Chapter 3

Almost mathematics on ringed sites

The main goal of this chapter is to "globalize" the results of Chapter 2. The two main cases of interest are almost coherent sheaves on schemes and "good" formal schemes. In order to treat those cases uniformly, we define the notion of almost sheaves in the most general set-up of ringed sites and check their basic properties. This is the content of Section 3.1. Sections 4.1 and 4.5 are devoted to establishing the foundations of almost coherent sheaves on schemes and formal schemes, respectively. In particular, we show that the notion of almost finitely generated (resp. presented, resp. coherent) module globalizes well on schemes and some "good" formal schemes. Then we discuss the derived category of almost sheaves and various functors on the derived categories of almost sheaves. Later in Chapter 4, we use this theory to establish foundations of almost coherent sheaves on schemes and formal schemes, respectively.

3.1 The category of $\mathcal{O}_X^a$-modules

We start this section by fixing a ring R with an ideal $\mathfrak{m}$ such that $\mathfrak{m} = \mathfrak{m}^2$ and $\widetilde{\mathfrak{m}} = \mathfrak{m} \otimes_R \mathfrak{m}$ is R-flat. We always do almost mathematics with respect to this ideal. The main goal of this section is to globalize the notion of almost mathematics to the case of ringed R-sites.

In this section, we fix a *ringed R-site* $(X, \mathcal{O}_X)$, i.e., a ringed site $(X, \mathcal{O}_X)$ where $\mathcal{O}_X$ is a sheaf of R-algebras on X. Note that any ringed site $(X, \mathcal{O}_X)$ is, in particular, a ringed $\mathcal{O}_X(X)$-site. The main goal of this section is to develop foundations of almost mathematics on ringed R-sites.

We note that, on each open $U \in X$, it makes sense to speak of almost $\mathcal{O}_X(U)$-modules with respect to the ideal $\mathfrak{m}\mathcal{O}_X(U)$; we refer to Lemma 2.1.11 for the details. In what follows, we extend the definition of almost modules to the category of $\mathcal{O}_X$-modules.

Definition 3.1.1. Let $(X, \mathcal{O}_X)$ be a ringed R-site, and let $\mathcal{F}$ be any $\mathcal{O}_X$-module. Then the *sheaf of almost section* $\widetilde{\mathfrak{m}} \otimes \mathcal{F}$ is the sheafification of the presheaf defined via the formula

$$U \mapsto \widetilde{\mathfrak{m}} \otimes_R \mathcal{F}(U).$$

Remark 3.1.2. This definition coincides with the tensor product $\underline{\widetilde{\mathfrak{m}}} \otimes_R \mathcal{F}$, where $\underline{\widetilde{\mathfrak{m}}}$ is the constant sheaf associated with the R-module $\mathfrak{m}$. Alternatively, we see that $\widetilde{\mathfrak{m}} \otimes \mathcal{F} \simeq \underline{\widetilde{\mathfrak{m}}}_X \otimes_{\mathcal{O}_X} \mathcal{F}$ where $\underline{\widetilde{\mathfrak{m}}}_X = \underline{\widetilde{\mathfrak{m}}} \otimes_R \mathcal{O}_X$.

We also note that flatness of the R-module $\widetilde{\mathfrak{m}}$ implies that the functor $- \otimes \widetilde{\mathfrak{m}}$ is exact and descends to a functor

$$- \otimes \widetilde{\mathfrak{m}} \colon \mathbf{D}(X) \to \mathbf{D}(X),$$

where $\mathbf{D}(X)$ is the derived category of $\mathcal{O}_X$-modules.

Definition 3.1.3. An $\mathcal{O}_X$-module $\mathcal{F}$ is *almost zero* if $\widetilde{\mathfrak{m}} \otimes \mathcal{F}$ is zero. We denote the category of almost zero $\mathcal{O}_X$-modules by Σ_X.

Remark 3.1.4. Since $\widetilde{\mathfrak{m}}$ is an R-flat module, we easily see that the category of almost zero $\mathcal{O}_X$-modules is a Serre subcategory of $\mathbf{Mod}_{\mathcal{O}_X} = \mathbf{Mod}_X$.

Lemma 3.1.5. *Let $(X, \mathcal{O}_X)$ be a ringed R-site, and let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Suppose that $\mathcal{U}$ is a base of topology on X. Then the following conditions are equivalent:*

(1) *$\mathcal{F} \otimes \widetilde{\mathfrak{m}}$ is the zero sheaf.*

(2) *For any $\varepsilon \in \mathfrak{m}$, $\varepsilon\mathcal{F} = 0$.*

(3) *For any $U \in \mathcal{U}$, the module $\widetilde{\mathfrak{m}} \otimes \mathcal{F}(U)$ is zero.*

(4) *For any $U \in \mathcal{U}$, the module $\mathfrak{m} \otimes \mathcal{F}(U)$ is zero.*

(5) *For any $U \in \mathcal{U}$, the module $\mathfrak{m}\big(\mathcal{F}(U)\big)$ is zero.*

Proof. We first show that (1) implies (2). We pick an element $\varepsilon \in \mathfrak{m} = \mathfrak{m}^2$ and write it as $\varepsilon = \sum x_i \cdot y_i$ for some $x_i, y_i \in \mathfrak{m}$. So the multiplication by ε map can be decomposed as

$$\mathcal{F} \xrightarrow{s \mapsto s \otimes \sum x_i \otimes y_i} \mathcal{F} \otimes \widetilde{\mathfrak{m}} \xrightarrow{m} \mathcal{F},$$

where the last map is induced by the multiplication map $\widetilde{\mathfrak{m}} \to R$. Then if $\mathcal{F} \otimes \widetilde{\mathfrak{m}} = 0$, the multiplication by ε map is zero for any $\varepsilon \in \mathfrak{m}$. Now (2) easily implies (5). Further, Lemma 2.1.1 ensures that (3), (4), and (5) are equivalent. Finally, (3) clearly implies (1). ∎

Lemma 3.1.6. *Let $(X, \mathcal{O}_X)$ be a ringed R-site, and let $\mathcal{F}$ be an almost zero $\mathcal{O}_X$-module. Then $\mathrm{H}^i(U, \mathcal{F}) \cong^a 0$ for any open*[1] *$U \in X$ and any $i \geq 0$.*

Proof. If $\mathcal{F}$ is almost zero, then $\varepsilon\mathcal{F} = 0$ for any $\varepsilon \in \mathfrak{m}$ by Lemma 3.1.5. Since the functors $\mathrm{H}^i(X, -)$ are R-linear, we conclude that $\varepsilon \mathrm{H}^i(U, \mathcal{F}) = 0$ for any open U and any $\varepsilon \in \mathfrak{m}$, $i \geq 0$. Thus Lemma 2.1.1 ensures that $\mathrm{H}^i(U, \mathcal{F}) \cong^a 0$. ∎

Definition 3.1.7. We say that a homomorphism $\varphi \colon \mathcal{F} \to \mathcal{G}$ of $\mathcal{O}_X$-modules is an *almost isomorphism* if $\mathrm{Ker}(\varphi)$ and $\mathrm{Coker}(\varphi)$ are almost zero.

[1] An open $U \in X$ is by definition an object $U \in \mathrm{Ob}(X)$ of the category underlying the site X.

Lemma 3.1.8. *A homomorphism* $\varphi\colon \mathcal{F} \to \mathcal{G}$ *of* $\mathcal{O}_X$*-modules is an almost isomorphism if and only if* $\varphi(U)\colon \mathcal{F}(U) \to \mathcal{G}(U)$ *is an almost isomorphism of* $\mathcal{O}_X(U)$*-modules for any open* $U \in X$.

Proof. The $\Leftarrow$ implication is clear from the definitions. We give a proof of the $\Rightarrow$ implication.

Suppose that φ is an almost isomorphism. We define the auxiliary $\mathcal{O}_X$-modules: $\mathcal{K} := \mathrm{Ker}(\varphi)$, $\mathcal{F}' := \mathrm{Im}(\varphi)$, $\mathcal{Q} := \mathrm{Coker}(\varphi)$. Lemma 3.1.6 implies that the maps

$$\mathcal{F}(U) \to \mathcal{F}'(U) \text{ and } \mathcal{F}'(U) \to \mathcal{G}(U)$$

are almost isomorphisms. In particular, the composition $\mathcal{F}(U) \to \mathcal{G}(U)$ must also be an almost isomorphism. ∎

Now we discuss the notion of almost $\mathcal{O}_X$-modules on a ringed R-site $(X, \mathcal{O}_X)$. This notion can be defined in two different ways: either as the quotient of the category of $\mathcal{O}_X$-modules by the Serre subcategory of almost zero modules or as modules over the almost structure sheaf $\mathcal{O}_X^a$. Now we need to explain these two notions in more detail.

Definition 3.1.9. We define the *category of almost* $\mathcal{O}_X$*-modules* as the quotient category

$$\mathbf{Mod}^a_{\mathcal{O}_X} := \mathbf{Mod}_{\mathcal{O}_X} / \Sigma_X.$$

Now we define the category $\mathbf{Mod}_{\mathcal{O}_X^a}$ of $\mathcal{O}_X^a$-modules that we will show to be equivalent to $\mathbf{Mod}^a_{\mathcal{O}_X}$. We recall that the almostification functor $(-)^a$ is exact and commutes with arbitrary products. This allows us to define the almost structure sheaf:

Definition 3.1.10. The *almost structure sheaf* $\mathcal{O}_X^a$ is the sheaf[2] of R^a-modules

$$\mathcal{O}_X^a\colon (\mathrm{Ob}(X))^{\mathrm{op}} \to \mathbf{Mod}_R^a$$

defined via the formula $U \mapsto \mathcal{O}_X(U)^a$.

Definition 3.1.11. We define the *category of* $\mathcal{O}_X^a$*-modules* $\mathbf{Mod}_{\mathcal{O}_X^a}$ as the category of modules over $\mathcal{O}_X^a \in \mathbf{Shv}(X, R^a)$ in the categorical sense. More precisely, the objects are sheaves of R^a-modules $\mathcal{F}$ with a map $\mathcal{F} \otimes_{R^a} \mathcal{O}_X^a \to \mathcal{F}$ over R^a satisfying the usual axioms for a module. Morphisms are defined in the evident way.

We now define the functor

$$(-)^a\colon \mathbf{Mod}_{\mathcal{O}_X} \to \mathbf{Mod}_{\mathcal{O}_X^a}$$

that sends a sheaf to its "almostification", i.e., it applies the functor $(-)^a\colon \mathbf{Mod}_R \to \mathbf{Mod}_R^a$ section-wise. Since the almostification functor $(-)^a$ is exact and commutes

[2]It is a sheaf exactly because $(-)^a$ is exact and commutes with arbitrary products.

with arbitrary product, it is evident that $\mathcal{F}^a$ is actually a sheaf for any $\mathcal{O}_X$-module $\mathcal{F}$. Moreover, it is clear that $\mathcal{F}^a \simeq 0$ for any almost zero $\mathcal{O}_X$-module $\mathcal{F}$. Thus, it induces the functor

$$(-)^a \colon \mathbf{Mod}^a_{\mathcal{O}_X} \to \mathbf{Mod}_{\mathcal{O}_X^a}.$$

The claim is that this functor induces the equivalence of categories. The first step towards the proof is to construct the right adjoint to $(-)^a \colon \mathbf{Mod}_{\mathcal{O}_X} \to \mathbf{Mod}_{\mathcal{O}_X^a}$. Our construction of the right adjoint functor will use the existence of the left adjoint functor. So we slightly postpone the proof of the equivalence mentioned above and first discuss adjoints to $(-)^a$.

We start with the definition of the left adjoint functor. The idea is to apply the functor $(-)_! \colon \mathbf{Mod}_{\mathcal{O}_X^a} \to \mathbf{Mod}_{\mathcal{O}_X}$ section-wise, though this strategy does not quite work as $(-)_!$ does not commute with infinite products.

Definition 3.1.12. We define the desired functor in two steps.

- First, $(-)_!^p \colon \mathbf{Mod}_{\mathcal{O}_X^a} \to \mathbf{Mod}^p_{\mathcal{O}_X}$ as[3]

$$\mathcal{F} \mapsto \big(U \mapsto \mathcal{F}(U)_!\big).$$

- With its help, $(-)_! \colon \mathbf{Mod}_{\mathcal{O}_X^a} \to \mathbf{Mod}_{\mathcal{O}_X}$ as the composition $(-)_! := (-)^\# \circ (-)_!^p$, where $(-)^\#$ is the sheafification functor.

Lemma 3.1.13. *Let $(X, \mathcal{O}_X)$ be a ringed R-site.*

(1) *The functor*

$$(-)_! \colon \mathbf{Mod}_{\mathcal{O}_X^a} \to \mathbf{Mod}_{\mathcal{O}_X}$$

is the left adjoint to the localization functor $(-)^a \colon \mathbf{Mod}_{\mathcal{O}_X} \to \mathbf{Mod}_{\mathcal{O}_X^a}$. In particular, we have a functorial isomorphism

$$\mathrm{Hom}_{\mathcal{O}_X^a}(\mathcal{F}, \mathcal{G}^a) \simeq \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}_!, \mathcal{G})$$

for any $\mathcal{F} \in \mathbf{Mod}_{\mathcal{O}_X^a}$, $\mathcal{G} \in \mathbf{Mod}_{\mathcal{O}_X}$.

