Chapter 2

Almost commutative algebra

This chapter is devoted to the study of almost coherent modules. We recall some basic definitions of almost mathematics in Section 2.1. Then we discuss the main properties of almost finitely generated and almost finitely presented modules in Section 2.5. These two sections closely follow the discussion of almost mathematics in [26]. Section 2.6 is dedicated to almost coherent modules and almost coherent rings. We show that almost coherent modules form a weak Serre subcategory of R-modules, and they coincide with almost finitely presented ones in the case of almost coherent rings. We discuss base change results in Section 2.8. Finally, we develop some topological aspects of almost finitely generated modules over "topologically universally adhesive rings" in Section 2.12.

2.1 The category of almost modules

We begin this section by recalling basic definitions of almost mathematics from [26]. We fix a "base" ring R with an ideal $\mathfrak{m}$ such that $\mathfrak{m}^2 = \mathfrak{m}$ and $\widetilde{\mathfrak{m}} = \mathfrak{m} \otimes_R \mathfrak{m}$ is flat. We always do almost mathematics with respect to $\mathfrak{m}$.

Lemma 2.1.1. *Let M be an R-module. Then the following are equivalent:*

(1) *The module $\mathfrak{m}M$ is the zero module.*

(2) *The module $\mathfrak{m} \otimes_R M$ is the zero module.*

(3) *The module $\widetilde{\mathfrak{m}} \otimes_R M$ is the zero module.*

(4) *The module M is annihilated by ε for every $\varepsilon \in \mathfrak{m}$.*

Proof. Note that the multiplication map $\mathfrak{m} \otimes_R \mathfrak{m} \to \mathfrak{m}$ is surjective as $\mathfrak{m}^2 = \mathfrak{m}$. This implies that we have surjections

$$\widetilde{\mathfrak{m}} \otimes_R M \twoheadrightarrow \mathfrak{m} \otimes_R M \twoheadrightarrow \mathfrak{m}M.$$

This shows that (3) implies (2), and (2) implies (1). It is clear that (2) implies (3), and (1) is equivalent to (4). So the only thing we are left to show is that (1) implies (2).

Suppose that $\mathfrak{m}M \simeq 0$. Pick an arbitrary basic element $a \otimes m \in \mathfrak{m} \otimes_R M$ with $a \in \mathfrak{m}$, $m \in M$. Since $\mathfrak{m}^2 = \mathfrak{m}$, there is a finite number of elements $y_1, \dots, y_k$, $x_1, \dots, x_k \in \mathfrak{m}$ such that

$$a = \sum_{i=1}^{k} x_i y_i.$$

Then we have an equality

$$a \otimes m = \sum_{i=1}^{k} x_i y_i \otimes m = \sum_{i=1}^{k} x_i \otimes y_i m = 0. \qquad \blacksquare$$

Definition 2.1.2. An R-module M is *almost zero*, if any of the equivalent conditions of Lemma 2.1.1 is satisfied for M.

Lemma 2.1.3. *Under the same assumption as above, the "multiplication" morphism* $\widetilde{\mathfrak{m}} \otimes_R \widetilde{\mathfrak{m}} \to \widetilde{\mathfrak{m}}$ *is an isomorphism.*

Proof. We consider a short exact sequence

$$0 \to \mathfrak{m} \to R \to R/\mathfrak{m} \to 0.$$

Note that $(R/\mathfrak{m}) \otimes_R \mathfrak{m} = \mathfrak{m}/\mathfrak{m}^2 = 0$, so we get a short exact sequence

$$0 \to \mathrm{Tor}_1^R(R/\mathfrak{m}, \mathfrak{m}) \to \widetilde{\mathfrak{m}} \to \mathfrak{m} \to 0.$$

Since $\mathrm{Tor}_1^R(R/\mathfrak{m}, \mathfrak{m})$ is almost zero, Lemma 2.1.1 says that after applying the functor $- \otimes_R \widetilde{\mathfrak{m}}$ we get an isomorphism

$$\widetilde{\mathfrak{m}} \otimes_R \widetilde{\mathfrak{m}} \simeq \mathfrak{m} \otimes_R \widetilde{\mathfrak{m}}.$$

Since $\widetilde{\mathfrak{m}}$ is R-flat, we also see that $\mathfrak{m} \otimes_R \widetilde{\mathfrak{m}}$ injects into $\widetilde{\mathfrak{m}}$. Moreover, it maps isomorphically onto its image $\mathfrak{m}\widetilde{\mathfrak{m}} = \widetilde{\mathfrak{m}}$ as $\mathfrak{m}^2 = \mathfrak{m}$. Taken together, it shows that

$$\widetilde{\mathfrak{m}} \otimes_R \widetilde{\mathfrak{m}} \simeq \widetilde{\mathfrak{m}}.$$

It is straightforward to see that the constructed isomorphism is the "multiplication" map. ■

We denote by Σ_R the category of almost zero R-modules considered as a full subcategory of $\mathbf{Mod}_R$.

Corollary 2.1.4. *The category* Σ_R *is a Serre subcategory of* $\mathbf{Mod}_R$.[1]

Proof. This follows directly from criterion (3) from Lemma 2.1.1, flatness of $\widetilde{\mathfrak{m}}$ and [68, Tag 02MP]. ■

This corollary allows us to define the quotient category[2] $\mathbf{Mod}_R^a := \mathbf{Mod}_R/\Sigma_R$ that we call as the category of almost R-modules. Note that the localization functor

$$(-)^a \colon \mathbf{Mod}_R \to \mathbf{Mod}_R^a$$

[1]We refer to [68, Tag 02MN] for the discussion of (weak) Serre categories.
[2]We refer to [68, Tag 02MS] for the discussion of quotient categories.

is an exact and essentially surjective functor. We refer to elements of $\mathbf{Mod}_R^a$ as almost R-modules or R^a-modules. We will usually denote them by M^a to distinguish almost R-modules from R-modules.

To simplify the exposition, we will use the notation $\mathbf{Mod}_R^a$ and $\mathbf{Mod}_{R^a}$ interchangeably.

Definition 2.1.5. A morphism $f : M \to N$ is *an almost isomorphism* (resp. *almost injection*, resp. *almost surjection*) if the corresponding morphism $f^a : M^a \to N^a$ is an isomorphism (resp. injection, resp. surjection) in $\mathbf{Mod}_R^a$.

Remark 2.1.6. For any R-module M, the natural morphism $\pi \colon \widetilde{\mathfrak{m}} \otimes_R M \to M$ is an almost isomorphism. Indeed, it suffices to show that

$$\widetilde{\mathfrak{m}} \otimes_R \operatorname{Ker} \pi \simeq 0 \text{ and } \widetilde{\mathfrak{m}} \otimes_R \operatorname{Coker} \pi \simeq 0.$$

Using R-flatness of $\widetilde{\mathfrak{m}}$, we can reduce the question to showing that the map

$$\widetilde{\mathfrak{m}} \otimes_R \pi \colon \widetilde{\mathfrak{m}} \otimes_R \widetilde{\mathfrak{m}} \otimes_R M \to \widetilde{\mathfrak{m}} \otimes_R M$$

is an isomorphism. This follows from Lemma 2.1.3.

Definition 2.1.7. Two R-modules M and N are called *almost isomorphic* if M^a is isomorphic to N^a in $\mathbf{Mod}_R^a$.

Lemma 2.1.8. *Let $f \colon M \to N$ be a morphism of R-modules, then the following hold:*

1. *The morphism f is an almost injection (resp. almost surjection, resp. almost isomorphism) if and only if* $\operatorname{Ker}(f)$ *(resp.* $\operatorname{Coker}(f)$*, resp. both* $\operatorname{Ker}(f)$ *and* $\operatorname{Coker}(f)$*) is an almost zero module.*
2. *We have a functorial bijection* $\operatorname{Hom}_R(\widetilde{\mathfrak{m}} \otimes_R M, N) \simeq \operatorname{Hom}_{\mathbf{Mod}_R^a}(M^a, N^a)$.
3. *Modules M and N are almost isomorphic (not necessarily via the morphism f) if and only if $\widetilde{\mathfrak{m}} \otimes_R M \simeq \widetilde{\mathfrak{m}} \otimes_R N$.*

Proof. (1) just follows from definition of the quotient category. (2) is discussed in detail in [26, page 12 (2.2.4)].

Next we show that (3) follows from (1) and (2). Remark 2.1.6 implies that M and N are almost isomorphic if $\widetilde{\mathfrak{m}} \otimes_R M \simeq \widetilde{\mathfrak{m}} \otimes_R N$.

Suppose that there is an almost isomorphism $\varphi \colon M^a \to N^a$. It has a representative $f \colon \widetilde{\mathfrak{m}} \otimes_R M \to N$ by (2). Now (1) together with the R-flatness of $\widetilde{\mathfrak{m}}$ implies that $\widetilde{\mathfrak{m}} \otimes_R f \colon \widetilde{\mathfrak{m}} \otimes_R \widetilde{\mathfrak{m}} \otimes_R M \to \widetilde{\mathfrak{m}} \otimes_R N$ is an isomorphism. Lemma 2.1.3 ensures that $\widetilde{\mathfrak{m}} \otimes_R \widetilde{\mathfrak{m}} \simeq \widetilde{\mathfrak{m}}$, so $\widetilde{\mathfrak{m}} \otimes_R f$ gives an isomorphism

$$\widetilde{\mathfrak{m}} \otimes_R f : \widetilde{\mathfrak{m}} \otimes_R M \to \widetilde{\mathfrak{m}} \otimes_R N.$$

∎

We now define the *functor of almost sections*

$$(-)_*\colon \mathbf{Mod}_R^a \to \mathbf{Mod}_R$$

via the formula

$$(M^a)_* := \mathrm{Hom}_{\mathbf{Mod}_R^a}(R^a, M^a) = \mathrm{Hom}_R(\widetilde{\mathfrak{m}}, M)$$

for any R^a-module M^a with an R-module representative M. The construction is clearly functorial in M^a, so it defines the functor $(-)_*\colon \mathbf{Mod}_R^a \to \mathbf{Mod}_R$.

The functor of almost sections will be the right adjoint to the almostification functor $(-)^a$. Before we discuss why this is the case, we need to define the unit and counit transformations.

We start with the unit of the adjunction. For any R-module M, there is a functorial morphism

$$\eta_{M,*}\colon M \to \mathrm{Hom}_R(\widetilde{\mathfrak{m}}, M) = M_*^a$$

that can easily be seen to be an almost isomorphism.

This allows us to define a functorial morphism

$$\varepsilon_{N^a,*}\colon (N_*^a)^a \to N^a$$

for any R^a-module N^a. Namely, the map $\eta_{N,*}\colon N \to N_*^a$ is an almost isomorphism, so we can invert it in the almost category and define

$$\varepsilon_{N^a,*} := (\eta_{N,*}^a)^{-1}\colon (N_*^a)^a \to N^a.$$

Now we define another functor

$$(-)_!\colon \mathbf{Mod}_R^a \to \mathbf{Mod}_R$$

that will be a left adjoint to the almostification functor $(-)^a$. Namely, we put

$$(M^a)_! := (M^a)_* \otimes_R \widetilde{\mathfrak{m}} \xleftarrow{\sim} M \otimes_R \widetilde{\mathfrak{m}}$$

for any R^a-module M^a with an R-module representative M. This construction is clearly functorial in M^a, so it does define a functor. Similarly to the discussion above, for any R-module M, we define the transformation

$$\varepsilon_{M,!}\colon (M^a)_! = \widetilde{\mathfrak{m}} \otimes_R M \to M$$

as the map induced by the natural morphism $\widetilde{\mathfrak{m}} \to R$. Clearly, $\varepsilon_{M,!}$ is an almost isomorphism for any M. Therefore, this actually allows us to define the morphism

$$\eta_{N^a,!}\colon N^a \to (\widetilde{\mathfrak{m}} \otimes_R N)^a \simeq (N_!^a)^a$$

as $\eta_{N^a,!} = (\varepsilon_{N,!}^a)^{-1}$. We summarize the main properties of these functors in the following lemma:

Lemma 2.1.9. *Let R and $\mathfrak{m}$ be as above. Then the following hold:*

(1) *The functor $(-)_*$ is the right adjoint to $(-)^a$. In particular, it is left exact.*

(2) *The unit of the adjunction is equal to $\eta_{M,*}$, the counit of the adjunction is equal to $\varepsilon_{N^a,*}$. In particular, both are isomorphisms.*

(3) *The functor $(-)_!$ is the left adjoint to the localization functor $(-)^a$.*

(4) *The functor $(-)_! \colon \mathbf{Mod}_R^a \to \mathbf{Mod}_R$ is exact.*

(5) *The unit of the adjunction is equal to $\eta_{N^a,!}$, the counit of the adjunction is equal to $\varepsilon_{M,!}$. In particular, both are almost isomorphisms.*

Proof. This is explained in [26, Proposition 2.2.13 and Proposition 2.2.21]. ■

Corollary 2.1.10. *Let R and $\mathfrak{m}$ be as above. Then $(-)^a \colon \mathbf{Mod}_R \to \mathbf{Mod}_R^a$ commutes with limits and colimits. In particular, $\mathbf{Mod}_R^a$ is complete and cocomplete, and filtered colimits and (arbitrary) products are exact in $\mathbf{Mod}_R^a$.*

Proof. The first claim follows from the fact that $(-)^a$ admits left and right adjoints. The second claim follows the first claim, exactness of $(-)^a$, and analogous exactness properties in $\mathbf{Mod}_R$. ■

The last thing we need to address in this section is how almost mathematics interacts with base change. We want to be able to talk about preservation of various properties of modules under a base change along a map $R \to S$. The issue here is to define the corresponding ideal $\mathfrak{m}_S$ as in the definition of almost mathematics. It turns out that the most naive ideal $\mathfrak{m}_S := \mathfrak{m}S$ does define an ideal of almost mathematics in S, but this is not entirely formal and crucially uses our choice of definition for an ideal of almost mathematics.

More precisely, if one starts with a flat ideal $\mathfrak{m} \subset R$, then the ideal $\mathfrak{m}_S \subset S$ is not necessarily flat. However, we show that flatness of $\widetilde{\mathfrak{m}}$ implies flatness of $\widetilde{\mathfrak{m}_S}$. For this reason, it is essential to not impose the stronger condition on $\mathfrak{m}$ to be R-flat in the foundations of almost mathematics.

Lemma 2.1.11. *Let $f \colon R \to S$ be a ring homomorphism, and let $\mathfrak{m}_S$ be the ideal $\mathfrak{m}S \subset S$. Then we have the equality $\mathfrak{m}_S^2 = \mathfrak{m}_S$ and the S-module $\widetilde{\mathfrak{m}_S} := \mathfrak{m}_S \otimes_S \mathfrak{m}_S$ is S-flat.*

Proof. The equality $\mathfrak{m}_S^2 = \mathfrak{m}_S$ follows from the analogous assumption on $\mathfrak{m}$ and the construction of $\mathfrak{m}_S$. Regarding the flatness issue, we claim that $\mathfrak{m}_S \otimes_S \mathfrak{m}_S \simeq (\mathfrak{m} \otimes_R S) \otimes_S (\mathfrak{m} \otimes_R S)$. That would certainly imply the desired flatness statement. To prove this claim, we look at the following short exact sequence:

$$0 \to \mathfrak{m} \to R \to R/\mathfrak{m} \to 0.$$

We apply $- \otimes_R S$ to get a short exact sequence

$$0 \to \mathrm{Tor}_1^R(R/\mathfrak{m}, S) \to \mathfrak{m} \otimes_R S \to \mathfrak{m}S \to 0.$$

We observe that $\mathrm{Tor}_1^R(R/\mathfrak{m}, S)$ is almost zero, so both $\mathrm{Tor}_1^R(R/\mathfrak{m}, S) \otimes_S \mathfrak{m}S$ and $\mathrm{Tor}_1^R(R/\mathfrak{m}, S) \otimes_S (\mathfrak{m} \otimes_R S)$ are zero modules due to Lemma 2.1.1. So we use the functors $- \otimes_S (\mathfrak{m} \otimes_R S)$ and $- \otimes_S \mathfrak{m}S$ to obtain isomorphisms

$$(\mathfrak{m} \otimes_R S) \otimes_S (\mathfrak{m} \otimes_R S) \simeq \mathfrak{m}S \otimes_R (\mathfrak{m} \otimes_R S) \simeq (\mathfrak{m}S) \otimes_S (\mathfrak{m}S).$$

Thus we get the desired equality. ■

Lemma 2.1.12. *Let $f : R \to S$ be a ring homomorphism, and let $F : \mathbf{Mod}_R \to \mathbf{Mod}_S$ be an R-linear functor (resp. let $F : \mathbf{Mod}_R^{\mathrm{op}} \to \mathbf{Mod}_S$ be an R-linear functor). Then F sends almost zero R-modules to almost zero S-modules.*

Proof. Suppose that M is an almost zero R-module, so $\varepsilon M = 0$ for any $\varepsilon \in \mathfrak{m}$. Then $\varepsilon F(M) = 0$ because F is R-linear, so $F(M)$ is almost zero by Lemma 2.1.1. ■

Corollary 2.1.13. *Let $f : R \to S$ be a ring homomorphism, and let $F : \mathbf{Mod}_R \to \mathbf{Mod}_S$ be a left or right exact R-linear functor (resp. let $F : \mathbf{Mod}_R^{\mathrm{op}} \to \mathbf{Mod}_S$ be a left or right exact R-linear functor). Then F preserves almost isomorphisms.*

Proof. We only show the case of a left exact functor $F : \mathbf{Mod}_R \to \mathbf{Mod}_S$, all other cases are analogous to this one. We choose any almost isomorphism $f : M' \to M''$ and wish to show that $F(f)$ is an almost isomorphism. For this, we consider the following exact sequences:

$$0 \to K \to M' \to M \to 0,$$
$$0 \to M \to M'' \to Q \to 0.$$

We know that K and Q are almost zero by our assumption on f. Now, the above short exact sequences induce the following exact sequences:

$$0 \to F(K) \to F(M') \to F(M) \to \mathrm{R}^1 F(K),$$
$$0 \to F(M) \to F(M'') \to F(Q).$$

Lemma 2.1.12 guarantees that $F(K)$, $\mathrm{R}^1 F(K)$, and $F(Q)$ are almost zero S-modules. Therefore, the morphisms $F(M') \to F(M)$ and $F(M) \to F(M'')$ are both almost isomorphisms. In particular, the composition $F(M') \to F(M'')$ is an almost isomorphism as well. ■

2.2 Basic functors on categories of almost modules

The category of almost modules admits certain natural functors induced from the category of R-modules. It has two versions of the Hom-functor and the tensor product functor. We summarize the properties of these functors in the following proposition:

Proposition 2.2.1. *Let $R, \mathfrak{m}$ be as above.*

(1) *We define the* tensor product *functor* $-\otimes_{R^a} -\colon \mathbf{Mod}_R^a \times \mathbf{Mod}_R^a \to \mathbf{Mod}_R^a$ *as*

$$(M^a, N^a) \mapsto (M_!^a \otimes_R N_!^a)^a.$$

Then there is a natural transformation of functors

$$\begin{array}{ccc} \mathbf{Mod}_R \times \mathbf{Mod}_R & \xrightarrow{\;-\otimes_R-\;} & \mathbf{Mod}_R \\ {\scriptstyle (-)^a\times(-)^a}\downarrow & \overset{\rho}{\nearrow\!\!\!\!\nearrow} & \downarrow{\scriptstyle (-)^a} \\ \mathbf{Mod}_R^a \times \mathbf{Mod}_R^a & \xrightarrow[\;-\otimes_{R^a}-\;]{} & \mathbf{Mod}_R^a \end{array}$$

that makes the diagram $(2,1)$-commutative. In particular, there is a functorial isomorphism $(M \otimes_R N)^a \simeq M^a \otimes_{R^a} N^a$ for any $M, N \in \mathbf{Mod}_R$.

(2) *There is a functorial isomorphism*

$$\mathrm{Hom}_{R^a}(M^a, N^a) \simeq \mathrm{Hom}_R(\widetilde{\mathfrak{m}} \otimes M, N),$$

for any $M, N \in \mathbf{Mod}_R$. In particular, there is a canonical structure of an R-module on the group $\mathrm{Hom}_{R^a}(M^a, N^a)$; thus it defines the functor

$$\mathrm{Hom}_{R^a}(-,-)\colon \mathbf{Mod}_{R^a}^{\mathrm{op}} \times \mathbf{Mod}_{R^a} \to \mathbf{Mod}_R.$$

(3) *We define the functor* $\mathrm{alHom}_{R^a}(-,-)\colon \mathbf{Mod}_{R^a}^{\mathrm{op}} \times \mathbf{Mod}_{R^a} \to \mathbf{Mod}_{R^a}$ *of* almost homomorphisms *as*

$$(M^a, N^a) \mapsto \mathrm{Hom}_{R^a}(M^a, N^a)^a.$$

Then there is a natural transformation of functors

$$\begin{array}{ccc} \mathbf{Mod}_R^{\mathrm{op}} \times \mathbf{Mod}_R & \xrightarrow{\;\mathrm{Hom}_R(-,-)\;} & \mathbf{Mod}_R \\ {\scriptstyle (-)^a\times(-)^a}\downarrow & \underset{\rho}{\swarrow\!\!\!\!\swarrow} & \downarrow{\scriptstyle (-)^a} \\ \mathbf{Mod}_{R^a}^{\mathrm{op}} \times \mathbf{Mod}_{R^a} & \xrightarrow[\;\mathrm{alHom}_{R^a}(-,-)\;]{} & \mathbf{Mod}_{R^a} \end{array}$$

that makes the diagram $(2,1)$-commutative. In particular, it yields an isomorphism $\mathrm{alHom}_{R^a}(M^a, N^a) \cong^a \mathrm{Hom}_R(M, N)^a$ *for any $M, N \in \mathbf{Mod}_R$.*

Proof. (1). We define

$$\rho_{M,N} \colon (M_!^a \otimes_R N_!^a)^a \to (M \otimes_R N)^a$$

to be the morphism induced by

$$M_!^a \simeq \widetilde{\mathfrak{m}} \otimes_R M \to M \text{ and } N_!^a \simeq \widetilde{\mathfrak{m}} \otimes_R N \to N.$$

It is clear that $\rho_{M,N}$ is functorial in both variables, so it defines a natural transformation of functors ρ. We also need to check that $\rho_{M,N}$ is an isomorphism for any M and N. This follows from the following two observations: $\rho_{M,N}$ is an isomorphism if and only if $\rho_{M,N} \otimes_R \widetilde{\mathfrak{m}}$ is an isomorphism; and $\rho_{M,N} \otimes_R \widetilde{\mathfrak{m}}$ is easily seen to be an isomorphism as $\widetilde{\mathfrak{m}} \otimes_R \widetilde{\mathfrak{m}} \to \widetilde{\mathfrak{m}}$ is an isomorphism.

(2) is just a reformulation of Lemma 2.1.8 (2).

In order to show (3), we need to define a functorial morphism

$$\rho_{M,N} \colon \mathrm{Hom}_R(M, N)^a \to \mathrm{alHom}_{R^a}(M^a, N^a).$$

We start by using the functorial identification from (2):

$$\mathrm{alHom}_{R^a}(M^a, N^a) \cong^a \mathrm{Hom}_R(\widetilde{\mathfrak{m}} \otimes M, N)^a.$$

Namely, we define $\rho_{M,N}$ as the morphism $\mathrm{Hom}_R(M, N)^a \to \mathrm{Hom}_R(\widetilde{\mathfrak{m}} \otimes M, N)^a$ induced by the map $\widetilde{\mathfrak{m}} \otimes M \to M$. This is clearly functorial in both variables, so it defines the natural transformation ρ.

