Chapter 7

Continuity results

In this chapter we embark on our final goal of the memoir, which is to prove that the family of compact quantum metric spaces $(C(\mathrm{SU}_q(2)), L_{t,q})_{t,q\in(0,1]}$ varies continuously in the quantum Gromov–Hausdorff distance; see Theorem D. The result in Corollary 6.4.2 shows that we may choose to work exclusively with the Lip-norm $L_{t,q}$, meaning that the domain equals the coordinate algebra $\mathcal{O}(\mathrm{SU}_q(2))$. Indeed, the corresponding continuity result for the Lip-norm $L^{\max}_{t,q}$ with domain equal to the Lipschitz algebra $\mathrm{Lip}_t(\mathrm{SU}_q(2))$ follows automatically. In effect, this allows us to circumvent a lot of analysis and work at a purely (Hopf-)algebraic level. We begin by providing a rough outline of the mains steps in the proof of continuity at a point $(t_0, q_0) \in (0, 1] \times (0, 1]$:

(1) We fine tune the result in Corollary 6.4.4 by showing that locally around (t_0, q_0) the fuzzy approximations approach quantum SU(2) in a uniform manner.

(2) Utilising the finite dimensionality of the fuzzy approximation we show that these vary continuously.

(3) Piecing together these approximation results, we arrive at the main continuity statement in Theorem 7.3.1 below.

7.1 Continuity of the fuzzy approximations

We begin by addressing point (2) in the above list.

Proposition 7.1.1. *Let $K, N \in \mathbb{N}_0$. The 2-parameter family of compact quantum metric spaces $(\mathrm{Fuzz}_N(B_q^K), L_{t,q})_{t,q\in(0,1]}$ varies continuously in the quantum Gromov–Hausdorff distance.*

Proof. Fix a $\delta \in (0, 1)$. We aim to apply [70, Theorem 11.2], and must therefore provide a fixed finite dimensional real vector space V with a distinguished vector e, a continuous family $(\|\cdot\|_{t,q})_{t,q\in[\delta,1]}$ of norms and a continuous family $(M_{t,q})_{t,q\in[\delta,1]}$ of seminorms such that $(V, e, \|\cdot\|_{t,q}, M_{t,q})$ is an order unit compact quantum metric space isomorphic to

$$\big(\mathrm{Fuzz}_N(B_q^K)_{\mathrm{sa}}, 1, \|\cdot\|, L_{t,q}\big) \quad \text{for all } t, q \in [\delta, 1].$$

We are going to apply the unital continuous field of C^*-algebras over $[\delta, 1]$ with total space $C(\mathrm{SU}_\bullet(2))$ and with fibre $C(\mathrm{SU}_q(2))$ for every $q \in [\delta, 1]$ which was introduced in Section 3.7. For each $q \in [\delta, 1]$, we recall that $\mathrm{ev}_q \colon C(\mathrm{SU}_\bullet(2)) \to C(\mathrm{SU}_q(2))$ denotes the unital $*$-homomorphism which evaluates at the point q.

We recall that $\mathcal{O}(\mathrm{SU}_\bullet(2)) \subseteq C(\mathrm{SU}_\bullet(2))$ denotes the smallest unital $*$-subalgebra containing $C([\delta, 1])$ and the generators $a_\bullet$ and $b_\bullet$. Notice also that $\mathcal{O}(\mathrm{SU}_\bullet(2))$ is a free $C([\delta, 1])$-module with basis given by the elements

$$\xi_\bullet^{klm} := \begin{cases} a_\bullet^k b_\bullet^l (b_\bullet^*)^m & k \geq 0 \\ b_\bullet^l (b_\bullet^*)^m (a_\bullet^*)^{-k} & k < 0, \end{cases}$$

for $k \in \mathbb{Z}$ and $l, m \in \mathbb{N}_0$.

For each $q \in [\delta, 1]$ we obtain a linear basis for the coordinate algebra $\mathcal{O}(\mathrm{SU}_q(2))$ by applying the evaluation map to the linearly independent subset $\{\xi_\bullet^{klm} \mid (k, l, m) \in \mathbb{Z} \times \mathbb{N}_0 \times \mathbb{N}_0\} \subseteq C(\mathrm{SU}_\bullet(2))$. In particular, we obtain that

$$\mathrm{ev}_q \colon \mathrm{span}_{\mathbb{C}}\big\{\xi_\bullet^{klm} \bigm| (k, l, m) \in \mathbb{Z} \times \mathbb{N}_0 \times \mathbb{N}_0\big\} \to \mathcal{O}(\mathrm{SU}_q(2))$$

is an isomorphism of vector spaces over $\mathbb{C}$. By an application of Corollary 6.2.8, we may choose a finite subset $J \subseteq \mathbb{Z} \times \mathbb{N}_0 \times \mathbb{N}_0$ satisfying that

$$\mathrm{ev}_q\big(\mathrm{span}_{\mathbb{C}}\big\{\xi_\bullet^{klm} \bigm| (k, l, m) \in J\big\}\big) = \mathrm{Fuzz}_N(B_q^K)$$