(2) *The functor $(-)_! \colon \mathbf{Mod}_{\mathcal{O}_X^a} \to \mathbf{Mod}_{\mathcal{O}_X}$ is exact.*

(3) *The counit morphism $(\mathcal{F}^a)_! \to \mathcal{F}$ is an almost isomorphism for any object $\mathcal{F} \in \mathbf{Mod}_{\mathcal{O}_X}$. The unit morphism $\mathcal{G} \to (\mathcal{G}_!)^a$ is an isomorphism for any object $\mathcal{G} \in \mathbf{Mod}_{\mathcal{O}_X^a}$. In particular, the functor $(-)^a$ is essentially surjective.*

Proof. (1) follows from Lemma 2.1.9 (3) and the adjunction between sheafication and the forgetful functor. More precisely, we have the following functorial isomorphisms:

$$\mathrm{Hom}_{\mathcal{O}_X^a}(\mathcal{F}, \mathcal{G}^a) \simeq \mathrm{Hom}_{\mathbf{Mod}^p_{\mathcal{O}_X}}(\mathcal{F}_!^p, \mathcal{G}) \simeq \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}_!, \mathcal{G}).$$

[3]$\mathbf{Mod}^p_{\mathcal{O}_X}$ stands for the category of modules over $\mathcal{O}_X$ in the category of presheaves.

We show (2). It is easy to see that $(-)_!$ is left exact from Lemma 2.1.9 (4) and the exactness of the sheafification functor. It is also right exact since it is a left adjoint functor to $(-)^a$.

Now we show (3). Lemma 2.1.9 (5) ensures that the kernel and cokernel of the counit map of presheaves $(\mathcal{F}^a)_!^p \to \mathcal{F}$ are annihilated by any $\varepsilon \in \mathfrak{m}$. Then the same holds after sheafification, proving that $(\mathcal{F}^a)_!^p \to \mathcal{F}$ is an almost isomorphism by Lemma 3.1.5.

We consider the unit map $\mathcal{G} \to (\mathcal{G}_!)^a$, we note that using the adjuction $((-)_!, (-)^a)$ section-wise, we can refine this map

$$\mathcal{G} \to (\mathcal{G}_!^p)^a \to (\mathcal{G}_!)^a.$$

It suffices to show that both maps are isomorphisms; the first map is an isomorphism by Lemma 2.1.9 (5). In particular, this implies that $(\mathcal{G}_!^p)^a$ is already a sheaf of almost R^a-modules, but then we see that the natural map $(\mathcal{G}_!^p)^a \to (\mathcal{G}_!)^a$ must also be an isomorphism as it coincides with the sheafification in the category of presheaves of R^a-modules. ■

Remark 3.1.14. In what follows, we denote the objects of $\mathrm{Mod}_{\mathcal{O}_X^a}$ by $\mathcal{F}^a$ to distinguish $\mathcal{O}_X$ and $\mathcal{O}_X^a$-modules. This notation does not cause any confusion as $(-)^a$ is essentially surjective.

Now we construct the right adjoint functor to $(-)^a$. The naive idea of applying $(-)_*$ section-wise works well in this case.

Definition 3.1.15. The functor of *almost sections* $(-)_*\colon \mathbf{Mod}_{\mathcal{O}_X^a} \to \mathbf{Mod}_{\mathcal{O}_X}$ is defined as

$$\mathcal{F}^a \mapsto \big(U \mapsto \mathrm{Hom}_R\big(\widetilde{\mathfrak{m}}, \mathcal{F}^a(U)_!\big) = \mathrm{Hom}_R(\widetilde{\mathfrak{m}}, \mathcal{F}(U))\big),$$

where the equality comes from Lemma 2.1.8 (2).

Remark 3.1.16. The functor $(-)_*$ is well defined, i.e., it defines a *sheaf* of $\mathcal{O}_X$-modules. This follows from the fact that $\mathrm{Hom}_R(\widetilde{\mathfrak{m}}, -)$ is left exact and commutes with arbitrary products.

Lemma 3.1.17. *Let $(X, \mathcal{O}_X)$ be a ringed R-site.*

(1) *The functor $(-)_*\colon \mathbf{Mod}_{\mathcal{O}_X^a} \to \mathbf{Mod}_{\mathcal{O}_X}$ is the right adjoint to the exact localization functor $(-)^a\colon \mathbf{Mod}_{\mathcal{O}_X} \to \mathbf{Mod}_{\mathcal{O}_X^a}$. In particular, it is left exact.*

(2) *The unit morphism $\mathcal{F} \to (\mathcal{F}^a)_*$ is an almost isomorphism for any object $\mathcal{F} \in \mathbf{Mod}_{\mathcal{O}_X}$. The counit morphism $(\mathcal{G}_*^a)^a \to \mathcal{G}^a$ is an isomorphism for any $\mathcal{G}^a \in \mathbf{Mod}_{\mathcal{O}_X^a}$.*

Proof. It is sufficient to check both claims section-wise. This, in turn, follows from Lemma 2.1.9 (1) and Lemma 2.1.9 (2) respectively. ■

Corollary 3.1.18. *The functor $(-)^a \colon \mathbf{Mod}_{\mathcal{O}_X} \to \mathbf{Mod}_{\mathcal{O}_X^a}$ commutes with limits and colimits. In particular, $\mathbf{Mod}_{\mathcal{O}_X^a}$ is complete and cocomplete, and filtered colimits and (finite) products are exact in $\mathbf{Mod}_{\mathcal{O}_X^a}$.*

Proof. The first claim follows from the fact that $(-)^a$ admits left and right adjoints. The second claim follows from the first claim, the exactness of $(-)^a$, and analogous exactness properties in $\mathbf{Mod}_R$. ∎

Corollary 3.1.19. *Let $(X, \mathcal{O}_X)$ be a ringed R-site. Then the functor*

$$(-)^a \colon \mathbf{Mod}_{\mathcal{O}_X} \to \mathbf{Mod}_{\mathcal{O}_X^a}$$

is exact.

Proof. The functor $(-)^a$ is exact as it has both left and right adjoints. ∎

Theorem 3.1.20. *Let $(X, \mathcal{O}_X)$ be a ringed R-site. Then the functor*

$$(-)^a \colon \mathbf{Mod}^a_{\mathcal{O}_X} \to \mathbf{Mod}_{\mathcal{O}_X^a}$$

is an equivalence of categories.

Proof. Lemma 3.1.17 implies that the functor $(-)^a \colon \mathbf{Mod}_{\mathcal{O}_X} \to \mathbf{Mod}_{\mathcal{O}_X^a}$ has a right adjoint functor $(-)_*$ such that the counit morphism $(-)^a \circ (-)_* \to \mathrm{id}$ is an isomorphism of functors. Moreover, the exactness of $(-)^a$ implies that a morphism $\varphi \colon \mathcal{F} \to \mathcal{G}$ is an almost isomorphism if and only if $\varphi^a \colon \mathcal{F}^a \to \mathcal{G}^a$ is an isomorphism. Thus, [27, Proposition 1.3] guarantees that the induced functor $(-)^a \colon \mathbf{Mod}^a_{\mathcal{O}_X} \to \mathbf{Mod}_{\mathcal{O}_X^a}$ is an equivalence. ∎

Remark 3.1.21. In what follows, we do not distinguish $\mathbf{Mod}_{\mathcal{O}_X^a}$ and $\mathbf{Mod}^a_{\mathcal{O}_X}$. Moreover, we sometimes denote both categories by $\mathbf{Mod}^a_X$ or $\mathbf{Mod}_{X^a}$ to simplify the notation.

3.2 Basic functors on categories of $\mathcal{O}_X^a$-modules

We discuss how to define certain basic functors on $\mathbf{Mod}^a_X$. Our main functors of interest are Hom, alHom, $\otimes$, f^*, and f_* (for any map f of ringed sites). We define their almost analogues and discuss the relation with their classical versions. As a by-product, we give a slightly more intrinsic definition of $(-)_* \colon \mathbf{Mod}^a_X \to \mathbf{Mod}_X$ along the lines of the definition of the $\mathbf{Mod}^a_R$-version of this functor. For the rest of the section, we fix a ring R with an ideal $\mathfrak{m}$ such that $\mathfrak{m} = \mathfrak{m}^2$ and $\widetilde{\mathfrak{m}} = \mathfrak{m} \otimes_R \mathfrak{m}$ is R-flat. We also fix an ringed R-site $(X, \mathcal{O}_X)$ that we also consider as a ringed $\mathcal{O}_X(X)$-site.

Definition 3.2.1. The global and local Hom functors are defined as follows:

- The *global Hom functor*

$$\mathrm{Hom}_{\mathcal{O}_X^a}(-,-)\colon \mathbf{Mod}_{X^a}^{\mathrm{op}} \times \mathbf{Mod}_{X^a} \to \mathbf{Mod}_{\mathcal{O}_X(X)}$$

  is defined as $(\mathcal{F}^a, \mathcal{G}^a) \mapsto \mathrm{Hom}_{\mathcal{O}_X^a}(\mathcal{F}^a, \mathcal{G}^a)$.

- The *local Hom functor*

$$\underline{\mathrm{Hom}}_{\mathcal{O}_X^a}(-,-)\colon \mathbf{Mod}_{X^a}^{\mathrm{op}} \times \mathbf{Mod}_{X^a} \to \mathbf{Mod}_X$$

  is defined as $(\mathcal{F}^a, \mathcal{G}^a) \mapsto \big(U \mapsto \mathrm{Hom}_{\mathcal{O}_U^a}(\mathcal{F}^a|_U, \mathcal{G}^a|_U)\big)$. The standard argument shows that this functor is well defined, i.e., $\underline{\mathrm{Hom}}_{\mathcal{O}_X^a}(\mathcal{F}, \mathcal{G})$ is indeed a sheaf of $\mathcal{O}_X$-modules.

Lemma 3.2.2. *Let $U \in \mathrm{Ob}(X)$, and let $\mathcal{F}^a, \mathcal{G}^a$ be $\mathcal{O}_X^a$-modules. Then the natural map*

$$\Gamma\big(U, \underline{\mathrm{Hom}}_{\mathcal{O}_X^a}(\mathcal{F}^a, \mathcal{G}^a)\big) \to \mathrm{Hom}_{\mathcal{O}_U^a}\big(\mathcal{F}^a|_U, \mathcal{G}^a|_U\big)$$

is an isomorphism of $\mathcal{O}_X(U)$-modules.

Proof. This is evident from the definition. ■

Lemma 3.2.3. *Let $(X, \mathcal{O}_X)$ be a ringed R-site. Then there is a functorial isomorphism of $\mathcal{O}_X$-modules*

$$\underline{\mathrm{Hom}}_{\mathcal{O}_X^a}(\mathcal{F}^a, \mathcal{G}^a) \xrightarrow{\sim} \underline{\mathrm{Hom}}_{\mathcal{O}_X}((\mathcal{F}^a)_!, \mathcal{G})$$

for $\mathcal{F}^a \in \mathbf{Mod}_X^a$ and $\mathcal{G} \in \mathbf{Mod}_X$.

Proof. Lemma 3.2.2 and Lemma 3.1.13 ensure that the desired isomorphism exists section-wise. It glues to a global isomorphism of sheaves since these section-wise isomorphisms are functorial in U. ■

Now we move on to show a promised more intrinsic definition of the functor $(-)_*$. As a warm-up, we need the following result:

Lemma 3.2.4. *Suppose that the ringed R-site $(X, \mathcal{O}_X)$ has a final object that (by slight abuse of notation) we denote by X. Then the evaluation map*

$$\mathrm{ev}_X\colon \mathrm{Hom}_{\mathcal{O}_X^a}\big(\mathcal{O}_X^a, \mathcal{G}^a\big) \to \mathrm{Hom}_{\mathcal{O}_X(X)^a}\big(\mathcal{O}_X^a(X), \mathcal{G}^a(X)\big)$$
$$\varphi \mapsto \varphi(X)$$

is an isomorphism of $\mathcal{O}_X(X)$-modules for any $\mathcal{G}^a \in \mathbf{Mod}_X^a$.

Proof. As $(-)^a$ is essentially surjective by Lemma 3.1.13 (3), there exists some $\mathcal{O}_X$-module $\mathcal{G}$ with almostification being equal to $\mathcal{G}^a$. Now we recall that the data of an $\mathcal{O}_X^a$-linear homomorphism $\varphi\colon \mathcal{O}_X^a \to \mathcal{G}^a$ is equivalent to the data of $\mathcal{O}_X(U)^a$-linear homomorphisms $\varphi_U \in \mathrm{Hom}_{\mathcal{O}_X(U)^a}\big(\mathcal{O}_X^a(U), \mathcal{G}^a(U)\big)$ for each open U in X such that the diagram

$$\begin{array}{ccc} \mathcal{O}_X(U)^a & \xrightarrow{\varphi_U} & \mathcal{G}(U)^a \\ {\scriptstyle r_{\mathcal{O}_X^a}|_V^U}\downarrow & & \downarrow{\scriptstyle r_{\mathcal{G}^a}|_V^U} \\ \mathcal{O}_X(V)^a & \xrightarrow[\varphi_V]{} & \mathcal{G}(V)^a \end{array}$$

commutes for any $V \subset U$. Now we note that an $\mathcal{O}_X(U)^a$-linear homomorphism φ_U uniquely determines an $\mathcal{O}_X(V)^a$-linear homomorphism φ_V in such a diagram. Indeed, this follows from the equality

$$\begin{aligned} &\mathrm{Hom}_{\mathcal{O}_X(V)^a}\big(\mathcal{O}_X(V)^a, \mathcal{G}(V)^a\big) \\ &\quad = \mathrm{Hom}_{\mathcal{O}_X(V)}\big(\widetilde{\mathfrak{m}} \otimes \mathcal{O}_X(V), \mathcal{G}(V)\big) \\ &\quad = \mathrm{Hom}_{\mathcal{O}_X(V)}\big(\widetilde{\mathfrak{m}} \otimes \mathcal{O}_X(U) \otimes_{\mathcal{O}_X(U)} \mathcal{O}_X(V), \mathcal{G}(V)\big) \\ &\quad = \mathrm{Hom}_{\mathcal{O}_X(U)}\big(\widetilde{\mathfrak{m}} \otimes \mathcal{O}_X(U), \mathcal{G}(V)\big) \\ &\quad = \mathrm{Hom}_{\mathcal{O}_X(U)^a}\big(\mathcal{O}_X(U)^a, \mathcal{G}(V)^a\big). \end{aligned}$$

Now we use the assumption that X is the final object to conclude that any homomorphism $\varphi\colon \mathcal{O}_X^a \to \mathcal{G}^a$ is uniquely defined by $\varphi(X)$. ■

Corollary 3.2.5. *Let $(X, \mathcal{O}_X)$ be an ringed R-site and let $U \in \mathrm{Ob}(X)$. Then the evaluation map*

$$\begin{aligned} \mathrm{ev}_U\colon \mathrm{Hom}_{\mathcal{O}_U^a}\big(\mathcal{O}_U^a, \mathcal{G}|_U^a\big) &\to \mathrm{Hom}_{\mathcal{O}_U(U)^a}\big(\mathcal{O}_U^a(U), \mathcal{G}^a(U)\big) \\ \varphi &\mapsto \varphi(U) \end{aligned}$$

is an isomorphism of $\mathcal{O}_X(U)$-modules for any $\mathcal{G}^a \in \mathbf{Mod}_X^a$.

