Chapter 4

Almost coherent sheaves on schemes and formal schemes

In this chapter, we develop a theory of almost coherent sheaves on schemes, and on a "nice" class of formal schemes.

4.1 Schemes. The category of almost coherent $\mathcal{O}_X^a$-modules

In this section we discuss the notion of almost quasi-coherent, almost finite type, almost finitely presented and almost coherent sheaves on an arbitrary scheme. One of the main goals is to show that almost coherent sheaves form a weak Serre subcategory in $\mathcal{O}_X$-modules. Another important result is the "approximation" Corollary 4.3.5; it will play a key role in reducing many global results about almost finitely presented $\mathcal{O}_X$-modules to the classical case of finitely presented $\mathcal{O}_X$-modules. In particular, we follow this approach in our proof of the almost proper mapping theorem in Section 5.1.

As always, we fix a ring R with an ideal $\mathfrak{m}$ such that $\mathfrak{m} = \mathfrak{m}^2$ and $\widetilde{\mathfrak{m}} = \mathfrak{m} \otimes_R \mathfrak{m}$ is R-flat. We always do almost mathematics with respect to this ideal. In what follows, X will always denote an R-scheme. Note that this implies that X is a locally spectral, ringed R-site, so the results of Chapter 3 and in particular Section 3.3 apply.

We begin with some definitions:

Definition 4.1.1. We say that an $\mathcal{O}_X^a$-module $\mathcal{F}^a$ is *almost quasi-coherent* if the associated $\mathcal{O}_X$-module $\mathcal{F}^a_! \simeq \widetilde{\mathfrak{m}} \otimes \mathcal{F}$ is quasi-coherent.

We say that an $\mathcal{O}_X$-module $\mathcal{F}$ is *almost quasi-coherent* if $\mathcal{F}^a$ is an almost quasi-coherent $\mathcal{O}_X^a$-module.

Remark 4.1.2. Any quasi-coherent $\mathcal{O}_X$-module is almost quasi-coherent.

Remark 4.1.3. We denote by $\mathbf{Mod}_{X^a}^{\mathrm{aqc}} \subset \mathbf{Mod}_{X^a}$ the full subcategory consisting of almost quasi-coherent $\mathcal{O}_X^a$-modules. It is straightforward[1] to see that the "almostification" functor induces an equivalence

$$\mathbf{Mod}_{X^a}^{\mathrm{aqc}} \simeq \mathbf{Mod}_X^{\mathrm{qc}} / (\Sigma_X \cap \mathbf{Mod}_X^{\mathrm{qc}}),$$

i.e., $\mathbf{Mod}_{X^a}^{\mathrm{aqc}}$ is equivalent to the quotient category of quasi-coherent $\mathcal{O}_X$-modules by the full subcategory of almost zero, quasi-coherent $\mathcal{O}_X$-modules.

[1] The proof is completely similar to the proof of Theorem 3.1.20 or Theorem 3.4.9.

Definition 4.1.4. We say that an $\mathcal{O}_X^a$-module $\mathcal{F}^a$ is *of almost finite type* (resp. *almost finitely presented*) if $\mathcal{F}^a$ is almost quasi-coherent, and there is a covering of X by open affines $\{U_i\}_{i\in I}$ such that $\mathcal{F}^a(U_i)$ is an almost finitely generated (resp. almost finitely presented) $\mathcal{O}_X^a(U_i)$-module.

We say that an $\mathcal{O}_X$-module $\mathcal{F}$ is *of almost finite type* (resp. *almost finitely presented*) if so is $\mathcal{F}^a$.

Remark 4.1.5. We denote by $\mathbf{Mod}_X^{\mathrm{qc,aft}}$ (resp. $\mathbf{Mod}_X^{\mathrm{qc,afp}}$) the full subcategory of $\mathbf{Mod}_X$ consisting of almost finite type (resp. almost finitely presented) quasi-coherent $\mathcal{O}_X$-modules. Similarly, we denote by $\mathbf{Mod}_{X^a}^{\mathrm{aft}}$ (resp. $\mathbf{Mod}_{X^a}^{\mathrm{afp}}$) the full subcategory of $\mathbf{Mod}_{X^a}$ consisting of almost finite type (resp. almost finitely presented) $\mathcal{O}_X^a$-modules. Again, it is straightforward to see that the "almostification" functors induce equivalences

$$\mathbf{Mod}_{X^a}^{\mathrm{aft}} \simeq \mathbf{Mod}_X^{\mathrm{qc,aft}}/(\Sigma_X \cap \mathbf{Mod}_X^{\mathrm{qc,aft}}),\ \mathbf{Mod}_{X^a}^{\mathrm{afp}} \simeq \mathbf{Mod}_X^{\mathrm{qc,afp}}/(\Sigma_X \cap \mathbf{Mod}_X^{\mathrm{qc,afp}}).$$

Remark 4.1.6. Recall that, in the usual theory of $\mathcal{O}_X$-modules, finite type $\mathcal{O}_X$-modules are usually not required to be quasi-coherent. However, it is more convenient for our purposes to put almost quasi-coherence in the definition of almost finite type modules.

The first thing that we need to check is that these notions do not depend on a choice of an affine covering.

Lemma 4.1.7. *Let $\mathcal{F}^a$ be an almost finite type (resp. almost finitely presented) $\mathcal{O}_X^a$-module on an R-scheme X. Then $\mathcal{F}^a(U)$ is an almost finitely generated (resp. almost finitely presented) $\mathcal{O}_X^a(U)$-module for any open affine $U \subset X$.*

Proof. First, Corollary 2.5.12 and Lemma 3.3.1 imply that for any open affine U, $\mathcal{F}^a(U)$ is almost finitely generated (resp. almost finitely presented) if and only if so is $(\widetilde{\mathfrak{m}} \otimes \mathcal{F}^a)(U)$. Thus, we can replace $\mathcal{F}^a$ by $\mathcal{F}^a_! \simeq \widetilde{\mathfrak{m}} \otimes \mathcal{F}$ to assume that $\mathcal{F}$ is an honest quasi-coherent $\mathcal{O}_X$-module.

Now Lemma 2.8.1 guarantees that the problem is local on X. So we can assume that $X = U$ is an affine scheme and we need to show that $\mathcal{F}(X)$ is almost finitely generated (resp. almost finitely presented).

We pick some covering $X = \bigcup_{i=1}^n U_i$ by open affines U_i such that $\mathcal{F}(U_i)$ is almost finitely generated (resp. almost finitely presented) as an $\mathcal{O}_X(U_i)$-module. We note that since $\mathcal{F}$ is quasi-coherent, we have an isomorphism

$$\mathcal{F}(U_i) \simeq \mathcal{F}(X) \otimes_{\mathcal{O}_X(X)} \mathcal{O}_X(U_i).$$

Now we see that the map $\mathcal{O}_X(X) \to \prod_{i=1}^n \mathcal{O}_X(U_i)$ is faithfully flat, and the module

$$\mathcal{F}(X) \otimes_{\mathcal{O}_X(X)} \left(\prod_{i=1}^n \mathcal{O}_X(U_i)\right) \simeq \left(\prod_{i=1}^n \mathcal{O}_X(U_i)\right) \otimes_{\mathcal{O}_X(X)} \mathcal{F}(X)$$

is almost finitely generated (resp. almost finitely presented) over $\prod_{i=1}^n \mathcal{O}_X(U_i)$. Thus, Lemma 2.10.5 guarantees that $\mathcal{F}(X)$ is almost finitely generated (resp. almost finitely presented) as an $\mathcal{O}_X(X)$-module. ■

Corollary 4.1.8. *Let $X = \operatorname{Spec} A$ be an affine R-scheme, and let $\mathcal{F}^a$ be an almost quasi-coherent $\mathcal{O}_X^a$-module. Then $\mathcal{F}^a$ is almost finite type (resp. almost finitely presented) if and only if $\Gamma(X, \mathcal{F}^a)$ is almost finitely generated (resp. almost finitely presented) A-module.*

Now we check that almost finite type and almost finitely presented $\mathcal{O}_X^a$-modules behave nicely in short exact sequences.

Lemma 4.1.9. *Let $0 \to \mathcal{F}'^a \xrightarrow{\varphi} \mathcal{F}^a \xrightarrow{\psi} \mathcal{F}''^a \to 0$ be an exact sequence of $\mathcal{O}_X^a$-modules.*

1. *If $\mathcal{F}^a$ is almost of finite type and $\mathcal{F}''^a$ is almost quasi-coherent, then $\mathcal{F}''^a$ is almost finite type.*
2. *If $\mathcal{F}'^a$ and $\mathcal{F}''^a$ are of almost finite type (resp. finitely presented), then so is $\mathcal{F}^a$.*
3. *If $\mathcal{F}^a$ is of almost finite type and $\mathcal{F}''^a$ is almost finitely presented, then $\mathcal{F}'^a$ is of almost finite type.*
4. *If $\mathcal{F}^a$ is almost finitely presented and $\mathcal{F}'^a$ is of almost finite type, then $\mathcal{F}''^a$ is almost finitely presented.*

Proof. First of all, we apply the exact functor $(-)_!$ to all $\mathcal{O}_X^a$-modules in question to assume the short sequence is an exact sequence of $\mathcal{O}_X$-modules and all $\mathcal{O}_X$-modules in this sequence are quasi-coherent. Note that we implicitly use here that quasi-coherent modules form a Serre subcategory of all $\mathcal{O}_X$-modules by [68, Tag 01IE]. Then we check the statement on the level of global sections on all open affine subschemes $U \subset X$ using that quasi-coherent sheaves have vanishing higher cohomology on affine schemes. That is done in Lemma 2.5.15. ■

Definition 4.1.10. An $\mathcal{O}_X^a$-module $\mathcal{F}^a$ is *almost coherent* if $\mathcal{F}^a$ is almost finite type, and for any open set U, any almost finite type $\mathcal{O}_U^a$-submodule $\mathcal{G}^a \subset (\mathcal{F}^a|_U)$ is an almost finitely presented $\mathcal{O}_U^a$-module.

We say that an $\mathcal{O}_X$-module $\mathcal{F}$ is *almost coherent* if $\mathcal{F}^a$ is an almost coherent $\mathcal{O}_X^a$-module.

Lemma 4.1.11. *Let X be an R-scheme, and let $\mathcal{F}^a$ be an $\mathcal{O}_X^a$-module. Then the following are equivalent:*

1. *$\mathcal{F}^a$ is almost coherent;*
2. *$\mathcal{F}^a$ is almost quasi-coherent, and the $\mathcal{O}_X^a(U)$-module $\mathcal{F}^a(U)$ is almost coherent for any open affine subscheme $U \subset X$;*

(3) *$\mathcal{F}^a$ is almost quasi-coherent, and there is a covering of X by open affine subschemes $\{U_i\}_{i \in I}$ such that $\mathcal{F}^a(U_i)$ is almost coherent for each i.*

In particular, if $X = \mathrm{Spec}\, A$ is an affine R-scheme and $\mathcal{F}^a$ is an almost quasi-coherent $\mathcal{O}_X^a$-module, then $\mathcal{F}^a$ is almost coherent if and only if $\mathcal{F}^a(X)$ is almost coherent as an A-module.

Proof. We start the proof by noting that we can replace $\mathcal{F}^a$ by $\mathcal{F}_!^a$ to assume that $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module.

First, we check that (1) implies (2). Given any affine open $U \subset X$ and any almost finitely generated almost submodule $M^a \subset \mathcal{F}(U)^a$, we define an almost subsheaf $(\widetilde{M^a})_! \subset (\mathcal{F}|_U)^a$. We see that $(\widetilde{M^a})_!$ must be an almost finitely presented $\mathcal{O}_U$-module, so Lemma 4.1.7 guarantees that $M_!^a$ is an almost finitely presented $\mathcal{O}_X(U)$-module. Therefore, any almost finitely generated submodule $M^a \subset \mathcal{F}(U)^a$ is almost finitely presented. This shows that $\mathcal{F}(U)$ is almost coherent.

Now we show that (2) implies (1). Suppose that $\mathcal{F}$ is almost quasi-coherent and $\mathcal{F}(U)$ is almost coherent for any open affine $U \subset X$. First of all, it implies that $\mathcal{F}$ is of almost finite type, since this notion is local by definition. Now suppose that we have an almost finite type almost $\mathcal{O}_X$-submodule $\mathcal{G} \subset (\mathcal{F}|_U)^a$ for some open U. It is represented by a homomorphism

$$\widetilde{\mathfrak{m}} \otimes \mathcal{G} \xrightarrow{g} \mathcal{F}$$

with $\mathcal{G}$ being an $\mathcal{O}_X$-module of almost finite type, and $\widetilde{\mathfrak{m}} \otimes \mathrm{Ker}\, g \simeq 0$. We want to show that $\mathcal{G}$ is almost finitely presented as an $\mathcal{O}_X$-module. This is a local question, so we can assume that U is affine. Then Lemma 3.3.1 implies that the natural morphism

$$g(U) : \widetilde{\mathfrak{m}} \otimes_R \mathcal{G}(U) \to \mathcal{F}(U)$$

defines an almost submodule of $\mathcal{F}(U)$. We conclude that $\widetilde{\mathfrak{m}} \otimes_R \mathcal{G}(U)$ is almost finitely presented by the assumption on $\mathcal{F}(U)$. Since the notion of almost finitely presented $\mathcal{O}_X$-module is local, we see that $\mathcal{G}$ is almost finitely presented.

Clearly, (2) implies (3), and it is easy to see that Corollary 2.10.6 guarantees that (3) implies (2). ■

Corollary 4.1.12. *Let X be an R-scheme.*

(1) *Any almost finite type $\mathcal{O}_X^a$-submodule of an almost coherent $\mathcal{O}_X^a$-module is almost coherent.*

(2) *Let $\varphi\colon \mathcal{F}^a \to \mathcal{G}^a$ be a homomorphism from an almost finite type $\mathcal{O}_X^a$-module to an almost coherent $\mathcal{O}_X^a$-module, then* $\mathrm{Ker}(\varphi)$ *is an almost finite type $\mathcal{O}_X^a$-module.*

(3) *Let $\varphi\colon \mathcal{F}^a \to \mathcal{G}^a$ be a homomorphism of almost coherent $\mathcal{O}_X^a$-modules, then* $\mathrm{Ker}(\varphi)$ *and* $\mathrm{Coker}(\varphi)$ *are almost coherent $\mathcal{O}_X^a$-modules.*

(4) *Given a short exact sequence of $\mathcal{O}_X^a$-modules*

$$0 \to \mathcal{F}'^a \to \mathcal{F}^a \to \mathcal{F}''^a \to 0,$$

if two out of three are almost coherent, so is the third.

Proof. The proofs of parts (1), (2) and (3) are quite straightforward. As usually, we apply $(-)_!$ to assume that all sheaves in question are quasi-coherent $\mathcal{O}_X$-modules. Then the question is local and it is sufficient to work on global sections over all affine open subschemes $U \subset X$. So the problem is reduced to Lemma 2.6.8.

