Chapter 1

Introduction

1.1 Motivation

The purpose of this work is two-fold. The first goal is to develop a sufficiently rich theory of almost coherent sheaves on schemes and a class of formal schemes. The second goal is to provide the reader with one interesting source of examples of almost coherent sheaves. Namely, we show that the complex of p-adic nearby cycles $\mathbf{R}\nu_*(\mathcal{E})$ has quasi-coherent, almost coherent cohomology sheaves for any admissible formal $\mathcal{O}_C$-scheme $\mathfrak{X}$ and $\mathcal{O}^+_{X^\diamond}/p$-vector bundle $\mathcal{E}$ (see Definition 6.5.1).

Before we discuss the content of each chapter in detail, we explain the motivation behind the work done in this memoir.

The first source of motivation comes from the work of P. Scholze on the finiteness of $\mathbf{F}_p$-cohomology groups of proper rigid-analytic varieties over p-adic fields (see [59]). The second source of motivation (clearly related to the first one) is the desire to set up a robust enough theory of almost coherent sheaves that is crucially used in our proof of Poincaré duality for $\mathbf{F}_p$-local systems on smooth and proper rigid-analytic varieties over p-adic fields in [71].

We start with the work of P. Scholze. In [59], he showed that there is an almost isomorphism

$$\mathrm{H}^i\left(X, \mathbf{F}_p\right) \otimes \mathcal{O}_C/p \simeq^a \mathrm{H}^i\left(X, \mathcal{O}^+_{X_{\text{ét}}}/p\right)$$

for any proper rigid-analytic variety X over a p-adic algebraically closed field C. This almost isomorphism allows us to reduce studying certain properties of $\mathrm{H}^i(X, \mathbf{F}_p)$ for a p-adic proper rigid-analytic space X to studying almost properties of the cohomology groups $\mathrm{H}^i(X, \mathcal{O}^+_{X_{\text{ét}}}/p)$, or the full complex $\mathbf{R}\Gamma(X, \mathcal{O}^+_{X_{\text{ét}}}/p)$. For instance, Scholze shows that $\mathrm{H}^i(X, \mathbf{F}_p)$ are finite groups by deducing it from almost coherence of $\mathrm{H}^i(X, \mathcal{O}^+_{X_{\text{ét}}}/p)$ over $\mathcal{O}_C/p$.

Scholze's argument does not involve any choice of an admissible formal model for X and is performed entirely on the generic fiber via an elaborate study of cancellations in certain spectral sequences. A different natural approach to studying $\mathbf{R}\Gamma(X, \mathcal{O}^+_{X_{\text{ét}}}/p)$ is to rewrite this complex as

$$\mathbf{R}\Gamma\left(X, \mathcal{O}^+_{X_{\text{ét}}}/p\right) \simeq \mathbf{R}\Gamma\left(\mathfrak{X}_0, \mathbf{R}t_*\mathcal{O}^+_{X_{\text{ét}}}/p\right)$$

for a choice of an admissible formal $\mathcal{O}_C$-model $\mathfrak{X}$ and the natural morphism of ringed sites

$$t\colon \left(X_{\text{ét}}, \mathcal{O}^+_{X_{\text{ét}}}/p\right) \to \left(\mathfrak{X}_{0,\text{Zar}}, \mathcal{O}_{\mathfrak{X}_0}\right)$$

with $\mathfrak{X}_0$ the mod-p fiber of $\mathfrak{X}$. Then we can separately study the complex $\mathbf{R}t_*(\mathcal{O}^+_{X_{\text{ét}}}/p)$ and the functor $\mathbf{R}\Gamma(\mathfrak{X}, -)$. In order to make this strategy work, we develop the notion of almost coherent sheaves on $\mathfrak{X}$ and $\mathfrak{X}_0$ and show its various properties similar to the properties of coherent sheaves. This occupies Chapters 2–5. While Chapters 6 and 7 are devoted to showing that the complex $\mathbf{R}t_*(\mathcal{O}^+_{X_{\text{ét}}}/p)$ has almost coherent cohomology groups, and to generalizing these finiteness results to all $\mathcal{O}^+/p$-vector bundles. Combining that with the almost proper mapping theorem (Theorem 1.2.9), we reprove [59, Lemma 5.8 and Theorem 5.1] in a slightly greater generality (allowing arbitrary Zariski-constructible coefficients as opposed to local systems).

Theorem 1.1.1 (Lemma 7.3.4, Lemma 7.3.7, and Lemma 6.7.10). *Let C be an algebraically closed p-adic non-archimedean field, let X be a proper rigid-analytic variety over C, and let $\mathcal{F}$ be a Zariski-constructible sheaf of $\mathbf{F}_p$-modules (see Definition 7.1.7). Then*

(1) $\mathrm{H}^i(X, \mathcal{F} \otimes_{\mathbf{F}_p} \mathcal{O}^+_{X_{\text{ét}}}/p)$ *is an almost finitely generated $\mathcal{O}_C/p$-module for $i \geq 0$;*

(2) *the natural morphism*

$$\mathrm{H}^i\left(X, \mathcal{F}\right) \otimes_{\mathbf{F}_p} \mathcal{O}_C/p \to \mathrm{H}^i\left(X, \mathcal{F} \otimes_{\mathbf{F}_p} \mathcal{O}^+_{X_{\text{ét}}}/p\right)$$

is an almost isomorphism for $i \geq 0$;

(3) $\mathrm{H}^i(X, \mathcal{F} \otimes_{\mathbf{F}_p} \mathcal{O}^+_{X_{\text{ét}}}/p)$ *is almost zero for $i > 2\dim X$.*

Theorem 1.1.2 (Lemma 7.3.6).[1] *In the notation of Theorem 1.1.1, we have*

(1) $\mathrm{H}^i(X, \mathcal{F})$ *is a finite group for $i \geq 0$;*

(2) $\mathrm{H}^i(X, \mathcal{F}) \simeq 0$ *for $i > 2\dim X$.*

Now we discuss the role of this memoir in our proof of Poincaré duality in [71]. We start with the precise formulation of this result.

Theorem 1.1.3 ([71]). *Let C be an algebraically closed p-adic non-archimedean field, let X be a rigid-analytic variety over C of pure dimension d, and let $\mathbf{L}$ be an $\mathbf{F}_p$-local system on $X_{\text{ét}}$. Then there is a canonical trace map*

$$t_X : \mathrm{H}^{2d}\left(X, \mathbf{F}_p(d)\right) \to \mathbf{F}_p$$

such that the induced pairing

$$\mathrm{H}^i(X, \mathbf{L}) \otimes \mathrm{H}^{2d-i}(X, \mathbf{L}^\vee(d)) \xrightarrow{-\cup-} \mathrm{H}^{2d}(X, \mathbf{F}_p(d)) \xrightarrow{t_X} \mathbf{F}_p$$

is perfect.

[1] Theorem 1.1.2 can also be easily deduced from the results of [7].