Proof. For the purpose of the proof, we can change the site X to the slicing site X/U of objects over U. Then U automatically becomes the final object in X/U, so we can apply Lemma 3.2.4 to finish the proof. ■

Now we are ready to prove a new description of the sheaf version of the functor $(-)_*$.

Lemma 3.2.6. *Let $(X, \mathcal{O}_X)$ be a ringed R-site. Then there is a functorial isomorphism of $\mathcal{O}_X$-modules*

$$\underline{\mathrm{Hom}}_{\mathcal{O}_X^a}(\mathcal{O}_X^a, \mathcal{F}^a) \to \mathcal{F}_*^a$$

for $\mathcal{F}^a \in \mathbf{Mod}_X^a$.

Proof. Lemma 3.2.2 and Corollary 3.2.5 imply the existence of an isomorphism of $\mathcal{O}_X(U)$-modules

$$\Gamma\big(U, \underline{\mathrm{Hom}}_{\mathcal{O}_X^a}(\mathcal{O}_X^a, \mathcal{F}^a)\big) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_U(U)^a}\big(\mathcal{O}_U^a(U), \mathcal{F}^a(U)\big)$$

that is functorial in both U and $\mathcal{F}^a$. We use the functorial isomorphism of $\mathcal{O}_X(U)$-modules

$$\mathrm{Hom}_{\mathcal{O}_U(U)^a}\big(\mathcal{O}_U(U)^a, \mathcal{F}^a(U)\big) \simeq \mathrm{Hom}_{R^a}\big(R^a, \mathcal{F}^a(U)\big) = (\mathcal{F}^a)_*(U)$$

to construct a functorial isomorphism

$$\Gamma\big(U, \underline{\mathrm{Hom}}_{\mathcal{O}_X^a}(\mathcal{O}_X^a, \mathcal{F}^a)\big) \xrightarrow{\sim} (\mathcal{F}^a)_*(U).$$

Functoriality in U ensures that it glues to the global isomorphism of $\mathcal{O}_X$-modules

$$\underline{\mathrm{Hom}}_{\mathcal{O}_X^a}\big(\mathcal{O}_X^a, \mathcal{F}^a\big) \xrightarrow{\sim} \mathcal{F}^a_*. \qquad \blacksquare$$

Now we discuss the functor of almost homomorphisms.

Definition 3.2.7. The global and local *alHom functors* are defined as follows:

- The *global alHom functor*

$$\mathrm{alHom}_{\mathcal{O}_X^a}(-,-)\colon \mathbf{Mod}_{X^a}^{\mathrm{op}} \times \mathbf{Mod}_{X^a} \to \mathbf{Mod}_{R^a}$$

  is defined as

$$\big(\mathcal{F}^a, \mathcal{G}^a\big) \mapsto \mathrm{Hom}_{\mathcal{O}_X^a}\big(\mathcal{F}^a, \mathcal{G}^a\big)^a \simeq \mathrm{Hom}_{\mathcal{O}_X}\big((\mathcal{F}^a)_!, \mathcal{G}\big)^a.$$

- The *local alHom functor*

$$\underline{\mathrm{alHom}}_{\mathcal{O}_X^a}(-,-)\colon \mathbf{Mod}_{X^a}^{\mathrm{op}} \times \mathbf{Mod}_{X^a} \to \mathbf{Mod}_{X^a}$$

  is defined as

$$(\mathcal{F}^a, \mathcal{G}^a) \mapsto \big(U \mapsto \mathrm{alHom}_{\mathcal{O}_U^a}(\mathcal{F}^a|_U, \mathcal{G}^a|_U)^a\big).$$

Remark 3.2.8. At this point we have not checked that $\underline{\mathrm{alHom}}_{\mathcal{O}_X^a}(\mathcal{F}^a, \mathcal{G}^a)$ is actually a sheaf. However, this follows from the following lemma:

Lemma 3.2.9. *The natural map*

$$\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\widetilde{\mathfrak{m}} \otimes \mathcal{F}, \mathcal{G})^a \to \underline{\mathrm{alHom}}_{\mathcal{O}_X^a}(\mathcal{F}^a, \mathcal{G}^a)$$

is an almost isomorphism of $\mathcal{O}_X^a$-modules for any $\mathcal{F}^a, \mathcal{G}^a \in \mathbf{Mod}_X^a$. In particular, $\underline{\mathrm{alHom}}_{\mathcal{O}_X^a}(\mathcal{F}^a, \mathcal{G}^a)$ is a sheaf of $\mathcal{O}_X^a$-modules.

Proof. This follows from the sequence of functorial in U isomorphisms:

$$\begin{aligned}\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\widetilde{\mathfrak{m}}\otimes\mathcal{F},\mathcal{G})(U)^a &\simeq^a \mathrm{Hom}_{\mathcal{O}_U}(\widetilde{\mathfrak{m}}\otimes\mathcal{F}|_U,\mathcal{G}|_U)^a\\ &\simeq^a \mathrm{alHom}_{\mathcal{O}_U^a}(\mathcal{F}^a|_U,\mathcal{G}^a|_U)\\ &\simeq^a \underline{\mathrm{alHom}}_{\mathcal{O}_X^a}(\mathcal{F}^a,\mathcal{G}^a)(U)\end{aligned}$$ ■

In order to make Definition 3.2.7 computable, we need to show that these functors can be computed by using any representative for $\mathcal{F}^a$ and $\mathcal{G}^a$.

Proposition 3.2.10. *Let $(X,\mathcal{O}_X)$ be a ringed R-site.*

(1) *There is a natural transformation of functors*

$$\begin{array}{ccc}\mathbf{Mod}_X^{\mathrm{op}}\times\mathbf{Mod}_X & \xrightarrow{\mathrm{Hom}_{\mathcal{O}_X}(-,-)} & \mathbf{Mod}_X\\ \big\downarrow{\scriptstyle (-)^a\times(-)^a} & \overset{\rho}{\swarrow} & \big\downarrow{\scriptstyle (-)^a}\\ \mathbf{Mod}_{X^a}^{\mathrm{op}}\times\mathbf{Mod}_{X^a} & \xrightarrow[\mathrm{alHom}_{\mathcal{O}_X^a}(-,-)]{} & \mathbf{Mod}_{X^a}\end{array}$$

that makes the diagram $(2,1)$-commutative. In particular, it yields an isomorphism $\mathrm{alHom}_{\mathcal{O}_X^a}(\mathcal{F}^a,\mathcal{G}^a)\simeq\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})^a$ *for any $\mathcal{F},\mathcal{G}\in\mathbf{Mod}_X$.*

(2) *There is a natural transformation of functors*

$$\begin{array}{ccc}\mathbf{Mod}_X^{\mathrm{op}}\times\mathbf{Mod}_X & \xrightarrow{\underline{\mathrm{Hom}}_{\mathcal{O}_X}(-,-)} & \mathbf{Mod}_X\\ \big\downarrow{\scriptstyle (-)^a\times(-)^a} & \overset{\rho}{\swarrow} & \big\downarrow{\scriptstyle (-)^a}\\ \mathbf{Mod}_{X^a}^{\mathrm{op}}\times\mathbf{Mod}_{X^a} & \xrightarrow[\underline{\mathrm{alHom}}_{\mathcal{O}_X^a}(-,-)]{} & \mathbf{Mod}_{X^a}\end{array}$$

that makes the diagram $(2,1)$-commutative. In particular, it yields an isomorphism $\underline{\mathrm{alHom}}_{\mathcal{O}_X^a}(\mathcal{F}^a,\mathcal{G}^a)\simeq\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})^a$ *for any $\mathcal{F},\mathcal{G}\in\mathbf{Mod}_X$.*

Proof. The proof is similar to the proof of Proposition 2.2.1 (3). The only new thing is that we need to prove an analogue of Corollary 2.1.13, that is, that the functors $\mathrm{alHom}_{\mathcal{O}_X}(-,\mathcal{G})$, $\underline{\mathrm{alHom}}_{\mathcal{O}_X}(-,\mathcal{G})$ preserve almost isomorphisms. It essentially boils down to showing that $\mathrm{Ext}^i_{\mathcal{O}_X}(\mathcal{K},\mathcal{G})\cong^a 0$ and $\underline{\mathrm{Ext}}^i_{\mathcal{O}_X}(\mathcal{K},\mathcal{G})\cong^a 0$ for any $\mathcal{K}\in\Sigma_X,\mathcal{G}\in\mathbf{Mod}_X$, and an integer $i\geq 0$.

Now Lemma 3.1.5 implies that $\varepsilon\mathcal{K}=0$ for any $\varepsilon\in\mathfrak{m}$. With that at hand, we see that $\mathrm{Ext}^i_{\mathcal{O}_X}(\mathcal{K},\mathcal{G})$ and $\underline{\mathrm{Ext}}^i_{\mathcal{O}_X}(\mathcal{K},\mathcal{G})$ are also annihilated by any $\varepsilon\in\mathfrak{m}$ since the functors $\mathrm{Ext}^i_{\mathcal{O}_X}(-,\mathcal{G})$, $\underline{\mathrm{Ext}}^i_{\mathcal{O}_X}(-,\mathcal{G})$ are R-linear. Thus, $\mathrm{Ext}^i_{\mathcal{O}_X}(\mathcal{K},\mathcal{G})$ and $\underline{\mathrm{Ext}}^i_{\mathcal{O}_X}(\mathcal{K},\mathcal{G})$ are almost zero by Lemma 2.1.1 and Lemma 3.1.5 respectively. ■

Definition 3.2.11. The *tensor product functor* $-\otimes_{\mathcal{O}_X^a}-\colon\mathbf{Mod}_X^a\times\mathbf{Mod}_X^a\to\mathbf{Mod}_X^a$ is defined as

$$(\mathcal{F}^a,\mathcal{G}^a)\mapsto\mathcal{F}_!^a\otimes_{\mathcal{O}_X}\mathcal{G}_!^a.$$

Proposition 3.2.12. *There is a natural transformation of functors*

$$\begin{array}{ccc} \mathbf{Mod}_X \times \mathbf{Mod}_X & \xrightarrow{-\otimes_{\mathcal{O}_X}-} & \mathbf{Mod}_X \\ \downarrow{\scriptstyle (-)^a \times (-)^a} & \overset{\rho}{\nearrow\!\!\!\nearrow} & \downarrow{\scriptstyle (-)^a} \\ \mathbf{Mod}_X^a \times \mathbf{Mod}_X^a & \xrightarrow[-\otimes_{\mathcal{O}_X^a}-]{} & \mathbf{Mod}_X^a \end{array}$$

that makes the diagram $(2,1)$*-commutative. In particular, there is a functorial isomorphism*

$$(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})^a \simeq \mathcal{F}^a \otimes_{\mathcal{O}_X^a} \mathcal{G}^a$$

for any $\mathcal{F}, \mathcal{G} \in \mathbf{Mod}_X$.

Proof. The proof is analogous to that of Propisition 2.2.1 (1). ∎

The tensor product is adjoint to $\underline{\mathrm{Hom}}$ as it happens in the case of R^a-modules. We give a proof of the local version of this statement.