The proof of part (4) is similar, but we need to invoke that given a short exact sequence of $\mathcal{O}_X$-modules

$$0 \to \mathcal{F}_!'^a \to \mathcal{F}_!^a \to \mathcal{F}_!''^a \to 0,$$

if two of these sheaves are quasi-coherent, so is the third one. This is proven in the affine case in [68, Tag 01IE], the general case reduces to the affine one. The rest of the argument is the same. ■

Definition 4.1.13. We define the categories $\mathbf{Mod}_X^{\mathrm{acoh}}$ (resp. $\mathbf{Mod}_X^{\mathrm{qc,acoh}}$, resp. $\mathbf{Mod}_{X^a}^{\mathrm{acoh}}$) as the full subcategory of $\mathbf{Mod}_X$ (resp. $\mathbf{Mod}_X$, resp. $\mathbf{Mod}_{X^a}$) consisting of the almost coherent $\mathcal{O}_X$-modules (resp. quasi-coherent almost coherent modules, resp. almost coherent almost $\mathcal{O}_X$-modules). As always, it is straightforward to see that the "almostification" functor induces the equivalence

$$\mathbf{Mod}_{X^a}^{\mathrm{acoh}} \simeq \mathbf{Mod}_X^{\mathrm{qc,acoh}} / \big(\Sigma_X \cap \mathbf{Mod}_X^{\mathrm{qc,acoh}}\big).$$

Moreover, Corollary 4.1.12 ensures that $\mathbf{Mod}_X^{\mathrm{acoh}} \subset \mathbf{Mod}_X$, $\mathbf{Mod}_X^{\mathrm{qc,acoh}} \subset \mathbf{Mod}_X$, and $\mathbf{Mod}_{X^a}^{\mathrm{acoh}} \subset \mathbf{Mod}_{X^a}$ are weak Serre subcategories.

The last thing that we discuss here is the notion of almost coherent schemes.

Definition 4.1.14. We say that an R-scheme X is *almost coherent* if the sheaf $\mathcal{O}_X$ is an almost coherent $\mathcal{O}_X$-module.

Lemma 4.1.15. *Let X be a coherent R-scheme. Then X is also almost coherent.*

Proof. The structure sheaf $\mathcal{O}_X$ is quasi-coherent by definition. Lemma 4.1.11 says that it suffices to show that $\mathcal{O}_X(U)$ is an almost coherent $\mathcal{O}_X(U)$-module for any open affine $U \subset X$. Since X is coherent, we conclude that $\mathcal{O}_X(U)$ is coherent as an $\mathcal{O}_X(U)$-module. Then Lemma 2.6.7 implies that it is actually almost coherent. ■

Lemma 4.1.16. *Let X be an almost coherent R-scheme. Then an $\mathcal{O}_X^a$-module $\mathcal{F}^a$ is almost coherent if and only if it is of almost finite presentation.*

Proof. The "only if" part is easy since any almost coherent $\mathcal{O}_X^a$-module is of almost finite presentation by definition. The "if" part is a local question, so we can assume that X is affine, then the claim follows from Lemma 2.6.14. ■

4.2 Schemes. Basic functors on almost coherent $\mathcal{O}_X^a$-modules

This section is devoted to study how certain functors defined in Section 3.2 interact with the notions of almost (quasi-) coherent $\mathcal{O}_X^a$-modules defined in the previous section.

As always, we fix a ring R with an ideal $\mathfrak{m}$ such that $\mathfrak{m} = \mathfrak{m}^2$ and $\widetilde{\mathfrak{m}} = \mathfrak{m} \otimes_R \mathfrak{m}$ is R-flat. We always do almost mathematics with respect to this ideal.

We start with the affine situation, i.e., $X = \operatorname{Spec} A$. In this case, we note that the functor $\widetilde{(-)}\colon \mathbf{Mod}_A \to \mathbf{Mod}_X^{\mathrm{qc}}$ sends almost zero A-modules to almost zero $\mathcal{O}_X$-modules. Thus, it induces the functor

$$\widetilde{(-)}\colon \mathbf{Mod}_{A^a} \to \mathbf{Mod}_{X^a}^{\mathrm{aqc}}.$$

Lemma 4.2.1. *Let $X = \operatorname{Spec} A$ be an affine R-scheme. Then for any $* \in \{\text{“ ”}, \mathrm{aft}, \mathrm{afp}, \mathrm{acoh}\}$, the functor $\widetilde{(-)}\colon \mathbf{Mod}_A \to \mathbf{Mod}_X^{\mathrm{qc}}$ induces equivalences $\widetilde{(-)}\colon \mathbf{Mod}_A^* \to \mathbf{Mod}_X^{\mathrm{qc},*}$. The quasi-inverse functor is given by $\Gamma(X, -)$.*

Proof. We note that the functor $\widetilde{(-)}\colon \mathbf{Mod}_A \to \mathbf{Mod}_X^{\mathrm{qc}}$ is an equivalence with the quasi-inverse $\Gamma(X, -)$. Now we note that Corollary 4.1.8 and Lemma 4.1.11 guarantee that a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ is almost finite type (resp. almost finitely presented, resp. almost coherent) if $\Gamma(X, \mathcal{F})$ is almost finitely generated (resp. almost finitely presented, resp. almost coherent) as an A-module. ■

Lemma 4.2.2. *Let $X = \operatorname{Spec} A$ be an affine R-scheme. Then for any $* \in \{\text{“ ”}, \mathrm{aft}, \mathrm{afp}, \mathrm{acoh}\}$, the functor $\widetilde{(-)}\colon \mathbf{Mod}_{A^a} \to \mathbf{Mod}_{X^a}^{\mathrm{aqc}}$ induces an equivalence $\widetilde{(-)}^a\colon \mathbf{Mod}_{A^a} \to \mathbf{Mod}_{X^a}^{\mathrm{aqc}}$ and restricts to further equivalences $\widetilde{(-)}^a\colon \mathbf{Mod}_{A^a}^* \to \mathbf{Mod}_{X^a}^*$. The quasi-inverse functor is given by $\Gamma(X, -)$.*

Proof. We note that $\widetilde{(-)}\colon \mathbf{Mod}_A \to \mathbf{Mod}_X^{\mathrm{qc}}$ induces an equivalence between almost zero A-modules and almost zero, quasi-coherent $\mathcal{O}_X$-modules. Thus, the claim follows from Lemma 4.2.1, Remark 4.1.3, Remark 4.1.5, Definition 4.1.13 and the analogous presentations of $\mathbf{Mod}_{A^a}^*$ as quotients of $\mathbf{Mod}_{A^a}$ for any $* \in \{\mathrm{aft}, \mathrm{afp}, \mathrm{acoh}\}$. ■

Now we show that the pullback functor preserves almost finite type and almost finitely presented $\mathcal{O}_X^a$-modules.

Lemma 4.2.3. *Let $f\colon X \to Y$ be a morphism of R-schemes.*

1. *Suppose that $X = \operatorname{Spec} B$, $Y = \operatorname{Spec} A$ are affine R-schemes. Then $f^*(\widetilde{M^a})$ is functorially isomorphic to $\widetilde{M^a \otimes_{A^a} B^a}$ for any $M^a \in \mathbf{Mod}_A^a$.*
2. *The functor f^* preserves almost quasi-coherence (resp. almost finite type, resp. almost finitely presented) for $\mathcal{O}$-modules.*
3. *The functor f^* preserves almost quasi-coherence (resp. almost finite type, resp. almost finitely presented) for $\mathcal{O}^a$-modules.*

Proof. (1) follows from Proposition 3.2.19 and the analogous result for quasi-coherent $\mathcal{O}_Y$-modules. More precisely, Proposition 3.2.19 provides us with the functorial isomorphism

$$f^*(\widetilde{M^a}) \simeq \left(f^*(\widetilde{M})\right)^a \simeq \left(\widetilde{M \otimes_A B}\right)^a \simeq \widetilde{M^a \otimes_{A^a} B^a}.$$

Now we note that (2) and (3) are local on X and Y, so we may and do assume that $X = \operatorname{Spec} B$, $Y = \operatorname{Spec} A$ are affine R-schemes. Furthermore, it clearly suffices to prove (2) as (3) follows formally from it.

Now Lemma 4.2.2 guarantees that any almost quasi-coherent $\mathcal{O}_X^a$-module is isomorphic to $\widetilde{M^a}$ for some A^a-module M^a. Furthermore, (1) ensures that $f^*(\widetilde{M^a}) \simeq \widetilde{M^a \otimes_{A^a} B^a}$ is an almost quasi-coherent $\mathcal{O}_X^a$-module. The other claims from (2) are proven similarly using Lemma 4.2.2 and Lemma 2.8.1. ■

Now we discuss that tensor product preserves certain finiteness properties of almost sheaves.

Lemma 4.2.4. *Let X be an R-scheme.*

1. *Suppose that $X = \operatorname{Spec} A$ is an affine R-scheme. Then $\widetilde{M^a} \otimes_{\mathcal{O}_X^a} \widetilde{N^a}$ is functorially isomorphic to $\widetilde{M^a \otimes_{A^a} N^a}$ for any $M^a, N^a \in \mathbf{Mod}_A^a$.*
2. *Let $\mathcal{F}^a, \mathcal{G}^a$ be two almost finite type (resp. almost finitely presented) $\mathcal{O}_X^a$-modules. Then the $\mathcal{O}_X^a$-module $\mathcal{F}^a \otimes_{\mathcal{O}_X^a} \mathcal{G}^a$ is almost finite type (resp. almost finitely presented). The analogous result holds for $\mathcal{O}_X$-modules $\mathcal{F}, \mathcal{G}$.*
3. *Let $\mathcal{F}^a$ be an almost coherent $\mathcal{O}_X^a$-module, and let $\mathcal{G}^a$ be an almost finitely presented $\mathcal{O}_X^a$-module. Then $\mathcal{F}^a \otimes_{\mathcal{O}_X^a} \mathcal{G}^a$ is an almost coherent $\mathcal{O}_X^a$-module. The analogous result holds for $\mathcal{O}_X$-modules $\mathcal{F}, \mathcal{G}$.*

Proof. The proof is similar to the proof of Lemma 4.2.3. The only difference is that one needs to use Proposition 3.2.12 in place of Proposition 3.2.19 to prove Part (1). Part (2) follows from Lemma 2.5.17, and Part (3) follows from Corollary 2.6.9. ■

We show that f_* preserves almost quasi-coherence of $\mathcal{O}^a$-modules for any quasi-compact and quasi-separated morphism f. Later on, we will be able to show that f_* also preserves almost coherence of $\mathcal{O}^a$-modules for certain proper morphisms.

Lemma 4.2.5. *Let $f: X \to Y$ be a quasi-compact and quasi-separated morphism of R-schemes.*

1. *The $\mathcal{O}_Y$-module $f_*(\mathcal{F})$ is almost quasi-coherent for any almost quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$.*
2. *The $\mathcal{O}_Y^a$-module $f_*(\mathcal{F}^a)$ is almost quasi-coherent for any almost quasi-coherent $\mathcal{O}_X^a$-module $\mathcal{F}^a$.*

Proof. Since $\mathcal{F}$ is almost quasi-coherent, we conclude that $\widetilde{\mathfrak{m}} \otimes \mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module. Thus $f_*(\widetilde{\mathfrak{m}} \otimes \mathcal{F})$ is a quasi-coherent $\mathcal{O}_Y$-module by [68, Tag 01LC]. Recall that the projection formula (Lemma 3.3.4) ensures that

$$f_*(\widetilde{\mathfrak{m}} \otimes \mathcal{F}) \simeq \widetilde{\mathfrak{m}} \otimes f_*\mathcal{F}.$$

Thus, we see that $\widetilde{\mathfrak{m}} \otimes f_*\mathcal{F} \simeq f_*(\mathcal{F}^a)_!$ is a quasi-coherent $\mathcal{O}_Y$-module. This shows that both $f_*(\mathcal{F})$ and $f_*(\mathcal{F}^a)$ are almost quasi-coherent over $\mathcal{O}_Y$ and $\mathcal{O}_Y^a$, respectively. This finishes the proof. ∎

Finally, we deal with the $\underline{\mathrm{Hom}}_{\mathcal{O}_X^a}(-,-)$ functor. We start with the following preparatory lemma:

Lemma 4.2.6. *Let X be an R-scheme.*

(1) *Suppose $X = \operatorname{Spec} A$ is an affine R-scheme. Then the canonical map*

$$\widetilde{\mathrm{Hom}_A(M, N)} \to \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\widetilde{M}, \widetilde{N}) \tag{4.2.1}$$

is an almost isomorphism of $\mathcal{O}_X$-modules for any almost finitely presented A-module M and any A-module N.

(2) *Suppose $X = \operatorname{Spec} A$ is an affine R-scheme. Then there is a functorial isomorphism*

$$\widetilde{\mathrm{alHom}_{A^a}(M^a, N^a)} \simeq \underline{\mathrm{alHom}}_{\mathcal{O}_X^a}(\widetilde{M^a}, \widetilde{N^a}) \tag{4.2.2}$$

of $\mathcal{O}_X^a$-modules for any almost finitely presented A^a-module M^a, and any A^a-module N^a. We also get a functorial almost isomorphism

$$\widetilde{\mathrm{Hom}_A(M, N)} \simeq^a \underline{\mathrm{Hom}}_{\mathcal{O}_X^a}(\widetilde{M^a}, \widetilde{N^a}) \tag{4.2.3}$$

of $\mathcal{O}_X$-modules for any almost finitely presented A-module M, and any A-module N.

(3) *Suppose $\mathcal{F}$ is an almost finitely presented $\mathcal{O}_X$-module and $\mathcal{G}$ is an almost quasi-coherent $\mathcal{O}_X$-module, then $\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is an almost quasi-coherent $\mathcal{O}_X$-module.*

(4) *Suppose $\mathcal{F}^a$ is an almost finitely presented $\mathcal{O}_X^a$-module and $\mathcal{G}^a$ is an almost quasi-coherent $\mathcal{O}_X^a$-module, then $\underline{\mathrm{Hom}}_{\mathcal{O}_X^a}(\mathcal{F}^a, \mathcal{G}^a)$ (resp. $\underline{\mathrm{alHom}}_{\mathcal{O}_X^a}(\mathcal{F}^a, \mathcal{G}^a)$) is an almost quasi-coherent $\mathcal{O}_X$-module (resp. $\mathcal{O}_X^a$-module).*

Proof. (1) Note the canonical morphism $\mathrm{Hom}_A(M, N) \to \mathrm{Hom}_{\mathcal{O}_X}(\widetilde{M}, \widetilde{N})$ for any A-modules M, N. This induces a morphism

$$\widetilde{\mathrm{Hom}_A(M, N)} \to \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\widetilde{M}, \widetilde{N}).$$

In order to show that it is an almost isomorphism for an almost finitely presented M, it suffices to show that the natural map

$$\mathrm{Hom}_A(M, N) \otimes_A A_f \to \mathrm{Hom}_{A_f}(M \otimes_A A_f, N \otimes_A A_f)$$

is an almost isomorphism for any $f \in A$. This follows from Lemma 2.9.11.

(2) follows easily from (1). Indeed, we apply the functorial isomorphism

$$\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})^a \simeq \underline{\mathrm{alHom}}_{\mathcal{O}_X^a}(\mathcal{F}^a, \mathcal{G}^a)$$

from Proposition 3.2.10 (2) to the almost isomorphism in Part (1) to get a functorial isomorphism

$$\widetilde{\mathrm{Hom}_A(M, N)}^a \simeq \underline{\mathrm{alHom}}_{\mathcal{O}_X^a}(\widetilde{M^a}, \widetilde{N^a}).$$

Now we use Proposition 2.2.1 (3) to get a functorial isomorphism

$$\mathrm{alHom}_{A^a}(M^a, N^a) \simeq \mathrm{Hom}_A(M, N)^a.$$

Applying the functor $\widetilde{(-)}$ to this isomorphism and composing it with the isomorphism above, we get a functorial isomorphism

$$\widetilde{\mathrm{alHom}_{A^a}(M^a, N^a)} \simeq \underline{\mathrm{alHom}}_{\mathcal{O}_X^a}(\widetilde{M^a}, \widetilde{N^a}).$$

The construction of the isomorphism (4.2.3) is similar and even easier.

