebraic spectral subspaces is a left comodule for $\mathcal{O}(\mathrm{SU}_q(2))$. In the last computation in the proof above we therefore have $(\phi_{\xi,\eta} \otimes 1)\Delta(x) \in B_q^K$, and hence the rest of the argument carries over to prove the remaining inequality. ■

As indicated above, we now wish to estimate how far the Berezin transform is from being a contraction for the Lip-norm $L_{t,q}^{\max}$. In general, there is no hope to commute the Berezin transform directly past the operation $u \cdot \partial_{t,q} \cdot u^*$ as we could when $t = q$. However, as Proposition 6.3.6 shows, the Berezin transform approximates the identity operator well on the Lip-unit ball, and this makes it possible to obtain strong estimates nevertheless. The analytic norm $\|\cdot\|_{t,q}$ introduced in Section 3.6 will be used as a tool in the analysis below, and we first provide an estimate on its values on the entries of the fundamental unitary $u \in \mathbb{M}_2(\mathcal{O}(\mathrm{SU}_q(2)))$. We denote these entries by u_{ij}, $i, j = 0, 1$.

Lemma 6.3.7. *For every $i, j \in \{0, 1\}$, it holds that*

$$\|u_{ij}\|_{t,q} = \|u_{ij}^*\|_{t,q} \leqslant q^{-\frac{1}{2}} + t^{-\frac{1}{2}} \quad \textit{and}$$
$$L_{t,q}^{\max}(u_{ij}) = L_{t,q}^{\max}(u_{ij}^*) \leqslant [1/2]_t + q^{-\frac{1}{2}}.$$

Proof. Let $i, j \in \{0, 1\}$ be given. Using that $\sigma_L(s^{\frac{1}{2}}, u_{ij}) = s^{j-\frac{1}{2}} u_{ij}$ for all $s \in (0, \infty)$, the result of the lemma follows from the estimate

$$\|u_{ij}^*\|_{t,q} = \|u_{ij}\|_{t,q} \leqslant \max\{q^{j-\frac{1}{2}} + t^{j-\frac{1}{2}}, q^{-j+\frac{1}{2}} + t^{-j+\frac{1}{2}}\} \leqslant q^{-\frac{1}{2}} + t^{-\frac{1}{2}}$$

together with the estimates

$$\begin{aligned} L_{t,q}^{\max}(u_{ij}^*) &= L_{t,q}^{\max}(u_{ij}) = \|\partial_t^V(u_{ij}) + \partial_q^H(u_{ij})\| \\ &\leqslant \|\partial_t^3(u_{ij})\| + \max\{q^{\frac{1}{2}}\|\partial_e(u_{ij})\|, q^{-1/2}\|\partial_f(u_{ij})\|\} \leqslant [1/2]_t + q^{-\frac{1}{2}}, \end{aligned}$$

where the last inequality follows from (3.3). ■

In the following lemma we recall that $\Pi_0^L : C(\mathrm{SU}_q(2)) \to C(S_q^2)$ denotes the spectral projection onto the Podleś sphere; see (3.17).

Lemma 6.3.8. *Let $x \in \ker(\Pi_0^L)$. We have the estimate*

$$\|x\| \leqslant \frac{\pi \cdot (t^{1/2} + t^{-1/2})}{\sqrt{3}} \cdot L_{t,q}^{\max}(x).$$

Proof. Without loss of generality, we may assume that $x \in \ker(\Pi_0^L) \cap \mathrm{Lip}_t(\mathrm{SU}_q(2))$ since the right-hand side of the desired inequality is equal to infinity otherwise. By Proposition 5.5.1 we then get that $x = \int_t^V \partial_t^V(x)$. It thus follows from Lemma 5.4.3 and the definition of $\int_t^V$ from (5.15) that

$$\|x\| \leqslant \|\int_t^V \| \cdot \|\partial_t^V(x)\| \leqslant \frac{\pi \cdot (t^{1/2} + t^{-1/2})}{\sqrt{3}} \cdot L_{t,q}^{\max}(x). \qquad ■$$

With the above auxiliary results at our disposal, we may now start estimating the error arising when commuting the Berezin transform past conjugation with the fundamental unitary. This will be relevant when estimating the $L_{t,q}^{\max}$-operator norm of the Berezin transform. An important point of the following lemma is that we are able to control the error term by means of a continuous function in t and q. For the statement, we recall that $\gamma := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Lemma 6.3.9. *Let $K \in \mathbb{N}_0$. There exists a continuous, positive function $g_K : (0, 1] \times (0, 1] \to (0, \infty)$ such that*

$$\|\beta_N^M(u\gamma x u^*) - u\beta_N^M(\gamma x)u^*\| \leqslant g_K(t, q) \cdot d_{t,q}^{\max}(\chi_N^M|_{B_q^K}, \epsilon|_{B_q^K}) \cdot L_{t,q}^{\max}(x)$$

for all $N, M \in \mathbb{N}_0$, all $t, q \in (0, 1]$ and all $x \in B_q^K \cap \ker(\Pi_0^L)$.