for all $q \in [\delta, 1]$. We apply the notation

$$W := \mathrm{span}_{\mathbb{C}}\big\{\xi_\bullet^{klm} \bigm| (k, l, m) \in J\big\} \subseteq C(\mathrm{SU}_\bullet(2))$$

and record that W becomes a finite-dimensional operator system (indeed, it holds that $\xi^* \in W$ whenever $\xi \in W$ and clearly $1 \in W$ as well). We put $V := W_{\mathrm{sa}}$ and record that the isomorphism

$$\mathrm{ev}_q \colon W \to \mathrm{Fuzz}_N(B_q^K)$$

induces an isomorphism of real vector spaces $\mathrm{ev}_q \colon V \to \mathrm{Fuzz}_N(B_q^K)_{\mathrm{sa}}$ for all $q \in [\delta, 1]$. For each $t, q \in [\delta, 1]$ we equip V with the unique order unit space structure such that $\mathrm{ev}_q \colon V \to \mathrm{Fuzz}_N(B_q^K)_{\mathrm{sa}}$ becomes an isomorphism of order unit spaces. We emphasise that this order unit space structure does not depend on the parameter $t \in [\delta, 1]$. Moreover, we may introduce the seminorm

$$M_{t,q} \colon V \to [0, \infty) \quad M_{t,q}(x_\bullet) := L_{t,q}\big(\mathrm{ev}_q(x_\bullet)\big).$$

In this fashion, we get that $(V, M_{t,q})$ becomes an order unit compact quantum metric space which is isometrically isomorphic to the order unit compact quantum metric space $(\mathrm{Fuzz}_N(B_q^K)_{\mathrm{sa}}, L_{t,q})$. We remark that the different order unit space structures on V yields a family of norms $(\| \cdot \|_{t,q})_{t,q \in [\delta,1]}$ on V. This family becomes continuous since we are dealing with a continuous field of C^*-algebras with total space $C(\mathrm{SU}_\bullet(2))$. Indeed, for each $t, q \in [\delta, 1]$ we record that $\|x_\bullet\|_{t,q} = \|\mathrm{ev}_q(x_\bullet)\|$. We therefore only need to show that the family of seminorms $(M_{t,q})_{t,q\in[\delta,1]}$ is continuous as well.

It then follows from the discussion in Section 3.7 that we have two $C([\delta,1])$-linear maps

$$\partial^1_\bullet \text{ and } \partial^2_\bullet\colon \mathcal{O}(\mathrm{SU}_\bullet(2)) \to \mathcal{O}(\mathrm{SU}_\bullet(2))$$

satisfying that $\mathrm{ev}_q \circ \partial^1_\bullet = \partial^1 \circ \mathrm{ev}_q$ and $\mathrm{ev}_q \circ \partial^2_\bullet = \partial^2 \circ \mathrm{ev}_q$. Moreover, for each $t \in (0,1]$ we may define the $C([\delta,1])$-linear map

$$\partial^3_{t,\bullet}\colon \mathcal{O}(\mathrm{SU}_\bullet(2)) \to \mathcal{O}(\mathrm{SU}_\bullet(2)) \quad \partial^3_{t,\bullet}(\xi^{klm}_\bullet) := \big[(k+l-m)/2\big]_t \cdot \xi^{klm}_\bullet.$$

By construction we obtain $\mathrm{ev}_q \circ \partial^3_{t,\bullet} = \partial^3_t \circ \mathrm{ev}_q$. Moreover, for each $x_\bullet \in \mathcal{O}(\mathrm{SU}_\bullet(2))$, we note that the map $(0,1] \to C(\mathrm{SU}_\bullet(2))$ defined by $t \mapsto \partial^3_{t,\bullet}(x_\bullet)$ is continuous with respect to the C^*-norm on $C(\mathrm{SU}_\bullet(2))$. For each $t \in (0,1]$, we may thus consider the $C([\delta,1])$-linear map

$$\partial_{t,\bullet}\colon \mathcal{O}(\mathrm{SU}_\bullet(2)) \to \mathbb{M}_2\big(C(\mathrm{SU}_\bullet(2))\big) \quad \partial_{t,\bullet} := \begin{pmatrix} \partial^3_{t,\bullet} & -\partial^2_\bullet \\ -\partial^1_\bullet & -\partial^3_{t,\bullet} \end{pmatrix}.$$

We notice that $\mathbb{M}_2(C(\mathrm{SU}_\bullet(2)))$ is again the total space of a continuous field of C^*-algebras over $[\delta,1]$, this time with fibres $\mathbb{M}_2(C(\mathrm{SU}_q(2)))$ for $q \in [\delta,1]$. For each $t,q \in [\delta,1]$ we moreover have that

$$M_{t,q}(x_\bullet) = \big\|\partial_{t,q}(\mathrm{ev}_q(x_\bullet))\big\| = \big\|\mathrm{ev}_q\big(\partial_{t,\bullet}(x_\bullet)\big)\big\|$$

for every $x_\bullet \in V$. From these observations we obtain that $(M_{t,q})_{t,q\in[\delta,1]}$ is a continuous family of seminorms on V.