(3) is a local question, so we can assume that $X = \mathrm{Spec}\, A$. We note that

$$\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \simeq^a \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\widetilde{\mathfrak{m}} \otimes \mathcal{F}, \widetilde{\mathfrak{m}} \otimes \mathcal{G})$$

by Proposition 3.2.10 (2). Thus, we can assume that both $\mathcal{F}$ and $\mathcal{G}$ are quasi-coherent. Then the claim follows from (1) and Lemma 4.2.1.

(4) is similarly just a consequence of (2) and Lemma 4.2.2. ∎

Corollary 4.2.7. *Let X be an R-scheme.*

(1) *Let $\mathcal{F}$ be an almost finitely presented $\mathcal{O}_X$-module, and let $\mathcal{G}$ be an almost coherent $\mathcal{O}_X$-module. Then $\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is an almost coherent $\mathcal{O}_X$-module.*

(2) *Let $\mathcal{F}^a$ be an almost finitely presented $\mathcal{O}_X^a$-module, and let $\mathcal{G}^a$ be an almost coherent $\mathcal{O}_X^a$-module. Then $\underline{\mathrm{Hom}}_{\mathcal{O}_X^a}(\mathcal{F}^a, \mathcal{G}^a)$ (resp. $\underline{\mathrm{alHom}}_{\mathcal{O}_X^a}(\mathcal{F}^a, \mathcal{G}^a)$) is an almost coherent $\mathcal{O}_X$-module (resp. $\mathcal{O}_X^a$-module).*

Proof. We start by observing that $\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \simeq^a \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\widetilde{\mathfrak{m}} \otimes \mathcal{F}, \widetilde{\mathfrak{m}} \otimes \mathcal{G})$ by Proposition 3.2.10 (2). Thus we can assume that both $\mathcal{F}$ and $\mathcal{G}$ are actually quasi-coherent. In that case we use Lemma 4.2.6 (1) and Lemma 4.1.11 to reduce the question to showing that $\underline{\mathrm{Hom}}_A(M, N)$ is almost coherent for any almost finitely presented M and almost coherent N. However, this has already been done in Corollary 2.6.9.

Part (2) follows from Part (1) thanks to the isomorphisms $\underline{\mathrm{Hom}}_{\mathcal{O}_X^a}(\mathcal{F}^a, \mathcal{G}^a) \simeq \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{F}_!^a, \mathcal{G})$ and $\underline{\mathrm{alHom}}_{\mathcal{O}_X^a}(\mathcal{F}^a, \mathcal{G}^a) \simeq \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})^a$. ∎

4.3 Schemes. Approximation of almost finitely presented $\mathcal{O}_X^a$-modules

One of the defects of our definition of almost finitely presented $\mathcal{O}_X$-modules is that it is (Zariski)-local on X; we require the existence of an approximation by finitely presented $\mathcal{O}_X$-modules only Zariski-locally on X. In particular, this definition is not well adapted to proving global statements such as the almost proper mapping theorem. We resolve this issue by showing that (on a quasi-compact quasi-separated scheme) any almost finitely presented $\mathcal{O}_X^a$-module can be *globally* approximated by finitely presented $\mathcal{O}_X$-modules.

As always, we fix a ring R with an ideal $\mathfrak{m}$ such that $\mathfrak{m} = \mathfrak{m}^2$ and $\widetilde{\mathfrak{m}} = \mathfrak{m} \otimes_R \mathfrak{m}$ is R-flat. We always do almost mathematics with respect to this ideal.

Lemma 4.3.1. *Let X be an R-scheme, and $\{\mathcal{G}_i^a\}_{i\in I}$ a filtered diagram of almost quasi-coherent $\mathcal{O}_X^a$-modules.*

(1) *The natural morphism*

$$\gamma^0_{\mathcal{F}} \colon \mathrm{colim}_I \, \underline{\mathrm{alHom}}_{\mathcal{O}_X}(\mathcal{F}^a, \mathcal{G}_i^a) \to \underline{\mathrm{alHom}}_{\mathcal{O}_X}(\mathcal{F}^a, \mathrm{colim}_I \, \mathcal{G}_i^a)$$

is injective for an almost finitely generated $\mathcal{O}_X^a$-module $\mathcal{F}^a$.

(2) *The natural morphism*

$$\gamma^0_{\mathcal{F}} \colon \mathrm{colim}_I \, \underline{\mathrm{alHom}}_{\mathcal{O}_X}(\mathcal{F}^a, \mathcal{G}_i^a) \to \underline{\mathrm{alHom}}_{\mathcal{O}_X}(\mathcal{F}^a, \mathrm{colim}_I \, \mathcal{G}_i^a)$$

is an almost isomorphism for an almost finitely presented $\mathcal{O}_X^a$-module $\mathcal{F}^a$.

Proof. The statement is local, so we can assume that $X = \mathrm{Spec}\, A$ is an affine scheme. Then Lemma 4.2.2 implies that $\mathcal{F}^a \simeq M^a$ and $\mathcal{G}_i^a \simeq N_i^a$ for an almost finitely generated (resp. almost finitely presented) A-module M. Then [68, Tag 009F] and Lemma 4.2.6 imply that it suffices to show that

$$\gamma^0_M \colon \mathrm{colim}_i \, \mathrm{alHom}_{A^a}(M^a, N_i^a) \to \mathrm{alHom}_{A^a}(M^a, \mathrm{colim}\, N_i^a)$$

is injective (resp. an isomorphism) in $\mathbf{Mod}_R^a$. But this is exactly Corollary 2.5.11. ∎

Corollary 4.3.2. *Let X be a quasi-compact and quasi-separated R-scheme, and $\{\mathcal{G}_i^a\}_{i\in I}$ a filtered diagram of almost quasi-coherent $\mathcal{O}_X^a$-modules.*

(1) *The natural morphism*

$$\gamma^0_{\mathcal{F}} \colon \mathrm{colim}_I \, \mathrm{alHom}_{\mathcal{O}_X}(\mathcal{F}^a, \mathcal{G}_i^a) \to \mathrm{alHom}_{\mathcal{O}_X}(\mathcal{F}^a, \mathrm{colim}_I \, \mathcal{G}_i^a)$$

is injective for an almost finitely generated $\mathcal{O}_X^a$-module $\mathcal{F}^a$.

(2) *The natural morphism*

$$\gamma^0_{\mathcal{F}} \colon \mathrm{colim}_I \, \mathrm{alHom}_{\mathcal{O}_X}(\mathcal{F}^a, \mathcal{G}_i^a) \to \mathrm{alHom}_{\mathcal{O}_X}(\mathcal{F}^a, \mathrm{colim}_I \, \mathcal{G}_i^a)$$

is an almost isomorphism for an almost finitely presented $\mathcal{O}_X^a$-module $\mathcal{F}^a$.

Proof. It formally follows from Lemma 3.2.25, Lemma 4.3.1, and [68, Tag 009F] (and Corollary 3.1.18). ∎

Definition 4.3.3. An $\mathcal{O}_X$-module $\mathcal{F}$ is *globally almost finitely generated* (resp. *globally almost finitely presented*) if, for every finitely generated ideal $\mathfrak{m}_0 \subset \mathfrak{m}$, there are a quasi-coherent finitely generated (resp. finitely presented) $\mathcal{O}_X$-module $\mathcal{G}$ and a morphism $f : \mathcal{G} \to \mathcal{F}$ such that $\mathfrak{m}_0(\operatorname{Ker} f) = 0$, $\mathfrak{m}_0(\operatorname{Coker} f) = 0$.

Lemma 4.3.4. *Let X be a quasi-compact and quasi-separated R-scheme, and $\mathcal{F}$ an almost adically quasi-coherent $\mathcal{O}_X$-module.*

(1) *If, for any filtered diagram of adically quasi-coherent $\mathcal{O}_X$-modules $\{\mathcal{G}_i\}_{i\in I}$, the natural morphism*

$$\operatorname{colim}_I \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}_i) \to \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}, \operatorname{colim}_I \mathcal{G}_i)$$

is almost injective, then $\mathcal{F}$ is globally almost finitely generated.

(2) *If, for any filtered system of adically quasi-coherent $\mathcal{O}_X$-modules $\{\mathcal{G}_i\}_{i\in I}$, the natural morphism*

$$\operatorname{colim}_I \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}_i) \to \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}, \operatorname{colim}_I \mathcal{G}_i)$$

is an almost isomorphism, then $\mathcal{F}$ is globally almost finitely presented.

Proof. Lemma 3.2.25 and Corollary 3.1.18 ensure that we can replace $\mathcal{F}$ with $\mathcal{F}_!{}^a$ without loss of generality. So, we may and do assume that $\mathcal{F}$ is quasi-coherent. Then the proof of Lemma 2.5.10 works essentially verbatim. We repeat it for the reader's convenience.

(1) Note that $\mathcal{F} \simeq \operatorname{colim}_I \mathcal{F}_i$ is a filtered colimit of its finitely generated $\mathcal{O}_X$-*submodules* (see [68, Tag 01PG]). Therefore, we see that

$$\operatorname{colim}_I \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F}/\mathcal{F}_i) \simeq^a \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}, \operatorname{colim}_I(\mathcal{F}/\mathcal{F}_i)) \simeq 0.$$

Consider an element α of the colimit that has a representative the quotient morphism $\mathcal{F} \to \mathcal{F}/\mathcal{F}_i$ (for some choice of i). Then, for every $\varepsilon \in \mathfrak{m}$, we have $\varepsilon\alpha = 0$ in $\operatorname{colim}_I \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F}/\mathcal{F}_i)$. Explicitly this means that there is $j \geq i$ such that $\varepsilon\mathcal{F} \subset \mathcal{F}_j$. Now note that this property is preserved by replacing j with any $j' > j$. Therefore, for any $\mathfrak{m}_0 = (\varepsilon_1, \dots, \varepsilon_n)$, we can find a finitely generated $\mathcal{O}_X$-submodule $\mathcal{F}_i \subset \mathcal{F}$ such that $\mathfrak{m}_0\mathcal{F} \subset \mathcal{F}_i$. Therefore, $\mathcal{F}$ is almost finitely generated.

(2) Fix any finitely generated sub-ideal $\mathfrak{m}_0 = (\varepsilon_1, \dots, \varepsilon_n) \subset \mathfrak{m}$. We use [68, Tag 01PJ] to write $\mathcal{F} \simeq \operatorname{colim}_\Lambda \mathcal{F}_\lambda$ as a filtered colimit of *finitely presented* $\mathcal{O}_X$-modules. By assumption, the natural morphism

$$\operatorname{colim}_\Lambda \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F}_\lambda) \to \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}, \operatorname{colim}_\Lambda \mathcal{F}_\lambda) = \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F})$$

is an almost isomorphism. In particular, $\varepsilon_i \mathrm{id}_{\mathcal{F}}$ lies in the image of this map for every $i = 1, \dots, n$. This means that, for every ε_i, there are $\lambda_i \in \Lambda$ and a morphism $g_i : \mathcal{F} \to \mathcal{F}_{\lambda_i}$ such that the composition satisfies

$$f_{\lambda_i} \circ g_i = \varepsilon_i \mathrm{id}_{\mathcal{F}},$$

where $f_{\lambda_i} : \mathcal{F}_{\lambda_i} \to \mathcal{F}$ is the natural morphism to the colimit. Note that the existence of such a g_i is preserved by replacing λ_i with any $\lambda_i' \geq \lambda_i$. Therefore, using that $\{\mathcal{F}_\lambda\}$ is a filtered diagram, we can find an index λ with maps

$$g_i : \mathcal{F} \to \mathcal{F}_\lambda$$

such that $f_\lambda \circ g_i = \varepsilon_i \mathrm{id}_{\mathcal{F}}$. Now we consider the morphism

$$G_i := g_i \circ f_\lambda - \varepsilon_i \mathrm{id}_{\mathcal{F}_\lambda} : \mathcal{F}_\lambda \to \mathcal{F}_\lambda.$$

Note that $\mathrm{Im}(G_i) \subset \mathrm{Ker}(f_\lambda)$ because

$$f_\lambda \circ g_i \circ f_\lambda - f_\lambda \varepsilon_i \mathrm{id}_{\mathcal{F}_\lambda} = \varepsilon_i f_\lambda - \varepsilon_i f_\lambda = 0.$$

We also have that $\varepsilon_i \,\mathrm{Ker}(f_\lambda) \subset \mathrm{Im}(G_i)$ because $G_i|_{\mathrm{Ker}(f_\lambda)} = \varepsilon_i \mathrm{id}$. So, $\sum_i \mathrm{Im}(G_i)$ is a quasi-coherent finitely generated $\mathcal{O}_X$-module such that

$$\mathfrak{m}_0(\mathrm{Ker}\, f_\lambda) \subset \sum_i \mathrm{Im}(G_i) \subset \mathrm{Ker}(f_\lambda).$$

Therefore, $f : \mathcal{F}' := \mathcal{F}_\lambda / (\sum_i \mathrm{Im}(G_i)) \to \mathcal{F}$ is a morphism such that $\mathcal{F}'$ is finitely presented, $\mathfrak{m}_0(\mathrm{Ker}\, f) = 0$, and $\mathfrak{m}_0(\mathrm{Coker}\, f) = 0$. Since $\mathfrak{m}_0 \subset \mathfrak{m}$ was an arbitrary finitely generated sub-ideal, we infer that $\mathcal{F}$ is globally almost finitely presented. ∎

Corollary 4.3.5. *Let X be a quasi-compact and quasi-separated R-scheme, and let $\mathcal{F}$ be an almost quasi-coherent $\mathcal{O}_X$-module. Then $\mathcal{F}$ is almost finitely presented (resp. almost finitely generated) if and only if for any finitely generated ideal $\mathfrak{m}_0 \subset \mathfrak{m}$ there is a morphism $f : \mathcal{G} \to \mathcal{F}$ such that $\mathcal{G}$ is a quasi-coherent finitely presented (resp. finitely generated) $\mathcal{O}_X$-module, $\mathfrak{m}_0(\mathrm{Ker}\, f) = 0$ and $\mathfrak{m}_0(\mathrm{Coker}\, f) = 0$.*

Proof. Corollary 4.3.2 ensures that $\mathcal{F}$ satisfies the conditions of Lemma 4.3.4. Now, Lemma 4.3.4 gives the desired result. ∎

Corollary 4.3.6. *Let X be a quasi-compact and quasi-separated R-scheme, and $\mathcal{F}^a$ an almost quasi-coherent $\mathcal{O}_X^a$-module.*

(1) *$\mathcal{F}^a$ is almost finitely generated if and only if, for every filtered diagram $\{\mathcal{G}_i^a\}_{i \in I}$ of almost quasi-coherent $\mathcal{O}_X^a$-modules, the natural morphism*

$$\mathrm{colim}_I \,\mathrm{alHom}_{\mathcal{O}_X^a}(\mathcal{F}^a, \mathcal{G}_i^a) \to \mathrm{alHom}_{\mathcal{O}_X^a}(\mathcal{F}^a, \mathrm{colim}_I \,\mathcal{G}_i^a)$$

is injective in $\mathbf{Mod}_R^a$.