Lemma 3.2.13. *Let* $(X, \mathcal{O}_X)$ *be a ringed* R*-site, and let* $\mathcal{F}^a, \mathcal{G}^a, \mathcal{H}^a$ *be* $\mathcal{O}_X^a$*-modules. Then there is a functorial isomorphism*

$$\underline{\mathrm{Hom}}_{\mathcal{O}_X^a}(\mathcal{F}^a \otimes_{\mathcal{O}_X^a} \mathcal{G}^a, \mathcal{H}^a) \simeq \underline{\mathrm{Hom}}_{\mathcal{O}_X^a}(\mathcal{F}^a, \underline{\mathrm{alHom}}_{\mathcal{O}_X^a}(\mathcal{G}^a, \mathcal{H}^a)).$$

After passing to the global sections, this gives the isomorphism

$$\mathrm{Hom}_{\mathcal{O}_X^a}(\mathcal{F}^a \otimes_{\mathcal{O}_X^a} \mathcal{G}^a, \mathcal{H}^a) \simeq \mathrm{Hom}_{\mathcal{O}_X^a}(\mathcal{F}^a, \underline{\mathrm{alHom}}_{\mathcal{O}_X^a}(\mathcal{G}^a, \mathcal{H}^a)).$$

And after passing to the almostifications, it gives an isomorphism

$$\underline{\mathrm{alHom}}_{\mathcal{O}_X^a}(\mathcal{F}^a \otimes_{\mathcal{O}_X^a} \mathcal{G}^a, \mathcal{H}^a) \simeq \underline{\mathrm{alHom}}_{\mathcal{O}_X^a}(\mathcal{F}^a, \underline{\mathrm{alHom}}_{\mathcal{O}_X^a}(\mathcal{G}^a, \mathcal{H}^a)).$$

Proof. We compute $\Gamma(U, \underline{\mathrm{Hom}}_{\mathcal{O}_X^a}(\mathcal{F}^a \otimes_{\mathcal{O}_X^a} \mathcal{G}^a, \mathcal{H}^a))$ by using Lemma 3.2.2 and the standard $(\otimes, \underline{\mathrm{Hom}})$ adjunction. Namely,

$$\begin{aligned}
&\Gamma\big(U, \underline{\mathrm{Hom}}_{\mathcal{O}_X^a}\big(\mathcal{F}^a \otimes_{\mathcal{O}_X^a} \mathcal{G}^a, \mathcal{H}^a\big)\big) \\
&\quad\simeq \mathrm{Hom}_{\mathcal{O}_U^a}\big(\mathcal{F}^a|_U \otimes_{\mathcal{O}_U^a} \mathcal{G}^a|_U, \mathcal{H}^a|_U\big) && \text{Lemma 3.2.2} \\
&\quad\simeq \mathrm{Hom}_{\mathcal{O}_U^a}\big((\mathcal{F}|_U \otimes_{\mathcal{O}_U} \mathcal{G}|_U)^a, \mathcal{H}^a|_U\big) && \text{Proposition 3.2.12} \\
&\quad\simeq \mathrm{Hom}_{\mathcal{O}_U}\big(\widetilde{\mathfrak{m}} \otimes (\mathcal{F}|_U \otimes_{\mathcal{O}_U} \mathcal{G}|_U), \mathcal{H}|_U\big) && \text{Lemma 3.1.13} \\
&\quad\simeq \mathrm{Hom}_{\mathcal{O}_U}\big((\widetilde{\mathfrak{m}} \otimes \mathcal{F}|_U) \otimes_{\mathcal{O}_U} (\widetilde{\mathfrak{m}} \otimes \mathcal{G}|_U), \mathcal{H}|_U\big) && \widetilde{\mathfrak{m}}^{\otimes 2} \simeq \widetilde{\mathfrak{m}} \\
&\quad\simeq \mathrm{Hom}_{\mathcal{O}_U}\big(\widetilde{\mathfrak{m}} \otimes \mathcal{F}|_U, \underline{\mathrm{Hom}}_{\mathcal{O}_U}\big(\widetilde{\mathfrak{m}} \otimes \mathcal{G}|_U, \mathcal{H}|_U\big)\big) && (\otimes, \underline{\mathrm{Hom}}) \text{ adjunction} \\
&\quad\simeq \mathrm{Hom}_{\mathcal{O}_U^a}\big(\mathcal{F}^a|_U, \underline{\mathrm{alHom}}_{\mathcal{O}_U}\big(\widetilde{\mathfrak{m}} \otimes \mathcal{G}|_U, \mathcal{H}|_U\big)\big) && \text{Lemma 3.1.13} \\
&\quad\simeq \Gamma\big(U, \underline{\mathrm{Hom}}_{\mathcal{O}_X^a}\big(\mathcal{F}^a, \underline{\mathrm{alHom}}_{\mathcal{O}_X^a}(\mathcal{G}^a, \mathcal{H}^a)\big)\big). && \text{Lemma 3.2.2}
\end{aligned}$$

Since these identifications are functorial in U, we can glue them to a global isomorphism

$$\underline{\mathrm{Hom}}_{\mathcal{O}_X^a}(\mathcal{F}^a \otimes_{\mathcal{O}_X^a} \mathcal{G}^a, \mathcal{H}^a) \simeq \underline{\mathrm{Hom}}_{\mathcal{O}_X^a}(\mathcal{F}^a, \underline{\mathrm{alHom}}_{\mathcal{O}_X^a}(\mathcal{G}^a, \mathcal{H}^a)).$$

This finishes the proof. ∎

Corollary 3.2.14. *Let $(X, \mathcal{O}_X)$ be a ringed R-site, and let $\mathcal{F}^a$ be an $\mathcal{O}_X^a$-module. Then the functor $- \otimes_{\mathcal{O}_X^a} \mathcal{F}^a$ is left adjoint to $\underline{\mathrm{alHom}}_{\mathcal{O}_X^a}(\mathcal{F}^a, -)$.*

For what follows, we fix a map $f: (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of ringed R-sites. We are going to define the almost version of the pullback and pushforward functors.

Definition 3.2.15. The *pullback functor* $f_a^*: \mathbf{Mod}_Y^a \to \mathbf{Mod}_X^a$ is defined as

$$\mathcal{F}^a \mapsto \left(f^*(\mathcal{F}_!^a)\right)^a.$$

In what follows, we will often abuse notation and simply write f^* instead of f_a^*. This is "allowed" by Proposition 3.2.19.

As always, we want to show that this functor can be actually computed by applying f^* to *any* representative of $\mathcal{F}^a$. The main ingredient is to show that f^* sends almost isomorphisms to almost isomorphisms. The following lemma shows slightly more, and will be quite useful later on:

Lemma 3.2.16. *Let $f: (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed R-sites. Then for any $\mathcal{O}_X$-module $\mathcal{F}$, there is a natural isomorphism $\varphi_f(\mathcal{F}): f^*(\widetilde{\mathfrak{m}} \otimes \mathcal{F}) \to \widetilde{\mathfrak{m}} \otimes f^*\mathcal{F}$ functorial in $\mathcal{F}$.*

Proof. We use Remark 3.1.2 to conclude that $\widetilde{\mathfrak{m}} \otimes \mathcal{F}$ is functorially isomorphic to $\widetilde{\mathfrak{m}}_Y \otimes_{\mathcal{O}_Y} \mathcal{F}$, where $\underline{\widetilde{\mathfrak{m}}}_Y := \underline{\widetilde{\mathfrak{m}}} \otimes_R \mathcal{O}_Y$. Now we note that $f^*(\underline{\widetilde{\mathfrak{m}}}_Y) \simeq \underline{\widetilde{\mathfrak{m}}}_X$ as can be easily seen from the very definitions (using that $\widetilde{\mathfrak{m}}$ is R-flat). Therefore, $\varphi_f(\mathcal{F})$ comes from the fact that the pullback functor commutes with the tensor product. More precisely, we define it as the composition

$$f^*(\widetilde{\mathfrak{m}} \otimes \mathcal{F}) \xrightarrow{\sim} f^*(\underline{\widetilde{\mathfrak{m}}}_Y \otimes_{\mathcal{O}_Y} \mathcal{F}) \xrightarrow{\sim} f^*(\underline{\widetilde{\mathfrak{m}}}_Y) \otimes_{\mathcal{O}_X} f^*(\mathcal{F}) \xrightarrow{\sim} \underline{\widetilde{\mathfrak{m}}}_X \otimes_{\mathcal{O}_X} f^*(\mathcal{F}).$$ ∎

We now also show a derived version of Lemma 3.2.16 that will be used later in the text.

Lemma 3.2.17. *Let $f: (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed R-sites. Then for any $\mathcal{F} \in \mathbf{D}(X)$, there is a natural isomorphism*

$$\varphi_f(\mathcal{F}): \mathbf{L}f^*(\widetilde{\mathfrak{m}} \otimes \mathcal{F}) \to \widetilde{\mathfrak{m}} \otimes \mathbf{L}f^*\mathcal{F}$$

functorial in $\mathcal{F}$.

Proof. Similarly, we use Remark 3.1.2 to say that $\widetilde{\mathfrak{m}} \otimes \mathcal{F}$ is functorially isomorphic to $\widetilde{\mathfrak{m}}_Y \otimes_{\mathcal{O}_Y} \mathcal{F}$, where $\underline{\widetilde{\mathfrak{m}}}_Y := \underline{\widetilde{\mathfrak{m}}} \otimes_R \mathcal{O}_Y$. We note that $\mathbf{L} f^*(\underline{\widetilde{\mathfrak{m}}}_Y) \simeq f^*(\underline{\widetilde{\mathfrak{m}}}_Y) \simeq \underline{\widetilde{\mathfrak{m}}}_X$ as $\widetilde{\mathfrak{m}}$ is R-flat. The rest of the proof is the same using the $\mathbf{L} f^*$ functorially commutes with the derived tensor product. ∎

Corollary 3.2.18. *Let $f\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed R-sites, and let $\varphi\colon \mathcal{F} \to \mathcal{G}$ be an almost isomorphism of $\mathcal{O}_Y$-modules. Then the homomorphism $f^*(\varphi)\colon f^*(\mathcal{F}) \to f^*(\mathcal{G})$ is an almost isomorphism.*

Proof. The question boils down to showing that the homomorphism

$$\widetilde{\mathfrak{m}} \otimes f^*(\mathcal{F}) \to \widetilde{\mathfrak{m}} \otimes f^*(\mathcal{G})$$

is an isomorphism. Lemma 3.2.16 ensures that it is sufficient to prove that the map

$$f^*(\widetilde{\mathfrak{m}} \otimes \mathcal{F}) \to f^*(\widetilde{\mathfrak{m}} \otimes \mathcal{G})$$

is an isomorphism. But this is clear because the map $\widetilde{\mathfrak{m}} \otimes \mathcal{F} \to \widetilde{\mathfrak{m}} \otimes \mathcal{G}$ is already an isomorphism. ∎

Proposition 3.2.19. *Let $f\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed R-sites. Then there is a natural transformation of functors*

$$\begin{array}{ccc} \mathbf{Mod}_Y & \xrightarrow{\ f^*\ } & \mathbf{Mod}_X \\ {\scriptstyle (-)^a}\downarrow & \overset{\rho}{\nearrow\!\!\!\!\nearrow} & \downarrow{\scriptstyle (-)^a} \\ \mathbf{Mod}_Y^a & \xrightarrow[\ f_a^*\ ]{} & \mathbf{Mod}_X^a \end{array}$$

that makes the diagram $(2,1)$-commutative. In particular, there is a functorial isomorphism $(f^*\mathcal{F})^a \simeq f_a^*(\mathcal{F}^a)$ for any $\mathcal{F} \in \mathbf{Mod}_Y$.

Proof. The proof is similar to that of Proposition 2.2.1. For any $\mathcal{F} \in \mathbf{Mod}_Y$, we define $\rho_{\mathcal{F}}\colon f^*(\widetilde{\mathfrak{m}} \otimes \mathcal{F})^a \to f^*(\mathcal{F})^a$ as the map induced by the natural homomorphism $\widetilde{\mathfrak{m}} \otimes \mathcal{F} \to \mathcal{F}$. It is clearly functorial in $\mathcal{F}$, and it is an isomorphism by Corollary 3.2.18. ∎

Definition 3.2.20. The *pushforward functor* $f_*^a\colon \mathbf{Mod}_X^a \to \mathbf{Mod}_Y^a$ is defined as

$$\mathcal{F}^a \mapsto \left(f_*(\mathcal{F}_!^a)\right)^a.$$

In what follows, we will often abuse the notation and simply write f_* instead of f_*^a. This is "allowed" by Proposition 3.2.24.

Definition 3.2.21. The *global sections functor* $\Gamma^a(X,-)\colon \mathbf{Mod}_X^a \to \mathbf{Mod}_R^a$ is defined as

$$\mathcal{F}^a \mapsto \Gamma\big(X, \mathcal{F}_!^a\big)^a.$$

In what follows, we will often abuse the notation and simply write Γ instead of Γ^a. This is also "allowed" by Proposition 3.2.24.

Remark 3.2.22. The global section functor can be realized as the pushforward along the map $(X, \mathcal{O}_X) \to (*, R)$.

Lemma 3.2.23. *Let $f\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed R-sites, and let $\varphi\colon \mathcal{F} \to \mathcal{G}$ be an almost isomorphism. Then the morphism $f_*(\varphi)\colon f_*(\mathcal{F}) \to f_*(\mathcal{G})$ is an almost isomorphism.*

Proof. The standard argument considering the kernel and cokernel of φ shows that it is sufficient to prove that $f_*\mathcal{K} \cong^a 0$, $\mathrm{R}^1 f_*\mathcal{K} \cong^a 0$ for any almost zero $\mathcal{O}_X$-module $\mathcal{K}$. This follows from R-linearity of f_* and Lemma 3.1.5. ∎

Proposition 3.2.24. *Let $f\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed R-spaces. Then there is a natural transformation of functors*

$$\begin{array}{ccc} \mathbf{Mod}_X & \xrightarrow{\ f_*\ } & \mathbf{Mod}_Y \\ {\scriptstyle (-)^a}\downarrow & \overset{\rho}{\nearrow} & \downarrow{\scriptstyle (-)^a} \\ \mathbf{Mod}_X^a & \xrightarrow[\ f_*^a\ ]{} & \mathbf{Mod}_Y^a \end{array}$$

that makes the diagram $(2,1)$-commutative. In particular, there is a functorial isomorphism $(f_*\mathcal{F})^a \simeq f_*^a(\mathcal{F}^a)$ for any $\mathcal{F} \in \mathbf{Mod}_X$. The same results hold for $\Gamma^a(X,-)$.