Proof. Without loss of generality we may focus on the case where $x \in \mathrm{Lip}_t(\mathrm{SU}_q(2)) \cap B_q^K \cap \ker(\Pi_0^L)$, since the right-hand side is otherwise equal to infinity. An application of Proposition 6.3.6 shows that the following inequalities hold for all $N, M \in \mathbb{N}_0$ and all $t, q \in (0,1]$; notice in this respect that $u_{ij} x u_{kj}^* \in B_q^K$ for all $i, j, k \in \{0,1\}$:

$$\begin{aligned} & \left\| \beta_N^M(u\gamma x u^*) - u\beta_N^M(\gamma x)u^* \right\| \\ & \leqslant \sum_{i,j,k=0}^{1} \left\| \beta_N^M(u_{ij} x u_{kj}^*) - u_{ij}\beta_N^M(x)u_{kj}^* \right\| \\ & \leqslant \sum_{i,j,k=0}^{1} \left\| \beta_N^M(u_{ij} x u_{kj}^*) - u_{ij} x u_{kj}^* \right\| + \sum_{i,j,k=0}^{1} \left\| u_{ij}(x - \beta_N^M(x))u_{kj}^* \right\| \\ & \leqslant d_{t,q}^{\max}\big(\chi_N^M|_{B_q^K}, \epsilon|_{B_q^K}\big) \cdot \sum_{i,j,k=0}^{1} \big(L_{t,q}^{\max}(u_{ij} x u_{kj}^*) + L_{t,q}^{\max}(x)\big). \end{aligned}$$

Applying Lemmas 4.3.7 and 6.3.7 we estimate that

$$\begin{aligned} L_{t,q}^{\max}(u_{ij} x u_{kj}^*) & \leqslant L_{t,q}^{\max}(u_{ij}) \cdot \|x\|_{t,q} \cdot \|u_{kj}\|_{t,q} + \|u_{ij}\|_{t,q} \cdot L_{t,q}^{\max}(x) \cdot \|u_{kj}\|_{t,q} \\ & \quad + \|u_{ij}\|_{t,q} \cdot \|x\|_{t,q} \cdot L_{t,q}^{\max}(u_{kj}) \\ & \leqslant 2\big([1/2]_t + q^{-\frac{1}{2}}\big)(q^{-\frac{1}{2}} + t^{-\frac{1}{2}}) \cdot \|x\|_{t,q} + (q^{-\frac{1}{2}} + t^{-\frac{1}{2}})^2 \cdot L_{t,q}^{\max}(x). \end{aligned}$$

The result of the lemma is now a consequence of Lemmas 3.6.5 and 6.3.8: indeed, we have that

$$\begin{aligned} \|x\|_{t,q} & \leqslant \sum_{m=-K}^{K} (t^{\frac{m}{2}} + q^{\frac{m}{2}}) \cdot \|x\| \\ & \leqslant \sum_{m=-K}^{K} (t^{\frac{m}{2}} + q^{\frac{m}{2}}) \cdot \frac{\pi \cdot (t^{1/2} + t^{-1/2})}{\sqrt{3}} \cdot L_{t,q}^{\max}(x). \end{aligned}$$ ■

Proposition 6.3.10. *Let $K \in \mathbb{N}_0$. Then, there exists a continuous positive function $h_K \colon (0,1] \times (0,1] \to (0,\infty)$ satisfying that*

(1) *$h_K(q,q) = 0$ for all $q \in (0,1]$ and;*

(2) *the following estimate holds*

$$L_{t,q}^{\max}\big(\beta_N^M(x)\big) \leqslant \big(1 + h_K(t,q) \cdot d_{t,q}^{\max}\big(\chi_N^M|_{B_q^K}, \epsilon|_{B_q^K}\big)\big) \cdot L_{t,q}^{\max}(x)$$

for all $N, M \in \mathbb{N}_0$, all $t, q \in (0,1]$ and all $x \in \mathrm{Lip}_t(\mathrm{SU}_q(2)) \cap B_q^K$.