(2) $\mathcal{F}^a$ *is almost finitely presented if and only if, for every filtered diagram* $\{\mathcal{G}_i^a\}_{i\in I}$ *of almost quasi-coherent* $\mathcal{O}_X^a$*-modules, the natural morphism*

$$\mathrm{colim}_I\ \mathrm{alHom}_{\mathcal{O}_X^a}(\mathcal{F}^a, \mathcal{G}_i^a) \to \mathrm{alHom}_{\mathcal{O}_X^a}(\mathcal{F}^a, \mathrm{colim}_I\ \mathcal{G}_i^a)$$

is an isomorphism in $\mathbf{Mod}_R^a$.

4.4 Schemes. Derived category of almost coherent $\mathcal{O}_X^a$-modules

The goal of this section is to define different versions of the "derived category of almost coherent sheaves". Namely, we define the categories $\mathbf{D}_{\mathrm{acoh}}(X)$, $\mathbf{D}_{\mathrm{qc,acoh}}(X)$, and $\mathbf{D}_{\mathrm{acoh}}(X)^a$. Then we show that many functors of interest preserve almost coherence in an appropriate sense.

Definition 4.4.1. We define $\mathbf{D}_{\mathrm{aqc}}(X)$ (resp. $\mathbf{D}_{\mathrm{aqc}}(X)^a$) to be the full triangulated subcategory of $\mathbf{D}(X)$ (resp. $\mathbf{D}(X)^a$) consisting of the complexes with almost quasi-coherent cohomology sheaves.

Definition 4.4.2. We define $\mathbf{D}_{\mathrm{acoh}}(X)$ (resp. $\mathbf{D}_{\mathrm{qc,acoh}}(X)$, resp. $\mathbf{D}_{\mathrm{acoh}}(X)^a$) to be the full triangulated subcategory of $\mathbf{D}(X)$ (resp. $\mathbf{D}(X)$, resp. $\mathbf{D}(X)^a$) consisting of the complexes with almost coherent (resp. quasi-coherent and almost coherent, resp. almost coherent) cohomology sheaves.

Remark 4.4.3. Definition 4.4.2 makes sense as the categories $\mathbf{Mod}_X^{\mathrm{acoh}}$, $\mathbf{Mod}_X^{\mathrm{qc,acoh}}$, and $\mathbf{Mod}_{X^a}^{\mathrm{acoh}}$ are weak Serre subcategories of $\mathbf{Mod}_X$, $\mathbf{Mod}_X$, and $\mathbf{Mod}_X^a$ respectively.

Now suppose that $X = \mathrm{Spec}\, A$ is an affine R-scheme. Then we note that the functor

$$\widetilde{(-)}\colon \mathbf{Mod}_A \to \mathbf{Mod}_X$$

is additive and exact, thus it can be easily derived to the functor

$$\widetilde{(-)}\colon \mathbf{D}(A) \to \mathbf{D}_{\mathrm{qc}}(X).$$

Lemma 4.4.4. *Let* $X = \mathrm{Spec}\, A$ *be an affine* R*-scheme. Then the functor*

$$\widetilde{(-)}\colon \mathbf{D}(A) \to \mathbf{D}_{\mathrm{qc}}(X)$$

is a t*-exact equivalence of triangulated categories*[2] *with the quasi-inverse given by* $\mathbf{R}\Gamma(X,-)$. *Moreover, these two functors induce the equivalence*

$$\widetilde{(-)}\colon \mathbf{D}^*_{\mathrm{acoh}}(A) \longleftrightarrow \mathbf{D}^*_{\mathrm{qc,acoh}}(X)\colon \mathbf{R}\Gamma(X,-)$$

for any $* \in \{\text{“ ”}, +, -, b\}$.

[2] Meant with respect to the standard t-structures.

Proof. The first part is just [68, Tag 06Z0]. In particular, it shows the isomorphism $\mathrm{H}^i(\mathbf{R}\Gamma(X, \mathcal{F})) \simeq \mathrm{H}^0(X, \mathcal{H}^i(\mathcal{F}))$ for any $\mathcal{F} \in \mathbf{D}_{qc}(X)$. Now Lemma 4.1.11 implies that $\mathcal{H}^i(\mathcal{F})$ is almost coherent if and only if so is $\mathrm{H}^0(X, \mathcal{H}^i(\mathcal{F}))$. So the functor $\mathbf{R}\Gamma(X, -)$ sends $\mathbf{D}^*_{\mathrm{qc,acoh}}(X)$ to $\mathbf{D}^*_{\mathrm{acoh}}(A)$.

We also observe that the functor $\widetilde{(-)}$ clearly sends $\mathbf{D}_{\mathrm{acoh}}(A)$ to $\mathbf{D}_{\mathrm{qc,acoh}}(X)$. Thus, we conclude that $\widetilde{(-)}$ and $\mathbf{R}\Gamma(X, -)$ induce an equivalence between $\mathbf{D}_{\mathrm{acoh}}(A)$ and $\mathbf{D}_{\mathrm{qc,acoh}}(X)$. The bounded versions follow from t-exactness of both functors. ∎

Lemma 4.4.5. *Let $X = \operatorname{Spec} A$ be an affine R-scheme. Then the almostification functor*

$$(-)^a \colon \mathbf{D}^*_{\mathrm{qc}}(X) \to \mathbf{D}^*_{\mathrm{aqc}}(X)^a$$

induces an equivalence $\mathbf{D}^*_{\mathrm{qc}}(X)/\mathbf{D}^*_{\mathrm{qc},\Sigma_X}(X) \xrightarrow{\sim} \mathbf{D}^*_{\mathrm{aqc}}(X)^a$ for any $* \in \{\text{“ ”}, +, -, b\}$. Similarly, the induced functor

$$\mathbf{D}^*_{\mathrm{qc,acoh}}(X)/\mathbf{D}^*_{\mathrm{qc},\Sigma_X}(X) \xrightarrow{\sim} \mathbf{D}^*_{\mathrm{acoh}}(X)^a$$

is an equivalence for any $* \in \{\text{“ ”}, +, -, b\}$.

Proof. The functor $(-)_! \colon \mathbf{D}^*_{\mathrm{aqc}}(X)^a \to \mathbf{D}^*_{\mathrm{qc}}(X)$ gives the left adjoint to $(-)^a$ such that $\mathrm{id} \to (-)_! \circ (-)^a$ is an isomorphism and the kernel of $(-)^a$ consists exactly of the morphisms f such that $\mathrm{cone}(f) \in \mathbf{D}_{\mathrm{qc},\Sigma_X}(X)$. Thus, the dual version of [27, Proposition 1.3] finishes the proof of the first claim. The proof of the second claim is similar once one notices that $\widetilde{M^a}$ is almost coherent for any almost coherent A^a-module M^a. The latter fact follows from Lemma 4.1.11. ∎

Lemma 4.1.11 ensures that $\mathbf{D}(A)^a \simeq \mathbf{D}(A)/\mathbf{D}_{\Sigma_A}(A)$. Since $\widetilde{(-)}$ clearly sends $\mathbf{D}_{\Sigma_A}(A)$ to $\mathbf{D}^*_{\mathrm{qc},\Sigma_X}(X)$, we conclude that it induces a functor

$$\widetilde{(-)} \colon \mathbf{D}^*(A)^a \to \mathbf{D}^*_{\mathrm{aqc}}(X)^a.$$

Theorem 4.4.6. *Let $X = \operatorname{Spec} A$ be an affine R-scheme. Then the functor*

$$\widetilde{(-)} \colon \mathbf{D}(A)^a \to \mathbf{D}_{\mathrm{aqc}}(X)^a$$

is a t-exact equivalence of triangulated categories with the quasi-inverse given by $\mathbf{R}\Gamma(X, -)$. Moreover, these two functors induce equivalences

$$\widetilde{(-)} \colon \mathbf{D}^*_{\mathrm{acoh}}(A)^a \rightleftarrows \mathbf{D}^*_{\mathrm{acoh}}(X)^a \colon \mathbf{R}\Gamma(X, -)$$

for any $* \in \{\text{“ ”}, +, -, b\}$.

Proof. We note that Lemma 4.4.4 ensures that $\widetilde{(-)}\colon \mathbf{D}^*_{\mathrm{qc,acoh}}(X) \to \mathbf{D}^*_{\mathrm{acoh}}(X)^a$ is an equivalence with quasi-inverse $\mathbf{R}\Gamma(X,-)$. Moreover, $(-)^a$ induces an equivalence between $\mathbf{D}_{\Sigma_A}(A)$ and $\mathbf{D}_{\mathrm{qc},\Sigma_X}(X)$; we leave the verification to the interested reader. Thus, Lemma 4.4.5 ensures that $\widetilde{(-)}$ gives an equivalence

$$\mathbf{D}(A)^a \simeq \mathbf{D}(A)/\mathbf{D}_{\Sigma_A}(A) \xrightarrow{\sim} \mathbf{D}_{\mathrm{qc}}(X)/\mathbf{D}_{\mathrm{qc},\Sigma_X}(X) \simeq \mathbf{D}_{\mathrm{aqc}}(X)^a.$$

Its quasi-inverse is given by the functor $\mathbf{R}\Gamma(X,-)\colon \mathbf{D}_{\mathrm{aqc}}(X)^a \to \mathbf{D}(A)^a$ by Proposition 3.5.23.

The version with almost coherent cohomology sheaves is similar to the analogous statement from Lemma 4.4.4. ∎

Lemma 4.4.7. *Let $f\colon X \to Y$ be a morphism of R-schemes.*

(1) *Suppose that both $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$ are affine R-schemes. Then $\mathbf{L}f^*(\widetilde{M^a})$ is functorially isomorphic to $\widetilde{M^a \otimes^L_{A^a} B^a}$ for any $M^a \in \mathbf{D}(A)^a$.*

(2) *The functor $\mathbf{L}f^*$ carries an object of $\mathbf{D}^*_{\mathrm{aqc}}(Y)$ to an object of $\mathbf{D}^*_{\mathrm{aqc}}(X)$ for $* \in \{\text{“ ”}, -\}$.*

(3) *The functor $\mathbf{L}f^*$ carries an object of $\mathbf{D}^*_{\mathrm{aqc}}(Y)^a$ to an object of $\mathbf{D}^*_{\mathrm{aqc}}(X)^a$ for $* \in \{\text{“ ”}, -\}$.*

(4) *Suppose that X and Y are almost coherent R-schemes. Then the functor $\mathbf{L}f^*$ carries an object of $\mathbf{D}^-_{\mathrm{qc,acoh}}(Y)$ (resp. $\mathbf{D}^-_{\mathrm{acoh}}(Y)$) to an object of $\mathbf{D}^-_{\mathrm{qc,acoh}}(X)$ (resp. $\mathbf{D}^-_{\mathrm{acoh}}(X)$).*

(5) *Suppose that X and Y are almost coherent R-schemes. Then the functor $\mathbf{L}f^*$ carries an object of $\mathbf{D}^-_{\mathrm{acoh}}(Y)^a$ to an object of $\mathbf{D}^-_{\mathrm{acoh}}(X)^a$.*

Proof. We start with Part (1). We use Proposition 3.5.20 to see the isomorphism $\mathbf{L}f^*\big(\widetilde{M^a}\big) \simeq \big(\mathbf{L}f^*\big(\widetilde{M}\big)\big)^a$. Proposition 2.4.16 says that $\big(\widetilde{M \otimes^L_A B}\big)^a \simeq \widetilde{M^a \otimes^L_{A^a} B^a}$, so it suffices to show $\mathbf{L}f^*\big(\widetilde{M}\big) \simeq \widetilde{M \otimes^L_A B}$. But this is a classical fact about quasi-coherent sheaves.

Now we show (2). We note that Lemma 3.2.17 implies that $\mathbf{L}f^*(\widetilde{\mathfrak{m}} \otimes \mathcal{F}) \simeq \widetilde{\mathfrak{m}} \otimes \mathbf{L}f^*(\mathcal{F})$ for any $\mathcal{F} \in \mathbf{D}(Y)$. Thus, we can replace $\mathcal{F}$ with $\widetilde{\mathfrak{m}} \otimes \mathcal{F}$ to assume that it is quasi-coherent. Then it is a standard fact that $\mathbf{L}f^*$ sends $\mathbf{D}^*_{\mathrm{qc}}(Y)$ to $\mathbf{D}^*_{\mathrm{qc}}(X)$ for $* \in \{\text{“ ”}, -\}$.

(3) follows from Part (2) by noting that $\mathbf{L}f^*(\mathcal{F}^a) \simeq (\mathbf{L}f^*(\mathcal{F}^a_!))^a$ according to Proposition 3.5.20.

To prove (4), we use again the isomorphism $\mathbf{L}f^*(\widetilde{\mathfrak{m}} \otimes \mathcal{F}) \simeq \widetilde{\mathfrak{m}} \otimes \mathbf{L}f^*(\mathcal{F})$ to assume that $\mathcal{F}$ is in $\mathbf{D}^-_{\mathrm{qc,acoh}}(X)$. Then Lemma 4.4.4 guarantees that there exists $M \in \mathbf{D}^-_{\mathrm{coh}}(A)$ such that $\widetilde{M} \simeq \mathcal{F}$. Thus Part (1) and Lemma 4.1.11 ensure that it is sufficient to show that $M^a \otimes^L_{A^a} B^a \simeq (M \otimes^L_A B)^a$ has almost finitely presented cohomology modules. This is exactly the content of Corollary 2.8.2.

(5) follows from (4) as $\mathbf{L}f^*(\mathcal{F}^a) \simeq (\mathbf{L}f^*(\mathcal{F}_!^a))^a$. ■

Lemma 4.4.8. *Let X be an R-scheme.*

(1) *Suppose that $X = \operatorname{Spec} A$ is an affine R-scheme. Then $\widetilde{M^a} \otimes^L_{\mathcal{O}_X^a} \widetilde{N^a}$ is functorially isomorphic to $\widetilde{M^a \otimes^L_{A^a} N^a}$ for any $M^a, N^a \in \mathbf{D}(A)^a$.*

(2) *Let $\mathcal{F}, \mathcal{G} \in \mathbf{D}^*_{\mathrm{aqc}}(X)$, then $\mathcal{F} \otimes^L_{\mathcal{O}_X} \mathcal{G} \in \mathbf{D}_{\mathrm{aqc}}(X)$ for $* \in \{\text{“ ”}, -\}$.*

(3) *Let $\mathcal{F}^a, \mathcal{G}^a \in \mathbf{D}^*_{\mathrm{aqc}}(X)^a$, then $\mathcal{F}^a \otimes^L_{\mathcal{O}_X^a} \mathcal{G}^a \in \mathbf{D}_{\mathrm{aqc}}(X)^a$ for $* \in \{\text{“ ”}, -\}$.*

(4) *Suppose that X is an almost coherent R-scheme, and let $\mathcal{F}, \mathcal{G} \in \mathbf{D}^-_{\mathrm{qc,acoh}}(X)$ (resp. $\mathbf{D}^-_{\mathrm{acoh}}(X)$). Then $\mathcal{F} \otimes^L_{\mathcal{O}_X} \mathcal{G} \in \mathbf{D}^-_{\mathrm{qc,acoh}}(X)$ (resp. $\mathbf{D}^-_{\mathrm{acoh}}(X)$).*

(5) *Suppose that X is an almost coherent R-scheme, and let $\mathcal{F}^a, \mathcal{G}^a \in \mathbf{D}^-_{\mathrm{acoh}}(X)^a$. Then $\mathcal{F}^a \otimes^L_{\mathcal{O}_X^a} \mathcal{G}^a \in \mathbf{D}^-_{\mathrm{acoh}}(X)^a$.*

Proof. The proof is basically identical to that of Lemma 4.4.7 and is left to the reader. We only mention that one has to use Proposition 2.6.18 in place of Corollary 2.8.2. ■

