Preface

This book is the result of a conference titled 'Arithmetic of L-functions', held at ICMAT (Madrid) in May 2023. The conference aimed at bringing together researchers to discuss this central and vibrant area of modern number theory, whose roots trace back to the groundbreaking work of Dirichlet and Kummer in the nineteenth century. In the second half of the twentieth century, the subject was revolutionised by the formulation and study of 'main conjectures' in Iwasawa theory, the Birch and Swinnerton-Dyer conjecture, and the 'Tamagawa number conjecture' due to Bloch and Kato. Through such conjectures and related results, the properties of L-functions informed the development of arithmetic geometry.

The contents of this volume can be broadly divided into those focusing on arithmetic over fields of characteristic zero, and those dealing with arithmetic in positive characteristic. To reflect this distinction, we have organised the chapters into two separate parts.

The topics covered in Part I include the Galois module structure of ideal class groups, reciprocity laws in Iwasawa theory, Euler systems, p-adic L-functions, and étale cohomology – each of which has had a remarkable importance in the study of p-adic Galois representations over the last few decades.

Chapter 1, by Burns and Sano, establishes relations of a functional equation type between Euler systems of p-adic representations. Upon specialisation, these relations show the existence of Euler systems that interpolate certain Dedekind zeta values of totally real fields.

A p-adic perspective also features in Chapter 2, by Büyükboduk, Casazza, Pal, and de Vera-Piquero. Largely expository, this chapter develops an approach to the Artin formalism for triple product p-adic L-functions, through the use of explicit reciprocity laws and formulas of 'Gross–Zagier type'.

Chapters 3 and 4, contributed by T. Dokchitser and V. Dokchitser, form a two-part study linking group theory with arithmetic applications. Chapter 3, purely group-theoretic in nature, proves a character formula for conjugacy classes in cosets of a normal subgroup in a finite group with an abelian quotient. This result finds an arithmetic application in Chapter 4, where the authors apply their formula to Weil representations over non-archimedean local fields, showing that the étale cohomology of abelian varieties over such fields is determined by the Euler factors they give rise to.

Following this are two chapters that study in detail the Galois module structure of the 'minus parts' of ideal class groups of CM fields. In Chapter 5, by Greither and Kataoka, this structure is explored up to a certain natural equivalence relation. One

particular question considered by the authors is how large a proportion of equivalence classes can be realised as classes of minus parts of class groups.

Chapter 6, by Kurihara, shifts the focus to the Fitting ideals of the Pontryagin duals of these minus parts of class groups, providing a comprehensive account of known results in this direction. Notably, the chapter includes a direct deduction of the relevant case of the equivariant Iwasawa main conjecture from recent work by Dasgupta, Kakde, Silliman, and Wang. This result also covers the case $p = 2$, and has not been available in the literature in this form.

Closely related is Chapter 7, by Mejías Gil and Nickel, which studies and unifies different constructions of global arithmetic complexes that naturally occur in the formulation of non-commutative Iwasawa main conjectures over (not necessarily totally real) number fields.

Part I concludes with a survey by Venjakob in Chapter 8 on the general topic of explicit reciprocity laws. The viewpoint taken here is primarily Iwasawa-theoretic, and particular emphasis is put on recent developments in the setting of Lubin–Tate towers.

Part II of this volume serves as an introduction to the emerging subject of special L-values in positive characteristic. This relatively recent direction within the general area of global function field arithmetic is concerned with the invariants of Galois representations valued in local fields of positive characteristic, as provided by Drinfeld modules or t-modules.

The first chapter in this part, Chapter 9 by Anglès, considers P-adic Carlitz–Goss zeta values, where P is a finite place of the rational function field over a finite field. Then, over real function fields, the author proves a P-adic class formula that relates such values to a P-adic regulator on the module of Taelman units of the Carlitz module.

The final two chapters are interrelated and concerned with ∞-adic (rather than P-adic) L-values of more general objects, from an equivariant point of view. Chapter 10, by Green and Popescu, proves an equivariant Tamagawa number formula for t-modules and, as an application, initiates the study of the arithmetic properties of equivariant L-values of Drinfeld modules at positive integers.

An important ingredient in the latter study is the main result of Chapter 11, contributed by Popescu and Ramachandran. This result fills a gap in the literature by comparing a common and convenient explicit interpretation of the L-values of Drinfeld modules with the formal definition of the special values of Goss L-functions.

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We hope that this volume will not only serve as a record of the work discussed at ICMAT, but also as a useful resource for researchers working on arithmetic of L-functions in the years to come.

London and Madrid, February 2025

Dominik Bullach
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Group photo taken on the last day of the conference.