We also need to check that $\rho_{M,N}$ is an isomorphism for any M and N. This boils down to the fact that $\mathrm{Hom}_R(-, N)$ sends almost isomorphisms to almost isomorphisms. This, in turn, follows from Corollary 2.1.13. ∎

Remark 2.2.2. It is straightforward to check that whenever N^a has a structure of an S^a-module for some R-algebra S, then the R^a-modules $\mathrm{alHom}_{R^a}(M^a, N^a)$ and $M^a \otimes_{R^a} N^a$ have functorial-in-M^a structures of S^a-modules. This implies that the functors $- \otimes_{R^a} N^a$, $\mathrm{alHom}_{R^a}(-, N^a)$ naturally land in $\mathbf{Mod}_S^a$, i.e., they define functors

$$- \otimes_{R^a} N^a \colon \mathbf{Mod}_R^a \to \mathbf{Mod}_S^a \text{ and } \mathrm{alHom}_{R^a}(-, N^a) \colon \mathbf{Mod}_R^{a,op} \to \mathbf{Mod}_S^a.$$

Similarly, $\mathrm{Hom}_{R^a}(-, N^a)$ defines a functor $\mathbf{Mod}_R^a \to \mathbf{Mod}_S$.

The functor of almost homomorphisms is quite important, as it turns out to be the *inner Hom functor*, i.e., it is right adjoint to the tensor product.

Lemma 2.2.3. *Let $f \colon R \to S$ be a ring homomorphism, and let M^a be an R^a-module and N^a, K^a be S^a-modules. Then there is a functorial S-linear isomorphism*

$$\mathrm{Hom}_{S^a}(M^a \otimes_{R^a} N^a, K^a) \simeq \mathrm{Hom}_{R^a}(M^a, \mathrm{alHom}_{S^a}(N^a, K^a)).$$

Proof. This is a consequence of the usual $\otimes$-Hom-adjunction, Proposition 2.2.1, and the fact that $\widetilde{\mathfrak{m}}^{\otimes 2} \simeq \widetilde{\mathfrak{m}}$. Indeed, we have the following sequence of functorial isomorphisms:

$$\begin{aligned}\mathrm{Hom}_{S^a}(M^a \otimes_{R^a} N^a, K^a) &\simeq \mathrm{Hom}_S(\widetilde{\mathfrak{m}} \otimes_R M \otimes_R N, K)\\ &\simeq \mathrm{Hom}_S((\widetilde{\mathfrak{m}} \otimes_R M) \otimes_R (\widetilde{\mathfrak{m}} \otimes_R N), K)\\ &\simeq \mathrm{Hom}_R(\widetilde{\mathfrak{m}} \otimes_R M, \mathrm{Hom}_S(\widetilde{\mathfrak{m}} \otimes_R N, K))\\ &\simeq \mathrm{Hom}_{R^a}(M, \mathrm{alHom}_{S^a}(N^a, K^a)).\end{aligned}$$

The first isomomorphism follows from Proposition 2.2.1 (1), (2), the second isomorphism follows from the observation $\widetilde{\mathfrak{m}}^{\otimes 2} \simeq \widetilde{\mathfrak{m}}$, the third isomorphism is just the classical $\otimes$-Hom-adjunction, and the last isomorphism is a consequence of Proposition 2.2.1 (2), (3). ■

Corollary 2.2.4. (1) *Let N be an R^a-module, then the functor $-\otimes_{R^a} N^a$ is left adjoint to the functor* $\mathrm{alHom}_{R^a}(N^a, -)$.

(2) *Let $R \to S$ be a ring homomorphism. Then the functor $-\otimes_{R^a} S^a \colon \mathbf{Mod}_R^a \to \mathbf{Mod}_S^a$ is left adjoint to the forgetful functor.*

Proof. Part (1) follows from Lemma 2.2.3 by taking S to be equal to R. Part (2) follows from Lemma 2.2.3 by taking N^a to be equal to S^a. ■

Definition 2.2.5. The following types of R^a-modules will be used throughout the memoir:

- An R^a-module M^a is *flat* if the functor $M^a \otimes_{R^a} - \colon \mathbf{Mod}_R^a \to \mathbf{Mod}_R^a$ is exact.
- An R^a-module M^a is *faithfully flat* if it is flat and $N^a \otimes_{R^a} M^a \simeq 0$ if and only if $N^a \simeq 0$.
- An R-module M is *almost flat* (resp. *almost faithfully flat*) if an R^a-module M^a is *flat* (resp. *faithfully flat*)
- An R^a-module I^a is *injective* if the functor $\mathrm{Hom}_{R^a}(-, I^a) \colon \mathbf{Mod}_R^{a,op} \to \mathbf{Mod}_R$ is exact.
- An R^a-module P^a is *almost projective* if the functor $\mathrm{alHom}_{R^a}(P^a, -) \colon \mathbf{Mod}_R^a \to \mathbf{Mod}_R^a$ is exact.

Lemma 2.2.6. *The functor $(-)^a \colon \mathbf{Mod}_R \to \mathbf{Mod}_R^a$ sends flat (resp. faithfully flat, resp. injective, resp. projective) R-modules to flat (resp. faithfully flat, resp. injective, resp. almost projective) R^a-modules.*

Proof. The case of flat modules is clear from Proposition 2.2.1 (1). Now suppose that M is a faithfully flat R-module. Recall that $M \otimes_R - \colon \mathbf{Mod}_R \to \mathbf{Mod}_R$ is an exact and faithful functor. Therefore, if $M \otimes_R N$ is almost zero, it implies that so is N. Thus Proposition 2.2.1 (1) ensures that M^a is almost faithfully flat.

The case of injective modules follows from the fact that $(-)^a$ admits an exact left adjoint functor $(-)_!$. The case of projective modules is clear from the definition. ■

Lemma 2.2.7. *The functor* $(-)_!\colon \mathbf{Mod}_R^a \to \mathbf{Mod}_R$ *sends flat* R^a*-modules to flat* R*-modules.*

Proof. This follows from the formula $M_!^a \otimes_R N \simeq (M^a \otimes_{R^a} N^a)_!$ for any R^a-module M^a and an R-module N. ■

Warning 2.2.8. If M^a is a faithfully flat R^a-module, the R-module $M_!^a$ may not be faithfully flat. For instance, R^a is a faithfully flat R^a-module, but $R_!^a = \widetilde{\mathfrak{m}}$ is not a faithfully flat R-module. For example, $\widetilde{\mathfrak{m}} \otimes_R R/\mathfrak{m} \simeq 0$.

Corollary 2.2.9. *Any bounded above complex* $C^{\bullet,a} \in \mathbf{Comp}^-(R^a)$ *admits a resolution* $P^{\bullet,a} \to C^{\bullet,a}$ *by a bounded above complex of almost projective modules.*

Proof. We consider the complex $C_!^{\bullet,a} \in \mathbf{Comp}^-(R)$; it admits a resolution by a complex of free modules $p\colon P^\bullet \to C_!^{\bullet,a}$. Now we apply $(-)^a$ to this morphism to obtain the maps

$$P^{\bullet,a} \xrightarrow{p^a} (C_!^{\bullet,a})^a \xleftarrow{\varepsilon} C^{\bullet,a}.$$

The map ε is an *isomorphism* in $\mathbf{Comp}(R^a)$ by Lemma 2.1.9, and p^a is a quasi-isomorphism by construction. Thus, $\varepsilon^{-1} \circ p^a\colon P^{\bullet,a} \to C^{\bullet,a}$ is a quasi-isomorphism in $\mathbf{Comp}(R^a)$. We conclude by noting that each term of $P^{\bullet,a}$ is almost projective by Lemma 2.2.6. ■

2.3 Derived category of almost modules

We define the derived category of almost modules in two different ways and show that these definitions coincide. Later we define certain derived functors on the derived category of almost modules. We pay some extra attention to showing that the functors in this section are well defined on unbounded derived categories.

Definition 2.3.1. We define the *derived category of almost* R*-modules* as $\mathbf{D}(R^a) := \mathbf{D}(\mathbf{Mod}_R^a)$.

We define the bounded version of the derived category of almost R-modules $\mathbf{D}^*(R^a)$ for $* \in \{+, -, b\}$ as the full subcategory consisting of bounded below (resp. bounded above, resp. bounded) complexes.

Definition 2.3.2. We define the *almost derived category of* R*-modules* as the Verdier quotient $\mathbf{D}(R)^a := \mathbf{D}(\mathbf{Mod}_R)/\mathbf{D}_{\Sigma_R}(\mathbf{Mod}_R)$.

We recall that Σ_R is the Serre subcategory of $\mathbf{Mod}_R$ that consists of almost zero modules, and $\mathbf{D}_{\Sigma_R}(\mathbf{Mod}_R)$ is the full triangulated category of elements in $\mathbf{D}(\mathbf{Mod}_R)$ with almost zero cohomology modules.

We note that the functor $(-)^a\colon \mathbf{Mod}_R \to \mathbf{Mod}_R^a$ is exact and additive. Thus, it can be derived to the functor $(-)^a\colon \mathbf{D}(R) \to \mathbf{D}(R^a)$. Similarly, the functor $(-)_!\colon \mathbf{Mod}_R^a \to \mathbf{Mod}_R$ is additive and exact, so it can be derived to the functor $(-)_!\colon \mathbf{D}(R^a) \to \mathbf{D}(R)$. The standard argument shows that $(-)_!$ is a left adjoint functor to the functor $(-)^a$ since this already happens on the level of abelian categories. Now we also want to derive the functor $(-)_*\colon \mathbf{Mod}_R^a \to \mathbf{Mod}_R$. In order to do this on the level of unbounded derived categories, we need to show that $\mathbf{D}(R^a)$ has enough K-injective objects.

Definition 2.3.3. A complex of R^a-modules $I^{\bullet,a}$ is *K-injective* if

$$\mathrm{Hom}_{K(R^a)}(C^{\bullet,a}, I^{\bullet,a}) = 0$$

for any acyclic complex $C^{\bullet,a}$ of R^a-modules.

Remark 2.3.4. We remind the reader that $K(R^a)$ stands for the homotopy category of R^a-modules.

Lemma 2.3.5. *The functor* $(-)^a\colon \mathbf{Comp}(R) \to \mathbf{Comp}(R^a)$ *sends* K*-injective* R*-complexes to* K*-injective* R^a*-complexes.*

Proof. We note that $(-)^a$ admits an exact left adjoint $(-)_!$ thus [68, Tag 08BJ] ensures that $(-)^a$ preserves K-injective complexes. ∎

Corollary 2.3.6. *Every object* $M^{\bullet,a} \in \mathbf{Comp}(R^a)$ *is quasi-isomorphic to a* K*-injective complex.*

Proof. We know that the complex $M^\bullet \in \mathbf{Comp}(R)$ is quasi-isomorphic to a K-injective complex $I^\bullet$ by [68, Tag 090Y] (or [68, Tag 079P]). Now we use Lemma 2.3.5 to say that $I^{\bullet,a}$ is a K-injective complex that is quasi-isomorphic to $M^{\bullet,a}$. ∎

As the first application of Corollary 2.3.6, we define the functor $(-)_*\colon \mathbf{D}(R^a) \to \mathbf{D}(R)$ as the derived functor of $(-)_*\colon \mathbf{Mod}_R^a \to \mathbf{Mod}_R$. This functor exists by [68, Tag 070K].

Lemma 2.3.7. (1) *The functors*

$$\mathbf{D}(R) \underset{(-)^a}{\overset{(-)_!}{\rightleftarrows}} \mathbf{D}(R^a)$$

are adjoint. Moreover, the unit (resp. counit) morphism

$$(M^a)_! \to M \ (\textit{resp. } N \to (N_!)^a)$$

is an almost isomorphism (resp. isomorphism) for any $M \in \mathbf{D}(R)$*,* $N \in \mathbf{D}(R^a)$*. In particular, the functor* $(-)^a$ *is essentially surjective.*

(2) *The functors*

$$\mathbf{D}(R) \underset{(-)_*}{\overset{(-)^a}{\rightleftarrows}} \mathbf{D}(R^a)$$

are adjoint. Moreover, the unit (resp. counit) morphism

$$M \to (M^a)_* \text{ (resp. } (N_*)^a \to N)$$

is an almost isomorphism (resp. isomorphism) for any $M \in \mathbf{D}(R)$, $N \in \mathbf{D}(R^a)$.

Proof. We start the proof by showing (1). First, we note that the functors $(-)_!$ and $(-)^a$ are adjoint by the discussion above. Now we show that the cone of the counit map is always in $\mathbf{D}_{\Sigma_R}(R)$. As both functors $(-)^a$ and $(-)_!$ are exact on the level of abelian categories, it suffices to show the claim for $M \in \mathbf{Mod}_R^a$. But then the statement follows from Lemma 2.1.9 (5). The same argument shows that the unit map $N \to (N_!)^a$ is an isomorphism for any $N \in \mathbf{D}(R^a)$.

Now we go to (2). We define the functor $(-)_*: \mathbf{D}(R^a) \to \mathbf{D}(R)$ as the right derived functor of the left exact additive functor $(-)_*: \mathbf{Mod}_R^a \to \mathbf{Mod}_R$. This functor exists by [68, Tag 070K] and Corollary 2.3.6. The functor $(-)_*$ is right adjoint to $(-)^a$ by [68, Tag 0DVC].

We check that the natural map $M \to (M^a)_*$ is an almost isomorphism for any $M \in \mathbf{D}(R)$. We choose some K-injective resolution $M \xrightarrow{\sim} I^\bullet$. Then Lemma 2.3.5 guarantees that $M^a \to I^{\bullet,a}$ is a K-injective resolution of the complex M^a. The map $M \to (M^a)_*$ has a representative

$$I^\bullet \to (I^{\bullet,a})_*.$$

This map is an almost isomorphism of complexes by Lemma 2.1.9 (2). Thus, the map $M \to (M^a)_*$ is an almost isomorphism. A similar argument shows that the counit map $(N_*)^a \to N$ is an (almost) isomorphism for any $N \in \mathbf{D}(R^a)$. ∎

Theorem 2.3.8. *The functor* $(-)^a: \mathbf{D}(R) \to \mathbf{D}(R^a)$ *induces an equivalence of triangulated categories* $(-)^a: \mathbf{D}(R)^a \to \mathbf{D}(R^a)$.

Proof. We recall that the Verdier quotient is constructed as the localization of $\mathbf{D}(R)$ along the morphisms $f: C \to C'$ such that $\mathrm{cone}(f) \in \mathbf{D}_{\Sigma_R}(R)$. For instance, this is the definition of Verdier quotient in [68, Tag 05RI]. Now we see that a morphism $f^a: C^a \to C'^a$ is invertible in $\mathbf{D}(R^a)$ if and only if $\mathrm{cone}(f) \in \mathbf{D}_{\Sigma_R}(R)$, by the definition of Σ_R and the exactness of $(-)^a$. Moreover, $(-)^a$ admits a right adjoint such that $(-)^a \circ (-)_* \to \mathrm{id}$ is an isomorphism of functors. Thus, we can apply [27, Proposition 1.3] to say that the induced functor $(-)^a: \mathbf{D}(R)^a \to \mathbf{D}(R^a)$ must be an equivalence. ∎

Remark 2.3.9. Theorem 2.3.8 shows that the two notions of the derived category of almost modules are the same. In what follows, we do not distinguish $\mathbf{D}(R^a)$ and $\mathbf{D}(R)^a$ anymore.

2.4 Basic functors on derived categories of almost modules

Now we can "derive" certain functors constructed in the previous section. We start by defining the derived versions of different Hom functors, after that we move to the case of the derived tensor product functor.

Definition 2.4.1. We define the *derived Hom* functor

$$\mathbf{R}\mathrm{Hom}_{R^a}(-,-)\colon \mathbf{D}(R^a)^{\mathrm{op}} \times \mathbf{D}(R^a) \to \mathbf{D}(R)$$

as it is done in [68, Tag 0A5W], using the fact that $\mathbf{Comp}(R^a)$ has enough K-injective complexes.

We define *Ext modules* via the following formula:

$$\mathrm{Ext}^i_{R^a}\left(M^a, N^a\right) := \mathrm{H}^i\left(\mathbf{R}\mathrm{Hom}_{R^a}(M^a, N^a)\right) \in \mathbf{Mod}_R,$$

for $M^a, N^a \in \mathbf{Mod}^a_R$.

Explicitly, for any $M^a, N^a \in \mathbf{D}(R^a)$, the complex $\mathbf{R}\mathrm{Hom}_{R^a}(M^a, N^a)$ is constructed as follows: We choose a representative $C^{\bullet,a} \to M^a$ and a K-injective resolution $N^a \to I^{\bullet,a}$. Then we set $\mathbf{R}\mathrm{Hom}_{R^a}(M^a, N^a) = \mathrm{Hom}^\bullet_{R^a}(C^{\bullet,a}, I^{\bullet,a})$. This construction is independent of the choices and is functional in both variables. We refer to [68, Tag 0A5W] for the details.

Remark 2.4.2. We see that [68, Tag 0A64] implies a functorial isomorphism

$$\mathrm{H}^i\left(\mathbf{R}\mathrm{Hom}_{R^a}(M^a, N^a)\right) \simeq \mathrm{Hom}_{\mathbf{D}(R)^a}\left(M^a, N^a[i]\right).$$

Lemma 2.4.3.

(1) *There are functorial isomorphisms*

$$\mathrm{Hom}_{\mathbf{D}(R)^a}(M^a, N^a) \simeq \mathrm{Hom}_{\mathbf{D}(R)}(M^a_!, N)$$

and

$$\mathbf{R}\mathrm{Hom}_{R^a}(M^a, N^a) \simeq \mathbf{R}\mathrm{Hom}_R(M^a_!, N)$$

for any $M, N \in \mathbf{D}(R)$.

(2) *For any chosen* $M^a \in \mathbf{Mod}^a_R$*, the functor* $\mathbf{R}\mathrm{Hom}_{R^a}(M^a, -)\colon \mathbf{D}(R)^a \to \mathbf{D}(R)$ *is isomorphic to the (right) derived functor of* $\mathrm{Hom}_{R^a}(M^a, -)$.

Proof. The first claim easily follows from the fact that $(-)^a$ is a right adjoint to the exact functor $(-)_!$. We leave the details to the reader.

The second claim follows from [68, Tag 070K] and Corollary 2.3.6. ∎

Definition 2.4.4. We define the *derived functor of almost homomorphisms*

$$\mathbf{R}\mathrm{alHom}_{R^a}(-,-)\colon \mathbf{D}(R^a)^{\mathrm{op}} \times \mathbf{D}(R^a) \to \mathbf{D}(R^a)$$

as

$$\mathbf{R}\mathrm{alHom}_{R^a}(M^a, N^a) := \mathbf{R}\mathrm{Hom}_{R^a}(M^a, N^a)^a = \mathbf{R}\mathrm{Hom}_R(M^a_!, N)^a.$$

We define the *almost Ext modules* as R^a-modules defined by

$$\mathrm{alExt}^i_{R^a}\left(M^a, N^a\right) := \mathrm{H}^i\left(\mathbf{R}\mathrm{alHom}_{R^a}(M^a, N^a)\right)$$

for $M^a, N^a \in \mathbf{Mod}^a_R$.

Definition 2.4.5. For $K^{\bullet,a}, L^{\bullet,a} \in \mathbf{Comp}(R^a)$, we define the *complex of almost homomorphisms* $\mathrm{alHom}^\bullet_{R^a}(K^{\bullet,a}, L^{\bullet,a})$ as follows:

$$\mathrm{alHom}^n_{R^a}(K^{\bullet,a}, L^{\bullet,a}) := \prod_{n=p+q} \mathrm{alHom}_{R^a}(K^{-q,a}, L^{p,a})$$

with the differentials

$$\mathrm{d}(f) = \mathrm{d}_{L^{\bullet,a}} \circ f - (-1)^n f \circ \mathrm{d}_{K^{\bullet,a}}.$$

Lemma 2.4.6. *Let $P^{\bullet,a}$ be a bounded above complex of R^a-modules with almost projective cohomology modules and let $M^{\bullet,a} \to N^{\bullet,a}$ be an almost quasi-isomorphism of bounded below complexes of R^a-modules. Then the natural morphism*

$$\mathrm{alHom}^\bullet_{R^a}(P^{\bullet,a}, M^{\bullet,a}) \to \mathrm{alHom}^\bullet_{R^a}(P^{\bullet,a}, N^{\bullet,a})$$

is an almost quasi-isomorphism.

Proof. We note that as in the case of the usual Hom-complexes, there are convergent[3] spectral sequences

$$\mathrm{E}_1^{i,j} = \mathrm{H}^j\left(\mathrm{alHom}^\bullet_{R^a}(P^{-i,a}, M^{\bullet,a})\right) \Rightarrow \mathrm{H}^{i+j}\left(\mathrm{alHom}^\bullet_{R^a}(P^{\bullet,a}, M^{\bullet,a})\right)$$
$$\mathrm{E'}_1^{i,j} = \mathrm{H}^j\left(\mathrm{alHom}^\bullet_{R^a}(P^{-i,a}, N^{\bullet,a})\right) \Rightarrow \mathrm{H}^{i+j}\left(\mathrm{alHom}^\bullet_{R^a}(P^{\bullet,a}, N^{\bullet,a})\right).$$

Moreover, there is a natural morphism of spectral sequences $\mathrm{E}_1^{i,j} \to \mathrm{E'}_1^{i,j}$. Thus, it suffices to show that the associated map on the first page is an almost isomorphism

[3]Here we use that $P^{\bullet,a}$ is bounded above, $M^{\bullet,a}$ and $N^{\bullet,a}$ are bounded below.

at each entry. For this, we use the fact that $\mathrm{alHom}_{R^a}(P^{-i,a}, -)$ is exact to rewrite the first page of this spectral sequence as

$$\mathrm{E}_1^{i,j} = \mathrm{alHom}_{R^a}\big(P^{-i,a}, \mathrm{H}^j(M^{\bullet,a})\big)$$

and the same for $\mathrm{E'}_1^{i,j}$. So the question boils down to showing that the natural morphisms

$$\mathrm{alHom}_{R^a}\big(P^{-i,a}, \mathrm{H}^j(M^{\bullet,a})\big) \to \mathrm{alHom}_{R^a}\big(P^{-i,a}, \mathrm{H}^j(N^{\bullet,a})\big)$$

are almost isomorphisms. But this is clear as $M^{\bullet,a} \to N^{\bullet,a}$ is an almost quasi-isomorphism. ∎

Lemma 2.4.7. *Let $P_1^{\bullet,a} \to P_2^{\bullet,a}$ be an almost quasi-isomorphism of bounded above complexes with almost projective cohomology modules and let $M^{\bullet,a}$ be a bounded below complex of R^a-modules. Then the natural morphism*

$$\mathrm{alHom}^\bullet_{R^a}(P_2^{\bullet,a}, M^{\bullet,a}) \to \mathrm{alHom}^\bullet_{R^a}(P_1^{\bullet,a}, M^{\bullet,a})$$

is an almost quasi-isomorphism.

Proof. We choose some injective resolution $M^{\bullet,a} \to I^{\bullet,a}$ of the bounded below complex $M^{\bullet,a}$. Then we have a commutative diagram

$$\begin{array}{ccc}
\mathrm{alHom}^\bullet_{R^a}(P_2^{\bullet,a}, M^{\bullet,a}) & \longrightarrow & \mathrm{alHom}^\bullet_{R^a}(P_1^{\bullet,a}, M^{\bullet,a}) \\
\downarrow & & \downarrow \\
\mathrm{alHom}^\bullet_{R^a}(P_2^{\bullet,a}, I^{\bullet,a}) & \longrightarrow & \mathrm{alHom}^\bullet_{R^a}(P_1^{\bullet,a}, M^{\bullet,a}).
\end{array}$$

The bottom horizontal arrow is an almost quasi-isomorphism by the standard categorical argument with injective resolutions. The vertical maps are almost quasi-isomorphism by Lemma 2.4.6. ∎

Proposition 2.4.8. (1) *There is a natural transformation of functors*

$$\begin{array}{ccc}
\mathbf{D}(R)^{\mathrm{op}} \times \mathbf{D}(R) & \xrightarrow{\mathbf{R}\mathrm{Hom}_R(-,-)} & \mathbf{D}(R) \\
\big\downarrow{\scriptstyle (-)^a \times (-)^a} & \overset{\rho}{\swarrow\!\!\!\swarrow} & \big\downarrow{\scriptstyle (-)^a} \\
\mathbf{D}(R^a)^{\mathrm{op}} \times \mathbf{D}(R^a) & \xrightarrow[\mathbf{R}\mathrm{alHom}_{R^a}(-,-)]{} & \mathbf{D}(R^a)
\end{array}$$

that makes the diagram $(2,1)$*-commutative. In particular,*

$$\mathbf{R}\mathrm{alHom}_{R^a}(M^a, N^a) \cong^a \mathbf{R}\mathrm{Hom}_R(M, N)^a$$

for any $M, N \in \mathbf{D}(R)$.