Lemma 4.4.9. *Let $f : X \to Y$ be a quasi-compact and quasi-separated morphism of R-schemes. Suppose that Y is quasi-compact.*

(1) *The functor $\mathbf{R}f_*$ carries $\mathbf{D}^*_{\mathrm{aqc}}(X)$ to $\mathbf{D}^*_{\mathrm{aqc}}(Y)$ for any $* \in \{\text{“ ”}, -, +, b\}$.*

(2) *The functor $\mathbf{R}f_*$ carries $\mathbf{D}^*_{\mathrm{aqc}}(X)^a$ to $\mathbf{D}^*_{\mathrm{aqc}}(Y)^a$ for any $* \in \{\text{“ ”}, -, +, b\}$.*

Proof. Proposition 3.5.23 tells us that $(\mathbf{R}f_*\mathcal{F})^a \simeq \mathbf{R}f_*\mathcal{F}^a$. Since $(\widetilde{\mathfrak{m}} \otimes \mathcal{F})^a \simeq \mathcal{F}^a$, we see that it suffices to show that the functor

$$\mathbf{R}f_*(\widetilde{\mathfrak{m}} \otimes -)$$

carries $\mathbf{D}^*_{\mathrm{aqc}}(X)$ to $\mathbf{D}^*_{\mathrm{aqc}}(Y)$ for any $* \in \{\text{“ ”}, -, +, b\}$. Since $\widetilde{\mathfrak{m}} \otimes \mathcal{F}$ is in $\mathbf{D}_{\mathrm{qc}}(X)$, we conclude that it is enough to show that $\mathbf{R}f_*(-)$ carries $\mathbf{D}^*_{\mathrm{qc}}(X)$ to $\mathbf{D}^*_{\mathrm{qc}}(Y)$ for any $* \in \{\text{“ ”}, -, +, b\}$. This is proven in [68, Tag 08D5]. ■

Before going to the case of the derived Hom-functors, we recall the construction of the functorial map

$$\psi : \widetilde{\mathbf{R}\mathrm{Hom}_A(M, N)} \to \mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{O}_X}\left(\widetilde{M}, \widetilde{N}\right)$$

for any $M \in \mathbf{D}^-(A)$, $N \in \mathbf{D}^+(A)$, and an affine scheme $X = \operatorname{Spec} A$. For this, we recall that the functor $\widetilde{(-)}$ is left adjoint to the global section functor $\Gamma(X, -)$ on the abelian level. Thus, after deriving these functors, we see that $\widetilde{(-)}$ is left adjoint to $\mathbf{R}\Gamma(X, -)$. Thus it follows that, for any $\mathcal{F} \in \mathbf{D}(X)$, there is a canonical morphism $\widetilde{\mathbf{R}\Gamma(X, \mathcal{F})} \to \mathcal{F}$. We apply it to $\mathcal{F} = \mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{O}_X}\left(\widetilde{M}, \widetilde{N}\right)$ to obtain the desired morphism

$$\psi : \widetilde{\mathbf{R}\mathrm{Hom}_A(M, N)} \to \mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{O}_X}\left(\widetilde{M}, \widetilde{N}\right).$$

Lemma 4.4.10. *Let X be an almost coherent R-scheme.*

(1) *Suppose* $X = \operatorname{Spec} A$ *is an affine* R*-scheme. The canonical map*

$$\psi : \widetilde{\mathbf{R}\mathrm{Hom}_A(M, N)} \to \mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{O}_X}\left(\widetilde{M}, \widetilde{N}\right)$$

is an almost isomorphism for $M \in \mathbf{D}^-_{\mathrm{acoh}}(A)$, $N \in \mathbf{D}^+(A)$.

(2) *Suppose* $X = \operatorname{Spec} A$ *is an affine* R*-scheme. There is a functorial isomorphism*

$$\widetilde{\mathbf{R}\mathrm{alHom}_{A^a}(M^a, N^a)} \simeq \mathbf{R}\underline{\mathrm{alHom}}_{\mathcal{O}_X^a}\left(\widetilde{M^a}, \widetilde{N^a}\right)$$

for $M^a \in \mathbf{D}^-_{\mathrm{acoh}}(A)^a$, $N^a \in \mathbf{D}^+(A)^a$. *We also get a functorial almost isomorphism*

$$\widetilde{\mathbf{R}\mathrm{Hom}_{A^a}(M^a, N^a)} \simeq^a \mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{O}_X^a}\left(\widetilde{M^a}, \widetilde{N^a}\right)$$

for $M \in \mathbf{D}^-_{\mathrm{acoh}}(A)$, $N \in \mathbf{D}^+(A)$.

(3) *Suppose* $\mathcal{F} \in \mathbf{D}^-_{\mathrm{acoh}}(X)$ *and* $\mathcal{G} \in \mathbf{D}^+_{\mathrm{aqc}}(X)$. *Then* $\mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \in \mathbf{D}^+_{\mathrm{aqc}}(X)$.

(4) *Suppose that* $\mathcal{F}^a \in \mathbf{D}^-_{\mathrm{acoh}}(X)^a$ *and* $\mathcal{G}^a \in \mathbf{D}^+_{\mathrm{aqc}}(X)^a$. *In this case we have both* $\mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{O}_X^a}(\mathcal{F}^a, \mathcal{G}^a) \in \mathbf{D}^+_{\mathrm{aqc}}(X)$ *and* $\mathbf{R}\underline{\mathrm{alHom}}_{\mathcal{O}_X^a}(\mathcal{F}^a, \mathcal{G}^a) \in \mathbf{D}^+_{\mathrm{aqc}}(X)^a$.

Proof. We start with (1). We use the convergent compatible spectral sequences

$$\mathrm{E}_2^{p,q} = \widetilde{\mathrm{Ext}_A^p(\mathrm{H}^{-q}(M), N)} \Rightarrow \widetilde{\mathrm{Ext}_A^{p+q}(M, N)}$$
$$\mathrm{E'}_2^{p,q} = \underline{\mathrm{Ext}}^p_{\mathcal{O}_X}\left(\widetilde{\mathrm{H}^{-q}(M)}, \widetilde{N}\right) \Rightarrow \underline{\mathrm{Ext}}^{p+q}_{\mathcal{O}_X}\left(\widetilde{M}, \widetilde{N}\right)$$

to reduce to the case when $M \in \mathbf{Mod}_A^{\mathrm{acoh}}$ is an A-module concentrated in degree 0. Similarly, we use the compatible spectral sequences

$$\mathrm{E}_2^{p,q} = \widetilde{\mathrm{Ext}_A^q(M, \mathrm{H}^p(N))} \Rightarrow \widetilde{\mathrm{Ext}_A^{p+q}(M, N)}$$
$$\mathrm{E'}_2^{p,q} = \underline{\mathrm{Ext}}^q_{\mathcal{O}_X}\left(\widetilde{M}, \widetilde{\mathrm{H}^p(N)}\right) \Rightarrow \underline{\mathrm{Ext}}^{p+q}_{\mathcal{O}_X}\left(\widetilde{M}, \widetilde{N}\right)$$

to assume that $N \in \mathbf{Mod}_A$. Thus, the claim boils down to showing that the natural map

$$\widetilde{\mathrm{Ext}_A^p(M, N)} \to \underline{\mathrm{Ext}}^p_{\mathcal{O}_X}\left(\widetilde{M}, \widetilde{N}\right)$$

is an almost isomorphism for any $M \in \mathbf{Mod}_A^{\mathrm{acoh}}$, $N \in \mathbf{Mod}_A$, and $p \geq 0$. Lemma 3.1.5 says that it is sufficient to show that the kernel and cokernel are annihilated by any finitely generated sub-ideal $\mathfrak{m}_0 \subset \mathfrak{m}$.

Recall that, for any $\mathcal{O}_X$-modules $\mathcal{F}$, $\mathcal{G}$, the sheaf $\underline{\mathrm{Ext}}^p_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is canonically isomorphic to the sheafification of the presheaf

$$U \mapsto \mathrm{Ext}^p_{\mathcal{O}_U}(\mathcal{F}|_U, \mathcal{G}|_U).$$

Thus, in order to show that the map $\widetilde{\mathrm{Ext}^p_A(M,N)} \to \underline{\mathrm{Ext}}^p_{\mathcal{O}_X}(\widetilde{M}, \widetilde{N})$ is an almost isomorphism, it suffices to show that

$$\mathrm{Ext}^p_A(M,N) \otimes_A A_f \to \mathrm{Ext}^p_{\mathcal{O}_{X_f}}(\widetilde{M_f}, \widetilde{N_f})$$

is an almost isomorphism. Now we use canonical isomorphisms

$$\begin{aligned}\mathrm{Ext}^p_{\mathcal{O}_{X_f}}(\widetilde{M_f}, \widetilde{N_f}) &\simeq \mathrm{Hom}_{\mathbf{D}(X_f)}(\widetilde{M_f}, \widetilde{N_f}[p]) \\ &\simeq \mathrm{Hom}_{\mathbf{D}(A_f)}(M_f, N_f[p]) \\ &\simeq \mathrm{Ext}^p_{A_f}(M_f, N_f),\end{aligned}$$

where the second isomorphism follows using that $\widetilde{(-)}$ induces a t-exact equivalence $\widetilde{(-)}\colon \mathbf{D}(A_f) \to \mathbf{D}_{\mathrm{qc}}(\mathrm{Spec}\, A_f)$. Thus, the question boils down to showing that the natural map

$$\mathrm{Ext}^p_A(M,N) \otimes_A A_f \to \mathrm{Ext}^p_{A_f}(M_f, N_f)$$

is an almost isomorphism. This follows from Lemma 2.9.12.

(2) formally follows from (1) by using Proposition 3.5.8 (1).

(3) is also a basic consequence of (2). Indeed, the claim is local, so we can assume that $X = \mathrm{Spec}\, A$ is an affine R-scheme. In that case, we use Theorem 4.4.6 to say that $\mathcal{F} \simeq \widetilde{M}$, $\mathcal{G} \simeq \widetilde{N}$ for some $M \in \mathbf{D}^-_{\mathrm{acoh}}(A)$, $N \in \mathbf{D}^+(A)$. Then $\mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \simeq \widetilde{\mathbf{R}\mathrm{Hom}_A(M,N)}$ by (2), and the latter complex has almost quasi-coherent cohomology sheaves by design.

(4) easily follows from (3) and the isomorphisms

$$\begin{aligned}\mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{O}_X^a}(\mathcal{F}^a, \mathcal{G}^a) &\simeq \mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{F}^a_!, \mathcal{G}) \\ \mathbf{R}\underline{\mathrm{alHom}}_{\mathcal{O}_X^a}(\mathcal{F}^a, \mathcal{G}^a) &\simeq \mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{F}^a_!, \mathcal{G})^a\end{aligned}$$

that come from Lemma 3.5.5 (1) and Definition 3.5.6. ■

Corollary 4.4.11. *Let X be an almost coherent R-scheme.*

(1) *Let $\mathcal{F} \in \mathbf{D}^-_{\mathrm{aqc,acoh}}(X)$, $\mathcal{G} \in \mathbf{D}^+_{\mathrm{aqc,acoh}}(X)$. Then $\mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \in \mathbf{D}^+_{\mathrm{aqc,acoh}}(X)$.*

(2) *Let $\mathcal{F}^a \in \mathbf{D}^-_{\mathrm{acoh}}(X)^a$, $\mathcal{G}^a \in \mathbf{D}^+_{\mathrm{acoh}}(X)^a$. Then $\mathbf{R}\underline{\mathrm{alHom}}_{\mathcal{O}_X^a}(\mathcal{F}^a, \mathcal{G}^a) \in \mathbf{D}^+_{\mathrm{acoh}}(X)^a$.*

Proof. The question is local on X, so we can assume that $X = \mathrm{Spec}\, A$ is affine. Then Lemma 4.4.10, Theorem 4.4.6, and Lemma 4.1.11 reduce both questions to showing that $\mathbf{R}\mathrm{Hom}_A(M,N) \in \mathbf{D}^+_{\mathrm{acoh}}(A)$ for $M \in \mathbf{D}^-_{\mathrm{acoh}}(A)$ and $N \in \mathbf{D}^+_{\mathrm{acoh}}(A)$. This is the content of Proposition 2.6.19. ■

Proposition 4.4.12. *Let $f: X \to Y$ be a quasi-compact quasi-separated morphism of R-schemes, $\mathcal{F}^a \in \mathbf{D}_{\mathrm{aqc}}(X)^a$, and $\mathcal{G} \in \mathbf{D}_{\mathrm{aqc}}(Y)^a$. Then the projection morphism (see the discussion before Proposition 3.5.27)*

$$\rho: \mathbf{R}f_*(\mathcal{F}^a) \otimes^L_{\mathcal{O}^a_Y} \mathcal{G}^a \to \mathbf{R}f_*\big(\mathcal{F}^a \otimes^L_{\mathcal{O}^a_X} \mathbf{L}f^*(\mathcal{G}^a)\big)$$

is an isomorphism in $\mathbf{D}(Y)^a$.

Proof. Proposition 3.5.14, Proposition 3.5.20, and Proposition 3.5.23 imply that we can replace $\mathcal{F}^a$ (resp. $\mathcal{G}^a$) with $\mathcal{F}^a_! \in \mathbf{D}_{\mathrm{qc}}(X)^a$ (resp. $\mathcal{G}^a_! \in \mathbf{D}_{\mathrm{qc}}(Y)^a$). So it suffices to show the analogous result for modules with *quasi-coherent* cohomology sheaves. This is proven in [68, Tag 08EU]. ∎

4.5 Formal schemes. The category of almost coherent $\mathcal{O}^a_{\mathfrak{X}}$-modules

In this section, we discuss the notion of almost coherent sheaves on "good" formal schemes. One of the main complications compared to the case of usual schemes is that there is no good notion of a "quasi-coherent" sheaf on a formal scheme. Namely, even though there is a notion of adically quasi-coherent sheaves on a large class of formal schemes due to [25, Section I.3], this notion does not behave well. In particular, this category is not a weak Serre subcategory of $\mathcal{O}_{\mathfrak{X}}$-modules for a "nice" formal scheme $\mathfrak{X}$.

Another (related) difficulty comes from the lack of the Artin–Rees lemma for not finitely generated modules. More precisely, many operations with adically quasi-coherent sheaves require taking completions, but this operation is usually not exact without the presence of the Artin–Rees lemma.

We deal with this by using a version of the Artin–Rees lemma (Lemma 2.12.6) for almost finitely generated modules over "good" rings. The presence of the Artin–Rees lemma suggests that it is reasonable to expect that we might have a good notion of adically quasi-coherent, almost coherent $\mathcal{O}_{\mathfrak{X}}$-modules on some "good" class of formal schemes.

We start by giving a set-up in which we can develop the theory of almost coherent sheaves.

Set-up 4.5.1. We fix a ring R with a finitely generated ideal I such that R is I-adically complete, I-adically topologically universally adhesive[3], and I-torsion free with an ideal $\mathfrak{m}$ such that $I \subset \mathfrak{m}$, $\mathfrak{m}^2 = \mathfrak{m}$ and $\widetilde{\mathfrak{m}} := \mathfrak{m} \otimes_R \mathfrak{m}$ is R-flat.

[3]This means that the algebra $R\langle X_1, \dots, X_n\rangle[T_1, \dots, T_m]$ is I-adically adhesive for any n and m.