The essential idea of the proof (at least for $\mathbf{L} = \underline{\mathbf{F}}_p$) is to use Theorem 1.1.1 to reduce Poincaré duality to almost duality for the complex $\mathbf{R}\Gamma(X, \mathcal{O}^+_{X_{\text{ét}}}/p)$. We study this complex via the isomorphism

$$\mathbf{R}\Gamma(X, \mathcal{O}^+_{X_{\text{ét}}}/p) \simeq \mathbf{R}\Gamma(\mathfrak{X}_0, \mathbf{R}t_*\mathcal{O}^+_{X_{\text{ét}}}/p).$$

Roughly, we separately show almost duality for the "nearby cycles functor" $\mathbf{R}t_*$, and then establish an almost version of Grothendieck duality for the $\mathcal{O}_C/p$-scheme $\mathfrak{X}_0$. Even to formulate these things precisely, one needs a good theory of almost (coherent) sheaves that globalizes the theory of almost (coherent) modules. For this theory to be useful, we have to establish that almost coherent sheaves share many properties similar to classical coherent sheaves *and* the "nearby cycles" $\mathbf{R}t_*\big(\mathcal{O}^+_{X_{\text{ét}}}/p\big)$ are almost coherent.

The main goal of Chapters 2–5 is to develop this general theory of almost (coherent) sheaves. In Chapter 6, we study $\mathcal{O}^+/p$, $\mathcal{O}^+$, and $\mathcal{O}$-vector bundles in different topologies. Chapter 7 is devoted to verifying that "nearby cycles" are almost coherent. That being said, we now discuss the content and main results of each section in more detail.

1.2 Foundations of almost mathematics (Chapters 2–5)

Section 2.1 defines the category of almost modules and studies its main properties. This section is very motivated by [26]. However, it seems that some results that we need later in the memoir are not present in [26], so we give an (almost) self-contained introduction to almost commutative algebra. We define the category of almost modules (see the discussion after Corollary 2.1.4), the almost tensor product functor $- \otimes_{R^a} -$ (see Proposition 2.2.1 (1)), the almost Hom functor $\mathrm{alHom}_{R^a}(-,-)$ (see Proposition 2.2.1 (3)), and the notion of almost finitely generated (see Definitions 2.5.1), almost finitely presented (see Definition 2.5.2), and almost coherent modules (see Definition 2.6.1). We show that almost coherent modules satisfy many natural properties similar to the properties of classical coherent modules. We summarize some of them in the following theorem:

Theorem 1.2.1 (Lemma 2.6.8, Propositions 2.6.18, 2.6.19, 2.6.20, Theorem 2.10.3, and Lemma 2.10.5). *Let R be a ring with an ideal $\mathfrak{m}$ such that $\widetilde{\mathfrak{m}} := \mathfrak{m} \otimes_R \mathfrak{m}$ is R-flat and $\mathfrak{m}^2 = \mathfrak{m}$.*

(1) *Almost coherent R^a-modules form a weak Serre subcategory of $\mathbf{Mod}^a_R$.*

(2) *Let R be an almost coherent ring (i.e., free rank-1 R-module is almost coherent), and M^a, N^a two objects in $\mathbf{D}^-_{\mathrm{acoh}}(R)^a$. Then $M^a \otimes^L_{R^a} N^a \in \mathbf{D}^-_{\mathrm{acoh}}(R)^a$.*

(3) *Let R be an almost coherent ring, and $M^a \in \mathbf{D}^-_{\mathrm{acoh}}(R)^a$, $N^a \in \mathbf{D}^+_{\mathrm{acoh}}(R)^a$. Then*

$$\mathbf{R}\mathrm{alHom}_{R^a}(M^a, N^a) \in \mathbf{D}^+_{\mathrm{acoh}}(R)^a.$$

(4) *Let R be an almost coherent ring, $M^a \in \mathbf{D}^-_{\mathrm{acoh}}(R)^a$, $N^a \in \mathbf{D}^+(R)^a$, and P^a an almost flat R^a-module. Then the natural map $\mathbf{R}\mathrm{Hom}_{R^a}(M^a, N^a) \otimes_{R^a} P^a \to \mathbf{R}\mathrm{Hom}_{R^a}(M^a, N^a \otimes_{R^a} P^a)$ is an almost isomorphism.*

(5) *Descent of almost modules along an almost faithfully flat morphism $R \to S$ is always effective.*

(6) *Let $R \to S$ be an almost faithfully flat map, and let M^a be an R^a-module. Suppose that $M^a \otimes_{R^a} S^a$ is almost finitely generated (resp. almost finitely presented, resp. almost coherent) S^a-module. Then so is M^a.*

If R is I-adically adhesive for some finitely generated ideal I (in the sense of Definition 2.12.1), we can show that almost finitely generated R-modules satisfy a (weak) version of the Artin–Rees lemma, and behave nicely with respect to the completion functor. These results will be crucial for globalizing the theory of almost coherent modules on formal schemes.

Lemma 1.2.2 (Lemma 2.12.6 and Lemma 2.12.7). *Let R be an I-adically adhesive ring with an ideal $\mathfrak{m}$ such that $I \subset \mathfrak{m}$, $\mathfrak{m}^2 = \mathfrak{m}$, and $\mathfrak{m} \otimes_R \mathfrak{m}$ is R-flat (see Setup 2.12.3). Let M be an almost finitely generated R-module. Then the following hold:*

(1) *For any R-submodule $N \subset M$, the induced topology on N coincides with the I-adic topology.*

(2) *The natural morphism $M \otimes_R \widehat{R} \to \widehat{M}$ is an isomorphism. In particular, if R is I-adically complete, then any almost finitely generated R-module is also I-adically complete.*

If R is a (topologically) finitely generated algebra over a perfectoid valuation ring K^+ (see Definition B.2), we can say even more. In this case, it turns out that R is almost noetherian (see Definition 2.7.1), so the theory simplifies significantly. Another useful result is that it suffices to check that a derived complete complex is almost coherent after taking the derived quotient by a pseudo-uniformizer ϖ. This is very handy in practice because it reduces many (subtle) integral questions to the torsion case, where there are no topological subtleties.

Theorem 1.2.3 (Theorem 2.11.5, Theorem 2.11.9, Theorem 2.13.2). *Let K^+ be a perfectoid valuation ring with a pseudo-uniformizer ϖ as in Lemma B.9, and let R be a K^+-algebra. Then the following hold:*

(1) *R is almost noetherian if R is (topologically) finite type over K^+.*

(2) *Suppose R is a topologically finite type K^+-algebra and M is a derived ϖ-adically complete object in $\mathbf{D}(R)$ such that $[M/\varpi] \in \mathbf{D}^{[c,d]}_{\mathrm{acoh}}(R/\varpi)$. Then $M \in \mathbf{D}^{[c,d]}_{\mathrm{acoh}}(R)$.*

We discuss the extension of almost mathematics to ringed sites in Chapter 3. The main goal is to generalize all constructions from almost mathematics to a general ringed site. We define the notion of almost $\mathcal{O}_X$-modules on a ringed site $(X, \mathcal{O}_X)$ (see Definition 3.1.9) and of $\mathcal{O}_X^a$-modules (see Definition 3.1.10), and show that they are equivalent:

Theorem 1.2.4 (Theorem 3.1.20). *Let R be as in Theorem 1.2.1 and $(X, \mathcal{O}_X)$ a ringed R-site. Then the functor*

$$(-)^a \colon \mathbf{Mod}^a_{\mathcal{O}_X} \to \mathbf{Mod}_{\mathcal{O}_X^a}$$

is an equivalence of categories.

We define the functors $- \otimes -$, $\mathrm{Hom}_{\mathcal{O}_X^a}(-,-)$, $\mathrm{alHom}_{\mathcal{O}_X^a}(-,-)$, $\underline{\mathrm{Hom}}_{\mathcal{O}_X^a}(-,-)$, $\underline{\mathrm{alHom}}_{\mathcal{O}_X^a}(-,-)$, f_*, f^* too on the category of $\mathcal{O}_X^a$-modules. We refer to Section 3.2 for an extensive discussion of these functors. Then we study the derived category of $\mathcal{O}_X^a$-modules and derived analogues of the functors mentioned above. This is done in Sections 3.4 and 3.5.