(2) *For any chosen* $M^a \in \mathbf{Mod}_R^a$, *the functor* $\mathbf{R}\mathrm{alHom}_{R^a}(M^a, -)\colon \mathbf{D}(R^a) \to \mathbf{D}(R^a)$ *is isomorphic to the (right) derived functor of* $\mathrm{alHom}_{R^a}(M^a, -)$.

(3) *For any chosen* $N^a \in \mathbf{Mod}_R^a$, *the functor* $\mathbf{R}\mathrm{alHom}_{R^a}(-, N^a)\colon \mathbf{D}^-(R^a)^{\mathrm{op}} \to \mathbf{D}(R^a)$ *is isomorphic to the (right) derived functor of* $\mathrm{alHom}_{R^a}(-, N^a)$.

Proof. In order to show Part (1), we construct functorial morphisms

$$\rho_{M,N}\colon \mathbf{R}\mathrm{Hom}_R(M, N)^a \to \mathbf{R}\mathrm{alHom}_{R^a}(M^a, N^a),$$

for any $M, N \in \mathbf{D}(R)$. We recall that there is a functorial identification

$$\mathbf{R}\mathrm{alHom}_{R^a}(M^a, N^a) \cong^a \mathbf{R}\mathrm{Hom}_R(M_!^a, N)^a \cong^a \mathbf{R}\mathrm{Hom}_R(\widetilde{\mathfrak{m}} \otimes_R M, N)^a.$$

So we define

$$\rho_{M,N}\colon \mathbf{R}\mathrm{Hom}_R(M, N)^a \to \mathbf{R}\mathrm{Hom}_R(\widetilde{\mathfrak{m}} \otimes_R M, N)^a$$

as the morphism induced by the canonical map $\widetilde{\mathfrak{m}} \otimes_R M \to M$. This is clearly functorial, so it defines the stated natural transformation of functors. The only thing we are left to show is that $\rho_{M,N}$ is an almost isomorphism for any $M, N \in \mathbf{D}(R)$.

We recall that $\mathbf{R}\mathrm{Hom}_R(M, N)$ is isomorphic to $\mathrm{Hom}_R^\bullet(C^\bullet, I^\bullet)$ for any choice of a K-injective resolution of $N \xrightarrow{\sim} I^\bullet$ and any resolution $M \xrightarrow{\sim} C^\bullet$. Since $\widetilde{\mathfrak{m}} \otimes_R C^\bullet$ is a resolution of $\widetilde{\mathfrak{m}} \otimes_R M$ due to the R-flatness of $\widetilde{\mathfrak{m}}$, we reduce the question to showing that the natural map

$$\alpha\colon \mathrm{Hom}_R^\bullet(C^\bullet, I^\bullet) \to \mathrm{Hom}_R^\bullet(\widetilde{\mathfrak{m}} \otimes_R C^\bullet, I^\bullet)$$

is an almost quasi-isomorphism of complexes. For this, it suffices to show that α is an isomorphism of complexes. Now note that the degree-n part of α is the map

$$\prod_{p+q=n} \mathrm{Hom}_R(C^{-q}, I^p) \to \prod_{p+q=n} \mathrm{Hom}_R(\widetilde{\mathfrak{m}} \otimes_R C^{-q}, I^p).$$

Since (infinite) products are exact in $\mathbf{Mod}_R^a$, and any (infinite) product of almost zero modules is almost zero, it is enough that we show that each particular map $\mathrm{Hom}_R(C^{-q}, I^p) \to \mathrm{Hom}_R(\widetilde{\mathfrak{m}} \otimes_R C^{-q}, I^p)$ is an almost isomorphism. This follows from Proposition 2.2.1 (3).

Part (2) is similar to Part (2) of Lemma 2.4.3.

Part (3) is also similar to Part (2) of Lemma 2.4.3, but there are some subtleties due to the fact that $\mathbf{Mod}_R^a$ does not have enough projective objects. We fix this issue by using [68, Tag 06XN] instead of [68, Tag 070K]. We apply it to the subset $\mathcal{P}$ of bounded above complexes with almost projective terms. This result is indeed applicable in our situation due to Corollary 2.2.9 and Lemma 2.4.7. ∎

Now we deal with the case of the derived tensor product functor.

Definition 2.4.9. We say that a complex $K^{\bullet,a}$ of R^a-modules is *almost K-flat* if the naive tensor product complex $C^{\bullet,a} \otimes^{\bullet}_{R^a} K^{\bullet,a}$ is acyclic for any acyclic complex $C^{\bullet,a}$ of R^a-modules.

Lemma 2.4.10. *The functor* $(-)^a\colon \mathbf{Comp}(R) \to \mathbf{Comp}(R^a)$ *sends* K*-flat* R*-complexes to almost* K*-flat* R^a*-complexes.*

Proof. Suppose that $C^{\bullet,a}$ is an acyclic complex of R^a-modules and $K^\bullet$ is a K-flat complex. Then we see that

$$\begin{aligned} C^{\bullet,a} \otimes^{\bullet}_{R^a} K^{\bullet,a} &\cong^a (C^\bullet \otimes^{\bullet}_R K^\bullet)^a \\ &\cong^a (\widetilde{\mathfrak{m}} \otimes_R C^\bullet \otimes^{\bullet}_R K^\bullet)^a \cong^a ((\widetilde{\mathfrak{m}} \otimes_R C^\bullet) \otimes^{\bullet}_R K^\bullet)^a. \end{aligned}$$

The latter complex is acyclic as $\widetilde{\mathfrak{m}} \otimes C^\bullet$ is acyclic and $K^\bullet$ is K-flat. ∎

Corollary 2.4.11. *Every object* $M^{\bullet,a} \in \mathbf{Comp}(R^a)$ *is quasi-isomorphic to an almost* K*-flat complex.*

Proof. We know that the complex $M^\bullet \in \mathbf{Comp}(R)$ is quasi-isomorphic to a K-flat complex $K^\bullet$ by [68, Tag 06Y4]. Now we use Lemma 2.4.10 to say that $K^{\bullet,a}$ is an almost K-flat complex that is quasi-isomorphic to $M^{\bullet,a}$. ∎

Definition 2.4.12. We define the *derived tensor product functor*

$$- \otimes^L_{R^a} - \colon \mathbf{D}(R)^a \times \mathbf{D}(R)^a \to \mathbf{D}(R)^a$$

by the rule $(M^a, N^a) \mapsto (M_! \otimes^L_R N_!)^a$ for any $M^a, N^a \in \mathbf{D}(R)^a$.

Proposition 2.4.13. (1) *There is a natural transformation of functors*

$$\begin{array}{ccc} \mathbf{D}(R) \times \mathbf{D}(R) & \xrightarrow{\ -\otimes^L_R-\ } & \mathbf{D}(R) \\ \Big\downarrow {\scriptstyle (-)^a \times (-)^a} & \overset{\rho}{\nearrow} & \Big\downarrow {\scriptstyle (-)^a} \\ \mathbf{D}(R)^a \times \mathbf{D}(R)^a & \xrightarrow[\ -\otimes^L_{R^a}-\ ]{} & \mathbf{D}(R)^a \end{array}$$

that makes the diagram $(2,1)$*-commutative. In particular, there is a functorial isomorphism* $(M \otimes^L_R N)^a \simeq M^a \otimes^L_{R^a} N^a$ *for any* $M, N \in \mathbf{D}(R)$.

(2) *For any chosen* $M^a \in \mathbf{Mod}^a_R$*, the functor* $M^a \otimes^L_{R^a} -\colon \mathbf{D}(R)^a \to \mathbf{D}(R)^a$ *is isomorphic to the (left) derived functor of* $M^a \otimes_{R^a} -$.

Proof. The proof of Part (1) is similar to that of Proposition 2.2.1 (1). We leave the details to the reader.

The proof of Part (2) is similar to that of Proposition 2.4.8 (2). The claim follows by applying [68, Tag 06XN] with $\mathcal{P}$ being the subset of almost K-flat complexes. This result is indeed applicable in our situation due to Corollary 2.4.11 and the almost version of [68, Tag 064L]. ∎

Lemma 2.4.14. *Let $M^a, N^a, K^a \in \mathbf{D}(R)^a$, then we have a functorial isomorphism*

$$\mathbf{R}\mathrm{Hom}_{R^a}(M^a \otimes^L_{R^a} N^a, K^a) \simeq \mathbf{R}\mathrm{Hom}_{R^a}(M^a, \mathbf{R}\mathrm{alHom}_{R^a}(N^a, K^a)).$$

In particular, the functors $\mathbf{R}\mathrm{alHom}_{R^a}(N^a, -)\colon \mathbf{D}(R)^a \longleftrightarrow \mathbf{D}(R)^a : - \otimes^L_{R^a} N^a$ *are adjoint.*

Proof. The claim follows from the following sequence of canonical identifications:

$$\begin{aligned}
&\mathbf{R}\mathrm{Hom}_{R^a}(M^a \otimes^L_{R^a} N^a, K^a) && \\
&\quad \simeq \mathbf{R}\mathrm{Hom}_R((\widetilde{\mathfrak{m}} \otimes_R M) \otimes^L_R (\widetilde{\mathfrak{m}} \otimes_R N), K) && \text{Lemma 2.4.3 (1)} \\
&\quad \simeq \mathbf{R}\mathrm{Hom}_R(\widetilde{\mathfrak{m}} \otimes_R M, \mathbf{R}\mathrm{Hom}_R(\widetilde{\mathfrak{m}} \otimes_R N, K)) && \text{[68, Tag 0A5W]} \\
&\quad \simeq \mathbf{R}\mathrm{Hom}_{R^a}(M^a, \mathbf{R}\mathrm{Hom}_R(\widetilde{\mathfrak{m}} \otimes_R N, K)^a) && \text{Lemma 2.4.3 (1)} \\
&\quad \simeq \mathbf{R}\mathrm{Hom}_{R^a}(M^a, \mathbf{R}\mathrm{alHom}_{R^a}(N^a, K^a)). && \text{Definition 2.4.4}
\end{aligned}$$

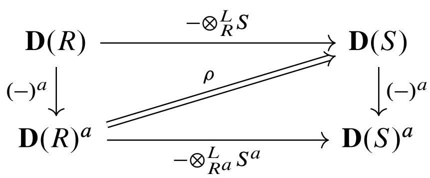
■

Definition 2.4.15. Let $f\colon R \to S$ be a ring homomorphism. We define the *base change functor*

$$- \otimes^L_{R^a} S^a \colon \mathbf{D}(R)^a \to \mathbf{D}(S)^a$$

by the rule $M^a \mapsto (M_! \otimes^L_R S)^a$ for any $M^a \in \mathbf{D}(R)^a$.

Proposition 2.4.16. (1) *There is a natural transformation of functors*



that makes the diagram $(2,1)$*-commutative. In particular, there is a functorial isomorphism* $(M \otimes^L_R S)^a \simeq M^a \otimes^L_{R^a} S^a$ *for any* $M \in \mathbf{D}(R)$.

(2) *The functor* $- \otimes^L_{R^a} S^a \colon \mathbf{D}(R)^a \to \mathbf{D}(S)^a$ *is isomorphic to the (left) derived functor of* $- \otimes^L_{R^a} S^a$.

Proof. The proof is identical to Proposition 2.4.13. ■

Lemma 2.4.17. *Let $R \to S$ be a ring homomorphism, let $M^a \in \mathbf{D}(R)^a$, and let $N^a \in \mathbf{D}(S)^a$. Then we have a functorial isomorphism*

$$\mathbf{R}\mathrm{Hom}_{S^a}(M^a \otimes^L_{R^a} S^a, N^a) \simeq \mathbf{R}\mathrm{Hom}_{R^a}(M^a, N^a).$$

In particular, the functors $\mathrm{Forget}\colon \mathbf{D}(S)^a \longleftrightarrow \mathbf{D}(R)^a : - \otimes^L_{R^a} S^a$ *are adjoint.*

Proof. The proof is similar to that of Lemma 2.4.14. ■

2.5 Almost finitely generated and almost finitely presented modules

In this section, we discuss the notions of almost finitely generated and almost finitely presented modules. Our discussion closely follows [26]. The main difference is that we avoid any use of "uniform structures" in our treatment; we think that it simplifies the exposition. We recall that we fixed some "base" ring R with an ideal $\mathfrak{m}$ such that $\mathfrak{m}^2 = \mathfrak{m}$ and $\widetilde{\mathfrak{m}} = \mathfrak{m} \otimes_R \mathfrak{m}$ is flat, and we always do almost mathematics with respect to this ideal.

Definition 2.5.1. An R-module M is called *almost finitely generated*, if for any $\varepsilon \in \mathfrak{m}$ there are an integer n_ε and an R-homomorphism

$$R^{n_\varepsilon} \xrightarrow{f} M$$

such that $\mathrm{Coker}(f)$ is killed by ε.

Definition 2.5.2. An R-module M is called *almost finitely presented*, if for any $\varepsilon, \delta \in \mathfrak{m}$ there are integers $n_{\varepsilon,\delta}$, $m_{\varepsilon,\delta}$ and a complex

$$R^{m_{\varepsilon,\delta}} \xrightarrow{g} R^{n_{\varepsilon,\delta}} \xrightarrow{f} M$$

such that $\mathrm{Coker}(f)$ is killed by ε and $\delta(\mathrm{Ker}\, f) \subset \mathrm{Im}\, g$.

Remark 2.5.3. Clearly, any almost finitely presented R-module is almost finitely generated.

Remark 2.5.4. A typical example of an almost finitely presented module that is not finitely generated is $M = \bigoplus_{n \geq 1} \mathcal{O}_C/p^{1/n}\mathcal{O}_C$ for an algebraically closed non-archimedean field C of mixed characteristic $(0, p)$.

The next few lemmas discuss basic properties of almost finitely generated and almost finitely presented modules. For example, it is not entirely obvious that these notions transfer across almost isomorphisms. We show that this is actually the case, so these notions descend to $\mathbf{Mod}_R^a$. We also show that almost finitely generated and almost finitely presented modules have many good properties that are similar to those of usual finitely generated and finitely presented modules.

Our first main goal is to get alternative criteria for a module to be almost finitely generated (resp. almost finitely presented) and show that this notion descends to the category of almost modules.

Lemma 2.5.5. *Let M be an R-module, then M is almost finitely generated if and only if for any finitely generated ideal $\mathfrak{m}_0 \subset \mathfrak{m}$ there is a morphism $R^n \xrightarrow{f} M$ such that $\mathfrak{m}_0(\mathrm{Coker}\, f) = 0$.*

Proof. The "if" part is clear, so we only need to deal with the "only if" part. We choose a set of generators $(\varepsilon_0, \dots, \varepsilon_n)$ for an ideal $\mathfrak{m}_0$. By assumption, we have R-morphisms

$$f_i \colon R^{n_{\varepsilon_i}} \to M$$

such that $\varepsilon_i(\operatorname{Coker} f_i) = 0$ for all i. Then the sum of these morphisms

$$f := \bigoplus_{i=1}^{n} f_i \colon R^{\sum n_{\varepsilon_i}} \to M$$

defines a map such that $\mathfrak{m}_0(\operatorname{Coker} f) = 0$. Since $\mathfrak{m}_0$ was an arbitrary morphism, this finishes the proof. ∎

Lemma 2.5.6. *Let M be an almost finitely presented R-module, and let $\varphi \colon R^n \to M$ be an R-homomorphism such that $\mathfrak{m}_1(\operatorname{Coker}\varphi) = 0$ for some ideal $\mathfrak{m}_1 \subset \mathfrak{m}$. Then for every finitely generated ideal $\mathfrak{m}_0 \subset \mathfrak{m}_1\mathfrak{m}$ there is morphism $\psi \colon R^m \to M$ such that*

$$R^m \xrightarrow{\psi} R^n \xrightarrow{\varphi} M$$

is a three-term complex and $\mathfrak{m}_0(\operatorname{Ker}\varphi) \subset \operatorname{Im}(\psi)$.

Proof. Since M is almost finitely presented, for any two elements $\varepsilon_1, \varepsilon_2 \in \mathfrak{m}$, we can find a complex

$$R^{m_2} \xrightarrow{g} R^{m_1} \xrightarrow{f} M$$

such that $\varepsilon_1(\operatorname{Coker} f) = 0$ and $\varepsilon_2(\operatorname{Ker} f) \subset \operatorname{Im} g$. Now we choose an element $\delta \in \mathfrak{m}_1$ and wish to define morphisms

$$\alpha \colon R^{m_1} \to R^n \text{ and } \beta \colon R^n \to R^{m_1}$$

such that $\varphi \circ \alpha = \delta f$ and $f \circ \beta = \varepsilon_1 \varphi$.

$$\begin{array}{ccccc} R^{m_2} & \xrightarrow{\quad g \quad} & R^{m_1} & \xrightarrow{\quad f \quad} & M \\ & & \beta \uparrow \downarrow \alpha & \nearrow_{\varphi} & \\ & & R^n & & \end{array}$$

To achieve this goal, we define α and β in the following way: we fix a basis $e_1, \dots, e_{m_1}$ of R^{m_1}, a basis $e'_1, \dots, e'_n$ of R^n, and then put α and β to be the unique R-linear morphisms such that

$$\alpha(e_i) = y_i \in R^n \quad \text{for some } y_i \text{ such that } \varphi(y_i) = \delta f(e_i),$$
$$\beta(e'_j) = x_j \in R^{m_1} \text{ for some } x_j \text{ such that } f(x_j) = \varepsilon_1 \varphi(e'_j).$$

It is clear that $\varphi \circ \alpha = \delta f$ and $f \circ \beta = \varepsilon_1 \varphi$ as it holds on the basis elements.

We define the morphism $\psi\colon R^n \oplus R^{m_2} \to R^n$ by the rule

$$\psi(x, y) = \alpha \circ \beta(x) - (\varepsilon_1\delta)x + \alpha \circ g(y).$$

Now we show that

$$\varphi \circ \psi = 0 \text{ and } \varepsilon_1\varepsilon_2\delta \operatorname{Ker}\varphi \subset \operatorname{Im}\psi.$$

We start by showing that $\varphi \circ \psi = 0$: it suffices to prove that

$$(\alpha \circ g)(y) \in \operatorname{Ker}\varphi \text{ for } y \in R^{m_2}, \text{ and } (\alpha \circ \beta)(x) - (\varepsilon_1\delta)x \in \operatorname{Ker}\varphi \text{ for } x \in R^n.$$

We note that we have an equality

$$(\varphi \circ \alpha \circ g)(y) = \delta(f \circ g)(y) = \delta 0 = 0,$$

so $(\alpha \circ g)(y) \in \operatorname{Ker}(\varphi)$. We also have an equality

$$\begin{aligned}(\varphi \circ (\alpha \circ \beta - \varepsilon_1\delta))(x) &= (\varphi \circ \alpha \circ \beta)(x) - \varepsilon_1\delta\varphi(x)\\ &= \delta(f \circ \beta)(x) - \varepsilon_1\delta\varphi(x)\\ &= \delta\varepsilon_1\varphi(x) - \varepsilon_1\delta\varphi(x)\\ &= 0.\end{aligned}$$

This shows that $(\alpha \circ \beta)(x) - (\varepsilon_1\delta)x \in \operatorname{Ker}(\varphi)$ as well.

We show that $(\varepsilon_1\varepsilon_2\delta)\operatorname{Ker}\varphi \subset \operatorname{Im}(\psi)$: we observe that for any $x \in \operatorname{Ker}\varphi$ we have $\beta(x) \subset \operatorname{Ker} f$ as $f \circ \beta = \varepsilon_1\varphi$. This implies that $\varepsilon_2\beta(x) \in \operatorname{Im} g$ since $\varepsilon_2 \operatorname{Ker} f \subset \operatorname{Im} g$. Thus, there is $y \in R^{m_2}$ such that $g(y) = \varepsilon_2\beta(x)$, so $(\alpha \circ g)(y) = \varepsilon_2\alpha \circ \beta(x)$. This shows that

$$\begin{aligned}\psi(-\varepsilon_2 x, y) &= -\varepsilon_2(\alpha \circ \beta)(x) + \varepsilon_1\varepsilon_2\delta x + (\alpha \circ g)(y)\\ &= -\varepsilon_2(\alpha \circ \beta)(x) + \varepsilon_1\varepsilon_2\delta x + \varepsilon_2(\alpha \circ \beta)(x) = \varepsilon_1\varepsilon_2\delta x.\end{aligned}$$

We conclude that $\varepsilon_1\varepsilon_2\delta x \in \operatorname{Im}(\psi)$ for any $x \in \operatorname{Ker}(\varphi)$.

Finally, we recall that $\mathfrak{m}_0$ is a finitely generated ideal, and that $\mathfrak{m}_0 \subset \mathfrak{m}_1\mathfrak{m} = \mathfrak{m}_1\mathfrak{m}^2 \subset \mathfrak{m}_1$. This means that we can find a finite set I, and a finite set of elements $\varepsilon_{i,1}, \varepsilon_{i,2} \in \mathfrak{m}, \delta_i \in \mathfrak{m}_1$ such that $\mathfrak{m}_0$ is contained in the ideal $J := (\varepsilon_{i,1}\varepsilon_{i,2}\delta_i)_{i\in I}$ (the ideal generated by all products $\varepsilon_{i,1}\varepsilon_{i,2}\delta_i$). The previous discussion implies that for each $i \in I$, we have a map $\psi_i : R^{k_i} \to R^n$ such that

$$\varphi \circ \psi_i = 0 \quad \text{and} \quad (\varepsilon_{i,1}\varepsilon_{i,2}\delta_i)(\operatorname{Ker}\varphi) \subset \operatorname{Im}\psi_i.$$

By passing to the homomorphism

$$\psi := \bigoplus_{i\in I} \psi_i\colon R^{\sum k_i} \to R^n,$$

we get a map ψ such that $\varphi \circ \psi = 0$ and $\mathfrak{m}_0(\operatorname{Ker}\varphi) \subset \operatorname{Im}(\psi)$. Therefore, ψ does the job. ■

Lemma 2.5.7. *Let M be an R-module. Then the following conditions are equivalent:*

(1) *The R-module M is almost finitely presented.*

(2) *For any finitely generated ideal $\mathfrak{m}_0 \subset \mathfrak{m}$ there exist a finitely presented R-module N and a homomorphism $f: N \to M$ such that $\mathfrak{m}_0(\operatorname{Ker} f) = 0$ and $\mathfrak{m}_0(\operatorname{Coker} f) = 0$.*

(3) *For any finitely generated ideal $\mathfrak{m}_0 \subset \mathfrak{m}$ there exist integers n, m and a three-term complex*
$$R^m \xrightarrow{g} R^n \xrightarrow{f} M$$
such that $\mathfrak{m}_0(\operatorname{Coker} f) = 0$ and $\mathfrak{m}_0(\operatorname{Ker} f) \subset \operatorname{Im} g$.

Proof. It is clear that (3) implies both (1) and (2).