Proof. We start out by choosing the continuous positive function $g_K \colon (0,1] \times (0,1] \to (0,\infty)$ according to Lemma 6.3.9. We then define the continuous positive function $h_K \colon (0,1] \times (0,1] \to (0,\infty)$ by putting

$$h_K(t,q) := 2 \cdot \sum_{m=1}^{K} \big| [m/2]_t - [m/2]_q \big| \cdot g_K(t,q)$$

and note that $h_K(q,q) = 0$ for all $q \in (0,1]$ as desired.

Let $N, M \in \mathbb{N}_0$ and $t, q \in (0,1]$ be given. Let moreover $x \in \mathrm{Lip}_t(\mathrm{SU}_q(2)) \cap B_q^K$, and remark that by (6.13), $x \in \mathrm{Lip}_q(\mathrm{SU}_q(2)) \cap B_q^K$ as well. We define the element $y = \partial_t^3(x) - \partial_q^3(x)$ and notice that $y \in B_q^K \cap \ker(\Pi_0^L)$ by Lemma 5.3.2. We moreover emphasise the identities

$$\partial_{t,q}(x) - \partial_{q,q}(x) = \partial_t^V(x) - \partial_q^V(x) = \gamma y, \quad \text{where } \gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Using Propositions 4.7.4 and 6.3.4 we now compute as follows:

$$\begin{aligned} & u \cdot \partial_{t,q}\big(\beta_N^M(x)\big) \cdot u^* \\ & = u \cdot (\partial_t^V - \partial_q^V)\big(\beta_N^M(x)\big) \cdot u^* + \delta\big(\beta_N^M(x)\big) \\ & = u \cdot \beta_N^M\big((\partial_t^V - \partial_q^V)(x)\big) \cdot u^* + \widetilde{\beta}_N^M\big(\delta(x)\big) \\ & = u \cdot \beta_N^M(\gamma y) \cdot u^* - \beta_N^M(u\gamma y u^*) + \beta_N^M\big(u \cdot (\partial_{t,q} - \partial_{q,q})(x) \cdot u^*\big) + \widetilde{\beta}_N^M\big(\delta(x)\big) \\ & = u \cdot \beta_N^M(\gamma y) \cdot u^* - \beta_N^M(u\gamma y u^*) + \widetilde{\beta}_N^M\big(u \cdot \partial_{t,q}(x) \cdot u^*\big). \end{aligned}$$

Combining the above computation with Lemma 6.3.9, recalling that $\widetilde{\beta}_N^M$ is a complete contraction by Lemma 6.2.10, we obtain that

$$\begin{aligned} L_{t,q}^{\max}\big(\beta_N^M(x)\big) & \leqslant \big\| u \cdot \beta_N^M(\gamma y) \cdot u^* - \beta_N^M(u\gamma y u^*)\big\| + L_{t,q}^{\max}(x) \\ & \leqslant g_K(t,q) \cdot d_{t,q}^{\max}\big(\chi_N^M|_{B_q^K}, \epsilon|_{B_q^K}\big) \cdot L_{t,q}^{\max}(y) + L_{t,q}^{\max}(x). \end{aligned}$$

The result of the present proposition follows from the equality $y = \sum_{m=-K}^{K}([m/2]_t - [m/2]_q) \cdot \Pi_m^L(x)$ so that

$$\begin{aligned} L_{t,q}^{\max}(y) & \leqslant \sum_{m=-K}^{K} \big|[m/2]_t - [m/2]_q\big| \cdot L_{t,q}^{\max}(\Pi_m^L(x)) \\ & \leqslant 2 \sum_{m=1}^{K} \big|[m/2]_t - [m/2]_q\big| \cdot L_{t,q}^{\max}(x), \end{aligned}$$

where the last inequality follows from Corollary 5.2.3. ∎

6.4 Approximation in the quantum Gromov–Hausdorff distance

As a result of the analysis carried out in this section, we shall see that the quantum Gromov–Hausdorff distance between the two compact quantum metric spaces $(C(\mathrm{SU}_q(2)), L_{t,q}^{\max})$ and $(C(\mathrm{SU}_q(2)), L_{t,q})$ is in fact equal to zero; cf. Corollary 6.4.2 below. When considering the quantum Gromov–Hausdorff convergence questions in Chapter 7, this result will allow us to work exclusively at the algebraic level, which will simplify matters significantly. We start out with a technical estimate, from which a number of our main results will follow.