The basic example of such a ring is a complete microbial[4] valuation ring R with algebraically closed fraction field K. We pick a pseudo-uniformizer ϖ and define $I := (\varpi)$, $\mathfrak{m} := \bigcup_{n=1}^{\infty}(\varpi^{1/n})$ for some compatible choice of roots of ϖ. We note that R is topologically universally adhesive by [24, Theorem 7.3.2].

We note that the assumptions in Set-up 4.5.1 imply that any finitely presented algebra over a topologically finitely presented R-algebra is coherent and I-adically adhesive. Coherence follows from [24, Proposition 7.2.2] and adhesiveness follows from the definition. In what follows, we will use those facts without saying.

In what follows, $\mathfrak{X}$ always means a topologically finitely presented formal R-scheme. We will denote by $\mathfrak{X}_k := \mathfrak{X} \times_{\operatorname{Spf} R} \operatorname{Spec} R/I^{k+1}$ the "reduction" schemes. They come equipped with a closed immersion $i_k \colon \mathfrak{X}_k \to \mathfrak{X}$. Also, given any $\mathcal{O}_{\mathfrak{X}}$-module $\mathcal{F}$, we will always denote its "reduction" $i_k^*\mathcal{F}$ by $\mathcal{F}_k$.

Definition 4.5.2. [25, Definition I.3.1.3] An $\mathcal{O}_{\mathfrak{X}}$-module $\mathcal{F}$ on a formal scheme $\mathfrak{X}$ of finite ideal type is called *adically quasi-coherent* if $\mathcal{F} \to \lim_n \mathcal{F}_n$ is an isomorphism and, for any open formal subscheme $\mathfrak{U} \subset \mathcal{F}$ and any ideal of definition $\mathcal{I}$ of finite type, the sheaf $\mathcal{F}/\mathcal{I}\mathcal{F}$ is a quasi-coherent sheaf on the scheme $(\mathfrak{U}, \mathcal{O}_{\mathfrak{U}}/\mathcal{I})$.

We denote by $\mathbf{Mod}_{\mathfrak{X}}^{\mathrm{qc}}$ the full subcategory of $\mathbf{Mod}_{\mathfrak{X}}$ consisting of the adically quasi-coherent $\mathcal{O}_{\mathfrak{X}}$-modules.

Definition 4.5.3. We say that an $\mathcal{O}_{\mathfrak{X}}^a$-module $\mathcal{F}^a$ is *almost adically quasi-coherent* if $\mathcal{F}_!^a \simeq \widetilde{\mathfrak{m}} \otimes \mathcal{F}$ is an adically quasi-coherent $\mathcal{O}_{\mathfrak{X}}$-module. We denote by $\mathbf{Mod}_{\mathfrak{X}^a}^{\mathrm{aqc}}$ the full subcategory of $\mathbf{Mod}_{\mathfrak{X}^a}$ consisting of the almost adically quasi-coherent $\mathcal{O}_{\mathfrak{X}}$-modules.

We say that an $\mathcal{O}_{\mathfrak{X}}$-module $\mathcal{F}$ is *almost adically quasi-coherent* if $\mathcal{F}^a$ is an almost quasi-coherent $\mathcal{O}_{\mathfrak{X}}^a$-module. We denote by $\mathbf{Mod}_{\mathfrak{X}}^{\mathrm{aqc}}$ the full subcategory of $\mathbf{Mod}_{\mathfrak{X}}$ consisting of the adically quasi-coherent $\mathcal{O}_{\mathfrak{X}}$-modules.

Remark 4.5.4. In general, we cannot say that every adically quasi-coherent $\mathcal{O}_{\mathfrak{X}}$-module $\mathcal{F}$ is almost adically quasi-coherent. The problem is that the sheaf $\widetilde{\mathfrak{m}} \otimes \mathcal{F}$ might not be complete, i.e., the map $\widetilde{\mathfrak{m}} \otimes \mathcal{F} \to \lim_k \widetilde{\mathfrak{m}} \otimes \mathcal{F}_k$ is a priori only an almost isomorphism.

Lemma 4.5.5. *Let $\mathfrak{X}$ be a topologically finitely presented formal R-scheme for R as in Set-up 4.5.1, and let $\mathcal{F}^a$ be an almost adically quasi-coherent $\mathcal{O}_{\mathfrak{X}}^a$-module. Then $\mathcal{F}_k^a$ is almost quasi-coherent for all k. Moreover, if an $\mathcal{O}_{\mathfrak{X}}^a$-module $\mathcal{G}^a$ is annihilated by some I^{n+1}, then $\mathcal{G}^a$ is almost adically quasi-coherent if and only if so is $\mathcal{G}_n^a$.*

[4] A valuation ring R is microbial if there is a non-zero topologically nilpotent element $\varpi \in R$. Any such element is called a pseudo-uniformizer.

Proof. To prove the first claim, it is sufficient to show that $\widetilde{\mathfrak{m}} \otimes \mathcal{F}_k$ is quasi-coherent provided that $\widetilde{\mathfrak{m}} \otimes \mathcal{F}$ is adically quasi-coherent. We use Corollary 3.2.18 to say that $\widetilde{\mathfrak{m}} \otimes \mathcal{F}_k \simeq (\widetilde{\mathfrak{m}} \otimes \mathcal{F})_k$ and the reduction of an adically quasi-coherent module is quasi-coherent. Therefore, each $\mathcal{F}_k^a$ is almost adically quasi-coherent.

Now if $\mathcal{G}$ is annihilated by I^{n+1}, then $\mathcal{G} = i_{n,*}\mathcal{G}_n$. We use the projection formula (Lemma 3.3.5) to see that $\widetilde{\mathfrak{m}} \otimes \mathcal{G} \simeq i_{n,*}(\mathcal{G}_n \otimes \widetilde{\mathfrak{m}})$. Clearly, $i_{n,*}$ sends quasi-coherent sheaves to adically quasi-coherent sheaves. So $\mathcal{G}^a$ is almost adically quasi-coherent if so is $\mathcal{G}_n^a$. ∎

Definition 4.5.6. We say that an $\mathcal{O}_{\mathfrak{X}}^a$-module $\mathcal{F}^a$ is *of almost finite type* (resp. *almost finitely presented*) if $\mathcal{F}^a$ is almost adically quasi-coherent, and there is a covering of $\mathfrak{X}$ by open affines $\{\mathfrak{U}_i\}_{i\in I}$ such that $\mathcal{F}^a(\mathfrak{U}_i)$ is an almost finitely generated (resp. almost finitely presented) $\mathcal{O}_{\mathfrak{X}}^a(\mathfrak{U}_i)$-module. We denote these categories by $\mathbf{Mod}_{\mathfrak{X}^a}^{\mathrm{aft}}$ and $\mathbf{Mod}_{\mathfrak{X}^a}^{\mathrm{afp}}$ respectively.

We say that an $\mathcal{O}_{\mathfrak{X}}$-module $\mathcal{F}$ is *of almost finite type* (resp. *almost finitely presented*) if so is $\mathcal{F}^a$. We denote these categories by $\mathbf{Mod}_{\mathfrak{X}}^{\mathrm{aft}}$ and $\mathbf{Mod}_{\mathfrak{X}}^{\mathrm{afp}}$ respectively.

Definition 4.5.7. We say that an $\mathcal{O}_{\mathfrak{X}}$-module $\mathcal{F}$ is *adically quasi-coherent of almost finite type* (resp. *adically quasi-coherent almost finitely presented*) if it is adically quasi-coherent and there is a covering of $\mathfrak{X}$ by open affines $\{\mathfrak{U}_i\}_{i\in I}$ such that $\mathcal{F}(\mathfrak{U}_i)$ is an almost finitely generated (resp. almost finitely presented) $\mathcal{O}_{\mathfrak{X}}(\mathfrak{U}_i)$-module. We denote these categories by $\mathbf{Mod}_{\mathfrak{X}}^{\mathrm{qc,aft}}$ and $\mathbf{Mod}_{\mathfrak{X}}^{\mathrm{qc,afp}}$ respectively.

Remark 4.5.8. If $\mathcal{F}^a$ is a finite type (resp. finitely presented) $\mathcal{O}_{\mathfrak{X}}^a$-module, then it follows that $(\mathcal{F}^a)_!$ is adically quasi-coherent of almost finite type (resp. almost finite presentation).

Remark 4.5.9. We note that, a priori, it is not clear if $\mathcal{F}^a$ is an almost finite type (resp. almost finitely presented) $\mathcal{O}_{\mathfrak{X}}^a$-module for an adically quasi-coherent almost finite type (resp. almost finitely presented) $\mathcal{O}_{\mathfrak{X}}$-module $\mathcal{F}$. The problem comes from the fact that we do not require $\widetilde{\mathfrak{m}} \otimes \mathcal{F}$ to be adically quasi-coherent in Definition 4.5.7. However, we will show in Lemma 4.5.10 that it is indeed automatic in this case.

Lemma 4.5.10. *Let $\mathfrak{X}$ be a topologically finitely presented formal R-scheme for R as in Set-up 4.5.1, and let $\mathcal{F}$ be an adically quasi-coherent $\mathcal{O}_{\mathfrak{X}}$-module of almost finite type (resp. of almost finite presentation). Then $\widetilde{\mathfrak{m}} \otimes \mathcal{F}$ is adically quasi-coherent. In particular, $\mathcal{F}$ is almost of finite type (resp. almost finitely presented).*

Proof. Corollary 2.5.12 and Lemma 3.3.1 imply that the only condition we really need to check is that $\widetilde{\mathfrak{m}} \otimes \mathcal{F}$ is adically quasi-coherent. Therefore, it suffices to prove the result for an adically quasi-coherent, almost finite type $\mathcal{O}_{\mathfrak{X}}$-module $\mathcal{F}$.

Since the question is local on $\mathfrak{X}$, we can assume that $\mathfrak{X} = \operatorname{Spf} A$ is affine and $M := \mathcal{F}(X)$ is almost finitely generated over A. Then we use [25, Theorem I.3.2.8] to conclude that $\mathcal{F}$ is isomorphic to M^Δ. We claim that $\widetilde{\mathfrak{m}} \otimes \mathcal{F}$ is isomorphic to $(\widetilde{\mathfrak{m}} \otimes_A M)^\Delta$ as that would imply that $\widetilde{\mathfrak{m}} \otimes \mathcal{F}$ is adically quasi-coherent by [25, Proposition I.3.2.2]. In order to show that $\widetilde{\mathfrak{m}} \otimes \mathcal{F}$ is isomorphic to $(\widetilde{\mathfrak{m}} \otimes_R M)^\Delta$, we need to check two things: for any open affine $\mathrm{Spf}\, B = \mathfrak{U} \subset \mathfrak{X}$ the B-module $(\widetilde{\mathfrak{m}} \otimes \mathcal{F})(\mathfrak{U})$ is I-adically complete, and then the natural map $(\widetilde{\mathfrak{m}} \otimes_R M)\widehat{\otimes}_A B \to (\widetilde{\mathfrak{m}} \otimes \mathcal{F})(\mathfrak{U})$ is an isomorphism.

We start with the first claim. Lemma 3.3.1 says that $(\widetilde{\mathfrak{m}} \otimes \mathcal{F})(\mathfrak{U})$ is isomorphic to $\widetilde{\mathfrak{m}} \otimes_R \mathcal{F}(\mathfrak{U})$. Since $\mathcal{F}$ is adically quasi-coherent, $\mathcal{F}(\mathfrak{U}) \simeq M\widehat{\otimes}_A B$, and therefore $(\widetilde{\mathfrak{m}} \otimes \mathcal{F})(\mathfrak{U}) \simeq \widetilde{\mathfrak{m}} \otimes_R (M\widehat{\otimes}_A B)$. Lemma 2.8.1 says that $M\otimes_A B$ is almost finitely generated over B, so it is already I-adically complete by Lemma 2.12.7. Therefore, we see that $\widetilde{\mathfrak{m}} \otimes_R \mathcal{F}(\mathfrak{U}) \simeq \widetilde{\mathfrak{m}} \otimes_R (M\otimes_A B)$, and the latter is almost finitely generated over B by Corollary 2.5.12. Thus, we use Lemma 2.12.7 once again to show its completeness.

Now we show that the natural morphism $(\widetilde{\mathfrak{m}} \otimes_R M)\widehat{\otimes}_A B \to (\widetilde{\mathfrak{m}} \otimes \mathcal{F})(\mathfrak{U})$ is an isomorphism. Again, using the same results as above, we can get rid of any completions and identify this map with the "identity" map

$$(\widetilde{\mathfrak{m}} \otimes_R M)\otimes_A B \to \widetilde{\mathfrak{m}} \otimes_R (M\otimes_A B).$$

This finishes the proof. ■

Lemma 4.5.11. *Let $\mathfrak{X}$ be a topologically finitely presented formal R-scheme for R as in Set-up 4.5.1, and let $\mathcal{F}^a$ be an almost finite type (resp. almost finitely presented) $\mathcal{O}^a_{\mathfrak{X}}$-module. Then the $\mathcal{O}^a_{\mathfrak{X}_k}$-module $\mathcal{F}^a_k$ is almost finite type (resp. almost finitely presented) for any integer k.*

Proof. Lemma 4.5.5 implies that each $\mathcal{F}^a_k$ is an almost quasi-coherent $\mathcal{O}_{X_k}$-module. So it is sufficient to find a covering of $\mathfrak{X}_k$ by open affines $\mathfrak{U}_{i,k}$ such that $\mathcal{F}^a_k(\mathfrak{U}_{i,k})$ is almost finitely generated (resp. almost finitely presented) over $\mathcal{O}^a_{\mathfrak{X}_k}(\mathfrak{U}_{i,k})$. We choose a covering of $\mathfrak{X}$ by open affines $\mathfrak{U}_i$ such that $\mathcal{F}^a(\mathfrak{U}_i)$ are almost finitely generated (resp. almost finitely presented) over $\mathcal{O}^a_{\mathfrak{X}}(\mathfrak{U}_i)$. Since the underlying topological spaces of $\mathfrak{X}$ and $\mathfrak{X}_k$ are the same, we conclude that $\mathfrak{U}_{i,k}$ form an affine open covering of $\mathfrak{X}_k$. Then using the vanishing result for higher cohomology groups of adically quasi-coherent sheaves on affine formal schemes of finite type [25, Theorem I.7.1.1] and Lemma 3.3.1, we deduce that

$$\mathcal{F}^a_k(\mathfrak{U}_{i,k}) \simeq \left(\widetilde{\mathfrak{m}} \otimes \mathcal{F}^a_k\right)(\mathfrak{U}_{i,k})^a \simeq \left(\widetilde{\mathfrak{m}} \otimes \mathcal{F}(\mathfrak{U}_i)/I^{k+1}\right)^a$$

is almost finitely generated (resp. almost finitely presented) over $\mathcal{O}_{\mathfrak{X}_k}(\mathfrak{U}_{i,k})$. ■

Lemma 4.5.12. *Let $\mathfrak{X}$ be a locally topologically finitely presented formal R-scheme for R as in Set-up 4.5.1, and let $\mathcal{F}^a$ be an almost finite type (resp. almost finitely presented) $\mathcal{O}^a_{\mathfrak{X}}$-module. Then $\mathcal{F}^a(\mathfrak{U})$ is an almost finitely generated (resp. almost finitely presented) $\mathcal{O}^a_{\mathfrak{X}}(\mathfrak{U})$-module for any open affine $\mathfrak{U} \subset \mathfrak{X}$.*

Proof. Corollary 2.5.12 and Lemma 3.3.1 guarantee that we can replace $\mathcal{F}$ with $\widetilde{\mathfrak{m}} \otimes \mathcal{F}$ for the purpose of the proof. Thus, we may and do assume that $\mathcal{F}$ is an adically quasi-coherent almost finitely generated (resp. almost finitely presented) $\mathcal{O}_{\mathfrak{X}}$-module. Then, using Lemma 2.8.1 and Lemma 2.12.7, we can use the argument as in the proof of Lemma 4.5.10 to show that the restriction of $\mathcal{F}$ to any open formal subscheme is still adically quasi-coherent of almost finite type (resp. finitely presented). Thus, we may and do assume that $\mathfrak{X} = \operatorname{Spf} A$ is an affine formal R-scheme.