We develop the theory of almost finitely presented and almost (quasi-)coherent sheaves on schemes and on a class of formal schemes in Section 4.1. The main goal is to show that these sheaves behave similarly to classical coherent sheaves in many aspects.

Roughly, we define almost finitely presented $\mathcal{O}_X^a$-modules as modules that, for any finitely generated sub-ideal $\mathfrak{m}_0 \subset \mathfrak{m}$, can be locally approximated by finitely presented $\mathcal{O}_X$-modules up to modules annihilated by $\mathfrak{m}_0$ (see Definition 4.1.4 for a precise definition). Sections 4.1–4.4 are mostly concerned with local properties of these sheaves. We summarize some of the main results below:

Theorem 1.2.5 (Corollary 4.1.12, Theorem 4.4.6, Lemmas 4.4.8, 4.4.7, 4.4.9, 4.4.10). *Let R be a ring with an ideal $\mathfrak{m}$ such that $\widetilde{\mathfrak{m}} := \mathfrak{m} \otimes_R \mathfrak{m}$ is R-flat and $\mathfrak{m}^2 = \mathfrak{m}$.*

(1) *For any R-scheme X, almost coherent $\mathcal{O}_X^a$-modules form a weak Serre subcategory of $\mathbf{Mod}^a_{\mathcal{O}_X}$.*

(2) *The functor*

$$\widetilde{(-)} \colon \mathbf{D}_*(R)^a \to \mathbf{D}_{\mathrm{aqc},*}(\mathrm{Spec}\, R)^a$$

is a t-exact equivalence of triangulated categories for $* \in \{\text{“ ”}\ \mathrm{acoh}\}$. Its quasi-inverse is given by $\mathbf{R}\Gamma(\mathrm{Spec}\, R, -)$. In particular, an almost quasi-coherent $\mathcal{O}^a_{\mathrm{Spec}\, R}$-module $\mathcal{F}^a$ is almost coherent if and only if $\mathcal{F}^a(\mathrm{Spec}\, R)$ is an almost coherent R^a-module.

(3) *The natural morphism* $\widetilde{M^a \otimes^L_{R^a} N^a} \to \widetilde{M^a} \otimes^L_{\mathcal{O}^a_{\operatorname{Spec} R}} \widetilde{N^a}$ *is an isomorphism for any* $M^a, N^a \in \mathbf{D}(R)^a$.

(4) *Let* $f\colon \operatorname{Spec} B \to \operatorname{Spec} A$ *be an* R*-morphism of affine schemes. Then* $\mathbf{L}f^*(\widetilde{M^a})$ *is functorially isomorphic to* $\widetilde{M^a \otimes^L_{A^a} B^a}$ *for any* $M^a \in \mathbf{D}(A)^a$.

(5) *Let* $f\colon X \to Y$ *be a quasi-compact and quasi-separated morphism of* R*-schemes. Suppose that* Y *is quasi-compact. Then* $\mathbf{R}f_*$ *carries* $\mathbf{D}^*_{\mathrm{aqc}}(X)^a$ *to* $\mathbf{D}^*_{\mathrm{aqc}}(Y)^a$ *for any* $* \in \{\text{“ ”}, -, +, b\}$.

(6) *Suppose that* R *is almost coherent. Then the natural maps*

$$\mathbf{R}\mathrm{alHom}_{R^a}\widetilde{(M^a, N^a)} \to \mathbf{R}\underline{\mathrm{alHom}}_{\mathcal{O}^a_{\operatorname{Spec} R}}(\widetilde{M^a}, \widetilde{N^a}),$$

$$\mathbf{R}\mathrm{Hom}_{R^a}\widetilde{(M^a, N^a)} \to \mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{O}^a_{\operatorname{Spec} R}}(\widetilde{M^a}, \widetilde{N^a})$$

are almost isomorphisms for $M^a \in \mathbf{D}^-_{\mathrm{acoh}}(R)^a$, $N^a \in \mathbf{D}^+(R)^a$.

We also show that, for a quasi-compact and quasi-separated scheme X, any almost finitely presented $\mathcal{O}^a_X$-module admits a global approximation by finitely presented $\mathcal{O}_X$-modules. This result is crucial for establishing *global* properties of almost finitely presented $\mathcal{O}^a_X$-modules, and it will be systematically used in Chapter 5.

Theorem 1.2.6 (Corollary 4.3.5). *Let* X *be a quasi-compact and quasi-separated* R*-scheme, and let* $\mathcal{F}$ *be an almost quasi-coherent* $\mathcal{O}_X$*-module. Then* $\mathcal{F}$ *is almost finitely presented (resp. almost finitely generated) if and only if for any finitely generated ideal* $\mathfrak{m}_0 \subset \mathfrak{m}$ *there is a morphism* $f\colon \mathcal{G} \to \mathcal{F}$ *such that* $\mathfrak{m}_0(\operatorname{Ker} f) = 0$, $\mathfrak{m}_0(\operatorname{Coker} f) = 0$, *and* $\mathcal{G}$ *is a quasi-coherent finitely presented (resp. finitely generated)* $\mathcal{O}_X$*-module.*

We now discuss the content of Sections 4.5–4.9. The main goal of these sections is to show an analogue of Theorem 1.2.5 for a class of formal schemes. To achieve this, we restrict our attention to the class of topologically finitely presented schemes over a topologically universally adhesive ring R (see Set-up 4.5.1). This, in particular, includes admissible formal schemes over a mixed characteristic, p-adically complete rank-1 valuation ring $\mathcal{O}_C$ with algebraically closed fraction field C.

One of the main difficulties in developing a good theory of almost coherent $\mathcal{O}^a_{\mathfrak{X}}$-modules on a formal scheme $\mathfrak{X}$ is that there is no good abelian theory of “quasi-coherent” on $\mathfrak{X}$. The theory of quasi-coherent sheaves is an important tool used in developing the theory of almost coherent sheaves on schemes that does not have an immediate counterpart in the world of formal schemes.

We overcome this issue in two different ways: we use the notion of adically quasi-coherent $\mathcal{O}_{\mathfrak{X}}$-modules introduced in [25] (see Definition 4.5.2) and the notion of derived quasi-coherent $\mathcal{O}_{\mathfrak{X}}$-modules introduced in [49] (see Definition 4.8.1). The first notion has the advantage that every adically quasi-coherent $\mathcal{O}_{\mathfrak{X}}$-module is an actual $\mathcal{O}_{\mathfrak{X}}$-module. However, these modules do not form a weak Serre subcategory inside $\mathbf{Mod}_{\mathcal{O}_{\mathfrak{X}}}$, so they are not always very useful in practice. The latter definition has the advantage that derived quasi-coherent $\mathcal{O}_{\mathfrak{X}}$-modules form a triangulated subcategory inside $\mathbf{D}(\mathfrak{X})$; it is quite convenient for certain purposes. However, derived quasi-coherent $\mathcal{O}_{\mathfrak{X}}$-modules are merely objects of $\mathbf{D}(\mathfrak{X})$ and not actual $\mathcal{O}_{\mathfrak{X}}$-modules in the classical sense. Therefore, we usually use adically quasi-coherent $\mathcal{O}_{\mathfrak{X}}$-modules when needed except for Section 4.8, where the notion of derived quasi-coherent $\mathcal{O}_{\mathfrak{X}}$-modules seems to be more useful for our purposes. In particular, it allows us to define the functor

$$(-)^{L\Delta} \colon \mathbf{D}_{\mathrm{acoh}}(A)^a \to \mathbf{D}_{\mathrm{acoh}}(\mathrm{Spf}\ A)^a$$

for any topologically finitely presented R-algebra A in a way that "extends" the classical functor $(-)^{\Delta} \colon \mathbf{Mod}_A^{\mathrm{acoh}} \to \mathbf{Mod}_{\mathcal{O}_{\mathfrak{X}}}$ (see Definition 4.8.7 and Lemma 4.8.13).