The assumptions in [70, Theorem 11.2] are thereby fulfilled, and since $\delta \in (0,1)$ was arbitrary this implies the claimed continuity result. ■

In the following subsection we address point (1) in the road map provided in the beginning of this chapter. This is the main technical step in the proof of Theorem D. The uniform fuzzy approximation which we are going to establish builds on a combination of the approximation results described in Chapter 6 and the continuity results obtained earlier for the Podleś sphere in [3, 25].

7.2 Uniformity of the fuzzy approximation

The core result of this section provides a uniform estimate on the Monge–Kantorovič distance between the states χ^M_N, $N, M \in \mathbb{N}_0$, and the counit ϵ. This estimate takes place on a fixed spectral band and the main part of the upper bound is given in terms of the Monge–Kantorovič distance between states on the Podleś sphere. One of the relevant states is the restriction of the counit while the remaining states on S_q^2 are all of the following form:

$$h_j\colon C(S_q^2) \to \mathbb{C} \quad h_j(x) := \langle j+1\rangle_q \cdot h\big((a^*)^j x a^j\big) \quad j \in \mathbb{N}_0. \tag{7.1}$$

We emphasise that the state h_j is the restriction of the state χ_j^0 (see (6.1)) to the Podleś sphere $C(S_q^2) \subseteq C(\mathrm{SU}_q(2))$. We are interested in the algebraic versions of the Monge–Kantorovič metrics on quantum SU(2) and the Podleś sphere defined by

$$d_{t,q}(\mu,\nu) := \sup\big\{|\mu(x)-\nu(x)| \,\big|\, x \in \mathcal{O}(\mathrm{SU}_q(2)), L_{t,q}(x) \leqslant 1\big\} \quad \text{for } \mu,\nu \in \mathcal{S}(C(\mathrm{SU}_q(2)))$$

and

$$d_q^0(\mu,\nu) := \sup\big\{|\mu(x)-\nu(x)| \,\big|\, x \in \mathcal{O}(S_q^2), L_q^0(x) \leqslant 1\big\} \quad \text{for } \mu,\nu \in \mathcal{S}(C(S_q^2)).$$

We recall from Proposition 5.1.2 that the seminorm $L_q^0\colon \mathcal{O}(S_q^2) \to [0,\infty)$ agrees with the restriction of the seminorm $L_{t,q}\colon \mathcal{O}(\mathrm{SU}_q(2)) \to [0,\infty)$ to $\mathcal{O}(S_q^2)$ for all values of $t \in (0,1]$.

Lemma 7.2.1. *Let $m \in \mathbb{Z}$ and $t,q \in (0,1]$. For every $x \in \mathcal{A}_q^m$ it holds that*

$$L_q^0\big((a^*)^m x\big) \leqslant (t^{\frac{1}{2}} + t^{-\frac{1}{2}} + 1) L_{t,q}(x) \quad \textit{for } m \geqslant 0 \quad \textit{and}$$
$$L_q^0\big(x a^{-m}\big) \leqslant (t^{\frac{1}{2}} + t^{-\frac{1}{2}} + 1) L_{t,q}(x) \quad \textit{for } m \leqslant 0.$$

Proof. We focus on the case where $m \geqslant 0$ since the remaining case follows by taking adjoints. Suppose thus that $m \geqslant 0$ and let $x \in \mathcal{A}_q^m$. We know that $(a^*)^m x \in \mathcal{A}_q^0$ and an application of Proposition 5.1.2 shows that $L_q^0((a^*)^m x) = L_{t,q}((a^*)^m x)$. In particular, we immediately obtain the relevant inequality for $m = 0$. We may thus assume that $m > 0$. Since $(a^*)^m = u_{00}^m$, it follows from (3.9) that

$$\partial_e((a^*)^m) = 0 \quad \text{and} \quad \partial_f((a^*)^m) = u_{01}^m \cdot \sqrt{q^{1-m}\langle m\rangle_q}.$$

As a consequence of these identities, we get the estimate

$$\big\|\partial_q^H((a^*)^m)\big\| = \big\|q^{-\frac{1}{2}}\partial_f((a^*)^m)\big\| \leqslant \sqrt{q^{-m}\cdot\langle m\rangle_q} \leqslant \sqrt{m}\cdot q^{-\frac{m}{2}}.$$

We moreover notice that Lemmas 5.3.4 and 5.4.2 imply the inequalities

$$\|x\| \leqslant \frac{1}{[m/2]_t} \cdot L_{t,q}(x) \leqslant \frac{t^{1/2}+t^{-1/2}}{m} \cdot L_{t,q}(x). \tag{7.2}$$