Proposition 6.4.1. *Let $\delta \in (0,1)$. For every $\varepsilon > 0$ there exists a $K_0 \in \mathbb{N}_0$ and a constant $C \geqslant 0$ such that*

$$\operatorname{dist}_Q\big((\beta_N^M(B_q^{K_0}), L_{t,q}); (C(\mathrm{SU}_q(2)), L_{t,q}^{\max})\big) \leqslant d_{t,q}^{\max}\big(\chi_N^M|_{B_q^{K_0}}, \epsilon|_{B_q^{K_0}}\big) \cdot C + \varepsilon$$

for all $N, M \in \mathbb{N}_0$ and all $t, q \in [\delta, 1]$. Moreover, if $X \subseteq C(\mathrm{SU}_q(2))$ is a sub-operator system such that $\mathrm{Dom}(L_{t,q}^{\max}) \cap X$ is norm-dense in X and $\beta_N^M(B_q^{K_0}) \subseteq X$, then it holds that

$$\operatorname{dist}_Q\big((X, L_{t,q}^{\max}); (C(\mathrm{SU}_q(2)), L_{t,q}^{\max})\big) \leqslant d_{t,q}^{\max}\big(\chi_N^M|_{B_q^{K_0}}, \epsilon|_{B_q^{K_0}}\big) \cdot C + \varepsilon$$

for all $t, q \in [\delta, 1]$.

Proof. Let $\varepsilon > 0$ be given and choose $K_0 \in \mathbb{N}_0$ such that $\varepsilon(\delta, K_0) \leqslant \varepsilon$; see (5.12) for the definition of $\varepsilon(\delta, K)$ for $K \in \mathbb{N}_0$. For every $N, M \in \mathbb{N}_0$ we remark that the seminorms $L_{t,q}$ and $L_{t,q}^{\max}$ agree on the sub-operator system $\beta_N^M(B_q^{K_0}) \subseteq C(\mathrm{SU}_q(2))$. This is a consequence of Lemmas 6.2.2 and 6.2.4. By Proposition 6.3.10, we may choose a constant $C_0 \geqslant 0$ such that

$$L_{t,q}^{\max}\big(\beta_N^M(x)\big) \leqslant \big(1 + C_0 \cdot d_{t,q}^{\max}\big(\chi_N^M|_{B_q^{K_0}}, \epsilon|_{B_q^{K_0}}\big)\big) \cdot L_{t,q}^{\max}(x) \tag{6.15}$$

for all $N, M \in \mathbb{N}_0$, all $t, q \in [\delta, 1]$ and all $x \in \mathrm{Lip}_t(\mathrm{SU}_q(2)) \cap B_q^{K_0}$. Combining Proposition 5.6.4 with Remark 5.6.5 we may choose the constant $C \geqslant 0$ such that

$$C_0 \cdot \operatorname{diam}\big(C(\mathrm{SU}_q(2)), L_{t,q}^{\max}\big) + 1 \leqslant C \quad \text{for all } t, q \in [\delta, 1].$$

Let now $N, M \in \mathbb{N}_0$ and $t, q \in [\delta, 1]$ be given. Define the unital map $\Phi := \beta_N^M \circ E_{K_0}^L \colon C(\mathrm{SU}_q(2)) \to \beta_N^M(B_q^{K_0})$ and note that Φ is positive since $E_{K_0}^L = \mathbb{M}(\gamma_{K_0})$ is a unital contraction (Lemma 5.4.4) and β_N^M is positive by construction. We then obtain from Propositions 5.5.3, 6.3.6 and Lemma 5.5.2 that

$$\begin{aligned}
\|x - \Phi(x)\| &\leqslant \|x - E_{K_0}^L(x)\| + \|E_{K_0}^L(x) - \beta_N^M(E_{K_0}^L(x))\| \\
&\leqslant \varepsilon(\delta, K_0) \cdot L_{t,q}^{\max}(x) + d_{t,q}^{\max}\big(\chi_N^M|_{B_q^{K_0}}, \epsilon|_{B_q^{K_0}}\big) \cdot L_{t,q}^{\max}(E_{K_0}^L(x)) \\
&\leqslant \big(\varepsilon + d_{t,q}^{\max}\big(\chi_N^M|_{B_q^{K_0}}, \epsilon|_{B_q^{K_0}}\big)\big) \cdot L_{t,q}^{\max}(x)
\end{aligned}$$