Theorem 1.2.7 (Lemma 4.5.23, Corollary 4.8.16, Lemmas 4.9.4, 4.9.3, 4.9.7). *Let R be a ring with a finitely generated ideal I such that R is I-adically complete, I-adically topologically universally adhesive, I-torsion free with an ideal $\mathfrak{m}$ such that $I \subset \mathfrak{m}$, $\mathfrak{m}^2 = \mathfrak{m}$ and $\widetilde{\mathfrak{m}} := \mathfrak{m} \otimes_R \mathfrak{m}$ is R-flat.*

(1) *For any topologically finitely presented formal R-scheme $\mathfrak{X}$, almost coherent $\mathcal{O}_{\mathfrak{X}}^a$-modules form a weak Serre subcategory of $\mathbf{Mod}_{\mathcal{O}_{\mathfrak{X}}}^a$.*

(2) *The functor*

$$\mathbf{R}\Gamma(\mathrm{Spf}\ R, -) \colon \mathbf{D}_{\mathrm{acoh}}(\mathrm{Spf}\ R)^a \to \mathbf{D}_{\mathrm{acoh}}(R)^a$$

is a t-exact equivalence of triangulated categories.

(3) *The natural morphism $(M^a \otimes^L_{R^a} N^a)^{L\Delta} \to (M^a)^{L\Delta} \otimes^L_{\mathcal{O}^a_{\mathrm{Spf}\ R}} (N^a)^{L\Delta}$ is an isomorphism for any $M^a, N^a \in \mathbf{D}_{\mathrm{acoh}}(R)^a$.*

(4) *Let $\mathfrak{f} \colon \mathrm{Spf}\ B \to \mathrm{Spf}\ A$ be a morphism of topologically finitely presented affine formal R-schemes. Then $\mathbf{L}\mathfrak{f}^*\big((M^a)^{L\Delta}\big)$ is functorially isomorphic to $(M^a \otimes^L_{A^a} B^a)^{L\Delta}$ for any $M^a \in \mathbf{D}_{\mathrm{acoh}}(A)^a$.*

(5) *The natural morphisms*

$$\big(\mathbf{R}\mathrm{alHom}_{R^a}(M^a, N^a)\big)^{L\Delta} \to \mathbf{R}\underline{\mathrm{alHom}}_{\mathcal{O}^a_{\mathrm{Spf}\ R}}\big((M^a)^{L\Delta}, (N^a)^{L\Delta}\big),$$

$$\big(\mathbf{R}\mathrm{Hom}_{R^a}(M^a, N^a)\big)^{L\Delta} \to \mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{O}^a_{\mathrm{Spf}\ R}}\big((M^a)^{L\Delta}, (N^a)^{L\Delta}\big)$$

are almost isomorphisms for $M^a \in \mathbf{D}^-_{\mathrm{acoh}}(R)^a$, $N^a \in \mathbf{D}^+_{\mathrm{acoh}}(R)^a$.

Similarly to the case of schemes, almost coherent sheaves on formal schemes satisfy the global approximation property:

Theorem 1.2.8 (Theorem 4.7.6). *Let R be as in Theorem 1.2.7, let $\mathfrak{X}$ be a finitely presented formal R-scheme, and let $\mathcal{F}$ be an almost finitely generated (resp. almost* *finitely presented) $\mathcal{O}_{\mathfrak{X}}$-module. Then, for any finitely generated ideal $\mathfrak{m}_0 \subset \mathfrak{m}$, there is an adically quasi-coherent, finitely generated (resp. finitely presented) $\mathcal{O}_{\mathfrak{X}}$-module $\mathcal{G}$ together with a map $\phi\colon \mathcal{G} \to \mathcal{F}$ such that $\mathfrak{m}_0(\operatorname{Coker}\phi) = 0$ and $\mathfrak{m}_0(\operatorname{Ker}\phi) = 0$.*

We discuss the global properties of almost coherent sheaves in Chapter 5. Namely, we generalize certain cohomological properties of classical coherent sheaves to the case of almost coherent sheaves. We start with the almost version of the proper mapping theorem:

Theorem 1.2.9 (Theorem 5.1.3). *Let R be a universally coherent*[2] *ring with an ideal $\mathfrak{m}$ such that $\widetilde{\mathfrak{m}} := \mathfrak{m} \otimes_R \mathfrak{m}$ is R-flat and $\mathfrak{m}^2 = \mathfrak{m}$. Let furthermore $f\colon X \to Y$ be a proper morphism of finitely presented R-schemes with quasi-compact Y. Then $\mathbf{R}f_*$ carries $\mathbf{D}^*_{\mathrm{acoh}}(X)^a$ to $\mathbf{D}^*_{\mathrm{acoh}}(Y)^a$ for $* \in \{\text{“ ”}, -, +, b\}$.*

The essential idea of the proof is to reduce Theorem 1.2.9 to the classical proper mapping theorem over a universally coherent base [25, Theorem I.8.1.3]. The key input to make this reduction work is Theorem 1.2.6.

We also prove a version of the almost proper mapping theorem for a morphism of formal schemes:

Theorem 1.2.10 (Theorem 5.1.6). *Let $\mathfrak{Y}$ be a topologically finitely presented formal R-scheme for R as in Set-up 4.5.1 and let $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{Y}$ be a proper, topologically finitely presented morphism. Then $\mathbf{R}\mathfrak{f}_*$ carries $\mathbf{D}^*_{\mathrm{acoh}}(\mathfrak{X})^a$ to $\mathbf{D}^*_{\mathrm{acoh}}(\mathfrak{Y})^a$ for $* \in \{\text{“ ”}, -, +, b\}$.*

Then we characterize quasi-coherent, almost coherent complexes on finitely presented, separated schemes over a universally coherent base ring R. This is an almost analogue of [68, Tag 0CSI]. We follow the same proof strategy but adjust it in certain places to make it work in the almost setting. This result is important for us as it will later play a crucial role in the proof of the formal GAGA theorem for almost coherent sheaves.

Theorem 1.2.11 (Theorem 5.2.3). *Let R be a universally coherent ring with an ideal $\mathfrak{m}$ such that $\widetilde{\mathfrak{m}} := \mathfrak{m} \otimes_R \mathfrak{m}$ is R-flat and $\mathfrak{m}^2 = \mathfrak{m}$. Let X be a separated, finitely presented R-scheme, and let $\mathcal{F} \in \mathbf{D}^-_{\mathrm{qc}}(X)$ be an object such that*

$$\mathbf{R}\mathrm{Hom}_X(\mathcal{P}, \mathcal{F}) \in \mathbf{D}^-_{\mathrm{acoh}}(R)$$

for any $\mathcal{P} \in \mathrm{Perf}(X)$. Then $\mathcal{F} \in \mathbf{D}^-_{\mathrm{qc,acoh}}(X)$.