Since $L_{t,q}((a^*)^m x) = L_q^0((a^*)^m x) = \|\partial_q^H((a^*)^m x)\|$, the result of the lemma now follows from Lemma 4.3.3 together with Lemma 3.6.3 and the estimate in (4.6):

$$\begin{aligned} L_{t,q}\big((a^*)^m x\big) &\leqslant \big\|\partial_q^H((a^*)^m)\big\| \cdot q^{m/2}\|x\| + q^{m/2}\big\|(a^*)^m\big\| \cdot \big\|\partial_q^H(x)\big\| \\ &\leqslant \frac{t^{1/2}+t^{-1/2}}{\sqrt{m}} \cdot L_{t,q}(x) + L_{t,q}(x) \leqslant (t^{\frac{1}{2}}+t^{-\frac{1}{2}}+1)\cdot L_{t,q}(x). \end{aligned}$$ ∎

Recall from Section 6.2 the linear functionals $\varphi_{r,s}\colon C(\mathrm{SU}_q(2)) \to \mathbb{C}$, $r, s \in \mathbb{N}_0$, given by

$$\varphi_{r,s}(x) = h\big((a^*)^s x a^r\big).$$

As noted in (6.3), for each $N, M \in \mathbb{N}_0$, the state χ_N^M appearing in the definition of the Berezin transform $\beta_N^M\colon C(\mathrm{SU}_q(2)) \to C(\mathrm{SU}_q(2))$ is then given by

$$\chi_N^M = \frac{1}{M+1} \sum_{s,r=N}^{N+M} \sqrt{\langle r+1\rangle_q \langle s+1\rangle_q} \cdot \varphi_{r,s}. \tag{7.3}$$

We first describe the linear functionals $\varphi_{r,s}$ in terms of the states h_j on the Podleś sphere introduced in (7.1).

Lemma 7.2.2. *Let $r, s \in \mathbb{N}_0$. For every $x \in C(\mathrm{SU}_q(2))$ it holds that*

$$\varphi_{r,s}(x) = \begin{cases} \frac{1}{\langle r+1\rangle_q} \cdot h_r\big((a^*)^{s-r} \cdot \Pi_{s-r}^L(x)\big) & s \geqslant r \\ \frac{1}{\langle s+1\rangle_q} \cdot h_s\big(\Pi_{s-r}^L(x) \cdot a^{r-s}\big) & r \geqslant s. \end{cases}$$

Proof. By continuity and linearity, we may assume that $x \in A_q^m$ for some $m \in \mathbb{Z}$. Since the Haar state $h\colon C(\mathrm{SU}_q(2)) \to \mathbb{C}$ vanishes on all but the zeroth spectral subspace and $(a^*)^s x a^r \in A_q^{m+r-s}$ we then have that $\varphi_{r,s}(x) \neq 0$ if and only if $m + r - s = 0$. Since $\Pi_{s-r}^L(x)$ also vanishes for $m + r - s \neq 0$, we may assume that $m = s - r$. For $m \geqslant 0$ we have that

$$\varphi_{r,s}(x) = h\big((a^*)^s x a^r\big) = h\big((a^*)^r (a^*)^m x a^r\big) = \frac{1}{\langle r+1\rangle_q} h_r\big((a^*)^m x\big).$$

Likewise, for $m \leqslant 0$ we get that

$$\varphi_{r,s}(x) = h\big((a^*)^s x a^r\big) = h\big((a^*)^s x a^{-m} a^s\big) = \frac{1}{\langle s+1\rangle_q} h_s(x a^{-m}).$$

This proves the present lemma. ∎

Inspired by Lemma 7.2.2, for each $m \in \mathbb{N}_0$, we now define the bounded operator $P_m\colon C(\mathrm{SU}_q(2)) \to C(S_q^2)$ by the formula

$$P_m(x) := \begin{cases} (a^*)^m \Pi_m^L(x) + \Pi_{-m}^L(x) a^m & m > 0 \\ \Pi_0^L(x) & m = 0. \end{cases}$$

Indeed, for every $s, r \in \mathbb{N}_0$ with $r < s$ we get from Lemma 7.2.2 that

$$\varphi_{s,r}(x) + \varphi_{r,s}(x) = \frac{1}{\langle r+1\rangle_q} \cdot h_r\big(P_{s-r}(x)\big) \quad \text{and} \quad \varphi_{r,r}(x) = \frac{1}{\langle r+1\rangle_q} \cdot h_r(P_0(x)), \tag{7.4}$$

for all $x \in C(\mathrm{SU}_q(2))$. Note also that $P_m(x^*) = P_m(x)^*$ since $\Pi_m(x^*) = \Pi_{-m}(x)^*$ for all $x \in C(\mathrm{SU}_q(2))$ and $m \in \mathbb{N}_0$. For each $N, M \in \mathbb{N}_0$ we may then express the state $\chi_N^M : C(\mathrm{SU}_q(2)) \to \mathbb{C}$ in terms of the bounded operators P_m, $m \in \mathbb{N}_0$, and the states $h_r : C(S_q^2) \to \mathbb{C}$, $r \in \mathbb{N}_0$:

Lemma 7.2.3. *Let $N, M \in \mathbb{N}_0$. For every $x \in C(\mathrm{SU}_q(2))$, it holds that*

$$\chi_N^M(x) = \frac{1}{M+1} \sum_{r=N}^{N+M} \sum_{m=0}^{N+M-r} \sqrt{\frac{\langle m+r+1\rangle_q}{\langle r+1\rangle_q}} \cdot h_r\big(P_m(x)\big).$$

Proof. Using (7.3) and (7.4), we obtain the desired result from the computation

$$\begin{aligned}(M+1)\cdot \chi_N^M &= \sum_{r=N}^{N+M} \langle r+1\rangle_q \cdot \varphi_{r,r} \\ &\quad + \sum_{r=N}^{N+M} \sum_{s=r+1}^{N+M} \sqrt{\langle r+1\rangle_q \langle s+1\rangle_q} \cdot (\varphi_{s,r} + \varphi_{r,s}) \\ &= \sum_{r=N}^{N+M} \sum_{s=r}^{N+M} \sqrt{\frac{\langle s+1\rangle_q}{\langle r+1\rangle_q}} \cdot (h_r \circ P_{s-r}).\end{aligned}$$

■

In order to estimate the distance between the counit ϵ and the state χ_N^M for different values of $N, M \in \mathbb{N}_0$ we introduce the linear functional $\psi_N^M : C(\mathrm{SU}_q(2)) \to \mathbb{C}$ defined by

$$\psi_N^M(x) = \frac{1}{M+1} \sum_{r=N}^{N+M} \sum_{m=0}^{N+M-r} \sqrt{\frac{\langle m+r+1\rangle_q}{\langle r+1\rangle_q}} \cdot \epsilon\big(P_m(x)\big)$$

for all $x \in C(\mathrm{SU}_q(2))$. The next two lemmas serve as preparation for Proposition 7.2.6, where we provide a uniform upper bound on the Monge–Kantorovič distance between the states ϵ and χ_N^M on a fixed spectral band.

Lemma 7.2.4. *Let $n \in \mathbb{Z}$. It holds that*

$$\big|\psi_N^M(x) - \epsilon(x)\big| \leqslant \left(\frac{1}{N+1} + \frac{1}{M+1}\right) \cdot (t^{\frac{1}{2}} + t^{-\frac{1}{2}}) \cdot L_{t,q}(x) \qquad (7.5)$$

for all $t, q \in (0,1]$, all $x \in \mathcal{A}_q^n$ and all $N, M \in \mathbb{N}_0$ with $M \geqslant |n|$.

Proof. Since ψ_N^M and ϵ respect the adjoint operation and $L_{t,q}$ is $*$-invariant, it suffices to treat the case $n \geqslant 0$. For $n = 0$, the estimate in (7.5) clearly holds since the left-hand side of the inequality is equal to zero. Suppose therefore that $n > 0$, and let

$q \in (0,1]$ be given. For each $r \in \mathbb{N}_0$ we start out by remarking that

$$\left| \frac{\sqrt{\langle n+r+1\rangle_q}}{\sqrt{\langle r+1\rangle_q}} - 1 \right| \leqslant \frac{\langle n+r+1\rangle_q}{\langle r+1\rangle_q} - 1 = q^{2(r+1)} \frac{\langle n\rangle_q}{\langle r+1\rangle_q} \leqslant n \cdot q^2 \frac{1}{\sum_{i=0}^r q^{-2i}} \leqslant \frac{n}{r+1}. \tag{7.6}$$

Fix now $N, M \in \mathbb{N}_0$ with $M \geqslant n$. From the above inequalities we obtain that

$$\begin{aligned} &\left| 1 - \frac{1}{M+1} \sum_{r=N}^{N+M-n} \frac{\sqrt{\langle n+r+1\rangle_q}}{\sqrt{\langle r+1\rangle_q}} \right| \\ &\leqslant \left| 1 - \frac{M+1-n}{M+1} \right| + \frac{1}{M+1} \sum_{r=N}^{N+M-n} \left| \frac{\sqrt{\langle n+r+1\rangle_q}}{\sqrt{\langle r+1\rangle_q}} - 1 \right| \\ &\leqslant \frac{n}{M+1} + \frac{n}{N+1}. \end{aligned}$$

Let furthermore $x \in \mathcal{A}_q^n$ be given. Since $\epsilon(a^*) = 1 = \epsilon(a)$ (and since ϵ is a unital $*$-homomorphism) we know that

$$\epsilon(P_m(x)) = \delta_{n,m} \cdot \epsilon(x) \quad \text{for all } m \in \mathbb{N}_0.$$

From this identity, we then get that

$$\begin{aligned} \psi_N^M(x) &= \frac{1}{M+1} \sum_{r=N}^{N+M} \sum_{m=0}^{N+M-r} \frac{\sqrt{\langle m+r+1\rangle_q}}{\sqrt{\langle r+1\rangle_q}} \cdot \epsilon(P_m(x)) \\ &= \frac{1}{M+1} \sum_{r=N}^{N+M-n} \frac{\sqrt{\langle n+r+1\rangle_q}}{\sqrt{\langle r+1\rangle_q}} \cdot \epsilon(x). \end{aligned}$$