Proof. We define $\rho_{\mathcal{F}}\colon f_*(\widetilde{\mathfrak{m}} \otimes \mathcal{F})^a \to f_*(\mathcal{F})^a$ as the map induced by the natural homomorphism $\widetilde{\mathfrak{m}} \otimes \mathcal{F} \to \mathcal{F}$. It is clearly functorial in $\mathcal{F}$, and it is an isomorphism by Lemma 3.2.23. ∎

Lemma 3.2.25. *Let $(X, \mathcal{O}_X)$ be a ringed R-site, and let $\mathcal{F}, \mathcal{G}$ be $\mathcal{O}_X^a$-modules. Then the natural morphism*

$$\Gamma\big(U, \underline{\mathrm{alHom}}_{\mathcal{O}_X^a}(\mathcal{F}^a, \mathcal{G}^a)\big) \to \mathrm{alHom}_{\mathcal{O}_U^a}(\mathcal{F}^a|_U, \mathcal{G}^a|_U)$$

is an isomorphism of R^a-modules for any $U \in \mathrm{Ob}(X)$.

Proof. The claim easily follows from Lemma 3.2.2, Proposition 3.2.10 (2), and Proposition 3.2.24. ∎

Lemma 3.2.26. *Let $f\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed R-sites, and let $\mathcal{F}^a \in \mathbf{Mod}^a_Y$, and $\mathcal{G}^a \in \mathbf{Mod}^a_X$. Then there is a functorial isomorphism of $\mathcal{O}_Y$-modules*

$$f_*\underline{\mathrm{Hom}}_{\mathcal{O}^a_X}(f^*(\mathcal{F}^a), \mathcal{G}^a) \simeq \underline{\mathrm{Hom}}_{\mathcal{O}^a_Y}(\mathcal{F}^a, f_*(\mathcal{G}^a)).$$

After passing to the global sections, this gives the isomorphism of $\mathcal{O}_Y(Y)$-modules

$$\mathrm{Hom}_{\mathcal{O}^a_X}(f^*(\mathcal{F}^a), \mathcal{G}^a) \simeq \mathrm{Hom}_{\mathcal{O}^a_Y}(\mathcal{F}^a, f_*(\mathcal{G}^a)).$$

And after passing to the almostifications, it gives the isomorphism of $\mathcal{O}^a_Y$-modules

$$f_*\underline{\mathrm{alHom}}_{\mathcal{O}^a_X}(f^*(\mathcal{F}^a), \mathcal{G}^a) \cong^a \underline{\mathrm{alHom}}_{\mathcal{O}^a_Y}(\mathcal{F}^a, f_*(\mathcal{G}^a)).$$

Proof. This is a combination of the classical (f^*, f_*)-adjunction, Lemma 3.1.13, Lemma 3.2.16, Proposition 3.2.19, and Proposition 3.2.24. Indeed, we choose $U \in \mathrm{Ob}(Y)$ and denote its pullback by $V := f^{-1}(U)$. We also define $\mathcal{F}^a_U := \mathcal{F}^a|_U$ and $\mathcal{G}^a_V := \mathcal{G}^a|_V$. The claim follows from the sequence of functorial isomorphisms

$$\begin{aligned}
&\Gamma\big(U, \underline{\mathrm{Hom}}_{\mathcal{O}^a_Y}(\mathcal{F}^a, f_*(\mathcal{G}^a))\big) && \\
&\simeq \mathrm{Hom}_{\mathcal{O}^a_U}\big(\mathcal{F}^a_U, f_*(\mathcal{G}^a_V)\big) && \text{Lemma 3.2.2}\\
&\simeq \mathrm{Hom}_{\mathcal{O}^a_U}\big(\mathcal{F}^a_U, f_*(\mathcal{G}_V)^a\big) && \text{Proposition 3.2.24}\\
&\simeq \mathrm{Hom}_{\mathcal{O}_U}\big(\widetilde{\mathfrak{m}} \otimes \mathcal{F}_U, f_*(\mathcal{G}_V)\big) && \text{Lemma 3.1.13}\\
&\simeq \mathrm{Hom}_{\mathcal{O}_V}\big(f^*(\widetilde{\mathfrak{m}} \otimes \mathcal{F}_U), \mathcal{G}_V\big) && (f^*, f_*)\text{-adjunction}\\
&\simeq \mathrm{Hom}_{\mathcal{O}_V}\big(\widetilde{\mathfrak{m}} \otimes f^*(\mathcal{F}_U), \mathcal{G}_V\big) && \text{Lemma 3.2.16}\\
&\simeq \mathrm{Hom}_{\mathcal{O}^a_V}\big(f^*(\mathcal{F}_U)^a, \mathcal{G}^a_V\big) && \text{Lemma 3.1.13}\\
&\simeq \mathrm{Hom}_{\mathcal{O}^a_V}\big(f^*(\mathcal{F}^a_U), \mathcal{G}^a_V\big) && \text{Proposition 3.2.19}\\
&\simeq \Gamma\big(U, f_*\underline{\mathrm{Hom}}_{\mathcal{O}^a_X}\big(f^*(\mathcal{F}^a), \mathcal{G}^a\big)\big). && \text{Lemma 3.2.2}
\end{aligned}$$

Since these identifications are functorial in U, we can glue them to a global isomorphism

$$f_*\underline{\mathrm{Hom}}_{\mathcal{O}^a_X}(f^*(\mathcal{F}^a), \mathcal{G}^a) \simeq \underline{\mathrm{Hom}}_{\mathcal{O}^a_Y}(\mathcal{F}^a, f_*(\mathcal{G}^a)).$$ ■

Corollary 3.2.27. *Let $f\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed R-sites. Then the functors* $f_*\colon \mathbf{Mod}^a_X \rightleftarrows \mathbf{Mod}^a_Y : f^*$ *are adjoint.*

3.3 Digression: The projection formula

In this section, we show that the tensor product $\widetilde{\mathfrak{m}} \otimes -$ behaves especially well on locally spectral spaces[4]. For instance, we show that we can explicitly describe sec-

[4]We refer to [68, Tag 08YF] and [70, Section 3] for a comprehensive discussion of (locally) spectral spaces.

tions of $\widetilde{\mathfrak{m}} \otimes \mathcal{F}$ on a basis of opens for such spaces, and verify a version of the projection formula for this tensor product.

Lemma 3.3.1. *Let $(X, \mathcal{O}_X)$ be a locally spectral, locally ringed R-space. Then for any spectral*[5] *open subset $U \subset X$, the natural morphism*

$$\widetilde{\mathfrak{m}} \otimes_R \mathcal{F}(U) \to (\widetilde{\mathfrak{m}} \otimes \mathcal{F})(U)$$

is an isomorphism of $\mathcal{O}_X(U)$-modules.

Proof. As spectral subspaces form a basis of topology on X, it suffices to show that the functor

$$U \to \widetilde{\mathfrak{m}} \otimes_R \mathcal{F}(U)$$

satisfies the sheaf condition on spectral open subsets. In particular, we can assume that X itself is spectral.

As any open spectral U is quasi-compact, we conclude that any open covering $U = \bigcup_{i \in I} U_i$ admits a refinement by a finite one. Thus, it is sufficient to check the sheaf condition for finite coverings of a spectral space by spectral open subspaces. Thus, we need to show that, for any finite covering $U = \bigcup_{i \in I} U_i$, the sequence

$$0 \to \widetilde{\mathfrak{m}} \otimes_R \mathcal{F}(U) \to \prod_{i=1}^{n} \left(\widetilde{\mathfrak{m}} \otimes_R \mathcal{F}(U_i)\right) \to \prod_{i,j=1}^{n} \left(\widetilde{\mathfrak{m}} \otimes_R \mathcal{F}(U_i \cap U_j)\right)$$

is exact. This follows from flatness of $\widetilde{\mathfrak{m}}$ and the fact that tensor product commutes with *finite* direct products. ∎

Now we want to show a version of the projection formula for the functor $\widetilde{\mathfrak{m}} \otimes -$, it will take some time to rigorously prove it. We recall that a map of locally spectral spaces is called *spectral*, if the pre-image of any spectral open subset is spectral.

Lemma 3.3.2. *Let $(X, \mathcal{O}_X)$ be a spectral locally ringed R-space. Then for any injective $\mathcal{O}_X$-module $\mathcal{I}$, the $\mathcal{O}_X$-module $\widetilde{\mathfrak{m}} \otimes \mathcal{I}$ is $\mathrm{H}^0(X, -)$-acyclic.*

Proof. We note that spectral open subspaces form a basis for the topology on X. Thus [68, Tag 01EV] and [68, Tag 0A36] imply that it suffices to show that

$$(\widetilde{\mathfrak{m}} \otimes \mathcal{I})(V) \xrightarrow{r_{\widetilde{\mathfrak{m}} \otimes \mathcal{I}}|_U^V} (\widetilde{\mathfrak{m}} \otimes \mathcal{I})(U)$$

is surjective for any inclusion of any *spectral* open subsets $U \hookrightarrow V$. Lemma 3.3.1 says that this map $r_{\widetilde{\mathfrak{m}} \otimes \mathcal{I}}|_U^V$ is identified with the map

$$\widetilde{\mathfrak{m}} \otimes_R \mathcal{I}(V) \xrightarrow{\widetilde{\mathfrak{m}} \otimes_R r_{\mathcal{I}}|_U^V} \widetilde{\mathfrak{m}} \otimes_R \mathcal{I}(U).$$

[5] We remind the reader that any quasi-compact quasi-separated open subset of a locally spectral space is spectral. This can be easily seen from the definitions.

But now we note that $r_{\mathcal{I}}|_U^V$ is surjective since any injective $\mathcal{O}_X$-module is flasque by [68, Tag 01EA], and therefore the map $\widetilde{\mathfrak{m}} \otimes_R r_{\mathcal{I}}|_U^V$ is surjective as well. ∎

Corollary 3.3.3. *Let $f\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a spectral morphism of locally spectral, locally ringed R-spaces, and let $\mathcal{I}$ be an injective $\mathcal{O}_X$-module. Then $\widetilde{\mathfrak{m}} \otimes \mathcal{I}$ is $f_*(-)$-acyclic.*

Proof. It suffices to show that for any open spectral $U \subset Y$, the higher cohomology groups

$$\mathrm{H}^i\big(X_U, (\widetilde{\mathfrak{m}} \otimes \mathcal{I})|_{X_U}\big)$$

vanish. This follows from Lemma 3.3.2 since X_U is spectral because both f and U are spectral. ∎

Lemma 3.3.4. *Let $f\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a spectral morphism of locally spectral, locally ringed R-spaces, and let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Then there is an isomorphism*

$$\beta : \widetilde{\mathfrak{m}} \otimes f_*\mathcal{F} \to f_*(\widetilde{\mathfrak{m}} \otimes \mathcal{F})$$

functorial in $\mathcal{F}$.

Proof. It suffices to define a morphism on a basis of spectral open subspaces $U \subset Y$. For any such $U \subset Y$, we define

$$\beta_U\colon (\widetilde{\mathfrak{m}} \otimes f_*\mathcal{F})(U) \to f_*(\widetilde{\mathfrak{m}} \otimes \mathcal{F})(U)$$

as the composition of isomorphisms

$$\begin{aligned}(\widetilde{\mathfrak{m}} \otimes f_*\mathcal{F})(U) &\xrightarrow{\alpha_U^{-1}} \widetilde{\mathfrak{m}} \otimes_R (f_*\mathcal{F})(U) = \widetilde{\mathfrak{m}} \otimes_R \mathcal{F}(X_U) \\ &\xrightarrow{\alpha_{X_U}} (\widetilde{\mathfrak{m}} \otimes \mathcal{F})(X_U) = f_*(\widetilde{\mathfrak{m}} \otimes \mathcal{F})(U)\end{aligned}$$

with α_U and α_{X_U} being isomorphisms from Lemma 3.3.1. Since the construction of α is functorial in U, we conclude that β defines a morphism of sheaves. It is an isomorphism because β_U is an isomorphism on a basis of Y. ∎

Lemma 3.3.5. *Let $f\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a spectral morphism of locally spectral, locally ringed R-spaces. Then for any $\mathcal{F} \in \mathbf{D}(X)$, there is a morphism*

$$\rho_f(\mathcal{F})\colon \widetilde{\mathfrak{m}} \otimes \mathbf{R}f_*\mathcal{F} \to \mathbf{R}f_*(\widetilde{\mathfrak{m}} \otimes \mathcal{F})$$

functorial in $\mathcal{F}$. This map is an isomorphism in either of the following cases:

- *the complex $\mathcal{F}$ is bounded below, i.e., $\mathcal{F} \in \mathbf{D}^+(X)$, or*
- *the space X is locally of uniformly bounded Krull dimension and $\mathcal{F} \in \mathbf{D}(X)$.*

Proof. We start the proof by constructing the map $\rho_f(\mathcal{F})$. Note that by adjunction it suffices to construct a map

$$\mathbf{L}f^*(\widetilde{\mathfrak{m}} \otimes \mathbf{R}f_*\mathcal{F}) \to \widetilde{\mathfrak{m}} \otimes \mathcal{F}.$$

We define this map as the composition

$$\mathbf{L}f^*(\widetilde{\mathfrak{m}} \otimes \mathbf{R}f_*\mathcal{F}) \xrightarrow{\varphi_f(\mathbf{R}f_*\mathcal{F})} \widetilde{\mathfrak{m}} \otimes \mathbf{L}f^*\mathbf{R}f_*\mathcal{F} \xrightarrow{\widetilde{\mathfrak{m}}\otimes\eta_{\mathcal{F}}} \widetilde{\mathfrak{m}} \otimes \mathcal{F},$$

where the first map is the isomorphism coming from Lemma 3.2.17 and the second map comes from the counit $\eta_{\mathcal{F}}$ of the $(\mathbf{L}f^*, \mathbf{R}f_*)$-adjunction.