Theorem 1.2.12 (Corollary 5.3.3). *Let R be as in Theorem 1.2.10, and let X be a finitely presented R-scheme. Then the functor*

$$\mathbf{L}c^*\colon \mathbf{D}^*_{\mathrm{acoh}}(X)^a \to \mathbf{D}^*_{\mathrm{acoh}}(\mathfrak{X})^a$$

induces an equivalence of categories for $* \in \{\text{“ ”}, +, -, b\}$.

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[2] Any finitely presented R-algebra A is coherent.

We note that the standard proof of the classical formal GAGA theorem via projective methods has no chance to work in the almost coherent situation (due to the lack of “finiteness” for almost coherent sheaves). Instead, we “explicitly” construct a pseudo-inverse to $\mathbf{L}c^*$ in the derived world by adapting an argument from the paper of J. Hall [31].

The last thing we discuss in Chapter 5 is the almost version of the Grothendieck duality. This is a crucial technical tool in our proof of Poincaré duality in [71]. So we develop some foundations of the $f^!$ functor in the almost world in this memoir. We summarize the main properties of this functor below:

Theorem 1.2.13 (Theorem 5.5.8). *Let R be as in Theorem 1.2.9, and FPS_R be the category of finitely presented, separated R-schemes. Then there is a well-defined pseudo-functor $(-)^!$ from FPS_R into the 2-category of categories such that*

(1) $(X)^! = \mathbf{D}^+_{\mathrm{aqc}}(X)^a$;
(2) *for a smooth morphism $f: X \to Y$ of pure relative dimension d, there is a natural isomorphism $f^! \simeq \mathbf{L}f^*(-)\otimes^L_{\mathcal{O}_X^a} \Omega^d_{X/Y}[d]$;*
(3) *for a proper morphism $f: X \to Y$, the pseudo-functor $f^!$ is right adjoint to $\mathbf{R}f_*: \mathbf{D}^+_{\mathrm{acoh}}(X)^a \to \mathbf{D}^+_{\mathrm{acoh}}(Y)^a$.*

1.3 $\mathcal{O}^+/p$, $\mathcal{O}^+$, and $\mathcal{O}$-vector bundles (Chapter 6)

The main goal of Chapter 6 is to study the categories of $\mathcal{O}^+/p$-vector bundles in the étale, quasi-proétale, and v-topologies. We also show that $\mathcal{O}^+/p$-vector bundles can be trivialized by some particular étale covers. These results will play a crucial role in Chapter 7. Also, as an application of our results, we give a new proof of the theorem of Kedlaya–Liu saying that, for a perfectoid space X, the categories of $\mathcal{O}$-vector bundles in the analytic and v-topologies are equivalent.

We formulate the results of this section more precisely below:

Theorem 1.3.1 (Corollary 6.6.9). *Let X be a strongly sheafy adic space (see Definition C.4.1) over $\mathrm{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)$. Then*

(1) *the categories $\mathrm{Vect}(X_{\text{ét}}; \mathcal{O}^+_{X_{\text{ét}}}/p)$, $\mathrm{Vect}(X^\diamond_{\mathrm{qp}}; \mathcal{O}^+_{X^\diamond_{\mathrm{qp}}}/p)$, and $\mathrm{Vect}(X^\diamond_v; \mathcal{O}^+_{X^\diamond}/p)$ are equivalent;*
(2) *these equivalences preserve cohomology groups;*
(3) *for any $\mathcal{O}^+_{X^\diamond}/p$-vector bundle $\mathcal{E}$ and a point $x \in X$, there exist an open affinoid subspace $x \in U_x \subset X$ and a finite étale surjective morphism $\widetilde{U}_x \to U_x$ such that $\mathcal{E}|_{\widetilde{U}_x^\diamond}$ is a free vector bundle.*

Theorem 1.3.1 (1), (2) is essentially due to B. Heuer (see [35, Section 2] for a similar result in a slightly different level of generality). However, Theorem 1.3.1 (3) does not seem to follow from [35] and is crucial for our arguments in Chapter 7.

We also prove a version of Theorem 1.3.1 for $\mathcal{O}^+$-vector bundles:

Theorem 1.3.2 (Theorem 6.8.4, Corollary 6.8.3). *Let X be a perfectoid space over* $\mathrm{Spa}\,(\mathbf{Q}_p, \mathbf{Z}_p)$. *Then*

(1) *the categories* $\mathrm{Vect}(X_{\text{ét}}; \mathcal{O}^+_{X_{\text{ét}}})$, $\mathrm{Vect}(X^\diamond_{\text{qp}}; \mathcal{O}^+_{X^\diamond_{\text{qp}}})$, *and* $\mathrm{Vect}(X^\diamond_v; \mathcal{O}^+_{X^\diamond})$ *are equivalent;*

(2) *these equivalences preserve cohomology groups;*

(3) *for any $\mathcal{O}^+_{X^\diamond}$-vector bundle $\mathcal{E}$ and a point $x \in X$, there exist an open affinoid subspace $x \in U_x \subset X$ and a finite étale surjective morphism $\widetilde{U}_x \to U_x$ such that $\mathcal{E}|_{\widetilde{U}^\diamond_x}$ is a free vector bundle.*

We also refer to Theorem 6.8.4 for a slightly more precise statement. As an application of our methods, we can also deduce the following theorem of Kedlaya–Liu:

Theorem 1.3.3 ([42, Theorem 3.5.8], [63, Lemma 17.1.8], [35, Theorem 4.27], Theorem 6.8.13). *Let X be a perfectoid space over* $\mathrm{Spa}\,(\mathbf{Q}_p, \mathbf{Z}_p)$.

(1) *The categories* $\mathrm{Vect}(X_{\text{an}}, \mathcal{O}_X)$, $\mathrm{Vect}(X_{\text{ét}}; \mathcal{O}_{X_{\text{ét}}})$, $\mathrm{Vect}(X^\diamond_{\text{qp}}; \mathcal{O}_{X^\diamond_{\text{qp}}})$, *as well as* $\mathrm{Vect}(X^\diamond_v; \mathcal{O}_{X^\diamond})$ *are equivalent. Furthermore, if* $X = \mathrm{Spa}\,(R, R^+)$ *is an affinoid perfectoid, all these categories are equivalent to the category of finite projective R-modules.*

(2) *These equivalences preserve cohomology groups.*

We note that the proof of Theorem 1.3.3 is quite different from the proofs of [42, Theorem 3.5.8] and [63, Lemma 17.1.8]. However, it is quite similar to the proof of [35, Theorem 4.27] (with appropriate simplifications). We also note that [35, Theorem 4.27] proves a stronger result that applies to G-torsors for any rigid group G. We also show that any $\mathcal{O}$-vector bundle in the v-topology admits an $\mathcal{O}^+$-lattice after a very explicit étale cover:

Theorem 1.3.4 (Corollary 6.8.14). *Let X denote a strongly sheafy adic space over* $\mathrm{Spa}\,(\mathbf{Q}_p, \mathbf{Z}_p)$, *and let $\mathcal{E}$ be an $\mathcal{O}_{X^\diamond}$-vector bundle. Then, for each $x \in X$, there are an open subspace $x \in U_x \subset X$, a finite étale surjective morphism $\widetilde{U}_x \to U_x$, and an $\mathcal{O}^+_{\widetilde{U}^\diamond_x}$-vector bundle $\mathcal{E}^+_x$ such that $\mathcal{E}^+_x\left[\frac{1}{p}\right] \simeq \mathcal{E}|_{\widetilde{U}_x}$.*

1.4 p-adic nearby cycles sheaves (Chapter 7)

The main goal of Chapter 7 is to give the main non-trivial example of almost coherent sheaves: the p-adic nearby cycles sheaves.