Combining the above estimates we get

$$\begin{aligned} \left| \psi_N^M(x) - \epsilon(x) \right| &= \left| \left(1 - \frac{1}{M+1} \sum_{r=N}^{N+M-n} \frac{\sqrt{\langle n+r+1\rangle_q}}{\sqrt{\langle r+1\rangle_q}} \right) \epsilon(x) \right| \\ &\leqslant \left(\frac{n}{M+1} + \frac{n}{N+1} \right) |\epsilon(x)|. \end{aligned}$$

Let finally $t \in (0,1]$ be given. The result of the lemma then follows from the above computations together with the estimate

$$|\epsilon(x)| \leqslant \|x\| \leqslant \frac{1}{[n/2]_t} L_{t,q}(x) \leqslant \frac{t^{1/2} + t^{-1/2}}{n} L_{t,q}(x),$$

see Lemmas 5.3.4 and 5.4.2. ∎

Lemma 7.2.5. *Let $n \in \mathbb{Z}$. The following inequality holds*

$$\begin{aligned}&\left|\chi_N^M(x) - \psi_N^M(x)\right| \\ &\quad \leqslant \left(1 + \frac{|n|}{N+1}\right)^{\frac{1}{2}} \cdot (t^{\frac{1}{2}} + t^{-\frac{1}{2}} + 1) \cdot \sup_{N \leqslant r \leqslant N+M} d_q^0\big(h_r, \epsilon|_{C(S_q^2)}\big) \cdot L_{t,q}(x)\end{aligned}$$

for all $t, q \in (0,1]$, all $x \in \mathcal{A}_q^n$ and all $N, M \in \mathbb{N}_0$ with $M \geqslant |n|$.

Proof. Let $t, q \in (0,1]$, $x \in \mathcal{A}_q^n$ and $N, M \in \mathbb{N}_0$ with $M \geqslant |n|$ be given. Since ψ_N^M and χ_N^M preserve the adjoint operation and $L_{t,q}$ is $*$-invariant we may, without loss of generality, assume that $n \geqslant 0$. As in (7.6) we have that

$$\frac{\langle n + r + 1\rangle_q}{\langle r + 1\rangle_q} \leqslant 1 + \frac{n}{r+1} \quad \text{for all } r \in \mathbb{N}_0.$$

Remark moreover that $P_m(x) = \delta_{n,m} \cdot (a^*)^n x$ for all $m \in \mathbb{N}_0$. An application of these observations together with Lemma 7.2.3 yield the following inequalities:

$$\begin{aligned}\left|\chi_N^M(x) - \psi_N^M(x)\right| &= \frac{1}{M+1}\left|\sum_{r=N}^{N+M-n} \frac{\sqrt{\langle n+r+1\rangle_q}}{\sqrt{\langle r+1\rangle_q}} \cdot \big(h_r(P_n(x)) - \epsilon(P_n(x))\big)\right| \\ &\leqslant \frac{1}{M+1}\sum_{r=N}^{N+M-n}\left(1 + \frac{n}{r+1}\right)^{\frac{1}{2}} d_q^0\big(h_r, \epsilon|_{C(S_q^2)}\big) \cdot L_q^0(P_n(x)) \\ &\leqslant \left(1 + \frac{n}{N+1}\right)^{\frac{1}{2}} \cdot \sup_{N \leqslant r \leqslant N+M} d_q^0\big(h_r, \epsilon|_{C(S_q^2)}\big) \cdot L_q^0(P_n(x)).\end{aligned}$$

The result of the present lemma now follows by noting that Lemma 7.2.1 entails the inequality $L_q^0(P_n(x)) \leqslant (t^{\frac{1}{2}} + t^{-\frac{1}{2}} + 1) \cdot L_{t,q}(x)$. ∎

Proposition 7.2.6. *Let $K \in \mathbb{N}_0$ and $\delta \in (0,1)$. There exist a constant $C > 0$ and a positive null sequence $(\varepsilon_{N,M})_{N,M=0}^{\infty}$ such that*

$$d_{t,q}\big(\chi_N^M|_{B_q^K}, \epsilon|_{B_q^K}\big) \leqslant C \cdot \sup_{N \leqslant r \leqslant N+M} d_q^0\big(h_r, \epsilon|_{C(S_q^2)}\big) + \varepsilon_{N,M}$$

for all $(t,q) \in [\delta, 1] \times (0,1]$ and all $N, M \in \mathbb{N}_0$ with $M \geqslant K$.

Proof. We define the constant $C > 0$ by putting

$$C := (2K+1) \cdot (1+K)^{\frac{1}{2}} \cdot (\delta^{\frac{1}{2}} + \delta^{-\frac{1}{2}} + 1),$$

and the null sequence $(\varepsilon_{N,M})_{N,M=0}^{\infty}$ by putting

$$\varepsilon_{N,M} := (2K+1) \cdot \left(\frac{1}{N+1} + \frac{1}{M+1}\right) \cdot (\delta^{\frac{1}{2}} + \delta^{-\frac{1}{2}})$$

for all $N, M \in \mathbb{N}_0$.