Now we show that $\rho_f(\mathcal{F})$ is an isomorphism for $\mathcal{F} \in \mathbf{D}^+(X)$. We choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$. In this case, we use Corollary 3.3.3 to note that β is the natural map

$$\widetilde{\mathfrak{m}} \otimes f_*(\mathcal{I}^\bullet) \to f_*(\widetilde{\mathfrak{m}} \otimes \mathcal{I}^\bullet)$$

that is an isomorphism by Lemma 3.3.4.

The last thing we need to show is that $\rho_f(\mathcal{F})$ is an isomorphism for any unbounded $\mathcal{F}$ when X is locally of uniformly bounded Krull dimension. The claim is local, so we may and do assume that both X and Y are spectral spaces. As X is quasi-compact (because it is spectral) and locally of finite Krull dimension, we conclude that X has finite Krull dimension, say $N := \dim X$. Then [57, Corollary 4.6] (or [68, Tag 0A3G]) implies that $\mathrm{H}^i(U, \mathcal{G}) = 0$ for any open spectral $U \subset X$, $\mathcal{G} \in \mathbf{Mod}_X$, and $i > N$. In particular, $\mathrm{R}^i f_*\mathcal{G} = 0$ for any $\mathcal{G} \in \mathbf{Mod}_X$, and $i > N$. Thus we see that the assumptions of [68, Tag 0D6U] are verified in this case (with $\mathcal{A} = \mathbf{Mod}_X$ and $\mathcal{A}' = \mathbf{Mod}_Y$), so the natural map

$$\mathcal{H}^j(\mathbf{R}f_*\mathcal{F}) \to \mathcal{H}^j(\mathbf{R}f_*(\tau^{\geq -n}\mathcal{F}))$$

is an isomorphism for any $\mathcal{F} \in \mathbf{D}(X)$, $j \geq N - n$. As $\widetilde{\mathfrak{m}}$ is R-flat, we get the commutative diagram

$$\begin{array}{ccc}
\mathcal{H}^j\left(\widetilde{\mathfrak{m}} \otimes \mathbf{R}f_*\mathcal{F}\right) & \xrightarrow{\mathcal{H}^j(\rho_{\mathcal{F}})} & \mathcal{H}^j\left(\mathbf{R}f_*(\widetilde{\mathfrak{m}} \otimes \mathcal{F})\right) \\
\sim\downarrow & & \downarrow\sim \\
\mathcal{H}^j\left(\widetilde{\mathfrak{m}} \otimes \mathbf{R}f_*(\tau^{\geq -n}\mathcal{F})\right) & \xrightarrow[\mathcal{H}^j(\rho_{\tau^{\geq -n}\mathcal{F}})]{} & \mathcal{H}^j\left(\mathbf{R}f_*(\widetilde{\mathfrak{m}} \otimes \tau^{\geq -n}\mathcal{F})\right)
\end{array}$$

with the vertical arrows being isomorphisms for $j \geq N - n$, and the bottom horizontal map is an isomorphism as $\tau^{\geq -n}\mathcal{F} \in \mathbf{D}^+(X)$. Thus, by choosing an appropriate $n \geq 0$, we see that $\mathcal{H}^j(\rho_{\mathcal{F}})$ is an isomorphism for any j; so $\rho_{\mathcal{F}}$ is an isomorphism itself. ∎

3.4 Derived category of $\mathcal{O}_X^a$-modules

This section is a global analogue of Section 2.3. We give two different definitions of the derived category of almost $\mathcal{O}_X$-modules and then show that they coincide.

For the rest of the section, we fix a ring R with an ideal $\mathfrak{m}$ such that $\mathfrak{m} = \mathfrak{m}^2$ and $\widetilde{\mathfrak{m}} = \mathfrak{m} \otimes_R \mathfrak{m}$ is R-flat. We also fix an ringed R-site $(X, \mathcal{O}_X)$.

Definition 3.4.1. By definition, the *derived category of $\mathcal{O}_X^a$-modules* is $\mathbf{D}(X^a) := \mathbf{D}(\mathbf{Mod}_X^a)$.

We define the bounded version of the derived category of almost R-modules $\mathbf{D}^*(X^a)$ for $* \in \{+, -, b\}$ as the full subcategory of $\mathbf{D}(X^a)$ consisting of bounded below (resp. bounded above, resp. bounded) complexes.

Definition 3.4.2. We define the *almost derived category of $\mathcal{O}_X$-modules* as the Verdier quotient[6] $\mathbf{D}(X)^a := \mathbf{D}(\mathbf{Mod}_X)/\mathbf{D}_{\Sigma_X}(\mathbf{Mod}_X)$.

Remark 3.4.3. We recall that Σ_X is the Serre subcategory of $\mathbf{Mod}_X$ that consists of the almost zero $\mathcal{O}_X$-modules.

We note that the functor $(-)^a \colon \mathbf{Mod}_X \to \mathbf{Mod}_X^a$ is exact and additive. Thus, it can be derived to the functor $(-)^a \colon \mathbf{D}(X) \to \mathbf{D}(X^a)$. Similarly, the functor $(-)_! \colon \mathbf{Mod}_X^a \to \mathbf{Mod}_X$ can be derived to the functor $(-)_! \colon \mathbf{D}(X^a) \to \mathbf{D}(X)$. The standard argument shows that $(-)_!$ is a left adjoint functor to the functor $(-)^a$ as this already happens on the level of abelian categories.

We also want to establish a derived version of the functor $(-)_*$. But since the functor is only left exact, we do need to do some work to derive it. Namely, we need to ensure that $\mathcal{O}_X^a$-modules admit enough K-injective complexes.

Definition 3.4.4. We say that a complex of $\mathcal{O}_X^a$-modules $I^{\bullet,a}$ is *K-injective* if the condition $\mathrm{Hom}_{K(\mathcal{O}_X^a)}(C^{\bullet,a}, I^{\bullet,a}) = 0$ is satisfied for any acyclic complex $C^{\bullet,a}$ of R^a-modules.

Remark 3.4.5. We remind the reader that $K(\mathcal{O}_X^a)$ stands for the homotopy category of $\mathcal{O}_X^a$-modules.

Lemma 3.4.6. *The functor $(-)^a \colon \mathbf{Comp}(\mathcal{O}_X) \to \mathbf{Comp}(\mathcal{O}_X^a)$ sends K-injective $\mathcal{O}_X^a$-complexes to K-injective $\mathcal{O}_X^a$-complexes.*

Proof. We note that $(-)^a$ admits an exact left adjoint $(-)_!$ thus [68, Tag 08BJ] ensures that $(-)^a$ preserves K-injective complexes. ∎

[6] We refer to [68, Tag 05RA] for an extensive discussion of Verdier quotients of triangulated categories.

Corollary 3.4.7. *Let $(X, \mathcal{O}_X)$ be a ringed R-site. Then every $\mathcal{F}^{\bullet,a} \in \mathbf{Comp}(\mathcal{O}_X^a)$ is quasi-isomorphic to a K-injective complex.*

Proof. The proof of Corollary 2.3.6 works verbatim with the only exception that one needs to use [68, Tag 079P] instead of [68, Tag 090Y]. ■

Similarly to the case of R^a-modules, we define the functor $(-)_*\colon \mathbf{D}(X^a) \to \mathbf{D}(X)$ as the derived functor of $(-)_*\colon \mathbf{Mod}_X^a \to \mathbf{Mod}_X$. This functor exists by [68, Tag 070K].

Lemma 3.4.8. *Let $(X, \mathcal{O}_X)$ be a ringed R-site.*

(1) *The functors $(-)^a\colon \mathbf{D}(X) \rightleftarrows \mathbf{D}(X^a) : (-)_!$ are adjoint. Moreover, the counit (resp. unit) morphism*

$$(\mathcal{F}^a)_! \to \mathcal{F} \text{ (resp. } \mathcal{G} \to (\mathcal{G}_!)^a\text{)}$$

is an almost isomorphism (resp. isomorphism) for any $\mathcal{F} \in \mathbf{D}(X), \mathcal{G} \in \mathbf{D}(X^a)$. In particular, the functor $(-)^a$ is essentially surjective.

(2) *The functor $(-)^a : \mathbf{D}(X) \to \mathbf{D}(X^a)$ also admits a right adjoint functor $(-)_*\colon \mathbf{D}(X^a) \to \mathbf{D}(X)$. Moreover, the unit (resp. counit) morphism*

$$\mathcal{F} \to (\mathcal{F}^a)_* \text{ (resp. } (\mathcal{G}_*)^a \to \mathcal{G}\text{)}$$

is an almost isomorphism (resp. isomorphism) for any $\mathcal{F} \in \mathbf{D}(X), \mathcal{G} \in \mathbf{D}(X^a)$.

Proof. The proof is similar to that of Lemma 2.3.7. ■

Theorem 3.4.9. *The functor $(-)^a\colon \mathbf{D}(X) \to \mathbf{D}(X^a)$ induces an equivalence of triangulated categories $(-)^a\colon \mathbf{D}(X)^a \to \mathbf{D}(X^a)$.*

Proof. The proof is similar to that of Theorem 2.3.8. ■

Remark 3.4.10. Theorem 3.4.9 shows that the two notions of the derived category of almost modules are the same. In what follows, we do not distinguish $\mathbf{D}(X^a)$ and $\mathbf{D}(X)^a$ anymore.

3.5 Basic functors on derived categories of $\mathcal{O}_X^a$-modules

Now we can "derive" certain functors constructed in Section 3.2. For the rest of the section, we fix a ringed R-site $(X, \mathcal{O}_X)$. The section follows the exposition of Section 2.4 very closely.

Definition 3.5.1. We define the *derived Hom* functors

$$\mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{O}_X^a}(-,-)\colon \mathbf{D}(X^a)^{\mathrm{op}} \times \mathbf{D}(X^a) \to \mathbf{D}(X),$$

and

$$\mathbf{R}\mathrm{Hom}_{\mathcal{O}_X^a}(-,-)\colon \mathbf{D}(X^a)^{\mathrm{op}} \times \mathbf{D}(X^a) \to \mathbf{D}(R)$$

as it is done in [68, Tag 08DH] and [68, Tag 0B6A], respectively.

Definition 3.5.2. We define the *global Ext-modules* as the R-modules

$$\mathrm{Ext}^i_{\mathcal{O}_X^a}\left(\mathcal{F}^a, \mathcal{G}^a\right) := \mathrm{H}^i\left(\mathbf{R}\mathrm{Hom}_{\mathcal{O}_X^a}(\mathcal{F}^a, \mathcal{G}^a)\right)$$

for $\mathcal{F}^a, \mathcal{G}^a \in \mathbf{Mod}_X^a$.

Finally, we define the *local Ext-sheaves* as the $\mathcal{O}_X$-modules $\underline{\mathrm{Ext}}^i_{\mathcal{O}_X^a}(\mathcal{F}^a, \mathcal{G}^a) := \mathcal{H}^i(\mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{O}_X^a}(\mathcal{F}^a, \mathcal{G}^a))$, for $\mathcal{F}^a, \mathcal{G}^a \in \mathbf{Mod}_X^a$.

Remark 3.5.3. We see that [68, Tag 0A64] implies that there is a functorial isomorphism

$$\mathrm{H}^i\left(\mathbf{R}\mathrm{Hom}_{\mathcal{O}_X^a}(\mathcal{F}^a, \mathcal{G}^a)\right) \simeq \mathrm{Hom}_{\mathbf{D}(R)^a}\left(\mathcal{F}^a, \mathcal{G}^a[i]\right)$$

for $\mathcal{F}^a, \mathcal{G}^a \in \mathbf{D}(X)^a$.

Remark 3.5.4. The standard argument shows that there is a functorial isomorphism

$$\mathbf{R}\Gamma\left(U, \mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{O}_X^a}(\mathcal{F}^a, \mathcal{G}^a)\right) \simeq \mathbf{R}\mathrm{Hom}_{\mathcal{O}_U^a}\left(\mathcal{F}^a|_U, \mathcal{G}^a|_U\right)$$

for any open $U \in X$, $\mathcal{F}^a, \mathcal{G}^a \in \mathbf{D}(X)^a$.

Now we show a local version of the $((-)_!, (-)^a)$-adjunction, and relate $\mathbf{R}\underline{\mathrm{Hom}}$ (resp. $\mathbf{R}\mathrm{Hom}$) to a certain derived functor. This goes in complete analogy with the situation in the usual (not almost) world.

Lemma 3.5.5. *Let $(X, \mathcal{O}_X)$ be a ringed R-site.*

(1) *There is a functorial isomorphism*

$$\mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{O}_X^a}(\mathcal{F}^a, \mathcal{G}^a) \simeq \mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{F}_!^a, \mathcal{G})$$

for any $\mathcal{F}^a \in \mathbf{D}(X)^a$ and $\mathcal{G} \in \mathbf{D}(X)$. In particular, this isomorphism induces functorial isomorphisms

$$\mathbf{R}\mathrm{Hom}_{\mathcal{O}_X^a}(\mathcal{F}^a, \mathcal{G}^a) \simeq \mathbf{R}\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}_!^a, \mathcal{G})$$

and

$$\mathrm{Hom}_{\mathbf{D}(X)^a}(\mathcal{F}^a, \mathcal{G}^a) \simeq \mathbf{R}\mathrm{Hom}_{\mathbf{D}(X)}(\mathcal{F}_!^a, \mathcal{G}).$$

(2) *For any chosen $\mathcal{F}^a \in \mathbf{Mod}_X^a$, the functor $\mathbf{R}\mathrm{Hom}_{\mathcal{O}_X^a}(\mathcal{F}^a, -)\colon \mathbf{D}(X)^a \to \mathbf{D}(R)$ is isomorphic to the (right) derived functor of $\mathrm{Hom}_{\mathcal{O}_X^a}(\mathcal{F}^a, -)$.*

(3) *For any chosen* $\mathcal{F}^a \in \mathbf{Mod}_X^a$, *the functor* $\mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{O}_X^a}(\mathcal{F}^a, -)\colon \mathbf{D}(X)^a \to \mathbf{D}(X)$ *is isomorphic to the (right) derived functor of* $\mathrm{Hom}_{\mathcal{O}_X^a}(\mathcal{F}^a, -)$.