We show that (1) implies (3). Since M is an almost finitely generated R-module, Lemma 2.5.5 guarantees that, for any finitely generated ideal $\mathfrak{m}' \subset \mathfrak{m}$, there exists a morphism $R^n \xrightarrow{f} M$ such that $\mathfrak{m}'(\operatorname{Coker} f) = 0$.

We know that $\mathfrak{m}_0 \subset \mathfrak{m} = \mathfrak{m}^2$; this easily implies that there is a finitely generated ideal $\mathfrak{m}_1 \subset \mathfrak{m}$ such that $\mathfrak{m}_0 \subset \mathfrak{m}_1\mathfrak{m} \subset \mathfrak{m}_1$. So, using $\mathfrak{m}' = \mathfrak{m}_1$, we can find a homomorphism $R^n \xrightarrow{\varphi} M$ such that $\mathfrak{m}_1(\operatorname{Coker}\varphi) = 0$. Lemma 2.5.6 claims that we can also find a homomorphism $\psi: R^m \to R^n$ such that
$$R^m \xrightarrow{\psi} R^n \xrightarrow{\varphi} M$$
is a three-term complex and $\mathfrak{m}_0(\operatorname{Ker}\varphi) \subset \operatorname{Im}\psi$. As $\mathfrak{m}_0 \subset \mathfrak{m}_1$ and $\mathfrak{m}_1(\operatorname{Coker}\varphi) = 0$, we get that $\mathfrak{m}_0(\operatorname{Coker}\varphi) = 0$ as well. This finishes the proof since $\mathfrak{m}_0$ was an arbitrary finitely generated sub-ideal of $\mathfrak{m}$.

Now we show that (2) implies (3). We pick an arbitrary finitely generated ideal $\mathfrak{m}_0 \subset \mathfrak{m}$, and we try to find a three-term complex
$$R^m \xrightarrow{g} R^n \xrightarrow{f} M$$
such that $\mathfrak{m}_0(\operatorname{Coker} f) = 0$ and $\mathfrak{m}_0(\operatorname{Ker} f) \subset \operatorname{Im}(g)$. To achieve this, we use the assumption in (2) to find a morphism $h: N \to M$ such that N is a finitely presented R-module, $\mathfrak{m}_0(\operatorname{Coker} h) = 0$, and $\mathfrak{m}_0(\operatorname{Ker} h) = 0$. Since N is finitely presented, we can find a short exact sequence
$$R^m \xrightarrow{g} R^n \xrightarrow{f'} N \to 0.$$
It is straightforward to see that a three-term complex
$$R^m \xrightarrow{g} R^n \xrightarrow{f := h \circ f'} M$$

satisfies the condition that $\mathfrak{m}_0(\operatorname{Coker} f) = 0$ and $\mathfrak{m}_0(\operatorname{Ker} f) \subset \operatorname{Im}(g)$. ■

Lemma 2.5.8. *Let M be an R-module, and suppose that for any finitely generated ideal $\mathfrak{m}_0 \subset \mathfrak{m}$ there exists a morphism $f\colon N \to M$ such that $\mathfrak{m}_0(\operatorname{Ker} f) = 0$, $\mathfrak{m}_0(\operatorname{Coker} f) = 0$, and N is almost finitely generated (resp. almost finitely presented). Then M is also almost finitely generated (resp. almost finitely presented).*

Proof. We give a proof only in the almost finitely presented case; the other case is easier. We pick an arbitrary finitely generated ideal $\mathfrak{m}_0 \subset \mathfrak{m}$ and another finitely generated ideal $\mathfrak{m}_1 \subset \mathfrak{m}$ such that $\mathfrak{m}_0 \subset \mathfrak{m}_1^2$. Then we use the assumption to get a morphism

$$f\colon N \to M$$

such that $\mathfrak{m}_1(\operatorname{Ker} f) = 0$, $\mathfrak{m}_1(\operatorname{Coker} f) = 0$ and N is an almost finitely presented R-module. Lemma 2.5.7 guarantees that there is a three-term complex

$$R^m \xrightarrow{h} R^n \xrightarrow{g} N$$

such that $\mathfrak{m}_1(\operatorname{Coker} g) = 0$ and $\mathfrak{m}_1(\operatorname{Ker} g) \subset \operatorname{Im} h$. Then we can consider a three-term complex

$$R^m \xrightarrow{h} R^n \xrightarrow{f' := f \circ g} M,$$

it is easily seen that $\mathfrak{m}_1^2(\operatorname{Coker} f') = 0$ and $\mathfrak{m}_1^2(\operatorname{Ker} f') \subset \operatorname{Im}(h)$. Since $\mathfrak{m}_0 \subset \mathfrak{m}_1^2$, we conclude that $\mathfrak{m}_0(\operatorname{Coker} f') = 0$ and $\mathfrak{m}_0(\operatorname{Ker} f') \subset \operatorname{Im}(h)$. This shows that M is almost finitely presented. ■

Lemma 2.5.9. *Let M be an R-module and let $\{N_i\}_{i \in I}$ be a filtered diagram of R-modules. Then*

(1) *the natural morphism*

$$\gamma_M^0\colon \operatorname{colim}_I \operatorname{Hom}_R(M, N_i) \to \operatorname{Hom}_R(M, \operatorname{colim}_I N_i)$$

is almost injective for an almost finitely generated M;

(2) *the natural morphism*

$$\gamma_M^0\colon \operatorname{colim}_I \operatorname{Hom}_R(M, N_i) \to \operatorname{Hom}_R(M, \operatorname{colim}_I N_i)$$

is an almost isomorphism and

$$\gamma_M^1\colon \operatorname{colim} \operatorname{Ext}_R^1(M, N_i) \to \operatorname{Ext}_R^1(M, \operatorname{colim} N_i)$$

is almost injective for an almost finitely presented M.

Proof. We give a proof for an almost finitely presented M; the case of an almost finitely generated M is similar.

Step 1: *The case of a finitely presented M.* In this case, γ^0_M is an isomorphism and γ^1_M is injective due to [68, Tag 064T] and [68, Tag 0G8W].

Step 2: *General case.* We fix a finitely generated ideal $\mathfrak{m}_0 \subset \mathfrak{m}$. Since $\mathfrak{m}_0 \subset \mathfrak{m} = \mathfrak{m}^4$, there is a finitely generated ideal $\mathfrak{m}_1$ such that $\mathfrak{m}_0 \subset \mathfrak{m}_1^4$. So we use Lemma 2.5.7 (2) to find a finitely presented module M' and a morphism $f: M' \to M$ such that $\mathrm{Ker}(f)$ and $\mathrm{Coker}(f)$ are annihilated by $\mathfrak{m}_1$. We denote the image of f by M'' and consider the short exact sequences

$$0 \to K \to M' \to M'' \to 0,$$
$$0 \to M'' \to M \to Q \to 0$$

with K and Q being annihilated by $\mathfrak{m}_1$. Applying the functors $\mathrm{colim}_I \mathrm{Hom}_R(-, N_i)$ and $\mathrm{Hom}_R(-, \mathrm{colim}_I N_i)$ and considering the associated long exact sequences, we see that

$$b_i : \mathrm{colim}_I \mathrm{Ext}^i_R(M, N_i) \to \mathrm{colim}_I \mathrm{Ext}^i_R(M', N_i)$$

and

$$c_i : \mathrm{Ext}^i_R(M, \mathrm{colim}_I N_i) \to \mathrm{Ext}^i_R(M', \mathrm{colim}_I N_i)$$

have kernels and cokernels annihilated by $\mathfrak{m}_1^2$ for any $i \geq 0$. Now we consider a commutative diagram

$$\begin{array}{ccc} \mathrm{colim}_I \mathrm{Ext}^i_R(M', N_i) & \xrightarrow{\gamma^i_{M'}} & \mathrm{Ext}^i_R(M', \mathrm{colim}_I N_i) \\ \uparrow{\scriptstyle b_i} & & \uparrow{\scriptstyle c_i} \\ \mathrm{colim}_I \mathrm{Ext}^i_R(M, N_i) & \xrightarrow[\gamma^i_M]{} & \mathrm{Ext}^i_R(M, \mathrm{colim}_I N_i) \end{array}$$

By Step 1, we know that $\gamma^i_{M'}$ is an isomorphism for $i = 0$ and injective for $i = 1$. Moreover, we know that b_i and c_i have kernels and cokernels annihilated by $\mathfrak{m}_1^2$. Then it is easy to see that $\mathrm{Coker}(\gamma^0_M)$, $\mathrm{Ker}(\gamma^0_M)$, and $\mathrm{Ker}(\gamma^1_M)$ are annihilated by $\mathfrak{m}_1^4$. In particular, they are annihilated by $\mathfrak{m}_0 \subset \mathfrak{m}_1^4$. Since $\mathfrak{m}_0$ was arbitrary finitely generated sub-ideal $\mathfrak{m}_0 \subset \mathfrak{m}$, we conclude that γ^0_M is an almost isomorphism and γ^1_M is almost injective. ■

Lemma 2.5.10. *Let M be an R-module.*

(1) *If, for any filtered diagram of R-modules $\{N_i\}_{i \in I}$, the natural morphism*

$$\mathrm{colim}_I \mathrm{Hom}_R(M, N_i) \to \mathrm{Hom}_R(M, \mathrm{colim}_I N_i)$$

is almost injective, then M is almost finitely generated.

(2) *If, for any filtered system of R-modules* $\{N_i\}$, *the natural morphism*

$$\operatorname{colim}_I \operatorname{Hom}_R(M, N_i) \to \operatorname{Hom}_R(M, \operatorname{colim}_I N_i)$$

is an almost isomorphism, then M *is almost finitely presented.*

Proof. (1) Note that $M \simeq \operatorname{colim}_I M_i$ is a filtered colimit of its finitely generated *submodules*. Therefore, we see that

$$\operatorname{colim}_I \operatorname{Hom}_R(M, M/M_i) \simeq^a \operatorname{Hom}_R(M, \operatorname{colim}_I (M/M_i)) \simeq 0.$$

Consider an element α of $\operatorname{colim}_I \operatorname{Hom}_R(M, M/M_i)$ that has a representative the quotient morphism $M \to M/M_i$ (for some choice of $i \in I$). Then, for every $\varepsilon \in \mathfrak{m}$, $\varepsilon\alpha = 0$ in $\operatorname{colim}_I \operatorname{Hom}_R(M, M/M_i)$. Explicitly, this means that there is $j \geq i$ such that $\varepsilon M \subset M_j$. Now we choose a surjection $R^{n_j} \to M_j$ to see that the composition $f: R^{n_j} \to M$ gives a map with $\varepsilon(\operatorname{Coker} f) = 0$. Now note that this property is preserved by replacing j with any $j' > j$. Therefore, for any $\mathfrak{m}_0 = (\varepsilon_1, \dots, \varepsilon_n)$, we can find a finitely generated submodule $M_i \subset M$ such that $\mathfrak{m}_0 M \subset M_i$. Therefore, M is almost finitely generated.

(2) Fix any finitely generated sub-ideal $\mathfrak{m}_0 = (\varepsilon_1, \dots, \varepsilon_n) \subset \mathfrak{m}$. We use [68, Tag 00HA] to write $M \simeq \operatorname{colim}_\Lambda M_\lambda$ as a filtered colimit of *finitely presented* R-modules. By assumption, the natural morphism

$$\operatorname{colim}_\Lambda \operatorname{Hom}_R(M, M_\lambda) \to \operatorname{Hom}_R(M, \operatorname{colim}_\Lambda M_\lambda) = \operatorname{Hom}_R(M, M)$$

is an almost isomorphism. In particular, $\varepsilon_i \mathrm{id}_M$ is in the image of this map for every $i = 1, \dots, n$. This means that, for every ε_i, there are an element $\lambda_i \in \Lambda$ and a morphism $g_i: M \to M_{\lambda_i}$ such that

$$f_{\lambda_i} \circ g_i = \varepsilon_i \mathrm{id}_M,$$

where $f_{\lambda_i}: M_{\lambda_i} \to M$ is the natural morphism to the colimit. Note that the existence of such a g_i is preserved by replacing λ_i with any $\lambda_i' \geq \lambda_i$. Therefore, using that $\{M_\lambda\}$ is a filtered diagram, we can find an index λ with maps

$$g_i: M \to M_\lambda$$

such that $f_\lambda \circ g_i = \varepsilon_i \mathrm{id}_M$. We consider the morphism

$$F_i := g_i \circ f_\lambda - \varepsilon_i \mathrm{id}_{M_\lambda}: M_\lambda \to M_\lambda.$$

We note that $\operatorname{Im}(F_i) \subset \operatorname{Ker}(f_\lambda)$ because

$$f_\lambda \circ g_i \circ f_\lambda - f_\lambda \varepsilon_i \mathrm{id}_{M_i} = \varepsilon_i f_\lambda - \varepsilon_i f_\lambda = 0.$$

We also have $\varepsilon_i \operatorname{Ker}(f_\lambda) \subset \operatorname{Im}(F_i)$ because $F_i|_{\operatorname{Ker}(f_\lambda)} = \varepsilon_i \mathrm{id}$. Therefore, $\sum_i \operatorname{Im}(F_i)$ is a finite R-module such that

$$\mathfrak{m}_0(\operatorname{Ker} f_\lambda) \subset \sum_i \operatorname{Im}(F_i) \subset \operatorname{Ker}(f_\lambda).$$

Therefore, $f\colon M' := M_\lambda / (\sum_i \operatorname{Im}(F_i)) \to M$ is a morphism such that its source M' is finitely presented, $\mathfrak{m}_0(\operatorname{Ker} f) = 0$, and $\mathfrak{m}_0(\operatorname{Coker} f) = 0$. Since $\mathfrak{m}_0 \subset \mathfrak{m}$ was an arbitrary finitely generated sub-ideal, we see that M is almost finitely presented. ■

Corollary 2.5.11. *Let M be an R-module. Then*

(1) *M is almost finitely generated if and only if the natural morphism*

$$\operatorname{colim}_I \operatorname{alHom}_R(M^a, N_i^a) \to \operatorname{alHom}_R(M^a, \operatorname{colim}_I N_i^a)$$

is injective in $\mathbf{Mod}_R^a$, for every filtered diagram $\{N_i^a\}_{i \in I}$ of R^a-modules;

(2) *M is almost finitely presented if and only if the natural morphism*

$$\operatorname{colim}_I \operatorname{alHom}_R(M^a, N_i^a) \to \operatorname{alHom}_R(M^a, \operatorname{colim}_I N_i^a)$$

is an isomorphism in $\mathbf{Mod}_R^a$, for every filtered diagram $\{N_i^a\}_{i \in I}$ of R^a-modules.

Proof. It formally follows from Lemma 2.5.9, Lemma 2.5.10, Proposition 2.2.1 (3), and Corollary 2.1.10. ■

Corollary 2.5.12. *Let M and N be two almost isomorphic R-modules (see Definition 2.1.7). Then M is almost finitely generated (resp. almost finitely presented) if and only if so is N.*

Proof. Corollary 2.5.11 implies that M is almost finitely generated (resp. almost finitely presented) if and only if $M_!^a$ is. Since $M_!^a \simeq N_!^a$, we get the desired result. ■

Corollary 2.5.13. *Let $R \to S$ be an almost isomorphism of rings. Then the forgetful functor $\mathbf{Mod}^*_{S^a} \to \mathbf{Mod}^*_{R^a}$ is an equivalence for $* \in \{\text{“ ”}, \text{aft}, \text{afp}\}$.*

Proof. Corollary 2.5.11 ensures that it suffices to prove the claim for $* = \text{“ ”}$ as the property of being almost finitely generated (resp. almost finitely presented) depends only on the category $\mathbf{Mod}_{R^a}$ and not on the ring R itself.

Corollary 2.2.4 (2) guarantees that the forgetful functor admits a right adjoint $- \otimes_{R^a} S^a \colon \mathbf{Mod}_R^a \to \mathbf{Mod}_S^a$. Therefore, it suffices to show that the natural morphisms

$$M^a \to M^a \otimes_{R^a} S^a$$

and

$$N^a \otimes_{R^a} S^a \to N^a$$

are isomorphisms for any $M \in \mathbf{Mod}_R^a$ and $N \in \mathbf{Mod}_S^a$. This is obvious from the fact that $R^a \to S^a$ is an isomorphism of R^a-modules. ■

Definition 2.5.14. We say that an R^a-module $M^a \in \mathbf{Mod}_R^a$ is *almost finitely generated* (resp. *almost finitely presented*) if its representative $M \in \mathbf{Mod}_R$ is almost finitely generated (resp. almost finitely presented). This definition does not depend on the choice of a representative due to Corollary 2.5.12.

We now want to establish certain good properties of almost finitely presented modules in short exact sequences. This will be crucial later in developing a good theory of almost coherent modules.

Lemma 2.5.15. *Let* $0 \to M' \xrightarrow{\varphi} M \xrightarrow{\psi} M'' \to 0$ *be an exact sequence of* R*-modules.*

(1) *If* M *is almost finitely generated, then so is* M''.
(2) *If* M' *and* M'' *are almost finitely generated (resp. finitely presented), then so is* M.
(3) *If* M *is almost finitely generated and* M'' *is almost finitely presented, then* M' *is almost finitely generated.*
(4) *If* M *is almost finitely presented and* M' *is almost finitely generated, then* M'' *is almost finitely presented.*

Proof. This can be easily deduced from Lemma 2.5.9 and Lemma 2.5.10 via the five lemma (or diagram chase). We only note that the Ext^1 part of Lemma 2.5.9 (2) is crucial to make the argument work. ■

Corollary 2.5.16. *Let* $0 \to M'^a \xrightarrow{\varphi} M^a \xrightarrow{\psi} M''^a \to 0$ *be an exact sequence of* R^a*-modules. Then all conclusions of Lemma 2.5.15 still hold.*

Proof. We use Lemma 2.1.9 (4), (5) to see that the sequence

$$0 \to (M'^a)_! \xrightarrow{\varphi_!} (M^a)_! \xrightarrow{\psi_!} (M''^a)_! \to 0$$

is exact and almost isomorphic to the original sequence. Moreover, Corollary 2.5.12 says that each of those modules $N^a_!$ is almost finitely generated (resp. almost finitely presented) if and only if so is the corresponding N^a. Thus, the problem is reduced to Lemma 2.5.15. ■

Lemma 2.5.17. *Let* M^a, N^a *be two almost finitely generated (resp. almost finitely presented)* R^a*-modules, then so is* $M^a \otimes_{R^a} N^a$*. Similarly,* $M \otimes_R N$ *is almost finitely generated (resp. almost finitely presented) for any almost finitely generated (resp. almost finitely presented)* R*-modules* M *and* N.

Proof. We show the claim only in the case of almost finitely presented modules; the case of almost finitely generated modules is significantly easier. Moreover, we use Proposition 2.2.1 (1) to reduce the question to showing that the tensor product of two almost finitely presented R-modules is almost finitely presented.

Step 1: *The case of finitely presented modules.* If both M and N are finitely presented, then this is a standard fact proven in [17, II Section 3.6, Proposition 6].

Step 2: *The case of M being finitely presented.* Now we deal with the case of a finitely presented R-module M and an almost finitely presented R-module N. We fix a finitely generated ideal $\mathfrak{m}_0 \subset \mathfrak{m}$ and a finitely generated ideal $\mathfrak{m}_1$ such that $\mathfrak{m}_0 \subset \mathfrak{m}_1^2$. Now we use Lemma 2.5.7 (2) to find a finitely presented module N' and a morphism $f : N' \to N$ such that $\mathrm{Ker}(f)$ and $\mathrm{Coker}(f)$ are annihilated by $\mathfrak{m}_0$. We denote the image of f by N'' and consider the short exact sequences

$$0 \to K \to N' \to N'' \to 0,$$
$$0 \to N'' \to N \to Q \to 0,$$

with K and Q being annihilated by $\mathfrak{m}_0$. After applying the functor $M \otimes_R -$, we get the following exact sequences:

$$M \otimes_R K \to M \otimes_R N' \to M \otimes_R N'' \to 0,$$
$$\mathrm{Tor}_1^R(M, Q) \to M \otimes_R N'' \to M \otimes_R N \to M \otimes_R Q \to 0.$$

We note that $M \otimes_R K, \mathrm{Tor}_1^R(M, Q)$, and $M \otimes_R Q$ are annihilated by $\mathfrak{m}_0$. Now it is straightforward to conclude that the map

$$M \otimes_R f : M \otimes N' \to M \otimes N$$

has kernel and cokernel annihilated by $\mathfrak{m}_1 \subset \mathfrak{m}_0^2$. Moreover, $M \otimes N'$ is a finitely presented module by Step 1. Since $\mathfrak{m}_1$ was an arbitrary finitely generated subideal of $\mathfrak{m}$, we conclude that $M \otimes N$ is almost finitely presented due to Lemma 2.5.7 (2).

Step 3: *The general case.* Repeat the argument of Step 2 once again using Step 2 in place of Step 1 at the end, and Lemma 2.5.8 in place of Lemma 2.5.7 (2). ∎

Lemma 2.5.18. *Let M be an almost finitely presented R-module, let N be any R-module, and let P be an almost flat R-module. Then the corresponding natural map* $\mathrm{Hom}_R(M, N) \otimes_R P \to \mathrm{Hom}_R(M, N \otimes_R P)$ *is an almost isomorphism.*

Similarly, $\mathrm{Hom}_{R^a}(M^a, N^a) \otimes_{R^a} P^a \to \mathrm{Hom}_{R^a}(M^a, N^a \otimes_{R^a} P^a)$ *is an almost isomorphism for any almost finitely presented R^a-module M^a, any R^a-module N^a, and an almost flat R^a-module P^a.*

Proof. Proposition 2.2.1 (1) and (3) ensure that it suffices to prove the claim for the case of honest R-modules M, N, and P.

Step 1: *The case of a finitely presented module* M. We choose a presentation of M:

$$R^n \to R^m \to M \to 0.$$

Then we use that P is almost flat to get a morphism of almost exact sequences:

$$\begin{array}{ccccccc} 0 \to & \mathrm{Hom}_R(M,N)\otimes_R P & \to & \mathrm{Hom}_R(R^m,N)\otimes_R P & \to & \mathrm{Hom}_R(R^n,N)\otimes_R P \\ & \downarrow & & \downarrow & & \downarrow \\ 0 \to & \mathrm{Hom}_R(M,N\otimes_R P) & \to & \mathrm{Hom}_R(R^m,N\otimes_R P) & \to & \mathrm{Hom}_R(R^n,N\otimes_R P). \end{array}$$

Clearly, the second and third vertical arrows are (almost) isomorphisms, so the first vertical arrow is an almost isomorphism as well.

Step 2: *The general case.* The case of an almost finitely presented module M follows from the finitely presented case by approximating M by finitely presented R-modules. This is similar to the strategy used in Lemma 2.5.17; we leave the details to the reader. ∎

The last ingredient we will need is the interaction between properties of an R-module M and its "reduction" M/I for some finitely generated ideal $I \subset \mathfrak{m}$. For example, we know that for an ideal $I \subset \mathrm{rad}(R)$ and a finite module M, Nakayama's lemma states that $M/I = 0$ if and only if $M = 0$. Another feature is that an I-adically complete module M is R-finite if and only if M/I is R/I-finite. It turns out that both facts have their "almost" analogues.

Lemma 2.5.19. *Let* $I \subset \mathfrak{m} \cap \mathrm{rad}(R)$ *be a finitely generated ideal. If* M *is an almost finitely generated* R*-module such that* $M/IM \simeq 0$*. Then* $M \simeq 0$*. If* $M/IM \cong^a 0$*, then* $M \cong^a 0$*.*

Proof. We use the definition of an almost finitely generated module to find a finite submodule N containing IM. If M/IM is isomorphic to the zero module, then inclusion $IM \subset N \subset M$ implies that $N = M$. Thus M is actually finitely generated, now we use the usual Nakayama's lemma to finish the proof.