We fix a p-adic perfectoid field K and a rigid-analytic variety X over K with an admissible formal $\mathcal{O}_K$-model $\mathfrak{X}$.

The rigid-analytic variety X comes with a morphism of ringed sites

$$\nu\colon \left(X_v^\diamond, \mathcal{O}^+_{X^\diamond}\right) \to \left(\mathfrak{X}_{\mathrm{Zar}}, \mathcal{O}_{\mathfrak{X}}\right)$$

and a morphism

$$\nu\colon \left(X_v^\diamond, \mathcal{O}^+_{X^\diamond}/p\right) \to \left(\mathfrak{X}_{0,\mathrm{Zar}}, \mathcal{O}_{\mathfrak{X}_0}\right),$$

where $\mathfrak{X}_0$ is the mod-p fiber of $\mathfrak{X}$, $X_v^\diamond$ is the v-site of the associated diamond (see Section 6.1), and $\mathcal{O}^+_{X^\diamond}$ is its integral "untilted" structure sheaf (see Definition 6.3.1).

The main goal of Chapter 7 is to show that the nearby cycles functor $\mathbf{R}\nu_*$ sends some class of $\mathcal{O}^+_{X^\diamond}/p$-sheaves to complexes of almost coherent $\mathcal{O}_{\mathfrak{X}_0}$-modules. More precisely, we show that, for any $\mathcal{O}^+_{X^\diamond}/p$-vector bundle $\mathcal{E}$, the complex $\mathbf{R}\nu_*\mathcal{E}$ has quasi-coherent and almost coherent cohomology sheaves. We also give a bound on its almost cohomological dimension.

Theorem 1.4.1 (Theorem 7.1.2). *Let $\mathfrak{X}$ be an admissible formal $\mathcal{O}_K$-scheme with adic generic fiber X of dimension d and mod-p fiber $\mathfrak{X}_0$, and let $\mathcal{E}$ be an $\mathcal{O}^+_{X^\diamond}/p$-vector bundle. Then*

(1) $\mathbf{R}\nu_*\mathcal{E} \in \mathbf{D}^+_{\mathrm{qc,acoh}}(\mathfrak{X}_0)$ *and* $(\mathbf{R}\nu_*\mathcal{E})^a \in \mathbf{D}^{[0,2d]}_{\mathrm{acoh}}(\mathfrak{X}_0)^a$*;*

(2) *if $\mathfrak{X} = \operatorname{Spf} A$ is affine, then the natural map*

$$\widetilde{\mathrm{H}^i\left(X_v^\diamond, \mathcal{E}\right)} \to \mathrm{R}^i\nu_*(\mathcal{E})$$

is an isomorphism for every $i \geq 0$;

(3) *the formation of $\mathrm{R}^i\nu_*(\mathcal{E})$ commutes with étale base change, i.e., for any étale morphism $\mathfrak{f}\colon\mathfrak{Y} \to \mathfrak{X}$ with adic generic fiber $f\colon Y \to X$, the natural morphism*

$$\mathfrak{f}_0^*\left(\mathrm{R}^i\nu_{\mathfrak{X},*}(\mathcal{E})\right) \to \mathrm{R}^i\nu_{\mathfrak{Y},*}\left(\mathcal{E}|_{Y^\diamond}\right)$$

is an isomorphism for any $i \geq 0$;

(4) *if $\mathfrak{X}$ has an open affine covering $\mathfrak{X} = \bigcup_{i\in I}\mathfrak{U}_i$ such that $\mathcal{E}|_{(\mathfrak{U}_{i,K})^\diamond}$ is small (see Definition 7.1.1), then*

$$(\mathbf{R}\nu_*\mathcal{E})^a \in \mathbf{D}^{[0,d]}_{\mathrm{acoh}}(\mathfrak{X}_0)^a;$$

(5) *there is an admissible blow-up $\mathfrak{X}' \to \mathfrak{X}$ such that $\mathfrak{X}'$ has an open affine covering $\mathfrak{X}' = \bigcup_{i\in I}\mathfrak{U}_i$ such that $\mathcal{E}|_{(\mathfrak{U}_{i,K})^\diamond}$ is small.*
In particular, there is a cofinal family of admissible formal models $\{\mathfrak{X}'_i\}_{i\in I}$ of X such that

$$(\mathbf{R}\nu_{\mathfrak{X}'_i,*}\mathcal{E})^a \in \mathbf{D}^{[0,d]}_{\mathrm{acoh}}(\mathfrak{X}'_{i,0})^a,$$

for each $i \in I$.

Remark 1.4.2. We note that Theorem 1.4.1 implies that the nearby cycles complex $\mathbf{R}\nu_*\mathcal{E}$ is quasi-coherent on the nose (as opposed to being almost quasi-coherent). This is quite unexpected to the author since all previous results on the cohomology groups of $\mathcal{O}^+/p$ were only available in the almost category.

Remark 1.4.3. We do not know if an admissible blow-up $\mathfrak{X}' \to \mathfrak{X}$ in the formulation of Theorem 1.4.1 is really necessary or just an artifact of the proof. More importantly, we do not know if, for every $\mathcal{O}^+_{X^\diamond}/p$-vector bundle $\mathcal{E}$, there is an admissible formal model $\mathfrak{X}$ such that the "nearby cycles" sheaf $\mathbf{R}\nu_{\mathfrak{X},*}\mathcal{E}$ lies in $\mathbf{D}^{[0,d]}_{\mathrm{acoh}}(\mathfrak{X}_0)^a$.

The proof of Theorem 1.4.1 crucially uses Theorem 1.3.1, and especially Theorem 1.3.1 (3).

Another family of sheaves for which we can establish a good behavior of "nearby cycles" is given by sheaves of the form $\mathcal{F} \otimes \mathcal{O}^+_{X^\diamond}/p$ for a Zariski-constructible étale sheaf of $\mathbf{F}_p$-modules (see Definition 7.1.7). Namely, in this case, we can get a better cohomological bound and show that nearby cycles almost commute with proper base change, as this happens in algebraic geometry.

Theorem 1.4.4 (Theorem 7.1.9 and Lemma 7.3.8). *Let $\mathfrak{X}$ be an admissible formal $\mathcal{O}_K$-scheme with adic generic fiber X of dimension d and mod-p fiber $\mathfrak{X}_0$, and let $\mathcal{F} \in \mathbf{D}^{[r,s]}_{zc}(X;\mathbf{F}_p)$. Then*

(1) *there is an isomorphism $\mathbf{R}t_*(\mathcal{F} \otimes \mathcal{O}^+_{X_{\text{ét}}}/p) \simeq \mathbf{R}\nu_*(\mathcal{F} \otimes \mathcal{O}^+_{X^\diamond}/p)$;*

(2) *$\mathbf{R}\nu_*(\mathcal{F} \otimes \mathcal{O}^+_{X^\diamond}/p) \in \mathbf{D}^+_{\mathrm{qc,acoh}}(\mathfrak{X}_0)$ and $\mathbf{R}\nu_*(\mathcal{F} \otimes \mathcal{O}^+_{X^\diamond}/p)^a \in \mathbf{D}^{[r,s+d]}_{\mathrm{acoh}}(\mathfrak{X}_0)^a$;*

(3) *if $\mathfrak{X} = \mathrm{Spf}\ A$ is affine, then the natural map*

$$\mathrm{H}^i\big(\widehat{X^\diamond_v, \mathcal{F} \otimes \mathcal{O}^+_{X^\diamond}}/p\big) \to \mathrm{R}^i\nu_*\big(\mathcal{F} \otimes \mathcal{O}^+_{X^\diamond}/p\big)$$

is an isomorphism for every $i \geq 0$;