So, now we have an affine topologically finitely presented formal R-scheme $\mathfrak{X} = \operatorname{Spf} A$, a finite[5] covering of $\mathfrak{X}$ by affines $\mathfrak{U}_i = \operatorname{Spf} A_i$, and an adically quasi-coherent $\mathcal{O}_{\mathfrak{X}}$-module $\mathcal{F}$ such that $\mathcal{F}(\mathfrak{U}_i)$ is almost finitely generated (resp. almost finitely presented) A_i-module. We want to show that $\mathcal{F}(\mathfrak{X})$ is an almost finitely generated (resp. almost finitely presented) A-module.

First, we deal with the *almost finitely generated case.* We note that Lemma 4.1.7, Lemma 4.5.11, and [25, Theorem I.7.1.1] imply that $\mathcal{F}(\mathfrak{X})/I$ is almost finitely generated. We know that $\mathcal{F}$ is adically quasi-coherent, so $\mathcal{F}(\mathfrak{X})$ must be an I-adically complete A-module. Therefore, Lemma 2.5.20 guarantees that $\mathcal{F}(X)$ is an almost finitely generated A-module.

Now we move to the *almost finitely presented case*. We already know that $\mathcal{F}(\mathfrak{X})$ is almost finitely generated over A. Thus, the standard argument with Lemma 2.12.7 implies that $\mathcal{F}(\mathfrak{U}_i) = \mathcal{F}(\mathfrak{X}) \otimes_A A_i$ for any i. Recall that [25, Proposition I.4.8.1] implies[6] that each $A \to A_i$ is flat. Since $\operatorname{Spf} A_i$ form a covering of $\operatorname{Spf} A$, we conclude that $A \to \prod_{i=1}^n A_i$ is faithfully flat. Now the result follows from faithfully flat descent for almost finitely presented modules, which is proven in Lemma 2.10.5. ■

Corollary 4.5.13. *Let $\mathfrak{X} = \operatorname{Spf} A$ be a topologically finitely presented affine formal R-scheme for R as in Set-up 4.5.1, and let $\mathcal{F}^a$ be an almost adically quasi-coherent $\mathcal{O}^a_{\mathfrak{X}}$-module. Then $\mathcal{F}^a$ is almost finite type (resp. almost finitely presented) if and only if $\mathcal{F}^a(\mathfrak{X})$ is an almost finitely generated (resp. almost finitely presented) A^a-module.*

Similarly, an adically quasi-coherent $\mathcal{O}_{\mathfrak{X}}$-module $\mathcal{F}$ is of almost finite type (resp. almost finitely presented) if and only if $\mathcal{F}(\mathfrak{X})$ is an almost finitely generated (resp. almost finitely presented) A-module.

Lemma 4.5.14. *Let $\mathfrak{X} = \operatorname{Spf} A$ be a topologically finitely presented affine formal R-scheme for R as in Set-up 4.5.1, let $\varphi\colon N \to M$ be a homomorphism of almost finitely generated A-modules. Then the following sequence:*

$$0 \to (\operatorname{Ker} \varphi)^\Delta \to N^\Delta \xrightarrow{\varphi^\Delta} M^\Delta \to (\operatorname{Coker} \varphi)^\Delta \to 0$$

is exact. Moreover, $\operatorname{Im}(\varphi)^\Delta \simeq \operatorname{Im}(\varphi^\Delta)$.

[5] We implicitly use here that $\mathfrak{X}$ is quasi-compact.

[6] Topologically universally adhesive rings are by definition “t. u. rigid-noetherian”.

Proof. We denote the kernel $\operatorname{Ker}\varphi$ by K, the image $\operatorname{Im}(\varphi)$ by M', and the cokernel $\operatorname{Coker}\varphi$ by Q.

We start with $\operatorname{Ker}\varphi^\Delta$: We note that $(\operatorname{Ker}\varphi^\Delta)(\mathfrak{X}) = K$, this induces a natural morphism $\alpha\colon K^\Delta \to \operatorname{Ker}\varphi^\Delta$. In order to show that it is an isomorphism, it suffices to check that it induces an isomorphism on values over a basis of principal open subsets. Now recall that for any A-module L, we have an equality $L^\Delta(\operatorname{Spf} A_{\{f\}}) = \widehat{L_f}$, where the completion is taken with respect to the I-adic topology. Thus, in order to check that α is an isomorphism, it suffices to show that $\widehat{K_f}$ is naturally identified with $(\operatorname{Ker}\varphi)(\operatorname{Spf} A_{\{f\}}) = \operatorname{Ker}(\widehat{N_f} \to \widehat{M_f})$. Using the Artin–Rees lemma (see Lemma 2.12.6) over the adhesive ring A_f, we conclude that the induced topologies on K_f and M'_f coincide with the I-adic ones. This implies that

$$\widehat{K_f} = \lim K_f/I^n K_f = \lim K_f/(I^n N_f \cap K_f)$$

and

$$\widehat{M'_f} = \lim M'_f/I^n M'_f = \lim M'_f/(I^n M_f \cap M'_f).$$

This guarantees that we have two exact sequences:

$$0 \to \widehat{K_f} \to \widehat{M_f} \to \widehat{M'_f} \to 0, \qquad 0 \to \widehat{M'_f} \to \widehat{N_f}.$$

In particular, we get that the natural map $\widehat{K_f} \to \operatorname{Ker}(\widehat{M_f} \to \widehat{N_f})$ is an isomorphism. This shows that $K^\Delta \simeq \operatorname{Ker}(\varphi^\Delta)$.

We prove the claim for $\operatorname{Im}\varphi^\Delta$: We note that since the category of $\mathcal{O}_\mathfrak{X}$-modules is abelian, we can identify $\operatorname{Im}\varphi^\Delta \simeq \operatorname{Coker}(K^\Delta \to N^\Delta)$. We observe that [25, Theorem I.7.1.1] and the established fact above that $\operatorname{Ker}\varphi$ is adically quasi-coherent imply that the natural map $\mathcal{F}(\mathfrak{U})/K^\Delta(\mathfrak{U}) \to (\operatorname{Im}\varphi^\Delta)(\mathfrak{U})$ is an isomorphism for any affine open formal subscheme $\mathfrak{U}$. In particular, we have $(\operatorname{Im}\varphi^\Delta)(\mathfrak{X}) = M/K = M'$. Therefore, we have a natural map $(M')^\Delta \to \operatorname{Im}\varphi^\Delta$ and we show that it is an isomorphism. It suffices to show that this map is an isomorphism on values over a basis of principal open subsets. Then we use the identification $\mathcal{F}(\mathfrak{U})/K^\Delta(\mathfrak{U}) \simeq (\operatorname{Im}\varphi)(\mathfrak{U})$ and the short exact sequence

$$0 \to \widehat{K_f} \to \widehat{M_f} \to \widehat{M'_f} \to 0,$$

to finish the proof.

We show the claim for $\operatorname{Coker}\varphi^\Delta$: The argument is identical to the argument for $\operatorname{Im}\varphi$ once we know that $\operatorname{Im}\varphi = \operatorname{Ker}(\mathcal{G} \to \operatorname{Coker}\varphi)$ is adically quasi-coherent. ∎

Corollary 4.5.15. *Let $\mathfrak{X} = \operatorname{Spf} A$ be a topologically finitely presented affine formal R-scheme for R as in Set-up 4.5.1, let M be an almost finitely generated A-module, and let N be any A-submodule of M. Then the following sequence is exact:*

$$0 \to N^\Delta \xrightarrow{\varphi^\Delta} M^\Delta \to (M/N)^\Delta \to 0.$$

Proof. We just apply Lemma 4.5.14 to the homomorphism $M \to M/N$ of almost finitely generated A-modules. ∎

Corollary 4.5.16. *Let $\mathfrak{X}$ be a topologically finitely presented formal R-scheme for R as in Set-up 4.5.1, and let $\varphi\colon \mathcal{F} \to \mathcal{G}$ be a morphism of adically quasi-coherent, almost finite type $\mathcal{O}_{\mathfrak{X}}$-modules. Then* $\operatorname{Ker}\varphi$ *is an adically quasi-coherent $\mathcal{O}_{\mathfrak{X}}$-module,* $\operatorname{Coker}\varphi$ *and* $\operatorname{Im}\varphi$ *are adically quasi-coherent $\mathcal{O}_{\mathfrak{X}}$-modules of almost finite type.*

Corollary 4.5.17. *Let $\mathfrak{X}$ be a topologically finitely presented formal R-scheme for R as in Set-up 4.5.1, and let $\varphi\colon \mathcal{F}^a \to \mathcal{G}^a$ be a morphism of almost finite type $\mathcal{O}^a_{\mathfrak{X}}$-modules. Then* $\operatorname{Ker}\varphi$ *is an almost adically quasi-coherent $\mathcal{O}^a_{\mathfrak{X}}$-module,* $\operatorname{Coker}\varphi$ *and* $\operatorname{Im}\varphi$ *are $\mathcal{O}^a_{\mathfrak{X}}$-modules of almost finite type.*

Proof. We apply the exact functor $(-)_!$ to the map φ and reduce the claim to Corollary 4.5.16. ∎

Now we are ready to show that almost finite type and almost finitely presented $\mathcal{O}_{\mathfrak{X}}$-modules share many good properties as we would expect. The only subtle thing is that we do not know whether an adically quasi-coherent quotient of an adically quasi-coherent, almost finite type $\mathcal{O}_{\mathfrak{X}}$-module is of almost finite type. The main extra complication here is that we need to be very careful with the adically quasi-coherent condition in the definition of almost finite type (resp. almost finitely presented) modules since that condition does not behave well in general.

Lemma 4.5.18. *Let $0 \to \mathcal{F}' \xrightarrow{\varphi} \mathcal{F} \xrightarrow{\psi} \mathcal{F}'' \to 0$ be an exact sequence of $\mathcal{O}_{\mathfrak{X}}$-modules, then the following assertions hold:*

(1) *If $\mathcal{F}$ is adically quasi-coherent of almost finite type and $\mathcal{F}'$ is adically quasi-coherent, then $\mathcal{F}''$ is adically quasi-coherent of almost finite type.*

(2) *If $\mathcal{F}'$ and $\mathcal{F}''$ are adically quasi-coherent of almost finite type (resp. almost finitely presented), then so is $\mathcal{F}$.*

(3) *If $\mathcal{F}$ is adically quasi-coherent of almost finite type, and $\mathcal{F}''$ is adically quasi-coherent, almost finitely presented, then $\mathcal{F}'$ is adically quasi-coherent of almost finite type.*

(4) *If $\mathcal{F}$ is adically quasi-coherent almost finitely presented, and $\mathcal{F}'$ is adically quasi-coherent of almost finite type, then $\mathcal{F}''$ is adically quasi-coherent, almost finitely presented.*

Proof. (1) Without loss of generality, we can assume that $\mathfrak{X} = \operatorname{Spf} A$ is an affine formal scheme. Then $\mathcal{F} \cong M^\Delta$ for some almost finitely generated A-module M, and $\mathcal{F}' \cong N^\Delta$ for some A-submodule $N \subset M$. Then Corollary 4.5.15 ensures that $\mathcal{F}'' \simeq (M/N)^\Delta$. In particular, it is adically quasi-coherent. The claim is then an easy consequence of the vanishing theorem [25, Theorem I.7.1.1] and Lemma 2.5.15 (1).

(2) The difficult part is to show that $\mathcal{F}$ is adically quasi-coherent. In fact, once we know that $\mathcal{F}$ is adically quasi-coherent, it is automatically of almost finite type (resp. almost finitely presented) by [25, Theorem I.7.1.1] and Lemma 2.5.15 (2).

In order to show that $\mathcal{F}$ is adically quasi-coherent, we may and do assume that $\mathfrak{X} = \operatorname{Spf} A$ is an affine formal R-scheme for some adhesive ring A. Then let us introduce A-modules $M' := \mathcal{F}'(\mathfrak{X})$, $M := \mathcal{F}(\mathfrak{X})$, and $M'' := \mathcal{F}''(\mathfrak{X})$. We have the natural morphism $M^{\Delta} \to \mathcal{F}$ and we show that it is an isomorphism. The vanishing theorem [25, Theorem I.7.1.1] implies that we have a short exact sequence:

$$0 \to M' \to M \to M'' \to 0.$$

Thus by Lemma 2.5.15 (2), M is almost finitely generated (resp. almost finitely presented). Then Lemma 4.5.14 gives that we have a short exact sequence

$$0 \to M'^{\Delta} \to M^{\Delta} \to M''^{\Delta} \to 0.$$

Using the vanishing theorem [25, Theorem I.7.1.1] once again, we get a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M'^{\Delta} & \longrightarrow & M^{\Delta} & \longrightarrow & M''^{\Delta} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{F}' & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{F}'' & \longrightarrow & 0, \end{array}$$

where the rows are exact, and the left and right vertical arrows are isomorphisms. That implies that the map $M^{\Delta} \to \mathcal{F}$ is an isomorphism.

(3) This easily follows from Lemma 2.5.15 (3), Corollary 4.5.16, and [25, Theorem I.7.1.1].

(4) This also easily follows from Lemma 2.5.15 (4), Corollary 4.5.16, and [25, Theorem I.7.1.1]. ■

We also give the almost version of Lemma 4.5.18:

Corollary 4.5.19. *Let* $0 \to \mathcal{F}'^{a} \xrightarrow{\varphi} \mathcal{F}^{a} \xrightarrow{\psi} \mathcal{F}''^{a} \to 0$ *be an exact sequence of* $\mathcal{O}_{\mathfrak{X}}^{a}$-*modules, then the following hold:*

(1) *If* $\mathcal{F}^{a}$ *is of almost finite type and* $\mathcal{F}'^{a}$ *is almost adically quasi-coherent, then* $\mathcal{F}''^{a}$ *is of almost finite type.*

(2) *If* $\mathcal{F}'^{a}$ *and* $\mathcal{F}''^{a}$ *are of almost finite type (resp. almost finitely presented), then so is* $\mathcal{F}^{a}$.

(3) *If* $\mathcal{F}^{a}$ *is of almost finite type and* $\mathcal{F}''^{a}$ *is almost finitely presented, then* $\mathcal{F}'^{a}$ *is of almost finite type.*

(4) *If $\mathcal{F}^a$ is of almost finitely presented and $\mathcal{F}'^a$ is of almost finite type, then $\mathcal{F}''^a$ is almost finitely presented.*

Definition 4.5.20. An $\mathcal{O}_{\mathfrak{X}}^a$-module $\mathcal{F}^a$ is *almost coherent* if $\mathcal{F}^a$ is almost finite type and for any open set $\mathfrak{U}$, any finite type $\mathcal{O}_{\mathfrak{X}}^a$-submodule $\mathcal{G}^a \subset (\mathcal{F}|_{\mathfrak{U}})^a$ is an almost finitely presented $\mathcal{O}_{\mathfrak{U}}$-module.

An $\mathcal{O}_{\mathfrak{X}}$-module $\mathcal{F}$ is *(adically quasi-coherent) almost coherent* if $\mathcal{F}^a$ is almost coherent (and $\mathcal{F}$ is adically quasi-coherent).