If M/IM is merely almost isomorphic to the zero module, then we see that the inclusion $IM \subset M$ is an almost isomorphism. In particular, $\mathfrak{m}M$ is almost isomorphic to IM. Using that $\mathfrak{m}^2 = \mathfrak{m}$, we obtain an *equality*

$$\mathfrak{m}M = \mathfrak{m}^2 M = \mathfrak{m}(IM) = I(\mathfrak{m}M).$$

Thus we can apply the argument from above to conclude that $\mathfrak{m}M = 0$. This finishes the proof as $\mathfrak{m}M \cong^a M$. ∎

Lemma 2.5.20. *Let R be I-adically complete for some finitely generated $I \subset \mathfrak{m}$. Then an I-adically complete R-module M is almost finitely generated if and only if M/IM is almost finitely generated.*

Proof. [26, Lemma 5.3.18] ■

2.6 Almost coherent modules and almost coherent rings

This section is devoted to the study of almost coherent modules which are "almost" analogues of classical coherent modules. We show that these modules form a weak Serre subcategory in $\mathbf{Mod}_R$. Then we study the special case of almost coherent modules over an almost coherent ring. In this case, we show that almost coherent modules are equivalent to almost finitely presented modules.

We recall that we fixed some "base" ring R with an ideal $\mathfrak{m}$ such that $\mathfrak{m}^2 = \mathfrak{m}$ and $\widetilde{\mathfrak{m}} = \mathfrak{m} \otimes_R \mathfrak{m}$ is flat, and we always do almost mathematics with respect to this ideal.

Definition 2.6.1. An (almost) R-module M is *almost coherent* if it is almost finitely generated and every almost finitely generated almost submodule $N^a \subset M^a$ is almost finitely presented.

Remark 2.6.2. An almost submodule $f: N^a \hookrightarrow M^a$ does not necessarily give rise to a submodule $N' \subset M$ for some $(N')^a \simeq N^a$. The most we can say is that there is an injection $f_!: (N^a)_! \hookrightarrow (M^a)_!$ whose almostification is equal to the morphism f (this follows from Lemma 2.1.8 (2)).

Lemma 2.6.3. *Let $R \to S$ be an almost isomorphism of rings. Then the forgetful functor $\mathbf{Mod}_{S^a}^{\mathrm{acoh}} \to \mathbf{Mod}_{R^a}^{\mathrm{acoh}}$ is an equivalence.*

Proof. This follows directly from Corollary 2.5.13 and Definition 2.6.1. ■

Lemma 2.6.4. *Let M^a be an almost R-module with a representative $M \in \mathbf{Mod}_R$. Then the following are equivalent:*

(1) *The almost module M^a is almost coherent.*

(2) *The R-module $(M^a)_*$ is almost finitely generated, and any almost finitely generated R-submodule of $(M^a)_*$ is almost finitely presented.*

(3) *The R-module $(M^a)_!$ is almost finitely generated, and any almost finitely generated R-submodule of $(M^a)_!$ is almost finitely presented.*

Proof. First of all, we note that Corollary 2.5.12 guarantees that M is almost finitely generated if and only if so is $(M^a)_*$. Second, Lemma 2.1.9 implies that the functor $(-)_*$ is left exact. Therefore, any almost submodule $N^a \subset M^a$ gives rise to an actual submodule $(N^a)_* \subset (M^a)_*$ that is almost isomorphic to N. In reverse, any submodule $N \subset (M^a)_*$ gives rise to an almost submodule of M^a. Hence, we see that all almost finitely generated almost submodules of M^a are almost finitely presented if and only if all actual almost finitely generated submodules of M_* are almost finitely presented (here we again use Corollary 2.5.12). This shows the equivalence of (1) and (2). The same argument shows that (1) is equivalent to (3). ■

Note that it is not that clear whether a coherent R-module is almost coherent. The issue is that in the definition of almost coherent modules we need to be able to handle all almost finitely generated almost submodules and not only finitely generated ones. The lemma below is a useful tool to deal with such problems; in particular, it turns out (Corollary 2.6.7) that all coherent modules are indeed almost coherent, but we do not know a direct way to see that.

Lemma 2.6.5. *Let M be an R-module. Then M is an almost coherent module if one of the following holds:*

(1) *For any finitely generated ideal $\mathfrak{m}_0 \subset \mathfrak{m}$, there exist a coherent R-module N and a morphism $f: N \to M$ such that $\mathfrak{m}_0(\mathrm{Ker}\, f) = 0$ and $\mathfrak{m}_0(\mathrm{Coker}\, f) = 0$.*

(2) *For any finitely generated ideal $\mathfrak{m}_0 \subset \mathfrak{m}$, there exist an* almost *coherent R-module N and a morphism $f: N \to M$ such that $\mathfrak{m}_0(\mathrm{Ker}\, f) = 0$ and $\mathfrak{m}_0(\mathrm{Coker}\, f) = 0$.*

Proof. We start the proof by noting that M comes with the natural almost isomorphism $M \to M^a_*$. Since both assumptions on M pass through this almost isomorphism, Lemma 2.6.4 implies that it suffices to show that $M_* := M^a_*$ is almost coherent.

Lemma 2.5.7 guarantees that M_* is almost finitely generated. Thus, we only need to check the second condition from Definition 2.6.1. So we pick an arbitrary almost finitely generated R-submodule $M_1 \subset M_*$ and wish to show that it is almost finitely presented. We choose an arbitrary finitely generated ideal $\mathfrak{m}_0 \subset \mathfrak{m}$ and another finitely generated ideal $\mathfrak{m}_1 \subset \mathfrak{m}$ such that $\mathfrak{m}_0 \subset \mathfrak{m}_1^2$.

We use Lemma 2.5.8 to find a morphism $\varphi: R^n \to M_1$ such that $\mathfrak{m}_1(\mathrm{Coker}\, \varphi) = 0$. We denote by $e_1, \dots, e_n$ the standard basis of R^n and by $x_i := \varphi(e_i)$ the image of e_i in M_1. We also choose a set of generators $(\varepsilon_1, \dots, \varepsilon_m)$ of the ideal $\mathfrak{m}_1$.

By assumption, there is a morphism $f: N \to M_*$ with a(n) (almost) coherent R-module N such that $\mathfrak{m}_1(\mathrm{Coker}\, f) = 0$ and $\mathfrak{m}_1(\mathrm{Ker}\, f) = 0$. This implies that $\varepsilon_i x_j$ is in the image of f for any $i = 1, \dots, m$, $j = 1, \dots, n$. Let us choose some $y_{i,j} \in N$ such that $f(y_{i,j}) = \varepsilon_i x_j$, and define an R-module N' as the submodule of N generated by all $y_{i,j}$. By construction, N' is a finite R-module. Since N is a (almost) coherent module, we conclude that N' is (almost) finitely presented.

We observe that $f' := f|_{N'}$ naturally lands in M_1, and we have $\mathfrak{m}_1(\operatorname{Ker} f') = 0$ and $\mathfrak{m}_1^2(\operatorname{Coker} f') = 0$. Since $\mathfrak{m}_0 \subset \mathfrak{m}_1^2$, this shows that the morphism

$$N' \xrightarrow{f'} M_1$$

has kernel and cokernel killed by $\mathfrak{m}_0$. Lemma 2.5.8 shows that M_1 is almost finitely presented. ■

Question 2.6.6. Does the converse of Lemma 2.6.5 hold?

Corollary 2.6.7. *Any coherent R-module M is almost coherent.*

The next thing we want to show is that almost coherent modules form a weak Serre subcategory of $\mathbf{Mod}_R$. This is an almost analogue of the corresponding statement in the classical case.

Lemma 2.6.8. *Let R and $\mathfrak{m}$ be as above. Then*

(1) *an almost finitely generated almost submodule of an almost coherent module is almost coherent;*

(2) *let $\varphi\colon N^a \to M^a$ be an almost homomorphism from an almost finitely generated R^a-module to an almost coherent R^a-module, then* $\operatorname{Ker}\varphi$ *is an almost finitely generated R^a-module;*

(3) *let $\varphi\colon N^a \to M^a$ be an injective almost homomorphism of almost coherent R^a-modules, then* $\operatorname{Coker}\varphi$ *is an almost coherent R^a-module;*

(4) *let $\varphi\colon N^a \to M^a$ be an almost homomorphism of almost coherent R^a-modules, then* $\operatorname{Ker}\varphi$ *and* $\operatorname{Coker}\varphi$ *are almost coherent R^a-modules;*

(5) *given a short exact sequence of R^a-modules $0 \to M'^a \to M^a \to M''^a \to 0$, if two out of three are almost coherent, so is the third.*

Proof. (1) This is evident from the definition of an almost coherent almost module.

(2) Let us define $N''^a := \operatorname{Im}\varphi$ and $N'^a := \operatorname{Ker}\varphi$, then Corollary 2.5.16 implies that N''^a is an almost finitely generated almost submodule of M^a. Furthermore, it is almost finitely presented since M^a is almost coherent. Thus, Corollary 2.5.16 implies that N' is almost finitely generated as well.

(3) We denote $\operatorname{Coker}\varphi$ by M''^a, then we have a short exact sequence

$$0 \to N^a \to M^a \to M''^a \to 0.$$

Corollary 2.5.16 implies that M''^a is almost finitely generated. Let us choose any almost finitely generated almost submodule $M_1''^a \subset M''^a$ and denote its pre-image in M^a by M_1^a. Then we have a short exact sequence

$$0 \to N^a \to M_1^a \to M_1''^a \to 0.$$

Corollary 2.5.16 guarantees that M_1^a is an almost finitely generated almost submodule of M^a. Since M^a is almost coherent, we see that M_1^a is an almost finitely presented R^a-module. Therefore, Corollary 2.5.16 implies that $M_1''^a$ is also almost finitely presented. Hence, the R^a-module M''^a is almost coherent.

(4) We know that $N'^a := \operatorname{Ker}\varphi$ is almost finitely generated by (2). Since N^a is almost coherent, we conclude that N'^a is almost coherent by (1). We define $N''^a := \operatorname{Im}\varphi$ and $M''^a := \operatorname{Coker}\varphi$, then we note that we have two short exact sequences

$$0 \to N'^a \to N^a \to N''^a \to 0,$$
$$0 \to N''^a \to M^a \to M''^a \to 0.$$

We observe that (3) shows that N''^a is almost coherent, then we use (3) once more to conclude that M''^a is also almost coherent.

(5) The only thing that we are left to show is that if M'^a and M''^a are almost coherent, so is M^a. It is almost finitely generated by Corollary 2.5.16. In order to check the second condition from Definition 2.6.1, we choose an almost finitely generated almost submodule $M_1^a \subset M^a$. Let us denote by $M_1''^a$ its image in M''^a, and by $M_1'^a$ the kernel of this map. So we have a short exact sequence

$$0 \to M_1'^a \to M_1^a \to M_1''^a \to 0.$$

Corollary 2.5.16 guarantees that $M_1''^a$ is an almost finitely generated almost submodule of the almost coherent R^a-module M''^a. Hence, (1) implies that $M_1''^a$ is almost coherent, in particular, it is almost finitely presented. Moreover, we use (2) to see that $M_1'^a$ is an almost finitely generated almost submodule of M'^a. Since M'^a is almost coherent, we conclude that $M_1'^a$ is almost finitely presented. Finally, Corollary 2.5.16 shows that M_1^a is almost finitely presented as well. This finishes the proof of almost coherence of the R^a-module M^a. ■

Corollary 2.6.9. *Let M^a be an almost finitely presented R^a-module and let N^a be an almost coherent R^a-module. Then $M^a \otimes_{R^a} N^a$ and $\mathrm{alHom}_{R^a}(M^a, N^a)$ are almost coherent.*

Proof. We use Proposition 2.2.1 (1) and (3) to reduce the question to showing that $M \otimes_R N$ and $\operatorname{Hom}_R(M, N)$ are almost coherent R-modules for any almost finitely presented R-module M and almost coherent R-module N.

Step 1: *The case of a finitely presented module M.* In this case, we choose a presentation of M as the quotient

$$R^n \to R^m \to M \to 0.$$

Then we have short exact sequences

$$N^n \to N^m \to M \otimes_R N \to 0$$

and

$$0 \to \mathrm{Hom}_R(M, N) \to N^m \to N^n.$$

We note that Lemma 2.6.8 (5) implies that N^m and N^n are almost coherent. Thus, Lemma 2.6.8 (5) guarantees that both $M \otimes_R N$ and $\mathrm{Hom}_R(M, N)$ are almost coherent as well.

Step 2: *The general case.* The argument is similar to the one used in Step 2 of the proof of Lemma 2.5.17. We approximate M by finitely presented R-modules. This gives us approximations of $M^a \otimes_{R^a} N^a$ and $\mathrm{alHom}_{R^a}(M^a, N^a)$ by almost coherent modules. Now Lemma 2.6.5 guarantees that these modules are almost coherent. We leave the details to the interested reader. ■

We define $\mathbf{Mod}_R^{\mathrm{acoh}}$ (resp. $\mathbf{Mod}_{R^a}^{\mathrm{acoh}}$) to be the strictly full[4] subcategory of $\mathbf{Mod}_R$ (resp. $\mathbf{Mod}_{R^a}$) consisting of almost coherent R-modules (resp. R^a-modules).

Corollary 2.6.10. *The category $\mathbf{Mod}_R^{\mathrm{acoh}}$ (resp. $\mathbf{Mod}_{R^a}^{\mathrm{acoh}}$) is a weak Serre subcategory of $\mathbf{Mod}_R$ (resp. $\mathbf{Mod}_{R^a}$).*

Corollary 2.6.10 and the discussion in [68, Tag 06UP] ensure that $\mathbf{D}_{\mathrm{acoh}}(R)$ and[5] $\mathbf{D}_{\mathrm{acoh}}(R)^a$ are strictly full saturated[6] triangulated subcategories of $\mathbf{D}(R)$ and $\mathbf{D}(R)^a$ respectively. We define $\mathbf{D}^+_{\mathrm{acoh}}(R) := \mathbf{D}_{\mathrm{acoh}}(R) \cap \mathbf{D}^+(R)$ and similarly for all other bounded versions.

Lemma 2.6.11. *Let $M \in \mathbf{D}(R)$ be a complex of R-modules. Then $M \in \mathbf{D}_{\mathrm{acoh}}(R)$ if one of the following holds:*

(1) *for every finitely generated ideal $\mathfrak{m}_0 \subset \mathfrak{m}$, there are $N \in \mathbf{D}_{\mathrm{coh}}(R)$ and a morphism $f : N \to M$ such that $\mathfrak{m}_0\big(\mathrm{H}^i(\mathrm{cone}(f))\big) = 0$ for every $i \in \mathbf{Z}$;*

(2) *for every finitely generated ideal $\mathfrak{m}_0 \subset \mathfrak{m}$, there are $N \in \mathbf{D}_{\mathrm{acoh}}(R)$ and a morphism $f : N \to M$ such that $\mathfrak{m}_0\big(\mathrm{H}^i(\mathrm{cone}(f))\big) = 0$ for every $i \in \mathbf{Z}$.*

Proof. This is an easy consequence of Lemma 2.6.5 applied together with the definition of $\mathbf{D}_{\mathrm{acoh}}(R)$. ■

The last part of this subsection is dedicated to the study of almost coherent rings and almost coherent modules over almost coherent rings. Recall that coherent modules over a coherent ring coincide with finitely presented ones. Similarly, we will show that almost coherent modules over an almost coherent ring turn out to be the same as almost finitely presented ones.

[4] A strictly full subcategory is a full subcategory that is closed under isomorphisms.

[5] These are, respectively, full subcategories of $\mathbf{D}(R)$ and $\mathbf{D}(R)^a$ of complexes with almost coherent cohomology modules.

[6] A strictly full subcategory $\mathcal{D}'$ of a triangulated category $\mathcal{D}$ is saturated if $X \oplus Y \in \mathcal{D}'$ implies $X, Y \in \mathcal{D}'$.

Definition 2.6.12. We say that a ring R is almost coherent if the rank-1 free module R is almost coherent as an R-module.

Lemma 2.6.13. *A coherent ring R is almost coherent.*

Proof. Apply Corollary 2.6.7 to the rank-1 free module R. ■

Lemma 2.6.14. *If R is an almost coherent ring, then any almost finitely presented R-module M is almost coherent.*

Proof. *Step* 1: If M is finitely presented over R, then we can write it as a cokernel of a map between free finite rank modules. A free finite rank module over an almost coherent ring is almost coherent due to Lemma 2.6.8 (5). A cokernel of a map of almost coherent modules is almost coherent due to Lemma 2.6.8 (4). Therefore, any finitely presented M is almost coherent.

Step 2: Suppose that M is merely almost finitely presented. Lemma 2.5.7 guarantees that, for any finitely generated $\mathfrak{m}_0 \subset \mathfrak{m}$, we can find a finitely presented module N and a map $f: N \to M$ such that $\mathrm{Ker}\, f$ and $\mathrm{Coker}\, f$ are annihilated by $\mathfrak{m}_0$. We know that N is almost coherent by Step 1. Therefore, Lemma 2.6.5 (2) implies that M is almost coherent as well. ■

Corollary 2.6.15. *Let R be an almost coherent ring. Then an R-module M is almost coherent if and only if it is almost finitely presented.*

Proof. The "only if" part is clear from the definition, the "if" part follows from Lemma 2.6.14. ■

Our next big goal is to show that bounded above almost coherent complexes over an almost coherent ring are exactly "almost pseudo-coherent complexes" in some precise way. More precisely, any element $M \in \mathbf{D}^-_{\mathrm{acoh}}(R)$ can be "approximated" up to any small torsion by complexes of finite free modules.

Proposition 2.6.16. *Let R be an almost coherent ring and let $M \in \mathbf{D}^-(R)$. Then $M \in \mathbf{D}^-_{\mathrm{acoh}}(R)$ if and only if, for every finitely generated ideal $\mathfrak{m}_0 \subset \mathfrak{m}$, there are a complex $F^\bullet$ of finite free R-modules, and a morphism*

$$f: F^\bullet \to M$$

such that $\mathfrak{m}_0\big(\mathrm{H}^i(\mathrm{cone}(f))\big) = 0$ for every $i \in \mathbf{Z}$. Moreover, if $M \in \mathbf{D}^{\le 0}_{\mathrm{coh}}(R)$, one can choose $F^\bullet \in \mathbf{Comp}^{\le 0}(R)$.

Proof. The "if" direction is Lemma 2.6.11. So we only need to prove the "only if" direction. For this direction, we fix a finitely generated ideal $\mathfrak{m}_0 \subset \mathfrak{m}$ and another finitely generated ideal $\mathfrak{m}_1 \subset \mathfrak{m}$ such that $\mathfrak{m}_0 \subset \mathfrak{m}_1^2$.

Without loss of generality, we may and do assume that $M \in \mathbf{D}^{\leq 0}(R)$, and then we choose a complex $M^\bullet \in \mathbf{Comp}^{\leq 0}(R)$ that represents M. Now we prove a slightly more precise claim:

Claim. *For every $n \in \mathbf{Z}$, there is a complex of finite free modules $F_n^\bullet$ with a morphism $f_n \colon F_n^\bullet \to M^\bullet$ such that*

(1) $F_n^\bullet \in \mathbf{Comp}^{[-n,0]}(R)$;

(2) $\sigma^{\geq -n+1} F_n^\bullet = F_{n-1}^\bullet$ *and* $\sigma^{\geq -n+1} f_n = f_{n-1}$, *where* $\sigma^{\geq n-1}$ *is the naive truncation;*

(3) *kernels and cokernels of* $\mathrm{H}^i(f_n)$ *are annihilated by* $\mathfrak{m}_1$ *for* $i \geq n+1$;

(4) *the cokernel of* $\mathrm{H}^n(f_n)$ *is annihilated by* $\mathfrak{m}_1$.

Proof of the claim. We argue by descending induction on n. If $n \geq 1$, $F^\bullet = 0$ works. Now we suppose that we can construct $F_n^\bullet$, and wish to construct $F_{n-1}^\bullet$. Consider the morphism f_n presented as a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & 0 & \longrightarrow & F_n^n & \xrightarrow{\mathrm{d}_F^n} & F_n^{n+1} & \xrightarrow{\mathrm{d}_F^{n+1}} & \cdots \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle f_n^n} & & \downarrow{\scriptstyle f_n^{n+1}} & & \downarrow \\ M^{n-2} & \xrightarrow[\mathrm{d}_M^{n-2}]{} & M^{n-1} & \xrightarrow[\mathrm{d}_M^{n-1}]{} & M^n & \xrightarrow[\mathrm{d}_M^{n}]{} & M^{n+1} & \xrightarrow[\mathrm{d}_M^{n+1}]{} & \cdots \end{array}$$

Firstly, $\mathrm{Ker}(\mathrm{d}_F^n)$ is almost coherent as a kernel between finitely presented modules over an almost coherent ring. Secondly, the R-module

$$B^n := \mathrm{Ker}\big(\mathrm{Ker}(\mathrm{d}_F^n) \to \mathrm{H}^n(M)\big)$$

is also almost coherent as a kernel between almost coherent modules. Therefore, there are a finite free R-module F'^{n-1} and a morphism

$$\mathrm{d}' \colon F'^{n-1} \to B^n$$

such that $\mathfrak{m}_1(\mathrm{Coker}\,\mathrm{d}') = 0$. Since $\mathrm{H}^{n-1}(M)$ is almost coherent, we can find a finite free R-module F''^{n-1} and a morphism

$$\lambda \colon F''^{n-1} \to \mathrm{H}^{n-1}(M)$$

such that $\mathfrak{m}_1(\mathrm{Coker}\,\lambda) = 0$. Let $\nu \colon F''^{n-1} \to Z^{n-1}(M^\bullet)$ be any lift of λ to the module of closed elements $Z^{n-1}(M^\bullet) = \mathrm{Ker}(\mathrm{d}_M^{n-1})$. We define

$$f''^{n-1} \colon F''^{n-1} \to M^{n-1}$$

to be the composition of ν with the inclusion $Z^{n-1}(M^\bullet) \to M^{n-1}$.

Now we wish to define $F^\bullet_{n-1}$ and f_{n-1}. We start with $F^\bullet_{n-1}$; we put $F^m_{n-1} = F^m_n$ if $m \geq n$, $F^m_{n-1} = 0$ if $m < n-1$, $F^{n-1}_{n-1} = F'^{n-1} \oplus F''^{n-1}$, and define the only non-evident differential

$$\mathrm{d}^{n-1}_F \colon F^{n-1}_{n-1} = F'^{n-1} \oplus F''^{n-1} \to F^n_n$$

to be zero on F''^{n-1} and equal to d' on F'^{n-1}. It is evident that $\mathrm{d}^n_F \circ \mathrm{d}^{n-1}_F = 0$, so this structure defines a complex $F^\bullet_{n-1}$ of finite free R-modules.