for all $x \in \mathrm{Lip}_t(\mathrm{SU}_q(2))$. Another application of Lemma 5.5.2 together with (6.15) moreover shows that

$$L_{t,q}^{\max}(\Phi(x)) \leqslant \big(1 + C_0 \cdot d_{t,q}^{\max}\big(\chi_N^M|_{B_q^{K_0}}, \epsilon|_{B_q^{K_0}}\big)\big) \cdot L_{t,q}^{\max}(x)$$

for all $x \in \mathrm{Lip}_t(\mathrm{SU}_q(2))$.

Using Corollary 2.2.5 we then see that

$$\begin{aligned}
&\operatorname{dist}_Q\big((\beta_N^M(B_q^{K_0}), L_{t,q}); (C(\mathrm{SU}_q(2)), L_{t,q}^{\max})\big) \\
&\quad \leqslant d_{t,q}^{\max}\big(\chi_N^M|_{B_q^{K_0}}, \epsilon|_{B_q^{K_0}}\big) \cdot \big(C_0 \cdot \operatorname{diam}\big(C(\mathrm{SU}_q(2)), L_{t,q}^{\max}\big) + 1\big) + \varepsilon \\
&\quad \leqslant d_{t,q}^{\max}\big(\chi_N^M|_{B_q^{K_0}}, \epsilon|_{B_q^{K_0}}\big) \cdot C + \varepsilon.
\end{aligned}$$

This proves the first part of the present proposition. The second part of our proposition now follows from Remark 2.2.6. ■

Corollary 6.4.2. *Let $t, q \in (0,1]$. The metrics $d_{t,q}$ and $d_{t,q}^{\max}$ agree on the state space $\mathcal{S}(C(\mathrm{SU}_q(2)))$. In particular, it holds that*

$$\operatorname{dist}_Q\big((C(\mathrm{SU}_q(2)), L_{t,q}); (C(\mathrm{SU}_q(2)), L_{t,q}^{\max})\big) = 0.$$

Proof. Let $\mu, \nu \in \mathcal{S}(C(\mathrm{SU}_q(2)))$. We trivially have that $d_{t,q}(\mu,\nu) \leqslant d_{t,q}^{\max}(\mu,\nu)$, so we need to prove the opposite inequality.

For every $K, N, M \in \mathbb{N}_0$ we recall that the seminorms $L_{t,q}$ and $L_{t,q}^{\max}$ agree on the sub-operator system $\beta_N^M(B_q^K) \subseteq C(\mathrm{SU}_q(2))$ (see Lemmas 6.2.2 and 6.2.4). We thereby obtain that the two metrics $d_{t,q}$ and $d_{t,q}^{\max}$ agree on the state space $\mathcal{S}(\beta_N^M(B_q^K))$.

Let $\varepsilon > 0$ be given. Combining the proof of Proposition 6.4.1 with Corollary 2.2.7 we may choose a $K_0 \in \mathbb{N}_0$ and a constant $C \geqslant 0$ such that

$$d_{t,q}^{\max}(\mu,\nu) \leqslant d_{t,q}^{\max}\big(\chi_N^M|_{B_q^{K_0}}, \epsilon|_{B_q^{K_0}}\big) \cdot C + \varepsilon/2 + d_{t,q}(\mu,\nu)$$

for all $N, M \in \mathbb{N}_0$. Next, by Theorem 5.6.1, $d_{t,q}^{\max}$ metrises the weak* topology on $\mathcal{S}(C(\mathrm{SU}_q(2)))$ and by Lemma 6.1.3 it follows that $\lim_{N,M\to\infty} d_{t,q}^{\max}(\chi_N^M, \epsilon) = 0$. We may thus choose $N, M \in \mathbb{N}_0$ such that

$$C \cdot d_{t,q}^{\max}\big(\chi_N^M|_{B_q^{K_0}}, \epsilon|_{B_q^{K_0}}\big) \leqslant C \cdot d_{t,q}^{\max}(\chi_N^M, \epsilon) \leqslant \varepsilon/2.$$

Combining these two estimates we obtain that

$$d_{t,q}^{\max}(\mu,\nu) \leqslant \varepsilon + d_{t,q}(\mu,\nu).$$

Since $\varepsilon > 0$ was arbitrary we have proved that $d_{t,q}^{\max}(\mu,\nu) = d_{t,q}(\mu,\nu)$.

