Preface

This book delves into the fascinating world of the Mumford–Shah functional, a cornerstone of image processing and a rich source of challenging mathematical problems. Introduced by Mumford and Shah in their seminal 1989 paper, this functional seeks to capture the essence of an image by decomposing it into a piecewise smooth function and a set of discontinuities. While its practical applications in image segmentation and reconstruction are undeniable, the functional also presents profound theoretical questions. Many of these regard the regularity of its minimizers: the central theme of this book is the exploration of these regularity questions, focusing specifically on the planar case. We embark on a comprehensive journey through the existing literature, revisiting classical results with fresh perspectives and incorporating new advancements. Our goal is to provide a self-contained and detailed account of the state of the art, accessible to both aspiring researchers seeking a thorough introduction and experts interested in the latest developments.

The Mumford–Shah conjecture, a long-standing open problem, predicts a remarkably simple structure for the discontinuity set of minimizers: a collection of smooth curves meeting at triple junctions forming equal angles. While this conjecture remains elusive in its full generality, significant progress has been made. We present a detailed exposition of the ε-regularity theory, a powerful tool that establishes the local regularity of the discontinuity set under certain assumptions. This theory allows us to understand the behavior of minimizers near points that resemble the conjectured structures. Beyond ε-regularity, we explore the classification of global minimizers, the properties of critical points, and various structural results that shed light on the nature of the discontinuity set.

Throughout the book we emphasize clarity and completeness, providing detailed proofs for all the presented theorems. We assume a solid background in measure theory and basic elliptic PDEs, while we provide supplementary material in the appendices to aid readers less familiar with specific technical aspects. We hope that this book will serve as a valuable resource for anyone intrigued by the beauty and complexity of the Mumford–Shah functional, inspiring further research and deeper understanding of this remarkable mathematical object.

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