Let now $(t, q) \in [\delta, 1] \times (0, 1]$ be given and let $x \in \mathcal{B}_q^K$ satisfy that $L_{t,q}(x) \leqslant 1$. For every $N, M \in \mathbb{N}_0$ with $M \geqslant K$, an application of Lemmas 7.2.4 and 7.2.5 then shows that

$$\begin{aligned} &\left|\chi_N^M(x) - \epsilon(x)\right| \\ &\quad\leqslant \left|\chi_N^M(x) - \psi_N^M(x)\right| + \left|\psi_N^M(x) - \epsilon(x)\right| \\ &\quad\leqslant \sum_{n=-K}^{K} \left|\chi_N^M(\Pi_n^L(x)) - \psi_N^M(\Pi_n^L(x))\right| + \sum_{n=-K}^{K} \left|\psi_N^M(\Pi_n^L(x)) - \epsilon(\Pi_n^L(x))\right| \\ &\quad\leqslant (2K+1)\left(1 + \frac{K}{N+1}\right)^{\frac{1}{2}} (t^{\frac{1}{2}} + t^{-\frac{1}{2}} + 1) \\ &\qquad \cdot \sup_{N \leqslant r \leqslant N+M} d_q^0\big(h_r, \epsilon|_{C(S_q^2)}\big) \cdot L_{t,q}(\Pi_n^L(x)) \\ &\qquad + (2K+1) \cdot \left(\frac{1}{N+1} + \frac{1}{M+1}\right) \cdot (t^{\frac{1}{2}} + t^{-\frac{1}{2}}) \cdot L_{t,q}(\Pi_n^L(x)) \\ &\quad\leqslant C \cdot \sup_{N \leqslant r \leqslant N+M} d_q^0\big(h_r, \epsilon|_{C(S_q^2)}\big) + \varepsilon_{N,M}, \end{aligned}$$

where the last estimate follows from Corollary 5.2.3. This proves the present proposition. ∎

The next proposition follows by an application of the estimate from Proposition 7.2.6 together with the core technical result from [3].

Proposition 7.2.7. *Let $\delta \in (0,1)$, $q_0 \in (0,1]$ and $K \in \mathbb{N}_0$. For every $\varepsilon > 0$ there exist an open interval I containing q_0 and $N_0, M_0 \in \mathbb{N}_0$ with $M_0 \geqslant K$ such that*

$$d_{t,q}\big(\chi_{N_0}^{M_0}|_{B_q^K}, \epsilon|_{B_q^K}\big) < \varepsilon$$

for all $q \in I \cap [\delta, 1]$ and all $t \in [\delta, 1]$.

Proof. By [3, Lemma 4.11], for every $r \in \mathbb{N}_0$, we may choose a continuous function $H_r\colon [\delta, 1] \to [0, \infty)$ such that

$$d_q^0\big(h_r, \epsilon|_{C(S_q^2)}\big) \leqslant H_r(q) \quad \text{for all } q \in [\delta, 1].$$

Moreover, by [3, Lemma 4.12] we may arrange that $\lim_{r\to\infty} H_r(q_0) = 0$.

Let us choose the constant $C > 0$ and the positive null sequence $(\varepsilon_{N,M})_{N,M=0}^{\infty}$ according to Proposition 7.2.6 with $\delta \in (0,1)$ and $K \in \mathbb{N}_0$ as given in the statement of the present proposition. Let now $\varepsilon > 0$ be given. Choose $N_0 \geqslant K$ such that $\varepsilon_{N,M} < \frac{\varepsilon}{2}$ for all $N, M \geqslant N_0$ and $H_r(q_0) < \frac{\varepsilon}{4C}$ for all $r \geqslant N_0$. Since the function H_r is continuous for all $r \in \mathbb{N}_0$, we may choose our open interval I containing q_0 such that

$$\left|H_r(q_0) - H_r(q)\right| < \frac{\varepsilon}{4C} \quad \text{for all } q \in I \cap [\delta, 1] \text{ and all } r \in \{N_0, N_0+1, \dots, 2N_0\}.$$

We now put $M_0 := N_0$ and it then follows from Proposition 7.2.6 that

$$d_{t,q}\big(\chi_{N_0}^{M_0}|_{B_q^K}, \epsilon|_{B_q^K}\big) \leqslant C \cdot \sup_{N_0 \leqslant r \leqslant 2N_0} H_r(q) + \varepsilon_{N_0,N_0} < \varepsilon$$

for all $q \in I \cap [\delta, 1]$ and all $t \in [\delta, 1]$. ∎

7.3 Continuity of quantum SU(2)

We are now ready to assemble all the information gathered in the previous sections to obtain a proof of our main continuity result, Theorem D from the introduction.

Theorem 7.3.1. *The* 2*-parameter family of compact quantum metric spaces*

$$\big(C(\mathrm{SU}_q(2)), L_{t,q}\big)_{(t,q)\in(0,1]\times(0,1]}$$

varies continuously in the quantum Gromov–Hausdorff distance.