(4) *the formation of $\mathrm{R}^i\nu_*(\mathcal{F} \otimes \mathcal{O}^+_{X^\diamond}/p)$ commutes with étale base change, i.e., for any étale morphism $\mathfrak{f}\colon \mathfrak{Y} \to \mathfrak{X}$ with adic generic fiber $f\colon Y \to X$, the natural morphism*

$$\mathfrak{f}_0^*\big(\mathrm{R}^i\nu_{\mathfrak{X},*}\big(\mathcal{F} \otimes \mathcal{O}^+_{X^\diamond}/p\big)\big) \to \mathrm{R}^i\nu_{\mathfrak{Y},*}\big(f^{-1}\mathcal{F} \otimes \mathcal{O}^+_{Y^\diamond}/p\big)$$

is an isomorphism for any $i \geq 0$;

(5) *if $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{Y}$ is a proper morphism of admissible formal $\mathcal{O}_K$-schemes with adic generic fiber $f\colon X \to Y$, then the natural morphism*

$$\mathbf{R}\nu_{\mathfrak{Y},*}\big(\mathbf{R}f_*\mathcal{F} \otimes \mathcal{O}^+_{Y^\diamond}/p\big) \to \mathbf{R}\mathfrak{f}_{0,*}\big(\mathbf{R}\nu_{\mathfrak{X},*}\big(\mathcal{F} \otimes \mathcal{O}^+_{X^\diamond}/p\big)\big)$$

is an almost isomorphism.

We also show an integral version of Theorem 1.4.1:

Theorem 1.4.5 (Theorem 7.1.11). *Let $\mathfrak{X}$ be an admissible formal $\mathcal{O}_K$-scheme with adic generic fiber X of dimension d, and let $\mathcal{E}$ be an $\mathcal{O}^+_{X^\diamond}$-vector bundle. Then*

(1) $\mathbf{R}\nu_*\mathcal{E} \in \mathbf{D}^+_{\mathrm{qc,acoh}}(\mathfrak{X})$ *and* $(\mathbf{R}\nu_*\mathcal{E})^a \in \mathbf{D}^{[0,2d]}_{\mathrm{acoh}}(\mathfrak{X})^a$*;*

(2) *if* $\mathfrak{X} = \mathrm{Spf}\ A$ *is affine, then the natural map*

$$\mathrm{H}^i\left(X^\diamondsuit_v, \mathcal{E}\right)^\Delta \to \mathrm{R}^i\nu_*(\mathcal{E})$$

is an isomorphism for every $i \geq 0$*;*

(3) *the formation of* $\mathrm{R}^i\nu_*(\mathcal{E})$ *commutes with étale base change, i.e., for any étale morphism* $\mathfrak{f}\colon\mathfrak{Y} \to \mathfrak{X}$ *with adic generic fiber* $f\colon Y \to X$*, the natural morphism*

$$\mathfrak{f}^*\left(\mathrm{R}^i\nu_{\mathfrak{X},*}(\mathcal{E})\right) \to \mathrm{R}^i\nu_{\mathfrak{Y},*}\left(\mathcal{E}|_{Y^\diamond}\right)$$

is an isomorphism for any $i \geq 0$*;*

(4) *if* $\mathfrak{X}$ *has an open affine covering* $\mathfrak{X} = \bigcup_{i\in I}\mathfrak{U}_i$ *such that* $\mathcal{E}|_{(\mathfrak{U}_{i,K})^\diamond}$ *is small (see Definition 7.1.10), then*

$$(\mathbf{R}\nu_*\mathcal{E})^a \in \mathbf{D}^{[0,d]}_{\mathrm{acoh}}(\mathfrak{X})^a;$$

(5) *there is an admissible blow-up* $\mathfrak{X}' \to \mathfrak{X}$ *such that* $\mathfrak{X}'$ *has an open affine covering* $\mathfrak{X}' = \bigcup_{i\in I}\mathfrak{U}_i$ *such that* $\mathcal{E}|_{(\mathfrak{U}_{i,K})^\diamond}$ *is small.*
In particular, there is a cofinal family of admissible formal models $\{\mathfrak{X}'_i\}_{i\in I}$ *of* X *such that*

$$(\mathbf{R}\nu_{\mathfrak{X}'_i,*}\mathcal{E})^a \in \mathbf{D}^{[0,d]}_{\mathrm{acoh}}(\mathfrak{X}'_i)^a,$$

for each $i \in I$*.*

Theorem 1.4.5 has an interesting consequence saying that v-cohomology groups of any $\mathcal{O}^+_{X^\diamond}$-vector bundle are almost coherent and almost vanish in degrees larger than $2\dim X$. This (together with Theorem 1.1.1) indicates that there should probably be stronger (almost) finiteness results for some bigger class $\mathcal{O}^+_{X^\diamond}$-modules.

Theorem 1.4.6 (Theorem 7.3.3). *Let K be a p-adic perfectoid field, let X be a proper rigid-analytic K-variety of dimension d, and let be $\mathcal{E}$ an $\mathcal{O}^+_{X^\diamond}$-vector bundle (resp. $\mathcal{O}^+_{X^\diamond}/p$-vector bundle). Then*

$$\mathbf{R}\Gamma(X^\diamondsuit_v, \mathcal{E}) \in \mathbf{D}^{[0,2d]}_{\mathrm{acoh}}(\mathcal{O}_K)^a.$$

We now explain the main steps of our proofs of Theorems 1.4.1 and 1.4.5 for $\mathcal{E} = \mathcal{O}^+_{X^\diamond}/p$ and $\mathcal{E} = \mathcal{O}^+_{X^\diamond}$ respectively:

Proof sketch. (1) We first show that the sheaves $\mathrm{R}^i\nu_*(\mathcal{O}^+_{X^\diamond}/p)$ are quasi-coherent. The main key input is that the cohomology groups of $\mathcal{O}^+_{X^\diamond}/p$-vector bundles vanish on strictly totally disconnected spaces (see Definition 6.2.5), and that each affinoid rigid-analytic variety admits a v-covering such that all terms of its Čech nerve are strictly totally disconnected.

(2) The same ideas can be used to show that the formation of $\mathrm{R}^i\nu_*(\mathcal{O}^+_{X^\diamond}/p)$ commutes with étale base change.

(3) We show next that the $\mathcal{O}_{\mathfrak{X}_0}$-modules $\mathrm{R}^i\nu_*\big(\mathcal{O}^+_{X^\diamond}/p\big)$ are almost coherent for smooth X. This is done in three steps: first, we find an admissible blow-up $\mathfrak{X}' \to \mathfrak{X}$ such that $\mathfrak{X}'$ has an open affine covering $\mathfrak{X}' = \bigcup_{i\in I}\mathfrak{U}_i$ such that each $\mathfrak{U}_i = \mathrm{Spf}\, A_i$ admits a finite rig-étale morphism to $\widehat{\mathbf{A}}^d_{\mathcal{O}_K}$, then we show that the cohomology groups $\mathrm{H}^i(\mathfrak{U}^\diamond_{i,K,v}, \mathcal{O}^+_{X^\diamond}/p)$ are almost coherent over A_i/pA_i, and after that we conclude almost coherence of $\mathrm{R}^i\nu_*\big(\mathcal{O}^+_{X^\diamond}/p\big)$.