We are only left to define f_{n-1}. We must put $f^m_{n-1} = f^m_n$ if $m > n-1$ and $f^m_{n-1} = 0$ if $m < n-1$, so the only question is to define f^{n-1}_{n-1}. By construction, we have $f^n_n(\mathrm{d}'F'^{n-1}) \subset \mathrm{d}^{n-1}_M M^{n-1}$, so we can find

$$f'_{n-1} \colon F'^{n-1} \to M^{n-1}$$

such that $\mathrm{d}^{n-1} \circ f'_{n-1} = f^n_n \circ \mathrm{d}'$. Thus we define

$$f^{n-1}_{n-1} \colon F^{n-1}_{n-1} = F'^{n-1} \oplus F''^{n-1} \to M^{n-1}$$

to be f'_{n-1} on F'^{n-1} and f''_{n-1} on F''^{n-1}. Then it is evident from the construction that $f^\bullet_{n-1}$ is a morphism of complexes, i.e. , it fits the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F^{n-1}_{n-1} & \xrightarrow{\mathrm{d}^{n-1}_F} & F^n_{n-1} & \xrightarrow{\mathrm{d}^n_F} & F^{n+1}_{n-1} & \xrightarrow{\mathrm{d}^{n+1}_F} & \cdots \\
\downarrow & & \downarrow{\scriptstyle f^{n-1}_{n-1}} & & \downarrow{\scriptstyle f^n_{n-1}} & & \downarrow{\scriptstyle f^{n+1}_{n-1}} & & \downarrow \\
M^{n-2} & \xrightarrow[\mathrm{d}^{n-2}_M]{} & M^{n-1} & \xrightarrow[\mathrm{d}^{n-1}_M]{} & M^n & \xrightarrow[\mathrm{d}^n_M]{} & M^{n+1} & \xrightarrow[\mathrm{d}^{n+1}_M]{} & \cdots
\end{array}$$

By construction, the kernel and cokernel of $\mathrm{H}^n(f_{n-1})$ are annihilated by $\mathfrak{m}_1$, and the cokernel of $\mathrm{H}^{n-1}(f_{n-1})$ is annihilated by $\mathfrak{m}_1$. So this finishes the proof of the claim. ■

Now the morphism $f \colon F^\bullet \to M^\bullet$ simply comes as the colimit of f_n, i.e.,

$$f = \operatorname{colim} f_n \colon F^\bullet := \operatorname{colim} F^\bullet_n \to M^\bullet.$$

It is not hard to see that the cohomology groups of $\operatorname{cone}(f)$ are annihilated by $\mathfrak{m}_0 \subset \mathfrak{m}_1^2$. ■

Corollary 2.6.17. *Let R be a coherent ring and $M \in \mathbf{D}^b(R)$. Then $M \in \mathbf{D}^b_{\mathrm{acoh}}(R)$ if and only if, for every finitely generated ideal $\mathfrak{m}_0 \subset \mathfrak{m}$, there is a complex $N \in \mathbf{D}^b_{\mathrm{coh}}(R)$ together with a morphism $f \colon N \to M$ such that $\mathfrak{m}_0(\mathrm{H}^i(\operatorname{cone}(f))) = 0$ for all i.*

Proof. The "if" direction is Lemma 2.6.11. So we only need to deal with the "only if" direction. Assume that $M \in \mathbf{D}^b(R)$. Then Proposition 2.6.16 implies that there are $F \in \mathbf{D}^-_{\mathrm{coh}}(R)$ and a morphism $f: F \to M$ such that $\mathfrak{m}_0(\mathrm{H}^i(\mathrm{cone}(f))) = 0$ for all i. Now we can replace F by $F' := \tau^{\geq a} F$ to get the desired approximation with $F' \in \mathbf{D}^b_{\mathrm{coh}}(R)$. ■

Proposition 2.6.18. *Let R be an almost coherent ring, and let M^a, N^a be objects in $\mathbf{D}^-_{\mathrm{acoh}}(R)^a$. Then $M^a \otimes^L_{R^a} N^a \in \mathbf{D}^-_{\mathrm{acoh}}(R)^a$.*

Proof. Proposition 2.4.13 ensures that it suffices to show that $M \otimes^L_R N \in \mathbf{D}^-_{\mathrm{acoh}}(R)$ for $M, N \in \mathbf{D}^-_{\mathrm{coh}}(R)$. Clearly, we can cohomologically shift both M and N to assume that they lie $\mathbf{D}^{\leq 0}_{\mathrm{coh}}(R)$.

Now we fix a finitely generated ideal $\mathfrak{m}_1 \subset \mathfrak{m}$ and use Proposition 2.6.16 to find an exact triangle

$$F^\bullet \to M \to Q,$$

where $F^\bullet \in \mathbf{D}^{\leq 0}(R)$ is a complex of finite free modules and $\mathrm{H}^i(Q)$ are all annihilated by $\mathfrak{m}_1$. Then it is easy to see that the kernel and cokernel of the map

$$\mathrm{H}^{-i}(F^\bullet \otimes^L_R N) \to \mathrm{H}^{-i}(M \otimes^L_R N)$$

are annihilated by $\mathfrak{m}_1^{i+1}$. Now we note that, clearly,

$$F^\bullet \otimes^L_R N \simeq F^\bullet \otimes^\bullet_R N$$

lies in $\mathbf{D}^-_{\mathrm{coh}}(R)$ because $F^\bullet$ is a complex of finite free modules. For each pair of an integer $i \geq 0$ and a finitely generated ideal $\mathfrak{m}_0 \subset \mathfrak{m} = \mathfrak{m}^{i+1}$, we can find another finitely generated ideal $\mathfrak{m}_1$ such that $\mathfrak{m}_0 \subset \mathfrak{m}_1^{i+1}$. Therefore, the map

$$\mathrm{H}^{-i}(F^\bullet \otimes^L_R N) \to \mathrm{H}^{-i}(M \otimes^L_R N)$$

is a morphism with an almost coherent source and $\mathfrak{m}_0$-torsion kernel and cokernel. Therefore, Lemma 2.6.5 (2) implies the claim. ■

Proposition 2.6.19. *Let R be an almost coherent ring, and let $M^a \in \mathbf{D}^-_{\mathrm{acoh}}(R)^a$, $N^a \in \mathbf{D}^+_{\mathrm{acoh}}(R)^a$. Then $\mathbf{R}\mathrm{alHom}_{R^a}(M^a, N^a) \in \mathbf{D}^+_{\mathrm{acoh}}(R)^a$.*

Proof. The proof is similar to that of Proposition 2.6.18. We use Proposition 2.4.8 and the same approximation argument to reduce to the case $M = F^\bullet$ is a bounded above complex of finite free modules. In this case, the claim is essentially obvious due to the explicit construction of the Hom-complex $\mathrm{Hom}^\bullet_R(F^\bullet, N)$. ■

Proposition 2.6.20. *Let R be an almost coherent ring, let $M \in \mathbf{D}^-_{\mathrm{acoh}}(R)$, let $N \in \mathbf{D}^+(R)$, and let P be an almost flat R-module. Then the natural map*

$$\mathbf{R}\mathrm{Hom}_R(M, N) \otimes_R P \to \mathbf{R}\mathrm{Hom}_R(M, N \otimes_R P)$$

is an almost isomorphism.

Similarly, $\mathbf{R}\mathrm{Hom}_{R^a}(M^a, N^a) \otimes^L_{R^a} P^a \to \mathbf{R}\mathrm{Hom}_{R^a}(M^a, N^a \otimes^L_{R^a} P^a)$ *is an almost isomorphism for any* $M^a \in \mathbf{D}^-_{\mathrm{acoh}}(R)^a$, $N^a \in \mathbf{D}^+(R)^a$, *and* P^a *an almost flat* R^a*-module.*

Proof. The proof is similar to that of the above lemmas. ∎

Corollary 2.6.21. *Let* R *be an almost coherent ring, let* M^a *be an object in* $\mathbf{D}^-_{\mathrm{acoh}}(R)^a$, *let* N^a *be an object in* $\mathbf{D}^+(R)^a$, *and let* P^a *be an almost flat* R^a*-module. Then the natural map*

$$\mathbf{R}\mathrm{alHom}_{R^a}(M^a, N^a) \otimes^L_{R^a} P^a \to \mathbf{R}\mathrm{alHom}_{R^a}(M^a, N^a \otimes_{R^a} P^a)$$

is an isomorphism in $\mathbf{D}(R^a)$.

2.7 Almost noetherian rings

The main goal of this section is to define the almost analogue of the noetherian property. We also verify some of its basic properties. Even though most of the basic facts about noetherian rings carry over to the almost world, we warn the reader that Hilbert's Nullstellensatz seems to be more subtle in the almost world (see Warning 2.7.9); we are able to establish it only in some very particular situations in Section 2.11.

As in the previous sections, we fix a ring R with an ideal $\mathfrak{m}$ such that $\mathfrak{m}^2 = \mathfrak{m}$ and $\widetilde{\mathfrak{m}} = \mathfrak{m} \otimes_R \mathfrak{m}$ is flat, and we always do almost mathematics with respect to this ideal.

Definition 2.7.1. A ring R is *almost noetherian* if every ideal $I \subset R$ is almost finitely generated.

The main goal is to show that every almost finitely generated module over an almost noetherian ring is almost finitely presented. In particular, an almost noetherian ring is almost coherent.

Lemma 2.7.2. *Let* R *be an almost noetherian ring, and* $M \subset R^n$ *an* R*-submodule. Then* M *is almost finitely generated.*

Proof. We argue by induction on n. The base of induction is $n = 1$, where the claim follows from the definition of an almost noetherian ring.

Suppose we know the claim for $n - 1$, so we deduce the claim for n. Denote by $R^{n-1} \subset R^n$ a free R-module spanned by the first $n - 1$ standard basis elements of R^n, and denote by $M' := M \cap R^{n-1}$ the intersection of M with R^{n-1}. Then we have a short exact sequence

$$0 \to M' \to M \to M'' \to 0,$$

where M'' is naturally an R-submodule of $R \simeq R^n/R^{n-1}$. By the induction hypothesis, M' is almost finitely generated. Then M'' is almost finitely generated by almost noetherianness of R. Thus, M is almost finitely generated by Lemma 2.5.15 (2). ∎

Lemma 2.7.3. *Let R be an almost noetherian ring. Then any almost finitely generated R-module M is almost finitely presented.*

Proof. Pick any finitely generated sub-ideal $\mathfrak{m}_0 \subset \mathfrak{m}$. By Lemma 2.5.5, there is an R-linear homomorphism

$$f: R^n \to M$$

such that $\mathfrak{m}_0(\operatorname{Coker} f) = 0$. Consider $N := \operatorname{Ker}(f)$. Lemma 2.7.2 ensures that N is also almost finitely generated, so there is an R-linear homomorphism

$$g': R^m \to N$$

such that $\mathfrak{m}_0(\operatorname{Coker} g') = 0$. Therefore, the composition

$$R^m \xrightarrow{g} R^n \xrightarrow{f} M$$

is a three-term complex with $\mathfrak{m}_0(\operatorname{Coker} f) = 0$ and $\mathfrak{m}_0(\operatorname{Ker} f) \subset \operatorname{Im}(g)$. Since $\mathfrak{m}_0$ was an arbitrary finitely generated sub-ideal in $\mathfrak{m}$, we conclude that M is almost finitely presented by Lemma 2.5.7 (3). ∎

Corollary 2.7.4. *A ring R is almost noetherian if and only if any almost finitely generated R-module M is almost finitely presented.*

Proof. If R is almost noetherian, then any almost finitely generated R-module is almost finitely presented due to Lemma 2.7.3.

Now we suppose that every almost finitely generated R-module is almost finitely presented, and we wish to show that R is almost noetherian. Consider an ideal $I \subset R$. Then R/I is clearly a finitely generated R-module, in particular, it is almost finitely generated. Therefore, it is almost finitely presented by our assumption on R. Now the short exact sequence

$$0 \to I \to R \to R/I \to 0$$

and Lemma 2.5.15 (3) imply that I is almost finitely generated. ∎

Corollary 2.7.5. *Let $R \to R'$ be an almost isomorphism of rings. Then R is almost noetherian if and only if R' is.*

Corollary 2.7.6. *Let R be an almost noetherian ring, and M an almost finitely generated R-module. Then any submodule $N \subset M$ is almost finitely generated.*

Proof. Consider the short exact sequence

$$0 \to N \to M \to M/N \to 0.$$

By construction, M/N is almost finitely generated and, therefore, almost finitely presented by Lemma 2.7.3. So Lemma 2.5.15 (3) implies that N is almost finitely generated. ■

Corollary 2.7.7. *Let R be an almost noetherian ring. Then R is almost coherent.*

Proof. Lemma 2.6.4 guarantees that it suffices to show that $R_! \simeq \widetilde{\mathfrak{m}}$ is almost finitely generated and every finitely generated submodule of $R_!$ is almost finitely presented. The first property is trivial since $R_!$ is almost isomorphic to R, and the second one follows from Lemma 2.7.3. ■

Corollary 2.7.8. *Let R be an almost noetherian ring. Then an R-module M (resp. an R^a-module M^a) is almost coherent if and only if it is almost finitely generated.*

Proof. It suffices to prove the claim for an honest R-module M. Corollary 2.7.7 and Corollary 2.6.15 imply that M is almost coherent if and only if it is almost finitely presented. Now Lemma 2.7.3 says that M is almost finitely presented if and only if it is almost finitely generated. This finishes the proof. ■

Warning 2.7.9. Unlike the case of usual noetherian rings, Hilbert's Nullstellensatz is more subtle in the almost world. In particular, we do not know if a polynomial algebra in a finite number of variables over an almost noetherian ring is almost noetherian. However, we will show that Hilbert's Nullstellensatz holds for perfectoid valuation rings in Section 2.11.

Example 2.7.10. Let $\mathbf{B}_I$ be the period ring from [22, Definition 1.6.2]. Then [69, Corollary 8.16] implies that the rings $\mathbf{B}_I^+$ are almost noetherian for any closed interval $I \subset (0, \infty)$. Another family of examples of almost noetherian rings will be constructed in Section 2.11.

2.8 Base change for almost modules

In this section, we discuss the behavior of almost modules with respect to base change. Recall that, for a ring homomorphism $\varphi: R \to S$, we always do almost mathematics on S-modules with respect to the ideal $\mathfrak{m}_S := \mathfrak{m}S$; look at Lemma 2.1.11 for details.

Lemma 2.8.1. *Let $\varphi: R \to S$ be a ring homomorphism, and let M^a be an almost finitely generated (resp. almost finitely presented) R^a-module. Then the S^a-module $M^a_S := M^a \otimes_{R^a} S^a$ is almost finitely generated (resp. almost finitely presented).*

Proof. The claim follows from Lemma 2.5.7 (2) and the fact that, for any finitely generated ideal $\mathfrak{m}'_0 \subset \mathfrak{m}_S$, there is a finitely generated ideal $\mathfrak{m}_0 \subset \mathfrak{m}$ such that $\mathfrak{m}'_0 \subset \mathfrak{m}_0 S$. We give a complete proof only in the case of finitely presented modules because the other case is an easier version of the same argument.

First, we note that it suffices to show that $M \otimes_R S$ is almost finitely presented. Now we note that, for any finitely generated ideal $\mathfrak{m}'_0 \subset \mathfrak{m}_S$, there is a finitely generated ideal $\mathfrak{m}_0 \subset \mathfrak{m}$ such that $\mathfrak{m}'_0 \subset \mathfrak{m}_0 S$. Therefore, it suffices to check the condition of Lemma 2.5.7 (2) only for ideals of the form $\mathfrak{m}_0 S$, where $\mathfrak{m}_0 \subset \mathfrak{m}$ is a finitely generated sub-ideal. Then we choose some finitely generated ideal $\mathfrak{m}_1 \subset \mathfrak{m}$ such that $\mathfrak{m}_0 \subset \mathfrak{m}_1^2$ and use Lemma 2.5.7 (2) to find a finitely presented module N and a map $f: N \to M$ such that $\mathfrak{m}_1(\mathrm{Ker}\, f) = \mathfrak{m}_1(\mathrm{Coker}\, f) = 0$. Consider an exact sequence

$$0 \to K \to N \xrightarrow{f} M \to Q \to 0$$

and denote the image of f by M'. Then we have the following exact sequences:

$$K \otimes_R S \to N \otimes_R S \to M' \otimes_R S \to 0,$$
$$\mathrm{Tor}_1^R(Q, S) \to M' \otimes_R S \to M \otimes_R S \to Q \otimes_R S.$$

Since $K \otimes_R S$, $\mathrm{Tor}_1^R(Q, S)$ and $Q \otimes_R S$ are killed by $\mathfrak{m}_1 S$, we conclude that $\mathrm{Coker}(f \otimes_R S)$ and $\mathrm{Ker}(f \otimes_R S)$ are annihilated by $\mathfrak{m}_1^2 S$. In particular, they are killed by $\mathfrak{m}_0 S$. Since $N \otimes_R S$ is finitely presented over S, Lemma 2.5.7 finishes the proof. ∎

Corollary 2.8.2. *Let $R \to S$ be a ring homomorphism of almost coherent rings, and let M^a be an object of $\mathbf{D}^-_{\mathrm{acoh}}(R)^a$. Then $M^a \otimes^L_{R^a} S^a \in \mathbf{D}^-_{\mathrm{acoh}}(S)^a$.*

Proof. The proof is similar to that of Proposition 2.6.18. We use Proposition 2.4.16 and a similar approximation argument based on Proposition 2.6.16 to reduce to the case $M \simeq F^\bullet$, where $F^\bullet$ is a bounded above complex of finite free modules. In this case, the claim is essentially obvious. ∎

Lemma 2.8.3. *Let S be an R-algebra that is finite (resp. finitely presented) as an R-module, and let M^a be an S^a-module. Then M^a is almost finitely generated (resp. almost finitely presented) over R^a if and only if it is almost finitely generated (resp. almost finitely presented) over S^a.*

Proof. As always, we first reduce the question to the case of an honest S-module M. Now we use the observation that it suffices to check the condition of Lemma 2.5.7 (2) only for the ideals of the form $\mathfrak{m}_0 S$ for some finitely generated ideal $\mathfrak{m}_0 \subset \mathfrak{m} \subset R$. Then the only non-trivial direction is to show that M is almost finitely presented over S if it is almost finitely presented over R. This is proven in a more general situation in Lemma 2.8.4. ∎

Lemma 2.8.4. *Let S be a possibly non-commutative R-algebra that is finite as a left (resp. right) R-module, and let M be a left (resp. right) S-module that is almost finitely presented over R. Then M is almost finitely presented over S (i.e., for every finitely generated ideal $\mathfrak{m}_0 \subset \mathfrak{m}$, there exist a finitely presented left (resp. right) S-module N and a map $N \to M$ such that* Ker f *and* Coker f *are annihilated by $\mathfrak{m}_0$).*

Remark 2.8.5. This lemma will actually be used for a non-commutative ring S in the proof of Theorem 5.2.1 that, in turn, will be used in the proof of formal GAGA for almost coherent sheaves Theorem 5.3.2. Namely, we will apply Lemma 2.8.4 to $S = \mathrm{End}_{\mathbf{P}^N}(\mathcal{O} \oplus \mathcal{O}(1) \oplus \cdots \oplus \mathcal{O}(N))$.

Besides this application, we will usually use Lemma 2.8.4 when R and S are almost coherent commutative rings. In this case, the proof of Lemma 2.8.4 can be significantly simplified.

Proof. We give a proof for left S-modules; the proof for right S-modules is the same. We start the proof by choosing some generators $x_1, \dots, x_n$ of S as an R-module. Then we pick a finitely generated ideal $\mathfrak{m}_0 \subset \mathfrak{m}$ and another finitely generated ideal $\mathfrak{m}_1$ such that $\mathfrak{m}_0 \subset \mathfrak{m}_1^2$. We also choose some generators $(\varepsilon_1, \dots, \varepsilon_k) = \mathfrak{m}_1$ and find a three-term complex

$$R^t \xrightarrow{g} R^m \xrightarrow{f} M$$

such that $\mathfrak{m}_1(\mathrm{Coker}\, f) = 0$ and $\mathfrak{m}_1(\mathrm{Ker}\, f) \subset \mathrm{Im}\, g$. Next we consider the images $y_i := f(e_i) \in M$ of the standard basis elements in R^m. Then we can find some $\beta_{i,j,s,r} \in R$ such that

$$\varepsilon_s x_i y_j = \sum_{r=1}^{m} \beta_{i,j,s,r} \cdot y_r \text{ with } \beta_{i,j,s,r} \in R$$

for any $s = 1, \dots, k;\ i = 1, \dots, n;\ j = 1, \dots, m$. Furthermore, we have t "relations"

$$\sum_{j=1}^{m} \alpha_{i,j} y_j = 0 \text{ with } \alpha_{i,j} \in R$$

such that for any relation $\sum_{i=1}^{m} b_i y_i = 0$ with $b_i \in R$ and any $\varepsilon \in \mathfrak{m}_1$, we have that the vector $\{\varepsilon b_i\}_{i=1}^{m} \in R^m$ lives in the R-subspace generated by vectors $\{\alpha_{i,j}\}_{i=1}^{m}$ for $j = 1, \dots, t$. Or, in other words, if $\sum_{j=1}^{m} \alpha_{i,j} y_j = 0$ then $\varepsilon(\sum_{j=1}^{m} \alpha_{i,j} e_j) \in \mathrm{Im}(g)$ for any $\varepsilon \in \mathfrak{m}_1$.

Now we are finally ready to define a three-term complex

$$S^{nmk+t} \xrightarrow{\psi} S^m \xrightarrow{\varphi} M.$$

We define the map φ to be the unique S-linear homomorphism such that $\varphi(e_i) = y_i$ for the standard basis in S^m. We define ψ as the unique S-linear homomorphism such that

$$\psi(f_{i,j,s}) = \varepsilon_s x_i e_j - \sum_{r=1}^{m} \beta_{i,j,s,r} \cdot e_r \text{ and } \psi(f_l') = \sum_{j=1}^{m} \alpha_{l,j} e_j$$

for the standard basis

$$\{f_{i,j,s}, f_l'\}_{i \le n, j \le m, s \le k, l \le t} \in S^{nmk+t}.$$

Then we clearly have that $\varphi \circ \psi = 0$ and that $\mathfrak{m}_1(\operatorname{Coker} \varphi) = 0$. We claim that $\mathfrak{m}_1^2(\operatorname{Ker} \varphi) \subset \operatorname{Im} \psi$.

Let $\varphi(\sum_{i=1}^{m} c_i e_i) = 0$ for some elements $c_i \in S$. We can write each

$$c_i = \sum_{j=1}^{n} r_{i,j} x_j \text{ with } r_{i,j} \in R \tag{2.8.1}$$

because $x_1, \ldots, x_n$ are R-module generators of S. Consequently, the condition that $\varphi(\sum_{i=1}^{m} c_i e_i) = 0$ is equivalent to $\sum_{i,j} r_{i,j} x_j y_i = 0$. Now recall next that for any $s = 1, \ldots, k$ we have

$$\varepsilon_s x_j y_i = \sum_{r=1}^{m} \beta_{j,i,s,r} \cdot y_r.$$

Therefore, multiplying equation (2.8.1) by ε_s, we get an equality

$$0 = \varepsilon_s \Big(\sum_{i,j} r_{i,j} x_j y_i\Big) = \sum_{i,j} r_{i,j} \Big(\sum_{r=1}^{m} \beta_{j,i,s,r} \cdot y_r\Big) = \sum_{r=1}^{m} \Big(\sum_{i,j} r_{i,j} \beta_{j,i,s,r}\Big) y_r.$$

This means that for any $s' = 1, \ldots, k$, the vector $\{\varepsilon_{s'}(\sum_{i,j} r_{i,j} \beta_{j,i,s,r})\}_{r=1}^{m} \in R^m$ lives in an R-subspace generated by vectors $\{\alpha_{i,j}\}_{i=1}^{m}$. In particular, for any r and s', $\varepsilon_{s'}(\sum_{i,j} r_{i,j} \beta_{j,i,s,r} e_r)$ is equal to $\psi(\text{some sum of } f_l')$ by the definition of ψ.