The fact that the quantum Gromov–Hausdorff distance between $(C(\mathrm{SU}_q(2)), L_{t,q}^{\max})$ and $(C(\mathrm{SU}_q(2)), L_{t,q})$ is equal to zero now follows from [70, Corollary 6.4] (see also the discussion near Theorem 2.2.3). ■

We also record a corollary which is an analogue to Corollary 6.4.2 for the spectral bands. Since the proof is similar but easier than the proof of Corollary 6.4.2 we are leaving it out.

Corollary 6.4.3. *Let $K \in \mathbb{N}_0$ and let $t, q \in (0,1]$. The metrics $d_{t,q}$ and $d^{\max}_{t,q}$ agree on the state space $\mathcal{S}(B^K_q)$. In particular, it holds that*

$$\mathrm{dist}_{\mathrm{Q}}\big((B^K_q, L^{\max}_{t,q}|_{B^K_q}); (B^K_q, L_{t,q}|_{B^K_q})\big) = 0.$$

Lastly, we single out the following consequence of Proposition 6.4.1, which shows that our fuzzy approximations do indeed approximate quantum SU(2) in the quantum Gromov–Hausdorff distance.

Corollary 6.4.4. *Let $t, q \in (0,1]$. It holds that*

$$\lim_{N,K\to\infty} \mathrm{dist}_{\mathrm{Q}}\big((\mathrm{Fuzz}_N(B^K_q), L_{t,q}); (C(\mathrm{SU}_q(2)), L^{\max}_{t,q})\big) = 0.$$

Proof. Let $\varepsilon > 0$ be given. By Proposition 6.4.1, there exist a $K_0 \in \mathbb{N}_0$ and a constant $C \geqslant 0$ such that

$$\mathrm{dist}_{\mathrm{Q}}\big((\beta^M_N(B^{K_0}_q), L_{t,q}); (C(\mathrm{SU}_q(2)), L^{\max}_{t,q})\big) \leqslant C \cdot d^{\max}_{t,q}\big(\chi^M_N|_{B^{K_0}_q}, \epsilon|_{B^{K_0}_q}\big) + \varepsilon/2$$

for all $N, M \in \mathbb{N}_0$. By Theorem 5.6.1 and Lemma 6.1.3 we may choose $N_0, M_0 \in \mathbb{N}_0$ with $M_0 \geqslant K_0$ such that

$$C \cdot d^{\max}_{t,q}\big(\chi^M_N|_{B^{K_0}_q}, \epsilon|_{B^{K_0}_q}\big) \leqslant C \cdot d^{\max}_{t,q}(\chi^M_N, \epsilon) < \varepsilon/2 \quad \text{for all } N \geqslant N_0 \text{ and } M \geqslant M_0.$$

For $N \geqslant N_0 + M_0$ and $K \geqslant K_0$ we obtain from Proposition 6.2.6 that

$$\beta^{M_0}_{N_0}(B^{K_0}_q) \subseteq \mathrm{Fuzz}_{N_0+M_0}(B^{K_0}_q) \subseteq \mathrm{Fuzz}_N(B^{K_0}_q) \subseteq \mathrm{Fuzz}_N(B^K_q),$$

and the last part of Proposition 6.4.1 therefore shows that

$$\mathrm{dist}_{\mathrm{Q}}\big((\mathrm{Fuzz}_N(B^K_q), L_{t,q}); (C(\mathrm{SU}_q(2)), L^{\max}_{t,q})\big) < \varepsilon,$$

for all $N \geqslant N_0 + M_0$ and all $K \geqslant K_0$. ∎

Remark 6.4.5. The case where $t = q = 1$ is of particular interest since $C(\mathrm{SU}_1(2)) = C(\mathrm{SU}(2))$ and the Lip-norm $L^{\max}_{1,1}$ computes the Lipschitz constant arising from twice the round metric d_{S^3} on $\mathrm{SU}(2) \cong S^3 \subseteq \mathbb{R}^4$; see Section 4.4 for details. Corollary 6.4.4 therefore provides a finite-dimensional approximation of $C(S^3)$ by subspaces invariant under the SU(2)-action (see Proposition 6.2.9). This yields an S^3-analogue of Rieffel's original result [71, Theorem 3.2] for the 2-sphere.