Proof. (1) Lemma 3.4.6 and the construction of derived homs ensure that

$$\mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{O}_X^a}(\mathcal{F}^a, \mathcal{G}^a) \simeq \underline{\mathrm{Hom}}^\bullet_{\mathcal{O}_X^a}(\mathcal{F}^{\bullet,a}, \mathcal{I}^{\bullet,a})$$
$$\mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{F}_!^a, \mathcal{G}) \simeq \underline{\mathrm{Hom}}^\bullet_{\mathcal{O}_X}(\mathcal{F}_!^{\bullet,a}, \mathcal{I}^\bullet),$$

where $\mathcal{G} \to \mathcal{I}^\bullet$ is a K-injective resolution. Now we recall the term-wise equalities

$$\underline{\mathrm{Hom}}^n_{\mathcal{O}_X^a}(\mathcal{F}^{\bullet,a}, \mathcal{I}^{\bullet,a}) = \prod_{p+q=n} \underline{\mathrm{Hom}}_{\mathcal{O}_X^a}(\mathcal{F}^{-q,a}, \mathcal{I}^{p,a})$$
$$\underline{\mathrm{Hom}}^n_{\mathcal{O}_X}(\mathcal{F}_!^{\bullet,a}, \mathcal{I}^\bullet) = \prod_{p+q=n} \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{F}_!^{-q,a}, \mathcal{I}^p).$$

Thus Lemma 3.2.3 produces term-wise isomorphisms

$$\kappa_n\colon \underline{\mathrm{Hom}}^n_{\mathcal{O}_X^a}(\mathcal{F}^{\bullet,a}, \mathcal{I}^{\bullet,a}) \to \underline{\mathrm{Hom}}^n_{\mathcal{O}_X}(\mathcal{F}_!^{\bullet,a}, \mathcal{I}^\bullet)$$

that commute with the differentials by inspection, therefore defining the desired isomorphism of complexes.

Parts (2) and (3) are identical to Lemma 2.4.3 (2). ∎

Definition 3.5.6. We define the *derived almost Hom* functors

$$\mathbf{R}\underline{\mathrm{alHom}}_{\mathcal{O}_X^a}(-,-)\colon \mathbf{D}(X^a)^{\mathrm{op}} \times \mathbf{D}(X^a) \to \mathbf{D}(X^a),$$
$$\mathbf{R}\mathrm{alHom}_{\mathcal{O}_X^a}(-,-)\colon \mathbf{D}(X^a)^{\mathrm{op}} \times \mathbf{D}(X^a) \to \mathbf{D}(R^a)$$

as

$$\mathbf{R}\underline{\mathrm{alHom}}_{\mathcal{O}_X^a}(\mathcal{F}^a, \mathcal{G}^a) := \mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{O}_X^a}(\mathcal{F}^a, \mathcal{G}^a)^a = \mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{F}_!^a, \mathcal{G})^a,$$
$$\mathbf{R}\mathrm{alHom}_{\mathcal{O}_X^a}(\mathcal{F}^a, \mathcal{G}^a) := \mathbf{R}\mathrm{Hom}_{\mathcal{O}_X^a}(\mathcal{F}^a, \mathcal{G}^a)^a = \mathbf{R}\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}_!^a, \mathcal{G})^a.$$

Definition 3.5.7. The *global almost Ext modules* are defined as the R^a-modules $\mathrm{alExt}^i_{\mathcal{O}_X^a}(\mathcal{F}^a, \mathcal{G}^a) := \mathrm{H}^i(\mathbf{R}\mathrm{alHom}_{\mathcal{O}_X^a}(\mathcal{F}^a, \mathcal{G}^a))$ for $\mathcal{F}^a, \mathcal{G}^a \in \mathbf{Mod}_X^a$.

We define the *local almost Ext sheaves* as the $\mathcal{O}_X^a$-modules $\underline{\mathrm{alExt}}^i_{\mathcal{O}_X^a}(\mathcal{F}^a, \mathcal{G}^a) := \mathcal{H}^i(\mathbf{R}\underline{\mathrm{alHom}}_{\mathcal{O}_X^a}(\mathcal{F}^a, \mathcal{G}^a))$ for $\mathcal{F}^a, \mathcal{G}^a \in \mathbf{Mod}_X^a$.

Proposition 3.5.8. *Let $(X, \mathcal{O}_X)$ be a ringed R-site.*

(1) *There is a natural transformation of functors*

$$\begin{array}{ccc}
\mathbf{D}(X)^{\mathrm{op}} \times \mathbf{D}(X) & \xrightarrow{\mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{O}_X}(-,-)} & \mathbf{D}(X) \\
\big\downarrow {\scriptstyle (-)^a \times (-)^a} & \overset{\rho}{\swarrow\!\!\!\swarrow} & \big\downarrow {\scriptstyle (-)^a} \\
\mathbf{D}(X^a)^{\mathrm{op}} \times \mathbf{D}(X^a) & \xrightarrow[\mathbf{R}\underline{\mathrm{alHom}}_{\mathcal{O}_X^a}(-,-)]{} & \mathbf{D}(X^a)
\end{array}$$

that makes the diagram $(2,1)$-commutative. In particular, it yields an isomprphism $\mathbf{R}\underline{\mathrm{alHom}}_{\mathcal{O}_X^a}(\mathcal{F}^a, \mathcal{G}^a) \simeq \mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})^a$ for any $\mathcal{F}, \mathcal{G} \in \mathbf{D}(X)$.

(2) *For any $\mathcal{F}^a \in \mathbf{Mod}_R^a$, the functor $\mathbf{R}\underline{\mathrm{alHom}}_{\mathcal{O}_X^a}(\mathcal{F}^a, -)\colon \mathbf{D}(X)^a \to \mathbf{D}(X)^a$ is isomorphic to the (right) derived functor of $\underline{\mathrm{alHom}}_{\mathcal{O}_X^a}(\mathcal{F}^a, -)$.*

(3) *The analogous results hold true for the functor $\mathbf{R}\mathrm{alHom}_{\mathcal{O}_X^a}(-,-)$.*

Proof. The proof is identical to that of Proposition 2.4.8. One only needs to use Proposition 3.2.10 in place of Proposition 2.2.1 (3). ■

Now we deal with the case of the derived tensor product functor. We will show that our definition of the derived tensor product functor makes $\mathbf{R}\underline{\mathrm{alHom}}_{\mathcal{O}_X^a}(-,-)$ into the inner Hom functor.

Definition 3.5.9. We say that a complex of $\mathcal{O}_X^a$-modules $\mathcal{F}^{\bullet,a}$ is *almost K-flat* if the naive tensor product complex $\mathcal{C}^{\bullet,a} \otimes^{\bullet}_{\mathcal{O}_X^a} \mathcal{F}^{\bullet,a}$ is acyclic for any acyclic complex $\mathcal{C}^{\bullet,a}$ of $\mathcal{O}_X^a$-modules.

Lemma 3.5.10. *The functor $(-)^a\colon \mathbf{Comp}(\mathcal{O}_X) \to \mathbf{Comp}(\mathcal{O}_X^a)$ sends K-flat $\mathcal{O}_X$-complexes to almost K-flat $\mathcal{O}_X^a$-complexes.*

Proof. The proof Lemma 2.4.10 applies verbatim. ■

Lemma 3.5.11. *Let $f\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed R-sites, and let $\mathcal{F}^{\bullet,a} \in \mathbf{Comp}(\mathcal{O}_Y^a)$ be an almost K-flat complex. Then $f^*(\mathcal{F}^{\bullet,a}) \in \mathbf{Comp}(\mathcal{O}_X^a)$ is almost K-flat.*

Proof. The proof of [68, Tag 06YW] works verbatim in this situation. ■

Corollary 3.5.12. *Every object $\mathcal{F}^{\bullet,a} \in \mathbf{Comp}(\mathcal{O}_X^a)$ is quasi-isomorphic to an almost K-flat complex.*

Proof. The proof of Corollary 2.4.11 applies verbatim with the only difference that one needs to use [68, Tag 06YF] in place of [68, Tag 06Y4]. ■

Definition 3.5.13. We define the *derived tensor product functor*

$$- \otimes^L_{\mathcal{O}_X^a} - \colon \mathbf{D}(X)^a \times \mathbf{D}(X)^a \to \mathbf{D}(X)^a$$

by the rule $(\mathcal{F}^a, \mathcal{G}^a) \mapsto (\mathcal{G}_! \otimes^L_{\mathcal{O}_X} \mathcal{G}_!)^a$ for any $\mathcal{F}^a, \mathcal{G}^a \in \mathbf{D}(X)^a$.

Proposition 3.5.14. (1) *There is a natural transformation of functors*

$$\begin{array}{ccc} \mathbf{D}(X) \times \mathbf{D}(X) & \xrightarrow{\ -\otimes^L_{\mathcal{O}_X} -\ } & \mathbf{D}(X) \\ \big\downarrow{\scriptstyle (-)^a \times (-)^a} & \overset{\rho}{\Longrightarrow} & \big\downarrow{\scriptstyle (-)^a} \\ \mathbf{D}(X)^a \times \mathbf{D}(X)^a & \xrightarrow[\ -\otimes^L_{\mathcal{O}_X^a} -\ ]{} & \mathbf{D}(X)^a \end{array}$$

that makes the diagram $(2,1)$*-commutative. In particular, there is a functorial isomorphism* $(\mathcal{F} \otimes^L_{\mathcal{O}_X} \mathcal{G})^a \simeq \mathcal{F}^a \otimes^L_{\mathcal{O}_X^a} \mathcal{G}^a$ *for any* $\mathcal{F}, \mathcal{G} \in \mathbf{D}(X)$.

(2) *For any chosen* $\mathcal{F}^a \in \mathbf{Mod}_X^a$, *the functor* $\mathcal{F}^a \otimes^L_{R^a} - \colon \mathbf{D}(X)^a \to \mathbf{D}(X)^a$ *is isomorphic to the (left) derived functor of* $\mathcal{F}^a \otimes_{\mathcal{O}_X^a} -$.

Proof. Again, the proof is identical to that of Proposition 3.5.14. The only non-trivial input that we need is the existence of sufficiently many K-flat complexes of $\mathcal{O}_X^a$-modules. But this is guaranteed by Corollary 3.5.12. ■

Remark 3.5.15. For any $\mathcal{F}^a, \mathcal{G}^a \in \mathbf{D}(X)^a$, there is a canonical morphism

$$\mathbf{R}\underline{\mathrm{alHom}}_{\mathcal{O}_X^a}(\mathcal{F}^a, \mathcal{G}^a) \otimes^L_{\mathcal{O}_X} \mathcal{F}^a \to \mathcal{G}^a$$

that, after the identifications from Proposition 3.5.8 and Proposition 3.5.14, is the almostification of the canonical morphism

$$\mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{F}_!^a, \mathcal{G}_!^a) \otimes^L_{\mathcal{O}_X} \mathcal{F}_!^a \to \mathcal{G}_!^a$$

from [68, Tag 0A8V].

Lemma 3.5.16. *Let* $(X, \mathcal{O}_X)$ *be a ringed* R*-site, and let* $\mathcal{F}^a, \mathcal{G}^a, \mathcal{H}^a \in \mathbf{D}(X)^a$. *Then we have a functorial isomorphism*

$$\mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{O}_X^a}\left(\mathcal{F}^a \otimes^L_{\mathcal{O}_X^a} \mathcal{G}^a, \mathcal{H}^a\right) \simeq \mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{O}_X^a}\left(\mathcal{F}^a, \mathbf{R}\underline{\mathrm{alHom}}_{R^a}(\mathcal{G}^a, \mathcal{H}^a)\right).$$

This induces functorial isomorphisms

$$\mathbf{R}\mathrm{Hom}_{\mathcal{O}_X^a}\left(\mathcal{F}^a \otimes^L_{\mathcal{O}_X^a} \mathcal{G}^a, \mathcal{H}^a\right) \simeq \mathbf{R}\mathrm{Hom}_{\mathcal{O}_X^a}\left(\mathcal{F}^a, \mathbf{R}\underline{\mathrm{alHom}}_{R^a}(\mathcal{G}^a, \mathcal{H}^a)\right),$$
$$\mathbf{R}\underline{\mathrm{alHom}}_{\mathcal{O}_X^a}\left(\mathcal{F}^a \otimes^L_{\mathcal{O}_X^a} \mathcal{G}^a, \mathcal{H}^a\right) \simeq \mathbf{R}\underline{\mathrm{alHom}}_{\mathcal{O}_X^a}\left(\mathcal{F}^a, \mathbf{R}\underline{\mathrm{alHom}}_{R^a}(\mathcal{G}^a, \mathcal{H}^a)\right),$$
$$\mathbf{R}\mathrm{alHom}_{\mathcal{O}_X^a}\left(\mathcal{F}^a \otimes^L_{\mathcal{O}_X^a} \mathcal{G}^a, \mathcal{H}^a\right) \simeq \mathbf{R}\mathrm{alHom}_{\mathcal{O}_X^a}\left(\mathcal{F}^a, \mathbf{R}\underline{\mathrm{alHom}}_{R^a}(\mathcal{G}^a, \mathcal{H}^a)\right).$$

Proof. The proof of the first isomorphism is very similar to that of Lemma 2.4.14. We leave the details to the interested reader. The second isomorphism comes from the fist one by applying the functor $\mathbf{R}\Gamma(X,-)$. The third and the fourth isomorphisms are obtained by applying $(-)^a$ to the first and the second isomorphisms respectively. Here, we implicitly use Proposition 3.5.8. ∎

Corollary 3.5.17. *Let $(X, \mathcal{O}_X)$ be a ringed R-site, and let $\mathcal{G}^a \in \mathbf{D}(X)^a$. Then the functors*

$$\mathbf{R}\underline{\mathrm{alHom}}_{\mathcal{O}_X^a}(\mathcal{G}^a, -)\colon \mathbf{D}(X)^a \leftrightarrows \mathbf{D}(X)^a \colon - \otimes^L_{\mathcal{O}_X^a} \mathcal{G}^a$$

are adjoint.