After unwinding all definitions, we get the following:

$$\begin{aligned}
&\varepsilon_{s'} \varepsilon_s \Big(\sum_{i=1}^{m} c_i e_i\Big) \\
&= \varepsilon_{s'} \varepsilon_s \Big(\sum_{i,j} r_{i,j} x_j e_i\Big) \\
&= \varepsilon_{s'} \Big(\sum_{i,j} r_{i,j} \Big(\varepsilon_s x_j e_i - \sum_r \beta_{j,i,s,r} e_r + \sum_r \beta_{j,i,s,r} e_r\Big)\Big) \\
&= \varepsilon_{s'} \Big(\sum_{i,j} r_{i,j} \Big(\varepsilon_s x_j e_i - \sum_r \beta_{j,i,s,r} e_r\Big)\Big) + \varepsilon_{s'} \Big(\sum_r \Big(\sum_{i,j} r_{i,j} \beta_{j,i,s,r}\Big) e_r\Big) \\
&= \psi\Big(\varepsilon_{s'} \sum_{i,j} r_{i,j} f_{j,i,s}\Big) + \psi\Big(\text{some sum of } f_l'\Big).
\end{aligned}$$

So we see that $\mathfrak{m}_1^2 \operatorname{Ker}(\varphi) \subset \operatorname{Im}\psi$. In particular, we have $\mathfrak{m}_0 \operatorname{Ker}(\varphi) \subset \operatorname{Im}\psi$. Now we replace the map $\varphi\colon S^n \to M$ with the induced map

$$\overline{\varphi}\colon \operatorname{Coker}(\psi) \to M$$

to get a map from a finitely presented left S-module such that $\operatorname{Ker}(\overline{\varphi})$ and $\operatorname{Coker}(\overline{\varphi})$ are annihillated by $\mathfrak{m}_0$. ∎

2.9 Almost faithfully flat algebras

In this section, we study the almost faithfully flat morphisms of *algebras*. This notion turns out to be quite subtle in the almost world due to the following two observations: The first observation is that, for an almost faithfully flat morphism $R \to S$, the $R_!^a$-module $S_!^a$ is always flat, but not necessarily faithfully flat (see Warning 6.1.8). Another observation is that $S_!^a$ usually does not have a structure of an R-algebra.

For these reasons, it is not evident how to relate almost faithful flatness of an R-algebra S to some classical faithful flatness. In order to make this possible, we replace the $(-)_!$-functor with another functor $(-)_{!!}$ that takes into account the R-algebra structure on S. This functor will send almost faithfully flat R-algebras into faithfully flat R-algebras, however, it will not, in general, send flat R-algebras into flat R-algebras. However, this functor will suffice for the purpose of studying almost faithfully flat morphisms.

In this section, we follow [26] pretty closely.

For the rest of the section, we fix a ring R with an ideal of almost mathematics $\mathfrak{m}$.

Definition 2.9.1. A homomorphism of R-algebras $A \to B$ is *almost flat* (resp. *almost faithfully flat*) if B^a is a flat (resp. faithfully flat) A^a-module (see Definition 2.2.5).

Lemma 2.9.2. *Any (faithfully) flat A-algebra B is almost (faithfully) flat.*

Proof. This follows directly from Lemma 2.2.6. ∎

Lemma 2.9.3. *Let A be an R-algebra and $f\colon A \to B$ a morphism of R-algebras. Then B is almost faithfully flat over A if and only if B^a is a flat A^a-module and $A^a \to B^a$ is universally injective, i.e., for any A^a-module M^a, the natural morphism $M^a \to M^a \otimes_{A^a} B^a$ is injective in $\mathbf{Mod}_A^a$.*

Proof. Suppose that B is almost faithfully flat. Then B^a is a flat A^a-module by definition. So we only need to show that $A^a \to B^a$ is universally injective. Pick any $M^a \in \mathbf{Mod}_A^a$ and consider the A^a-module

$$N^a := \operatorname{Ker}(M^a \to M^a \otimes_{A^a} B^a).$$

Flatness of B^a implies that the morphism

$$N^a \otimes_{A^a} B^a \to M^a \otimes_{A^a} B^a$$

is injective. Now we also note that the morphism

$$N^a \otimes_{A^a} B^a \to M^a \otimes_{A^a} B^a$$

is equal to zero by our choice of N^a. This implies that $N^a \otimes_{A^a} B^a \simeq 0$. Since B^a is faithfully flat over A^a, we conclude that $N^a \simeq 0$.

Now we suppose that B^a is a flat A^a-module and $A^a \to B^a$ is universally injective. Thus, for any A^a-module M^a, we have an injection $M^a \to M^a \otimes_{A^a} B^a$. So if $M^a \otimes_{A^a} B^a \simeq 0$, we conclude that $M^a \simeq 0$. Thus B^a is faithfully flat over A^a. ∎

Corollary 2.9.4. *Let A be an R-algebra and $f: A \to B$ a morphism of R-algebras. Then B is almost faithfully flat over A if and only if B^a and* $\mathrm{Coker}(f^a)$ *are flat A^a-modules.*

Proof. By Lemma 2.9.3, it suffices to show that f^a is universally injective if and only if $\mathrm{Coker}(f^a)$ is A^a-flat. Next we observe that, for any A^a-module M^a, we have the isomorphism $\mathrm{Ker}(M^a \to M^a \otimes_{A^a} B^a) \simeq \mathrm{H}^{-1}\big(M^a \otimes^L_{A^a} \mathrm{Coker}(f^a)\big)$. In particular,

$$\mathrm{H}^{-1}\big(M^a \otimes^L_{A^a} \mathrm{Coker}(f^a)\big) \simeq 0$$

for any A^a-module M^a if and only if the functor $- \otimes_{A^a} \mathrm{Coker}(f^a): \mathbf{Mod}^a_A \to \mathbf{Mod}^a_A$ is exact. In other words, $A^a \to B^a$ is universally injective if and only if $\mathrm{Coker}(f^a)$ is flat over A^a. ∎

Now we define the functor $(-)_{!!}: \mathbf{Alg}_R \to \mathbf{Alg}_R$. We start by constructing an R-algebra structure on $R \oplus A^a_! = R \oplus (\widetilde{\mathfrak{m}} \otimes_R A)$ by defining the multiplication as

$$(r \oplus a) \cdot (r' \oplus a') = (rr') \oplus (ra' + r'a + aa')$$

and the summation law coordinate-wise. One easily checks that this is a well-defined (unital, commutative) R-algebra structure on $R \oplus A^a_!$. We consider the R-submodule I_A of $R \oplus A_!$ generated by elements of the form $(mn, -m \otimes n \otimes 1_A)$ for $m, n \in \mathfrak{m}$.

Lemma 2.9.5. *The R-module $I_A \subset R \oplus A^a_!$ is an ideal.*

Proof. It suffices to show that, for any element $(r, x \otimes y \otimes a)$ in $R \oplus A^a_!$, the product

$$(r \oplus x \otimes y \otimes a) \cdot (mn \oplus -m \otimes n \otimes 1_A)$$

lies in I_A for any $m, n \in \mathfrak{m}$. By definition,

$$\begin{aligned}(r \oplus x \otimes y \otimes a) &\cdot (mn \oplus -m \otimes n \otimes 1_A)\\ &= (rmn) \oplus (-rm \otimes n \otimes 1_A + xm \otimes yn \otimes a - xm \otimes yn \otimes a)\\ &= r(mn \oplus -m \otimes n \otimes 1_A) \in I_A.\end{aligned}$$

∎

Definition 2.9.6. The functor $(-)_{!!}\colon \mathbf{Alg}_R \to \mathbf{Alg}_R$ is defined as

$$A \mapsto (R \oplus A^a_!)/I_A$$

with the induced R-algebra structure.

For any R-algebra A, there is a functorial R-algebra homomorphism $R \oplus A^a_! \to A$ defined by

$$r \oplus (m \otimes n \otimes a) \mapsto r + mna.$$

Clearly, this homomorphism is zero on I_A, so it descends to an R-algebra homomorphism $\eta\colon A_{!!} \to A$.

Lemma 2.9.7. (1) *For any R-algebra A, the natural morphism $\eta\colon A_{!!} \to A$ is an almost isomorphism.*

(2) *A morphism of R-algebras $f\colon A \to B$ is almost injective (as a morphism of R-modules) if and only if $f_{!!}\colon A_{!!} \to B_{!!}$ is injective.*

(3) *For any morphism of R-algebras $f\colon A \to B$, there is a canonical isomorphism of $A_{!!}$-modules* $\mathrm{Coker}(f_{!!}) \simeq \mathrm{Coker}(f)_!$.

(4) *The functor $(-)_{!!}\colon \mathbf{Alg}_R \to \mathbf{Alg}_R$ commutes with tensor products.*

Proof. (1) We recall that the morphism $A_! \to A$ is an almost isomorphism. In particular, it is almost surjective. Thus, $A_{!!} \to A$ is also almost surjective. Now we check almost injectivity. Suppose $\eta(\overline{a}) = 0$ where $a = r \oplus \sum_{i=1}^k m_i \otimes n_i \otimes a_i \in R \oplus \widetilde{\mathfrak{m}} \otimes A$ and $\overline{a} \in A_{!!}$ is the class of a in $A_{!!}$. Then the condition $\eta(\overline{a}) = 0$ implies that there is an equality

$$r + \sum_{i=1}^k m_i n_i a_i = 0$$

in A. In particular, for every $\varepsilon \in \mathfrak{m}$, we have $\varepsilon r = \sum_{i=1}^k (-m_i)(\varepsilon n_i a_i)$ in A. Thus, we see that

$$\begin{aligned}\varepsilon a = \varepsilon r \oplus \sum_{i=1}^k m_i \otimes n_i \otimes \varepsilon a_i &= \sum_{i=1}^k (-m_i)(\varepsilon n_i a_i) \oplus \sum_{i=1}^k m_i \otimes n_i \varepsilon a_i \otimes 1_A \\ &= \sum_{i=1}^k \big((-m_i)(\varepsilon n_i a_i) \oplus m_i \otimes \varepsilon n_i a_i \otimes 1_A\big) \in I_A.\end{aligned}$$

Therefore, $\varepsilon \overline{a} = 0$ for every $\varepsilon \in \mathfrak{m}$. In particular, η is almost injective.

(2) and (3) Consider a commutative diagram

$$\begin{array}{ccc} A_{!!} & \xrightarrow{f_{!!}} & B_{!!} \\ {\scriptstyle\eta_A}\downarrow & & \downarrow{\scriptstyle\eta_B} \\ A & \xrightarrow[f]{} & B. \end{array}$$

Since η_A and η_B are almost isomorphisms, we see that f is almost injective if and only if $f_{!!}$ is almost injective. So we are left to show that $f_{!!}$ is injective if f is almost injective, and $\mathrm{Coker}(f_{!!}) = \mathrm{Coker}(f)_!$. For this, we consider a commutative diagram of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I_A & \longrightarrow & R \oplus A_! & \longrightarrow & A_{!!} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \mathrm{id}\oplus f_!} & & \downarrow{\scriptstyle f_{!!}} & & \\ 0 & \longrightarrow & I_B & \longrightarrow & R \oplus B_! & \longrightarrow & B_{!!} & \longrightarrow & 0. \end{array}$$

Clearly, α is surjective, $\mathrm{Ker}(\mathrm{id} \oplus f_!) = \mathrm{Ker}(f_!) = \mathrm{Ker}(f)_!$, and $\mathrm{Coker}(\mathrm{id} \oplus f_!) = \mathrm{Coker}(f_!) = \mathrm{Coker}(f)_!$. Thus, the Snake lemma implies that

$$\mathrm{Ker}(f)_! \to \mathrm{Ker}(f_{!!})$$

is surjective and

$$\mathrm{Coker}(f_{!!}) \to \mathrm{Coker}(f)_!$$

is an isomorphism. Thus $f_{!!}$ is injective if f is almost injective, and $\mathrm{Coker}(f_{!!}) = \mathrm{Coker}(f)_!$.

(4) This is an elementary but pretty tedious computation. We leave it to the interested reader. ■

Corollary 2.9.8. *For any R-algebra A, the forgetful functor* $\mathbf{Mod}^*_{A^a} \to \mathbf{Mod}^*_{A^a_{!!}}$ *is an equivalence for* $* \in \{\text{“ ”}, \mathrm{aft}, \mathrm{afp}, \mathrm{acoh}\}$.

Proof. For $* =$ “ ”, the claim follows from Lemma 2.9.7 (1), Corollary 2.5.13, and Lemma 2.6.3. ■

Corollary 2.9.9. *Let $f : A \to B$ be an almost faithfully flat morphism of R-algebras. Then $f_{!!} : A_{!!} \to B_{!!}$ is faithfully flat.*

Proof. Let us denote by Q the cokernel of f as an A-module. Then Lemma 2.9.3 and Lemma 2.9.7 (2), (3) ensure that $f_{!!} : A_{!!} \to B_{!!}$ is injective and $\mathrm{Coker}(f_{!!}) = \mathrm{Coker}(f)_!$. Now Corollary 2.9.4 and Lemma 2.2.7 applied to $A^a_{!!} \simeq A^a$ imply that $\mathrm{Coker}(f_{!!}) = \mathrm{Coker}(f)_!$ is a flat $A_{!!}$-module. This already implies that B is a flat $A_{!!}$-module as an extension of two flat $A_{!!}$-modules. To see that it is faithfully flat, we note that flatness of $\mathrm{Coker}(f_{!!})$ implies that

$$M \to M \otimes_{A_{!!}} B_{!!}$$

is injective for any $A_{!!}$-module M. So $M \otimes_{A_{!!}} B_{!!} \simeq 0$ if and only if $M \simeq 0$. In other words, $B_{!!}$ is a faithfully flat $A_{!!}$-module. ■

Warning 2.9.10. The functor $(-)_{!!}$ does not send flat A-algebras to flat $A_{!!}$-algebras. See [26, Remark 3.1.3].

For future reference, we also show that the base change functor interacts especially well with the Hom-functor in the almost flat situation.

Lemma 2.9.11. *Let $R \to S$ be an almost flat morphism of rings, let M be an almost finitely presented R-module, and let N be an R-module. Then the natural map*

$$\mathrm{Hom}_R(M, N) \otimes_R S \to \mathrm{Hom}_S(M \otimes_R S, N \otimes_R S)$$

is an almost isomorphism.

Proof. This follows from the classical $\otimes$-Hom adjunction and Lemma 2.5.18. ∎

Lemma 2.9.12. *Let R be an almost coherent ring, let $R \to S$ be an almost flat map of rings, and let $M \in \mathbf{D}^-_{\mathrm{acoh}}(R)$, $N \in \mathbf{D}^+(R)$. Then the natural map*

$$\mathbf{R}\mathrm{Hom}_R(M, N) \otimes^L_R S \to \mathbf{R}\mathrm{Hom}_S(M \otimes^L_R S, N \otimes^L_R S)$$

is an almost isomorphism.

Proof. We recall that we always have a canonical isomorphism $\mathbf{R}\mathrm{Hom}_R(K, L) \simeq \mathbf{R}\mathrm{Hom}_S(K \otimes^L_R S, L)$ for any $K \in \mathbf{D}^-(R)$ and any $L \in \mathbf{D}^+(S)$. This implies that it suffices to show that the natural map

$$\mathbf{R}\mathrm{Hom}_R(M, N) \otimes^L_R S \to \mathbf{R}\mathrm{Hom}_R(M, N \otimes^L_R S)$$

is an almost isomorphism. This follows from Proposition 2.6.20. ∎

2.10 Almost faithfully flat descent

The main goal of this section is to show almost faithfully flat descent for almost modules.

For the rest of the section, we fix a ring R with an ideal of almost mathematics $\mathfrak{m}$.

In this section, for any morphism $A \to B$ of R-algebras, we denote the tensor product functor $- \otimes_{A^a} B^a$ simply by

$$f^*\colon \mathbf{Mod}^a_A \to \mathbf{Mod}^a_B.$$

In particular, if $A \to B$ is a morphism of R-algebras, the canonical "co-projection" morphisms $p_i\colon B \to B \otimes_A B$ induce morphisms

$$p_i^*\colon \mathbf{Mod}^a_B \to \mathbf{Mod}^a_{B \otimes_A B}$$

for $i \in \{1, 2\}$. The same applies to the "co-projections"

$$p_{i,j}^*\colon \mathbf{Mod}^a_{B \otimes_A B} \to \mathbf{Mod}^a_{B \otimes_A B \otimes_A B}$$

for $i \neq j \in \{1, 2, 3\}$.

Definition 2.10.1. The *almost descent category* $\mathbf{Desc}^a_{B/A}$ for a morphism of R-algebras $A \to B$ is the category whose objects are pairs (M^a, ϕ), where $M^a \in \mathbf{Mod}^a_B$ and

$$\phi \colon p_1^*(M^a) \to p_2^*(M^a)$$

in an isomorphism of $(B \otimes_A B)^a$-modules such that $p_{1,3}^*(\phi) = p_{2,3}^*(\phi) \circ p_{1,2}^*(\phi)$. Morphisms between (M^a, ϕ_M) and (N^a, ϕ_N) are defined to be B^a-linear homomorphisms $f \colon M^a \to N^a$ such that the diagram

$$\begin{array}{ccc} p_1^*(M^a) & \xrightarrow{\phi_M} & p_2^*(M^a) \\ {\scriptstyle p_1^*(f)}\downarrow & & \downarrow{\scriptstyle p_2^*(f)} \\ p_1^*(N^a) & \xrightarrow[\phi_N]{} & p_2^*(N^a) \end{array}$$

commutes.

Remark 2.10.2. Explicitly, an object of the descent category $\mathbf{Desc}^a_{B/A}$ is a B^a-module M^a with a $(B \otimes_A B)^a$-linear homomorphism $\phi \colon M^a \otimes_{A^a} B^a \to B^a \otimes_{A^a} M^a$ satisfying the “cocycle condition”.

There is a natural functor

$$\operatorname{Ind} \colon \mathbf{Mod}^a_A \to \mathbf{Desc}^a_{B/A}$$

that sends M^a to $f^*(M^a) = M^a \otimes_{A^a} B^a$ where we make the canonical identification $\phi \colon p_1^* f^*(M^a) \simeq p_2^* f^*(M^a)$ coming from the equality $f \circ p_1 = f \circ p_2$.

To define the functor in the other direction, we note that we have the natural B^a-module morphisms $\iota_i \colon M^a \to p_i^*(M^a)$ for $i \in \{1, 2\}$. Explicitly, they are defined as morphisms induced by $\iota_1(m) = m \otimes 1$ and $\iota_2(m) = 1 \otimes m$. Therefore, given a descent datum $(M^a, \phi) \in \mathbf{Desc}^a_{B/A}$, we can define an A^a-module

$$\operatorname{Ker}\big(M^a, \phi\big) := \operatorname{Ker}\big(M^a \xrightarrow{i_1 - \phi^{-1} i_2} M^a \otimes_{A^a} B^a\big)$$

that is functorial in $\mathbf{Desc}^a_{B/A}$. Therefore, this defines a functor

$$\operatorname{Ker} \colon \mathbf{Desc}^a_{B/A} \to \mathbf{Mod}^a_A.$$

We show that Ker and Ind are quasi-inverse to each other and induce an equivalence between $\mathbf{Desc}^a_{B/A}$ and $\mathbf{Mod}^a_A$ for an almost faithfully flat morphism $f \colon A \to B$.

Theorem 2.10.3. *Let $f \colon A \to B$ be an almost faithfully flat morphism. Then*

$$\operatorname{Ind} \colon \mathbf{Mod}^a_A \to \mathbf{Desc}^a_{B/A}$$

is an equivalence, and its quasi-inverse is given by $\operatorname{Ker} \colon \mathbf{Desc}^a_{B/A} \to \mathbf{Mod}^a_A$.

Proof. Corollary 2.9.8 and Corollary 2.9.9 imply that we may replace f with $f_{!!}$ to assume that f is faithfully flat. Then the claim follows from the classical faithfully flat descent (see [14, Theorem 6.1/4]) and the observation that the classical versions of Ind and Ker carry almost isomorphisms to almost isomorphisms. ∎

On a similar note, we show that the Amitsur complex for an almost faithfully flat morphism is acyclic.

Lemma 2.10.4. *Let $f: A \to B$ be an almost faithfully flat morphism of R-algebras, and $M \in \mathbf{Mod}_B^a$. Then the Amitsur complex*

$$0 \to M^a \to M^a \otimes_{A^a} B^a \to M^a \otimes_{A^a} B^a \otimes_{A^a} B^a \to \cdots$$

is an exact complex of $\mathbf{Mod}_B^a$-modules (see the discussion around [68, Tag 023K] for the precise definition of differentials in this complex).

Proof. Corollary 2.9.8 and Corollary 2.9.9 imply that we may replace f with $f_{!!}$ to assume that f is faithfully flat. Then the claim follows from [68, Tag 023M]. ∎

Now we show that some properties of A^a-modules can be verified after a faithfully flat base change.

Lemma 2.10.5. *Let $f: A \to B$ be an almost faithfully flat morphism of R-algebras, and let M^a be an A^a-module. Then M^a is an almost finitely generated (resp. almost finitely presented) A^a-module if and only if $M^a \otimes_{A^a} B^a$ is an almost finitely generated (resp. almost finitely presented) B^a-module.*

Proof. Corollary 2.9.8 and Corollary 2.9.9 imply that we may replace f with $f_{!!}$ to assume that f is a faithfully flat morphism. Then a standard argument reduces the questions to the case of an honest A-module M, i.e., we show that an A-module M is almost finitely generated (resp. almost finitely presented) if so is the B-module $M \otimes_A B$.

We start with the almost finitely generated case. So we assume that $M \otimes_A B$ is almost finitely generated over B and wish to show that M is almost finitely generated over A. Our assumption implies that, for any $\varepsilon \in \mathfrak{m}$, we can choose a morphism $g: B^n \to M \otimes_A B$ such that $\varepsilon(\operatorname{Coker} g) = 0$. Let us consider the standard basis $e_1, \dots, e_n$ of B^n, and write

$$g(e_i) = \sum_j m_{i,j} \otimes b_{i,j} \text{ with } m_{i,j} \in M, b_{i,j} \in B.$$

We define the A-module F as the finite free A-module with the basis $e_{i,j}$. Then we define the morphism

$$h: F \to M$$

as the unique A-linear homomorphism with $h(e_{i,j}) = m_{i,j}$. It is easy to see that $\varepsilon(\mathrm{Coker}(h \otimes_A B)) = 0$. Since f is faithfully flat, this implies that $\varepsilon(\mathrm{Coker}\, h) = 0$. We conclude that M is almost finitely generated as ε was an arbitrary element of $\mathfrak{m}$.

Now we deal with the almost finitely presented case. We pick some finitely generated ideal $\mathfrak{m}_0 \subset \mathfrak{m}$, and another finitely generated ideal $\mathfrak{m}_1 \subset \mathfrak{m}$ such that $\mathfrak{m}_0 \subset \mathfrak{m}_1\mathfrak{m}$. We try to find a three-term complex

$$A^m \xrightarrow{g} A^n \xrightarrow{f} M$$

such that $\mathfrak{m}_0(\mathrm{Coker}\, f) = 0$ and $\mathfrak{m}_0(\mathrm{Ker}\, f) \subset \mathrm{Im}\, g$.

The almost finitely generated case established above implies that M is almost finitely generated. In particular, we have some morphism

$$A^n \xrightarrow{f} M$$

such that $\mathfrak{m}_1(\mathrm{Coker}\, f) = 0$, thus $\mathfrak{m}_1(\mathrm{Coker}(f \otimes_A B)) = 0$ as well. Therefore, we can apply Lemma 2.5.6 to find a homomorphism $g' \colon B^m \to B^n$ satisfying the conditions $\mathfrak{m}_0(\mathrm{Ker}(f \otimes_A B)) \subset \mathrm{Im}(g')$ and $(f \otimes_A B) \circ g' = 0$. This implies that g' lands inside $\mathrm{Ker}(f \otimes_A B) = \mathrm{Ker}(f) \otimes_A B$ due to A-flatness of B.