As noted earlier, since

$$\mathrm{dist_Q}\big((C(\mathrm{SU}_q(2)), L_{t,q}), (C(\mathrm{SU}_q(2)), L_{t,q}^{\max})\big) = 0$$

by Corollary 6.4.2, the above theorem also holds true for $L_{t,q}^{\max}$ instead of $L_{t,q}$.

Proof. Let $(t_0, q_0) \in (0, 1] \times (0, 1]$ and $\varepsilon > 0$ be given and put $\delta := \min\{q_0/2, t_0/2\}$. By Proposition 6.2.6, $\beta_N^M(B_q^K) \subseteq \mathrm{Fuzz}_{N+M}(B_q^K)$ for all $N, M, K \in \mathbb{N}_0$ with $M \geqslant K$. Applying Proposition 6.4.1, we may choose $K_0 \in \mathbb{N}_0$ and a constant $C > 0$ such that

$$\begin{aligned}&\mathrm{dist_Q}\big((\mathrm{Fuzz}_{N+M}(B_q^{K_0}), L_{t,q}); (C(\mathrm{SU}_q(2)), L_{t,q}^{\max})\big)\\ &\quad\leqslant d_{t,q}^{\max}\big(\chi_N^M|_{B_q^{K_0}}, \epsilon|_{B_q^{K_0}}\big) \cdot C + \frac{\varepsilon}{6}\end{aligned}$$

for all $N, M \in \mathbb{N}_0$ with $M \geqslant K_0$ and all $t, q \in [\delta, 1]$. By Corollary 6.4.3 and Proposition 7.2.7, there exist $N_0, M_0 \in \mathbb{N}_0$ with $M_0 \geqslant K_0$ and an open interval I with $q_0 \in I$ such that

$$d_{t,q}^{\max}\big(\chi_{N_0}^{M_0}|_{B_q^{K_0}}, \epsilon|_{B_q^{K_0}}\big) = d_{t,q}\big(\chi_{N_0}^{M_0}|_{B_q^{K_0}}, \epsilon|_{B_q^{K_0}}\big) < \frac{\varepsilon}{6C}$$

for all $q \in I \cap [\delta, 1]$ and all $t \in [\delta, 1]$. Hence

$$\mathrm{dist_Q}\big((\mathrm{Fuzz}_{N_0+M_0}(B_q^{K_0}), L_{t,q}); (C(\mathrm{SU}_q(2)), L_{t,q}^{\max})\big) < \frac{\varepsilon}{3}$$

for all $q \in I \cap [\delta, 1]$ and all $t \in [\delta, 1]$. Note, at this point, that $V := [\delta, 1] \times (I \cap [\delta, 1]) \subseteq (0, 1] \times (0, 1]$ is a neighbourhood of the point (t_0, q_0). By Proposition 7.1.1, the compact quantum metric spaces $(\mathrm{Fuzz}_{N_0+M_0}(B_q^{K_0}), L_{t,q})_{t,q\in(0,1]}$ vary continuously in the quantum Gromov–Hausdorff distance, so we may choose a neighbourhood U of $(t_0, q_0) \in (0,1] \times (0,1]$ such that

$$\mathrm{dist}_{\mathrm{Q}}\big((\mathrm{Fuzz}_{N_0+M_0}(B_q^{K_0}), L_{t,q}); (\mathrm{Fuzz}_{N_0+M_0}(B_{q_0}^{K_0}), L_{t_0,q_0})\big) < \frac{\varepsilon}{3}$$

for all $(t,q) \in U$. An application of the triangle inequality for the quantum Gromov–Hausdorff distance [70, Theorem 4.3], now yields the estimate

$$\mathrm{dist}_{\mathrm{Q}}\big((C(\mathrm{SU}_q(2)), L_{t,q}^{\max}); (C(\mathrm{SU}_{q_0}(2)), L_{t_0,q_0}^{\max})\big) < \varepsilon$$

for all $(t,q) \in U \cap V$, thus completing the proof. ■

As a last remark we single out the following special case of the above theorem: As the deformation parameter q tends to 1, the quantum metric spaces $\mathrm{SU}_q(2)$ converge towards SU(2) equipped with its classical round metric rescaled with a factor 2. To make this statement precise, recall from Section 4.4, that we denote by d_{S^3} the usual round metric on $\mathrm{SU}(2) \cong S^3$. We then have the Lip-norm L_{Lip} which to any continuous function $f : \mathrm{SU}(2) \to \mathbb{C}$ assigns the Lipschitz constant with respect to the rescaled metric $2 \cdot d_{S^3}$. Comparing with Theorem 4.4.1, the special case of Theorem 7.3.1 then reads as follows:

Corollary 7.3.2. *As* $(t,q) \in (0,1] \times (0,1]$ *tends to* $(1,1)$*, the quantum metric spaces* $(C(\mathrm{SU}_q(2)), L_{t,q})$ *converge to* $(C(\mathrm{SU}(2)), L_{\mathrm{Lip}})$ *in quantum Gromov–Hausdorff distance.*