Now we discuss the almost analogues of derived pullbacks and derived pushforwards. We start with the derived pullbacks:

Definition 3.5.18. Let $f\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed R-sites. We define the *derived pullback functor*

$$\mathbf{L}f^*\colon \mathbf{D}(Y)^a \to \mathbf{D}(X)^a$$

as the derived functor of the right exact, additive functor $f^*\colon \mathbf{Mod}_Y^a \to \mathbf{Mod}_X^a$.

Remark 3.5.19. We need to explain why the desired derived functor exists and how it can be computed. It turns out that it can be constructed by choosing K-flat resolutions, the argument for this is identical to [68, Tag 06YY]. We only emphasize that three main inputs are Lemma 3.5.11, Lemma 3.5.10, and an almost analogue of [68, Tag 06YG].

Proposition 3.5.20. *Let $f\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed R-sites. Then there is a natural transformation of functors*

$$\begin{array}{ccc} \mathbf{D}(Y) & \xrightarrow{\mathbf{L}f^*} & \mathbf{D}(X) \\ \downarrow{\scriptstyle (-)^a} & \overset{\rho}{\nearrow\!\!\!\!\nearrow} & \downarrow{\scriptstyle (-)^a} \\ \mathbf{D}(Y)^a & \xrightarrow[\mathbf{L}f^*]{} & \mathbf{D}(X)^a \end{array}$$

that makes the diagram $(2,1)$-commutative. In particular, there is a functorial isomorphism $(\mathbf{L}f^*\mathcal{F})^a \simeq \mathbf{L}f^*(\mathcal{F}^a)$ for any $\mathcal{F} \in \mathbf{D}(Y)$.

Proof. We construct the natural transformation $\rho\colon \mathbf{L}f^* \circ (-)^a \Rightarrow (-)^a \circ \mathbf{L}f^*$ as follows: Pick any object $\mathcal{F} \in \mathbf{D}(Y)$ and its K-flat representative $\mathcal{K}^\bullet$, then $\mathcal{K}^\bullet$ is adapted to compute the usual derived pullback $\mathbf{L}f^*$. Lemma 3.5.11 ensures $\mathcal{K}^{\bullet,a}$ is also adapted to compute the almost version of the derived pullback $\mathbf{L}f^*$. So we define the morphism

$$\rho_{\mathcal{F}}\colon (f^*(\widetilde{\mathfrak{m}} \otimes \mathcal{K}^\bullet))^a \to f^*(\mathcal{K}^\bullet)^a$$

as the natural morphism induced by $\widetilde{\mathfrak{m}} \otimes \mathcal{K}^\bullet \to \mathcal{K}^\bullet$. This map is clearly functorial, so it defines a transformation of functors ρ. To show that it is an isomorphism of functors, it suffices to show that the map

$$f^*(\widetilde{\mathfrak{m}} \otimes \mathcal{K}^\bullet) \to f^*(\mathcal{K}^\bullet)$$

is an almost isomorphism of complexes for any K-flat complex $K^\bullet$. But this is clear as $\widetilde{\mathfrak{m}} \otimes \mathcal{K}^\bullet \to \mathcal{K}^\bullet$ is an almost isomorphism, and Corollary 3.2.18 ensures that f^* preserves almost isomorphisms. ∎

Definition 3.5.21. Let $f\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed R-sites. We define the *derived pushforward functor*

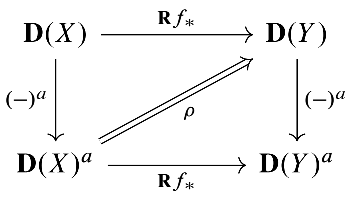
$$\mathbf{R}f_*\colon \mathbf{D}(X)^a \to \mathbf{D}(Y)^a$$

as the derived functor of the left exact, additive functor $f_*\colon \mathbf{Mod}_X^a \to \mathbf{Mod}_Y^a$.

We define the *derived global sections functor* $\mathbf{R}\Gamma(U, -)\colon \mathbf{D}(X)^a \to \mathbf{D}(R)^a$ in a similar way for any open $U \subset X$.

Remark 3.5.22. This functor exists by abstract nonsense (i.e., [68, Tag 070K]) as the category $\mathbf{Mod}_X^a$ has enough K-injective complexes by Corollary 3.4.7.

Proposition 3.5.23. *Let $f\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed R-sites. Then there is a natural transformation of functors*



that makes the diagram $(2,1)$-commutative. In particular, there is a functorial isomorphism $(\mathbf{R}f_*\mathcal{F})^a \simeq \mathbf{R}f_*(\mathcal{F}^a)$ for any $\mathcal{F} \in \mathbf{D}(X)$. The analogous results hold for the functor $\mathbf{R}\Gamma(U, -)$.

Proof. The proof is very similar to that of Proposition 3.5.20. The main essential ingredients are: $(-)^a$ sends K-injective complexes to K-injective complexes, and f_* preserves almost isomorphisms. These two results were shown in Lemma 3.4.6 and Lemma 3.2.23. ∎

Lemma 3.5.24. *Let $(X, \mathcal{O}_X)$ be a ringed R-site, let $\mathcal{F}$ be an $\mathcal{O}_X^a$-module, and let $U \in X$ be an open object. Then we have a canonical isomorphism*

$$\mathbf{R}\Gamma\big(U, \mathbf{R}\underline{\mathrm{alHom}}_{\mathcal{O}_X^a}(\mathcal{F}^a, \mathcal{G}^a)\big) \simeq \mathbf{R}\mathrm{alHom}_{\mathcal{O}_U^a}\big(\mathcal{F}^a|_U, \mathcal{G}^a|_U\big)$$

Proof. This easily follows from Remark 3.5.4, Proposition 3.5.8, and finally Proposition 3.5.23. ■

Lemma 3.5.25. *Let $f\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed R-sites. Then there is a functorial isomorphism*

$$\mathbf{R}f_*\mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{O}_X^a}(\mathbf{L}f^*\mathcal{F}^a, \mathcal{G}^a) \simeq \mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{O}_Y^a}(\mathcal{F}^a, \mathbf{R}f_*\mathcal{G}^a)$$

for $\mathcal{F}^a \in \mathbf{D}(Y)^a$, $\mathcal{G}^a \in \mathbf{D}(X)^a$. This isomorphism induces isomorphisms

$$\mathbf{R}f_*\mathbf{R}\underline{\mathrm{alHom}}_{\mathcal{O}_X^a}(\mathbf{L}f^*\mathcal{F}^a, \mathcal{G}^a) \simeq \mathbf{R}\underline{\mathrm{alHom}}_{\mathcal{O}_Y^a}(\mathcal{F}^a, \mathbf{R}f_*\mathcal{G}^a),$$
$$\mathbf{R}\mathrm{Hom}_{\mathcal{O}_X^a}(\mathbf{L}f^*\mathcal{F}^a, \mathcal{G}^a) \simeq \mathbf{R}\mathrm{Hom}_{\mathcal{O}_Y^a}(\mathcal{F}^a, \mathbf{R}f_*\mathcal{G}^a),$$
$$\mathbf{R}\mathrm{alHom}_{\mathcal{O}_X^a}(\mathbf{L}f^*\mathcal{F}^a, \mathcal{G}^a) \simeq \mathbf{R}\mathrm{alHom}_{\mathcal{O}_Y^a}(\mathcal{F}^a, \mathbf{R}f_*\mathcal{G}^a).$$

Proof. It is a standard exercise to show that the first isomorphism implies all other isomorphisms by applying certain functors to it, so we deal only with the first one. The proof of the first one is also quite standard and similar to Lemma 3.2.26, but we spell it out for the reader's convenience. The desired isomorphism comes from a sequence of canonical identifications:

$$\begin{aligned}
&\mathbf{R}f_*\mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{O}_X^a}\big(\mathbf{L}f^*(\mathcal{F}^a), \mathcal{G}^a\big) && \\
&\simeq \mathbf{R}f_*\mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{O}_X^a}\big(\mathbf{L}f^*(\mathcal{F})^a, \mathcal{G}^a\big) && \text{Proposition 3.5.20}\\
&\simeq \mathbf{R}f_*\mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{O}_X}\big(\widetilde{\mathfrak{m}} \otimes \mathbf{L}f^*(\mathcal{F}), \mathcal{G}\big) && \text{Lemma 3.5.5 (1)}\\
&\simeq \mathbf{R}f_*\mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{O}_X}\big(\mathbf{L}f^*(\widetilde{\mathfrak{m}} \otimes \mathcal{F}), \mathcal{G}\big) && \text{Lemma 3.2.17}\\
&\simeq \mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{O}_Y}\big(\widetilde{\mathfrak{m}} \otimes \mathcal{F}, \mathbf{R}f_*(\mathcal{G})\big) && \text{classical}\\
&\simeq \mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{O}_Y^a}\big(\mathcal{F}^a, \mathbf{R}f_*(\mathcal{G})^a\big) && \text{Lemma 3.5.5 (1)}\\
&\simeq \mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{O}_Y^a}\big(\mathcal{F}^a, \mathbf{R}f_*(\mathcal{G}^a)\big) && \text{Proposition 3.5.23.}
\end{aligned}$$
■

Corollary 3.5.26. *Let $f\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed R-sites. Then the functors $\mathbf{R}f_*(-)\colon \mathbf{D}(X)^a \longleftrightarrow \mathbf{D}(Y)^a \colon \mathbf{L}f^*(-)$ are adjoint.*

Now we discuss the projection formula in the world of almost sheaves. Suppose $f\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of ringed R-sites, $\mathcal{F}^a \in \mathbf{D}(X)^a$, and $\mathcal{G}^a \in \mathbf{D}(Y)^a$. We wish to construct the projection morphism

$$\rho\colon \mathbf{R}f_*(\mathcal{F}^a) \otimes^L_{\mathcal{O}_Y^a} \mathcal{G}^a \to \mathbf{R}f_*\big(\mathcal{F}^a \otimes^L_{\mathcal{O}_X^a} \mathbf{L}f^*(\mathcal{G}^a)\big).$$

By Corollary 3.5.26, it is equivalent to constructing a morphism

$$\pi\colon \mathbf{L}f^*\big(\mathbf{R}f_*(\mathcal{F}^a)\otimes^L_{\mathcal{O}_Y^a}\mathcal{G}^a\big)\to\mathcal{F}^a\otimes^L_{\mathcal{O}_X^a}\mathbf{L}f^*(\mathcal{G}^a).$$

We define π as the composition of the natural isomorphism

$$\mathbf{L}f^*\big(\mathbf{R}f_*(\mathcal{F}^a)\otimes^L_{\mathcal{O}_Y^a}\mathcal{G}^a\big)\simeq\mathbf{L}f^*\big(\mathbf{R}f_*(\mathcal{F}^a)\big)\otimes^L_{\mathcal{O}_X^a}\mathbf{L}f^*(\mathcal{G}^a)$$

and the morphism

$$\mathbf{L}f^*\big(\mathbf{R}f_*(\mathcal{F}^a)\big)\otimes^L_{\mathcal{O}_X^a}\mathbf{L}f^*(\mathcal{G}^a)\xrightarrow{\varepsilon_{\mathcal{F}^a}\otimes\mathrm{id}}\mathcal{F}^a\otimes^L_{\mathcal{O}_X^a}\mathbf{L}f^*(\mathcal{G}^a)$$

induced by the counit of the $(\mathbf{L}f^*,\mathbf{R}f_*)$-adjunction.

Proposition 3.5.27. *Let $f\colon(X,\mathcal{O}_X)\to(Y,\mathcal{O}_Y)$ be a morphism of ringed R-sites, $\mathcal{F}^a\in\mathbf{D}(X)^a$, and $\mathcal{G}\in\mathbf{D}(Y)$ a perfect complex. Then the projection morphism*

$$\rho\colon\mathbf{R}f_*(\mathcal{F}^a)\otimes^L_{\mathcal{O}_Y^a}\mathcal{G}^a\to\mathbf{R}f_*\big(\mathcal{F}^a\otimes^L_{\mathcal{O}_X^a}\mathbf{L}f^*(\mathcal{G}^a)\big)$$

is an isomorphism in $\mathbf{D}(Y)^a$.

Proof. The claim is local on Y, so we may assume that $\mathcal{G}$ is isomorphic to a bounded complex of finite free $\mathcal{O}_Y$-modules. Then an easy argument with naive filtrations reduces the question to the case when $\mathcal{G}=\mathcal{O}_Y^n$. This case is essentially obvious. ■