Now we do the same trick as above: we write

$$g(e_i) = \sum_j m_{i,j} \otimes b_{i,j} \text{ with } m_{i,j} \in \mathrm{Ker}(f), b_{i,j} \in B,$$

we define an R-module F as a finite free A-module with a basis $e_{i,j}$, and then we define the morphism

$$g \colon F \to \mathrm{Ker}(f)$$

as the unique A-linear morphism such that $g(e_{i,j}) = m_{i,j}$. With that, we can see that $\mathfrak{m}_0(\mathrm{Ker}(f \otimes_A B)) \subset \mathrm{Im}(g \otimes_A B)$. Since B is faithfully flat, we conclude that $\mathfrak{m}_0(\mathrm{Ker}\, f) \subset \mathrm{Im}(g)$ as well. This shows that a three-term complex

$$F \xrightarrow{g} A^n \xrightarrow{f} M$$

does the job. Therefore, M is an almost finitely presented A-module. ■

Corollary 2.10.6. *Let $f \colon A \to B$ be an almost faithfully flat morphism of R-algebras, and let M^a be an A^a-module. Suppose that $M^a \otimes_{A^a} B^a$ is an almost coherent B^a-module. Then so is M^a.*

Proof. This follows directly from Lemma 2.6.3 and Lemma 2.10.5. ■

Lemma 2.10.7. *Let $f \colon A \to B$ be an almost faithfully flat morphism of R-algebras, and let M^a be an A^a-module. Then M^a is a flat (resp. faithfully flat) A^a-module if and only if $M^a \otimes_{A^a} B^a$ is a flat (resp. faithfully flat) B^a-module.*

Proof. The classical proof works verbatim in the almost world. We leave the details to the reader. ∎

2.11 (Topologically) Finite type K^+-algebras

This section is devoted to the proof that (topologically) finite type algebras over a perfectoid valuation ring K^+ are almost noetherian. We refer to Appendix B for the relevant background on perfectoid valuation rings.

For the rest of the section, we fix a perfectoid valuation ring K^+ (see Definition B.2) with perfectoid fraction field K, associated rank-1 valuation ring $\mathcal{O}_K = K^\circ$ (see Remark B.3), and ideal of topologically nilpotent elements $\mathfrak{m} = K^{\circ\circ} \subset K^+$. Lemma B.12 ensures that $\mathfrak{m}$ is flat over K^+ and $\widetilde{\mathfrak{m}} \simeq \mathfrak{m}^2 = \mathfrak{m}$. Therefore, it makes sense to do almost mathematics with respect to the pair $(K^+, \mathfrak{m})$. In what follows, we always do almost mathematics on K^+-modules with respect to this ideal.

Warning 2.11.1. The ideal $\mathfrak{m} \subset K^+$ is not the maximal ideal of K^+. Instead, it is the maximal ideal of the associated rank-1 valuation ring $\mathcal{O}_K$.

Lemma 2.11.2. *Let K^+ be a perfectoid valuation ring. Then the natural inclusion $\iota\colon K^+ \to \mathcal{O}_K$ is an almost isomorphism.*

Proof. Clearly, the map $\iota\colon K^+ \to \mathcal{O}_K$ is injective, so it suffices to show that its cokernel is almost zero, i.e., annihilated by any $\varepsilon \in \mathfrak{m}$. Pick an element $x \in \mathcal{O}_K$, then $\varepsilon x \in \mathfrak{m} \subset K^+$. Therefore, we conclude that $\varepsilon(\operatorname{Coker} \iota) = 0$ finishing the proof. ∎

The first main result of this section is that any (topologically) finite type algebra over K^+ is almost noetherian.

Lemma 2.11.3. *Let K^+ be a perfectoid valuation ring, and $n \geq 0$ an integer. Then the Tate algebra $K^+\langle T_1, \dots, T_n\rangle$ is almost noetherian.*

Proof. First, we note that $\mathcal{O}_K\langle T_1, \dots, T_n\rangle \simeq K^+\langle T_1, \dots, T_n\rangle \otimes_{K^+} \mathcal{O}_K$. Therefore, Lemma 2.11.2 implies that the natural morphism

$$K^+\langle T_1, \dots, T_n\rangle \to \mathcal{O}_K\langle T_1, \dots, T_n\rangle$$

is an almost isomorphism. So Corollary 2.7.5 ensures that it suffices to show that $\mathcal{O}_K\langle T_1, \dots, T_n\rangle$ is almost noetherian.

Pick any ideal $I \subset \mathcal{O}_K\langle T_1, \dots, T_n\rangle = K\langle T_1, \dots, T_n\rangle^\circ$ and $0 \neq \varepsilon \in \mathfrak{m}$. Now [43, Satz 5.1] (or [11, Lemma 6.4/5]) applied to $B = K\langle T_1, \dots, T_n\rangle$, $E = \mathcal{O}_K\langle T_1, \dots, T_n\rangle$, $E' = I$, and $\alpha = |\varepsilon|_K$ guarantees that there is a finite submodule $E'' \subset I$ such that $\varepsilon I \subset E''$. Since ε was an arbitrary element of $\mathfrak{m}$, we conclude that I is indeed almost finitely generated. ∎

Corollary 2.11.4. *Let K^+ be a perfectoid valuation ring, $\varpi \in \mathfrak{m}$, and $n \geq 0$ an integer. Then the polynomial algebra $(K^+/\varpi^m)[T_1, \dots, T_n]$ is almost noetherian for any $m \geq 1$.*

Proof. It easily follows from Lemma 2.11.3, Corollary 2.7.4, and Lemma 2.8.3. ∎

Theorem 2.11.5. *Let K^+ be a perfectoid valuation ring, and A a topologically finite type K^+-algebra. Then A is almost noetherian.*

Proof. Since A is topologically finite type over K^+, there exists a surjection

$$f : K^+\langle T_1, \dots, T_n\rangle \to A \to 0.$$

Pick an ideal $I \subset A$ and consider its pre-image $J = f^{-1}(I)$. Then J is almost finitely generated over $K^+\langle T_1, \dots, T_n\rangle$ by Lemma 2.11.3. Therefore, Lemma 2.5.15 (1) ensures that I is almost finitely generated over $K^+\langle T_1, \dots, T_n\rangle$. Finally, Lemma 2.8.3 ensures that I is almost finitely generated over A. ∎

Now we are going to show that any finite type K^+-algebra is almost noetherian. Before doing this, we need a couple of preliminary lemmas.

Lemma 2.11.6. *Let R be a rank-1 valuation ring with a non-zero topologically nilpotent element $\varpi \in R$, and M a finite $R[T_1, \dots, T_n]$-module. Then $M[\varpi^\infty] = M[\varpi^c]$ for some $c \geq 0$.*

Proof. The $R[T_1, \dots, T_n]$-module $M' := M/M[\varpi^\infty]$ is finitely generated. Furthermore, M' is R-flat because it is torsion free (and R is a valuation ring). Therefore, [68, Tag 053E] ensures that M' is finitely presented over $R[T_1, \dots, T_n]$. Thus, we conclude that $M[\varpi^\infty]$ is finitely generated. In particular, $M[\varpi^\infty] = M[\varpi^c]$ for some N. ∎

Lemma 2.11.7. *Let R be a rank-1 valuation ring with a non-zero topologically nilpotent element $\varpi \in R$, M a finite $R[T_1, \dots, T_n]$-module, and $N \subset M$ an $R[T_1, \dots, T_n]$-submodule. Then there is a non-negative integer c such that*

$$N \cap \varpi^{m+c} M = \varpi^m (N \cap \varpi^c M)$$

for every $m \geq 0$.

Proof. Lemma 2.11.6 ensures that there is a suitable c such that $(M/N)[\varpi^\infty] = (M/N)[\varpi^c]$. Therefore, [25, Lemma 0.8.2.14] guarantees that, indeed,

$$N \cap \varpi^{m+c} M = \varpi^m (N \cap \varpi^c M)$$

for every $m \geq 0$. ∎

Lemma 2.11.8. *Let K^+ be a perfectoid valuation ring, and $n \geq 0$ an integer. Then the polynomial algebra $K^+[T_1, \dots, T_n]$ is almost noetherian.*

Proof. Similarly to the proof of Lemma 2.11.3, it suffices to treat the case $K^+ = \mathcal{O}_K$ a perfectoid valuation ring of rank-1 with a pseudo-uniformizer ϖ.

Now we fix an ideal $I \subset A := \mathcal{O}_K[T_1, \dots, T_n]$ and wish to show that I is almost finitely generated. Recall that the polynomial algebra $K[T_1, \dots, T_n]$ is noetherian by Hilbert's Nullstellensatz. Therefore, the ideal

$$I\left[\frac{1}{\varpi}\right] \subset K[T_1, \dots, T_n]$$

is finitely generated. So we can choose a finitely generated sub-ideal $J \subset I$ such that any element of I/J is annihilated by a power of ϖ, i.e., $(I/J)[\varpi^\infty] = I/J$. Clearly I/J is a submodule of a finite A-module A/J, so Lemma 2.11.6 easily implies that

$$I/J = (I/J)[\varpi^\infty] = (I/J)[\varpi^c]$$

for some $c \geq 0$. In other words, $\varpi^c I \subset J$. Now we use Lemma 2.11.7 to get an integer c' such that

$$I \cap \varpi^{c'} A \subset \varpi^c I \subset J.$$

We note that $I/(I \cap \varpi^{c'} A)$ is an ideal in $A/\varpi^{c'} A$, and therefore it is almost finitely generated over $A/\varpi^{c'} A$ by Corollary 2.11.4. Lemma 2.8.3 guarantees that it is also almost finitely generated over A.

The inclusion $I \cap \varpi^{c'} A \subset J$ implies that I/J is a quotient of an almost finitely generated A-module $I/(I \cap \varpi^{c'} A)$, and so is also almost finitely generated. Finally, the short exact sequence

$$0 \to J \to I \to I/J \to 0$$

and Lemma 2.5.15 (2) imply that I is almost finitely generated as well. ∎

Theorem 2.11.9. *Let K^+ be a perfectoid valuation ring, and A a finite type K^+-algebra. Then A is almost noetherian.*

Proof. It follows from Lemma 2.11.8, similarly to how Theorem 2.11.5 follows from Lemma 2.11.3. ∎

2.12 Almost finitely generated modules over adhesive rings

This section discusses some basic aspects of almost finitely generated modules over adhesive rings. The results of this section will be crucial in defining and verifying certain good properties of adically quasi-coherent, almost coherent sheaves on "good" formal schemes in Section 4.5. One of the essential ingredients that we will need later is the "weak" version of the Artin–Rees lemma (Lemma 2.12.6) and Lemma 2.12.7. Recall that these properties are already known for finite modules over adhesive rings. This is explained in a beautiful paper [24]. The main goal of this section is to extend these results to the case of almost finitely generated modules.

That being said, let us introduce the set-up for this section. We start with the definition of an adhesive ring:

Definition 2.12.1. [24, Definition 7.1.1] An adically topologized ring R endowed with the adic topology defined by a finitely generated ideal $I \subset R$ is said to be *(I-adically) adhesive* if it is noetherian outside[7] I and satisfies the following condition: for any finitely generated R-module M, its I^∞-torsion part $M[I^\infty]$ is finitely generated.

Remark 2.12.2. Following the convention of [24], we do not require a ring R with adic topology to be either I-adically complete or separated.

Set-up 2.12.3. We fix an I-adically adhesive ring R with an ideal $\mathfrak{m}$ such that $I \subset \mathfrak{m}$, $\mathfrak{m}^2 = \mathfrak{m}$ and $\widetilde{\mathfrak{m}} := \mathfrak{m} \otimes_R \mathfrak{m}$ is flat. We always do almost mathematics with respect to the ideal $\mathfrak{m}$.

The main example of an adhesive ring is a (topologically) finitely presented algebra over a complete microbial valuation ring. This follows from [24, Proposition 7.2.2] and [24, Theorem 7.3.2]. For example, any topologically finitely presented algebra over a complete rank-1 valuation ring is adhesive.

Lemma 2.12.4. *Let R be as in Set-up 2.12.3, and let M be an I-torsionfree almost finitely generated module. Then M is almost finitely presented. Similarly, any saturated submodule*[8] *of an almost finitely generated R-module is almost finitely generated.*

Proof. As M is almost finitely generated, we can find a finitely generated submodule $N \subset M$ that contains $\mathfrak{m}_0 M$ for a choice of a finitely generated ideal $\mathfrak{m}_0 \subset \mathfrak{m}$. Since N is a submodule of M, it is itself I-torsion free. Then [24, Proposition 7.1.2] shows that N is finitely presented. Then Lemma 2.5.7 (2) implies that M is almost finitely presented.

Now let M be an almost finitely generated R-module, and let $M' \subset M$ be a saturated submodule. Then M/M' is almost finitely generated by Lemma 2.5.15 (1) and it is I-torsion free. Therefore, it is almost finitely presented by the argument above. Then Lemma 2.5.15 (3) guarantees that M' is almost finitely generated. ∎

[7]By definition, this means that the scheme Spec $A \setminus V(I)$ is noetherian.
[8]A submodule $N \subset M$ is *saturated* if $M/N[I^\infty] = 0$.

Lemma 2.12.5. *Let R be as in Set-up 2.12.3, and let M be an almost finitely generated R-module. Then the I^∞-torsion module $M[I^\infty]$ is bounded (i.e., there is an integer n such that $M[I^n] = M[I^\infty]$).*

Proof. Since M is almost finitely generated and the ideal $I \subset \mathfrak{m}$ is finitely generated, we conclude that there exists a finitely generated submodule $N \subset M$ such that $IM \subset N$. Then $I(M[I^\infty]) \subset N[I^\infty]$, and $N[I^\infty]$ is finitely generated by adhesiveness of the ring R. In particular, there is an integer n such that $N[I^\infty]$ is annihilated by I^n. This implies that any element of $M[I^\infty]$ is annihilated by I^{n+1}. ■

Lemma 2.12.6. *Let R be as in Set-up 2.12.3, and let M be an almost finitely generated R-module. Suppose that $N \subset M$ is a submodule of M. For any integer n, there is an integer m such that $N \cap I^m M \subset I^n N$. In particular, the induced topology on the module N coincides with the I-adic one.*

Proof. If M is finitely generated, then this is [24, Theorem 4.2.2]. In general, we use the definition of almost finitely generated module to find a submodule $M' \subset M$ such that M' is finitely generated and $IM \subset M'$. We define $N' := N \cap M'$ as the intersection of those modules. Then the established "weak" form of the Artin–Rees lemma for finitely generated R-modules provides us with an integer m such that $N' \cap I^m M' \subset I^n N'$. In particular, we have

$$I^{m+1} M \cap N' \subset I^m M' \cap N' \subset I^n N' \subset I^n N.$$

Then we conclude that

$$I^{m+2} M \cap N \subset I^{m+1} M \cap M' \cap N \subset I^{m+1} M \cap N' \subset I^n N.$$

Since n was arbitrary, we conclude the claim. ■

Lemma 2.12.7. *Let R be as in Set-up 2.12.3, and let M be an almost finitely generated R-module. Then the natural morphism $M \otimes_R \widehat{R} \to \widehat{M}$ is an isomorphism. In particular, any almost finitely generated module over a complete adhesive ring is complete.*

Proof. We know that the claim holds for finitely generated modules by [24, Proposition 4.3.4]. Now we deal with the almost finitely generated case. We choose a finitely generated submodule $N \subset M$ such that $IM \subset N$. Lemma 2.12.6 implies that the induced topology on N coincides with the I-adic topology on N. Thus the short exact sequence

$$0 \to N \to M \to M/N \to 0$$

remains exact after completion. Since $R \to \widehat{R}$ is flat by [24, Proposition 4.3.4], we conclude that we have a morphism of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N \otimes_R \widehat{R} & \longrightarrow & M \otimes_R \widehat{R} & \longrightarrow & (M/N) \otimes_R \widehat{R} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \varphi_N} & & \downarrow{\scriptstyle \varphi_M} & & \downarrow{\scriptstyle \varphi_{M/N}} & & \\ 0 & \longrightarrow & \widehat{N} & \longrightarrow & \widehat{M} & \longrightarrow & \widehat{M/N} & \longrightarrow & 0. \end{array}$$

Note that φ_N is an isomorphism as N is finitely generated, and $\varphi_{M/N}$ is isomorphism since it is an I-torsion module so $M/N \simeq (M/N) \otimes_R \widehat{R} \simeq \widehat{M/N}$. The five lemma implies that φ_M is an isomorphism as well. ∎

Corollary 2.12.8. *Let R be as in Set-up 2.12.3, and let $M \in \mathbf{D}_{\mathrm{acoh}}(R)$. Suppose that R is I-adically complete. Then M is I-adically derived complete.*[9]

Proof. First of all, we note that [68, Tag 091P] implies that M is derived complete if and only if so are $\mathrm{H}^i(M)$ for any integer i. So it suffices to show that any almost coherent R-module is derived complete. Lemma 2.12.7 gives that any such module is classically complete, and [68, Tag 091T] ensures that any classically complete module is derived complete. ∎

2.13 Modules over topologically finite type K^+-algebras

The main goal of this section is to show that almost finite presentation of derived complete modules over a topologically finite type K^+-algebras can be checked modulo the pseudo-uniformizer.

For the rest of the section we fix a valuation perfectoid ring K^+ (see Definition B.2) with perfectoid fraction field K, associated rank-1 valuation ring $\mathcal{O}_K = K^\circ$ (see Remark B.3), and ideal of topologically nilpotent elements $\mathfrak{m} = K^{\circ\circ} \subset K^+$ with a pseudo-uniformizer $\varpi \in \mathfrak{m}$ as in Lemma B.9 (in particular, $\mathfrak{m} = \bigcup_n \varpi^{1/p^n} K^+$). Lemma B.12 ensures that $\mathfrak{m}$ is flat over K^+ and $\widetilde{\mathfrak{m}} \simeq \mathfrak{m}^2 = \mathfrak{m}$. Therefore, it makes sense to do almost mathematics with respect to the pair $(K^+, \mathfrak{m})$. In what follows, we always do almost mathematics on K^+-modules with respect to this ideal.

Lemma 2.13.1. *Let R be a topologically finite type K^+-algebra, and M an R-module that is ϖ-adically derived complete. Suppose that $M/\varpi M$ is almost coherent, then M is almost coherent as well.*

[9] Look at [68, Tag 091N] for the definition of derived completeness (or Definition A.1 in case of a principal ideal I).

Proof. Theorem 2.11.5 ensures that R is almost noetherian, and so Corollary 2.7.8 implies that it suffices to check that M is almost finitely generated. Recall that $\mathfrak{m} = \bigcup_n \varpi^{1/p^n} K^+$ for a pseudo-uniformizer ϖ as in Lemma B.9.

The assumption on M says that $M/\varpi M$ is almost coherent. Therefore, there is a morphism

$$\overline{g}\colon (R/\varpi R)^c \to M/\varpi M$$

such that $\varpi^{1/p}(\mathrm{Coker}\,\overline{g}) = 0$. We denote its cokernel by $\overline{Q} := \mathrm{Coker}(\overline{g})$. Now we lift $\overline{g}$ to a morphism

$$g\colon R^c \to M$$

and denote is cokernel by $Q := \mathrm{Coker}(g)$.

Step 1: *Q is annihilated by $\varpi^{1/p}$.* Suppose that $\varpi^{1/p} Q \neq 0$, so there is $x_0 \in Q$ such that $\varpi^{1/p} x_0 \neq 0$. Firstly, we note that $Q/\varpi \simeq \overline{Q}$ is annihilated by $\varpi^{1/p}$, so

$$\varpi^{1/p} x_0 = \varpi x_1 = \varpi^{1-1/p}(\varpi^{1/p} x_1).$$

Now we apply the same thing to x_1 to get

$$\varpi^{1/p} x_0 = \varpi^{1-1/p}(\varpi^{1/p} x_1) = (\varpi^{1-1/p})^2(\varpi^{1/p} x_2).$$

Continue the process to get a sequence of elements $x_n \in Q$ such that

$$\varpi^{1-1/p}(\varpi^{1/p} x_n) = \varpi^{1/p} x_{n-1}.$$

The sequence $\{\varpi^{1/p} x_i\}$ gives an element of

$$T^0(Q, \varpi^{1-1/p}) := \lim_n (\cdots \xrightarrow{\varpi^{1-1/p}} Q \xrightarrow{\varpi^{1-1/p}} Q)$$

that is non-trivial because $\varpi^{1/p} x_0 \neq 0$. Now we note that R^c is derived ϖ-adically complete since R is classically ϖ-adically complete by [11, Corollary 7.3/9] and any classically complete module is derived complete by [68, Tag 091T]. Therefore, Q is ϖ-adically derived complete as a cokernel of derived complete modules (see [68, Tag 091U]). Now [68, Tag 091S], Remark A.2, and [68, Tag 091Q] imply that $T^0(Q, \varpi^{1-1/p})$ must be zero leading to the contradiction.

Step 2: *M is almost coherent.* Note that $\overline{Q} \simeq Q/\varpi Q$ and Q is $\varpi^{1/p}$-torsion, so $\overline{Q} \simeq Q$. We know that $\overline{Q}$ is almost finitely generated over $R/\varpi R$ because it is a quotient of an almost finitely generated module $M/\varpi M$. Therefore, $Q \simeq \overline{Q}$ is almost finitely generated over R by Lemma 2.8.3. Now M is an extension of a finite R-module $\mathrm{Im}(g)$ by an almost finitely generated R-module Q, so it is also almost finitely generated by Lemma 2.5.15 (2). In particular, it is almost coherent since R is almost noetherian. ∎

Theorem 2.13.2. *Let R be a topologically finite type K^+-algebra, and $M \in \mathbf{D}(R)$ a ϖ-adically derived complete complex. Suppose that $[M/\varpi] \in \mathbf{D}^{[c,d]}_{\mathrm{acoh}}(R/\varpi)$, then $M \in \mathbf{D}^{[c,d]}_{\mathrm{acoh}}(R)$.*

Proof. Lemma A.3 guarantees that $M \in \mathbf{D}^{[c,d]}(R)$, so we only need to show that cohomology groups of M are almost coherent over R.

We argue by induction on $d - c$. If $c = d$, then $\mathrm{H}^d(M)/\varpi \simeq \mathrm{H}^d([M/\varpi])$ is almost coherent. Therefore, $M \simeq \mathrm{H}^d(M)[-d]$ is almost coherent by Lemma 2.13.1.

If $d > c$, we consider an exact triangle

$$\tau^{\leq d-1}M \to M \to \mathrm{H}^d(M)[-d].$$

We see that both $\tau^{\leq d-1}M$ and $\mathrm{H}^d(M)$ are derived complete by [68, Tag 091P] and [68, Tag 091S]. Moreover, we know that $\mathrm{H}^d(M)/\varpi \simeq \mathrm{H}^d([M/\varpi])$ is almost coherent. Therefore, $\mathrm{H}^d(M)$ is almost coherent by Lemma 2.13.1. Finally,

$$[\tau^{\leq d-1}M/\varpi] \simeq \mathrm{cone}\Big([M/\varpi] \to [\mathrm{H}^d(M)/\varpi][-d]\Big)[1]$$

is a (shifted) cone of a morphism in $\mathbf{D}^b_{\mathrm{acoh}}(R/\varpi)$, therefore, $[\tau^{\leq d-1}M/\varpi]$ also lies in $\mathbf{D}^b_{\mathrm{acoh}}(R/\varpi)$. By the induction hypothesis, we conclude that $\tau^{\leq d-1}M \in \mathbf{D}^{[c,d-1]}_{\mathrm{acoh}}(R)$. So $M \in \mathbf{D}^{[c,d]}_{\mathrm{acoh}}(R)$. ∎

Corollary 2.13.3. *Let R be a topologically finite type K^+-algebra, and $M \in \mathbf{D}(R)$ a ϖ-adically derived complete complex. Suppose that $[M^a/\varpi] \in \mathbf{D}^{[c,d]}_{\mathrm{acoh}}(R/\varpi)^a$, then $M^a \in \mathbf{D}^{[c,d]}_{\mathrm{acoh}}(R)^a$.*

Proof. Note that $\mathfrak{m} \otimes M$ is derived complete by Lemma A.4. So the claim follows from Theorem 2.13.2 applied to $\mathfrak{m} \otimes M$. ∎