The first step is the combination of [15, Proposition 3.7] and Theorem D.4. The first result allows us to choose an admissible blow-up $\mathfrak{X}' \to \mathfrak{X}$ with an open affine covering $\mathfrak{X}' = \bigcup_{i\in I}\mathfrak{U}_i$ such that each $\mathfrak{U}_i$ admits a rig-étale morphism $\mathfrak{U}_i \to \widehat{\mathbf{A}}^d_{\mathcal{O}_K}$. Then Theorem D.4 guarantees that we can change these morphisms so that they become *finite* and rig-étale.

The second step follows the strategy presented in [59]. We construct an explicit affinoid perfectoid cover of $\mathfrak{U}_i$ that is a $\mathbf{Z}_p(1)^d$-torsor. So we can reduce studying $\mathrm{H}^i(\mathfrak{U}^\diamond_{i,K,v}, \mathcal{O}^+_{X^\diamond}/p)$ to studying cohomology groups of $\mathbf{Z}_p(1)^d$ that can be explicitly understood via the Koszul complex.

The last step is the consequence of the almost proper mapping theorem in Theorem 1.2.9 and the already obtained results.

(4) The next step is to show that $\mathrm{R}^i\nu_*\big(\mathcal{O}^+_{X^\diamond}/p\big)$ is almost coherent for a general X. This is done by choosing a proper hypercovering by smooth spaces $X_\bullet$ and then using a version of cohomological v-descent to conclude almost coherence of the p-adic nearby cycles sheaves. As an important technical tool, we use the theory of diamonds developed in [61].

(5) Next we show that $\mathbf{R}\nu_*\big(\mathcal{O}^+_{X^\diamond}/p\big)$ is almost concentrated in degrees $[0,d]$. This claim is quite subtle. The key input is the version of the purity theorem [10, Theorem 10.11] that implies that any *finite* (but not necessarily étale) adic space over an affinoid perfectoid space has a diamond that is isomorphic to a diamond of an affinoid perfectoid space. This allows us to reduce the question of cohomological bounds of $\mathbf{R}\nu_*\big(\mathcal{O}^+_{X^\diamond}/p\big)^a$ to the question about the cohomological dimension of the pro-finite group $\mathbf{Z}_p(1)^d$. This can be explicitly understood via the Koszul complex again.

(6) Finally, we show Theorem 1.4.5 by reducing it to Theorem 1.4.1. The key input is Theorem 1.2.3 that allows us to check finiteness mod-p. ■

1.5 Notation

A *non-archimedean field* K is always assumed to be complete. A non-archimedean field K is called *p-adic* if its ring of power-bounded elements $\mathcal{O}_K = K^\circ$ is a ring of mixed characteristic $(0, p)$.

We follow [68, Tag 02MN] for the definition of a (weak) Serre subcategory of an abelian category $\mathcal{A}$.

For an ringed R-site $(X, \mathcal{O}_X)$, an element of the derived category $\mathcal{F} \in \mathbf{D}(X)$, and an element $\varpi \in R$, we denote by $[\mathcal{F}/\varpi]$ the cone of the multiplication by ϖ-morphism, i.e.,

$$[\mathcal{F}/\varpi] := \mathrm{cone}(\mathcal{F} \xrightarrow{\varpi} \mathcal{F}).$$

Namely, we say that a non-empty full subcategory $\mathcal{C}$ of an abelian category $\mathcal{A}$ is a *Serre subcategory* if, for any exact sequence $A \to B \to C$ with $A, C \in \mathcal{C}$, we have $B \in \mathcal{C}$. We say that $\mathcal{C}$ is a *weak Serre subcategory* if, for any exact sequence

$$A_0 \to A_1 \to A_2 \to A_3 \to A_4$$

with $A_0, A_1, A_3, A_4 \in \mathcal{C}$, we have $A_2 \in \mathcal{C}$. Look at [68, Tag 02MP] and [68, Tag 0754] for an alternative way to describe (weak) Serre subcategories.

If $\mathcal{C}$ is a Serre subcategory of an abelian category $\mathcal{A}$, we define the *quotient category* as a pair $(\mathcal{A}/\mathcal{C}, F)$ of an abelian category $\mathcal{A}/\mathcal{C}$ and an exact functor

$$F \colon \mathcal{A} \to \mathcal{A}/\mathcal{C}$$

such that, for any exact functor $G \colon \mathcal{A} \to \mathcal{B}$ to an abelian category $\mathcal{B}$ with $\mathcal{C} \subset \operatorname{Ker} G$, there is a factorization $G = H \circ F$ for a unique exact functor $H \colon \mathcal{A}/\mathcal{C} \to \mathcal{B}$. The quotient category always exists by [68, Tag 02MS].

If $\mathcal{B}$ is a full triangulated subcategory of a triangulated category $\mathcal{D}$, we define the *Verdier quotient* as a pair $(\mathcal{D}/\mathcal{B}, F)$ of a triangulated category $\mathcal{D}/\mathcal{B}$ and an exact functor

$$F \colon \mathcal{D} \to \mathcal{D}/\mathcal{B}$$

such that, for any exact functor $G \colon \mathcal{D} \to \mathcal{D}'$ to a pre-triangulated category $\mathcal{D}'$ with $\mathcal{B} \subset \operatorname{Ker} G$, there is a factorization $G = H \circ F$ for a unique exact functor $H \colon \mathcal{D}/\mathcal{B} \to \mathcal{D}'$. The Verdier quotient always exists by [68, Tag 05RJ].

We say that a diagram of categories

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{f} & \mathcal{B} \\ {\scriptstyle h}\downarrow & \overset{\alpha}{\nearrow\!\!\!\!\nearrow} & \downarrow{\scriptstyle g} \\ \mathcal{C} & \xrightarrow[k]{} & \mathcal{D} \end{array}$$

is *$(2,1)$-commutative* if $\alpha \colon k \circ h \Rightarrow g \circ f$ is a natural isomorphism of functors.

For an abelian group M and commuting endomorphisms $f_1, \dots, f_n$, we define the *Koszul complex*

$$K(M; f_1, \dots, f_n) := M \to M \otimes_{\mathbf{Z}} \mathbf{Z}^n \to M \otimes_{\mathbf{Z}} \wedge^2(\mathbf{Z}^n) \to \cdots \to M \otimes_{\mathbf{Z}} \wedge^n(\mathbf{Z}^n)$$

viewed as a chain complex in cohomological degrees $0, \dots, n$. The differential

$$d^k : M \otimes_{\mathbf{Z}} \wedge^k(\mathbf{Z}^n) \simeq \bigoplus_{1 \le i_1 < \cdots < i_k \le n} M \to M \otimes_{\mathbf{Z}} \wedge^{i+1}(\mathbf{Z}^n) \simeq \bigoplus_{1 \le j_1 < \cdots < j_{k+1} \le n} M$$

from M in spot $i_1 < \cdots < i_k$ to M in spot $j_1 < \cdots < j_{k+1}$ is nonzero only if $\{i_1, \dots, i_k\} \subset \{j_1, \dots, j_{k+1}\}$, in which case it is given by $(-1)^{m-1} f_{j_m}$, where $m \in \{1, \dots, k+1\}$ is the unique integer such that $j_m \notin \{i_1, \dots, i_k\}$.

If M is an R-module and f_i are elements of R, the complex $K(M; f_1, \dots, f_n)$ is a complex of R-modules and can be identified with

$$M \to M \otimes_R R^n \to M \otimes_R \wedge^2(R^n) \to \cdots \to M \otimes_R \wedge^n(R^n).$$