Remark 4.5.21. We note that Lemma 4.5.10 ensures that any adically quasi-coherent almost coherent $\mathcal{O}_X$-module $\mathcal{F}$ is almost coherent.

Lemma 4.5.22. *Let $\mathcal{F}^a$ be an $\mathcal{O}_{\mathfrak{X}}^a$-module on a topologically finitely presented formal R-scheme $\mathfrak{X}$. Then the following are equivalent:*

(1) *$\mathcal{F}^a$ is almost coherent;*

(2) *$\mathcal{F}^a$ is almost quasi-coherent and the $\mathcal{O}_{\mathfrak{X}}^a(\mathfrak{U})$-module $\mathcal{F}^a(\mathfrak{U})$ is almost coherent for any open affine formal subscheme $\mathfrak{U} \subset \mathfrak{X}$;*

(3) *$\mathcal{F}^a$ is almost quasi-coherent and there is a covering of $\mathfrak{X}$ by open affine subschemes $\{\mathfrak{U}_i\}_{i \in I}$ such that $\mathcal{F}^a(\mathfrak{U}_i)$ is almost coherent for each i.*

In particular, an $\mathcal{O}_{\mathfrak{X}}^a$-module $\mathcal{F}^a$ is almost coherent if and only if it is almost finitely presented.

Proof. The proof that these three notions are equivalent is identical to the proof of Lemma 4.5.22 modulo facts that we have already established in this chapter, especially Lemma 4.5.14.

As for the last claim, we recall that $\mathfrak{X}$ is topologically finitely presented over a topologically universally adhesive ring, so $\mathcal{O}_{\mathfrak{X}}(\mathfrak{U})$ is coherent for any open affine $\mathfrak{U}$ [25, Proposition 0.8.5.23, Lemma I.1.7.4, Proposition I.2.3.3]. Then the equivalence is proved by Lemma 2.6.13 and Corollary 2.6.15. ∎

Although Lemma 4.5.22 says that the notion of almost coherence coincides with the notion of almost finite presentation, it shows that almost coherence is morally "the correct" definition. In what follows, we prefer to use the terminology of almost coherent sheaves as it is shorter and gives a better intuition from our point of view.

Lemma 4.5.23. (1) *Any almost finite type $\mathcal{O}_{\mathfrak{X}}^a$-submodule of an almost coherent $\mathcal{O}_{\mathfrak{X}}^a$-module is almost coherent.*

(2) *Let $\varphi\colon \mathcal{F}^a \to \mathcal{G}^a$ be a homomorphism from an almost finite type $\mathcal{O}_{\mathfrak{X}}^a$-module to an almost coherent $\mathcal{O}_{\mathfrak{X}}^a$-module. Then* $\operatorname{Ker}\varphi$ *is an almost finite type $\mathcal{O}_{\mathfrak{X}}^a$-module.*

(3) *Let $\varphi\colon \mathcal{F}^a \to \mathcal{G}^a$ be a homomorphism of almost coherent $\mathcal{O}_{\mathfrak{X}}^a$-modules. Then* $\operatorname{Ker}\varphi$ *and* $\operatorname{Coker}\varphi$ *are almost coherent $\mathcal{O}_{\mathfrak{X}}$-modules.*

(4) *Given a short exact sequence of* $\mathcal{O}^a_{\mathfrak{X}}$*-modules*

$$0 \to \mathcal{F}'^a \to \mathcal{F}^a \to \mathcal{F}''^a \to 0,$$

if two out of three are almost coherent, then so is the third one.

Remark 4.5.24. There is also an evident version of this lemma for adically quasi-coherent almost coherent $\mathcal{O}_{\mathfrak{X}}$-modules.

Proof. The proof is identical to the proof of Corollary 4.1.12 once we have established Corollary 4.5.16 and the equivalence of almost coherent and almost finitely presented $\mathcal{O}_{\mathfrak{X}}$-modules from Lemma 4.5.22. ■

Corollary 4.5.25. *Let* $\mathfrak{X}$ *be a topologically finitely presented formal* R*-scheme for* R *as in Set-up 4.5.1. Then the category* $\mathbf{Mod}^{\mathrm{acoh}}_{\mathfrak{X}}$ *(resp.* $\mathbf{Mod}^{\mathrm{qc,acoh}}_{\mathfrak{X}}$*,* $\mathbf{Mod}^{\mathrm{acoh}}_{\mathfrak{X}^a}$*) of almost coherent* $\mathcal{O}_{\mathfrak{X}}$*-modules (resp. adically quasi-coherent, almost coherent* $\mathcal{O}_{\mathfrak{X}}$*-modules, resp. almost coherent* $\mathcal{O}^a_{\mathfrak{X}}$*-modules) is a weak Serre subcategory of* $\mathbf{Mod}_{\mathfrak{X}}$ *(resp.* $\mathbf{Mod}_{\mathfrak{X}}$*, resp.* $\mathbf{Mod}^a_{\mathfrak{X}}$*).*

4.6 Formal schemes. Basic functors on almost coherent $\mathcal{O}^a_{\mathfrak{X}}$-modules

In this section, we study the interaction between the functors defined in Section 3.2 and the notion of almost (quasi-)coherent $\mathcal{O}^a_{\mathfrak{X}}$-modules. The exposition follows Section 4.2 very closely.

We start with an affine situation, i.e., $\mathfrak{X} = \operatorname{Spf} A$. In this case, we note that the functor $(-)^{\Delta}\colon \mathbf{Mod}_A \to \mathbf{Mod}^{\mathrm{qc}}_{\mathfrak{X}}$ sends almost zero A-modules to almost zero $\mathcal{O}_{\mathfrak{X}}$-modules. Thus, it induces a functor

$$(-)^{\Delta}\colon \mathbf{Mod}_{A^a} \to \mathbf{Mod}_{\mathfrak{X}^a}.$$

Lemma 4.6.1. *Let* $\mathfrak{X} = \operatorname{Spf} A$ *be an affine formal* R*-scheme for* R *as in Set-up 4.5.1. Then* $(-)^{\Delta}\colon \mathbf{Mod}_A \to \mathbf{Mod}^{\mathrm{qc}}_{\mathfrak{X}}$ *induces an equivalence* $(-)^{\Delta}\colon \mathbf{Mod}^*_A \to \mathbf{Mod}^{\mathrm{qc},*}_{\mathfrak{X}}$ *for any* $* \in \{\mathrm{aft}, \mathrm{acoh}\}$*. The quasi-inverse functor is given by* $\Gamma(\mathfrak{X}, -)$*.*

Proof. We note first that the functor $(-)^{\Delta}\colon \mathbf{Mod}_A \to \mathbf{Mod}^{\mathrm{qc}}_{\mathfrak{X}}$ induces an equivalence between the category of I-adically complete A-modules and adically quasi-coherent $\mathcal{O}_{\mathfrak{X}}$-modules by [25, Theorem I.3.2.8]. Recall that almost finite type modules are complete due to Lemma 2.12.7. Thus, it suffices to show that an adically quasi-coherent $\mathcal{O}_{\mathfrak{X}}$-module is almost finitely generated (resp. almost coherent) if and only if so is $\Gamma(X, \mathcal{F})$. Now this follows from Corollary 4.5.13 and Lemma 4.5.22. ■

Lemma 4.6.2. *Let* $\mathfrak{X} = \operatorname{Spf} A$ *be an affine formal* R*-scheme for* R *as in Set-up 4.5.1. Then* $(-)^{\Delta}\colon \mathbf{Mod}_A \to \mathbf{Mod}^{\mathrm{qc}}_{\mathfrak{X}}$ *induces equivalences* $(-)^{\Delta}\colon \mathbf{Mod}^*_{A^a} \to \mathbf{Mod}^*_{\mathfrak{X}^a}$*, for any* $* \in \{\mathrm{aft}, \mathrm{acoh}\}$*. The quasi-inverse functor is given by* $\Gamma(X, -)$*.*

Proof. The proof is analogous to Lemma 4.2.2 once Lemma 4.6.1 is verified. ∎

Now we recall that, for any R-scheme X, we can define the I-adic completion of X as a colimit $\operatorname{colim}(X_k, \mathcal{O}_{X_k})$ of the reductions $X_k := X \times_R \operatorname{Spec} R/I^{k+1}$ in the category of formal schemes. We refer to [25, Section 1.4(c)] for more details. This completion comes with a map of locally ringed spaces

$$c : \widehat{X} \to X.$$

In the affine case, we note that $\widehat{\operatorname{Spec} A} = \operatorname{Spf} \widehat{A}$ for any R-algebra A.[7] We study properties of the completion map for a (topologically) finitely presented R-algebra A.

Lemma 4.6.3. *Let $X = \operatorname{Spec} A$ be an affine R-scheme for R as in Set-up 4.5.1. Suppose that A is either finitely presented or topologically finitely presented over R. Then the morphism $c : \widehat{X} \to X$ is flat, and there is a functorial isomorphism $M^\Delta \cong c^*(\widetilde{M})$ for any almost finitely generated A-module M.*

Proof. The flatness assertion is proven in [25, Proposition I.1.4.7 (2)]. Now the natural map

$$M \to \mathrm{H}^0(\mathfrak{X}, c^*(\widetilde{M}))$$

induces the map $M^\Delta \to c^*(\widetilde{M})$. To show that it is an isomorphism, it suffices to show that the map

$$\widehat{M_f} \to M_f \otimes_{A_f} \widehat{A_f}$$

is an isomorphism for any $f \in A$. This follows from Lemma 2.12.7, as each such A_f is I-adically adhesive. ∎

Corollary 4.6.4. *Let X be a locally finitely presented R-scheme for a ring R as in Set-up 4.5.1. Then the morphism $c : \widehat{X} \to X$ is flat and c^* sends almost finite type $\mathcal{O}_X^a$-modules (resp. almost coherent $\mathcal{O}_X^a$-modules) to almost finite type $\mathcal{O}_{\mathfrak{X}}^a$-modules (resp. almost coherent $\mathcal{O}_{\mathfrak{X}}^a$-modules).*

Similarly, c^* sends quasi-coherent almost finite type $\mathcal{O}_X$-modules (resp. quasi-coherent almost coherent $\mathcal{O}_X$-modules) to adically quasi-coherent almost finite type $\mathcal{O}_{\mathfrak{X}}$-modules (resp. adically quasi-coherent almost coherent $\mathcal{O}_{\mathfrak{X}}$-modules)

Proof. The statement is local, so we can assume that $X = \operatorname{Spec} A$. Then the claim follows from Lemma 4.6.3. ∎

Now we show that the pullback functor preserves almost finite type and almost coherent $\mathcal{O}_{\mathfrak{X}}^a$-modules.

[7] We note that $\widehat{A}$ is I-adically complete due to [68, Tag 05GG].

Lemma 4.6.5. *Let $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{Y}$ be a morphism of locally finitely presented formal R-schemes for R as in Set-up 4.5.1.*

(1) *Suppose that $\mathfrak{X} = \operatorname{Spf} B$ and $\mathfrak{Y} = \operatorname{Spf} A$ are affine formal R-schemes. Then $\mathfrak{f}^*(M^\Delta)$ is functorially isomorphic to $(M \otimes_A B)^\Delta$, for any $M \in \mathbf{Mod}_A^{\mathrm{aft}}$.*

(2) *Suppose again that $\mathfrak{X} = \operatorname{Spf} B$ and $\mathfrak{Y} = \operatorname{Spf} A$ are affine formal R-schemes. Then $\mathfrak{f}^*(M^{a,\Delta})$ is functorially isomorphic to $(M^a \otimes_{A^a} B^a)^\Delta$, for any $M^a \in \mathbf{Mod}_A^{a,\mathrm{aft}}$.*

(3) *The functor $\mathfrak{f}^*$ sends $\mathbf{Mod}_{\mathfrak{Y}}^{\mathrm{qc,aft}}$ (resp. $\mathbf{Mod}_{\mathfrak{Y}}^{\mathrm{qc,acoh}}$) to $\mathbf{Mod}_{\mathfrak{X}}^{\mathrm{aft}}$ (resp. $\mathbf{Mod}_{\mathfrak{X}}^{\mathrm{qc,acoh}}$).*

(4) *The functor $\mathfrak{f}^*$ sends $\mathbf{Mod}_{\mathfrak{Y}^a}^{\mathrm{aft}}$ (resp. $\mathbf{Mod}_{\mathfrak{Y}^a}^{\mathrm{acoh}}$) to $\mathbf{Mod}_{\mathfrak{X}^a}^{\mathrm{aft}}$ (resp. $\mathbf{Mod}_{\mathfrak{X}^a}^{\mathrm{acoh}}$).*

Proof. We prove (1), the other parts follow from this (as in the proof of Lemma 4.2.3).

We consider a commutative diagram

$$\begin{array}{ccc} \operatorname{Spf} B & \xrightarrow{c_B} & \operatorname{Spec} B \\ {\scriptstyle \mathfrak{f}}\downarrow & & \downarrow{\scriptstyle f} \\ \operatorname{Spf} A & \xrightarrow[c_A]{} & \operatorname{Spec} Am \end{array}$$

where the map $f\colon \operatorname{Spec} B \to \operatorname{Spec} A$ is the map induced by $\mathfrak{f}^\#\colon A \to B$. Then we have that $M^\Delta \simeq c_A^* \widetilde{M}$ by Lemma 4.6.3. Therefore,

$$\mathfrak{f}^*(M^\Delta) \simeq c_B^*\big(f^*\widetilde{M}\big) \simeq c_B^*\big(\widetilde{M \otimes_A B}\big) \simeq (M \otimes_A B)^\Delta$$

where the last isomorphism follows from Lemma 4.6.3. ∎

The next thing we discuss is the interaction of tensor products and almost coherent sheaves.

Lemma 4.6.6. *Let $\mathfrak{X}$ be a topologically finitely presented formal R-scheme for R as in Set-up 4.5.1.*

(1) *Suppose that $\mathfrak{X} = \operatorname{Spf} A$ is affine. Then $M^\Delta \otimes_{\mathcal{O}_{\mathfrak{X}}} N^\Delta$ is functorially isomorphic to $(M \otimes_A N)^\Delta$ for any $M, N \in \mathbf{Mod}_A^{\mathrm{aft}}$.*

(2) *Suppose that $\mathfrak{X} = \operatorname{Spf} A$ is affine. Then $M^{a,\Delta} \otimes_{\mathcal{O}_{\mathfrak{X}}^a} N^{a,\Delta}$ is functorially isomorphic to $(M^a \otimes_{A^a} N^a)^\Delta$ for any $M^a, N^a \in \mathbf{Mod}_{A^a}^{\mathrm{aft}}$.*

(3) *Let $\mathcal{F}, \mathcal{G}$ be two adically quasi-coherent almost finite type (resp. almost finitely presented) $\mathcal{O}_{\mathfrak{X}}$-modules. Then the $\mathcal{O}_X$-module $\mathcal{F} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{G}$ is adically quasi-coherent of almost finite type (resp. almost finitely presented).*

(4) *Let $\mathcal{F}^a, \mathcal{G}^a$ be two almost finite type (resp. almost coherent) $\mathcal{O}_{\mathfrak{X}}^a$-modules. Then the $\mathcal{O}_{\mathfrak{X}}^a$-module $\mathcal{F}^a \otimes_{\mathcal{O}_{\mathfrak{X}}^a} \mathcal{G}^a$ is of almost finite type (resp. almost coherent). The analogous result holds for $\mathcal{O}_X$-modules $\mathcal{F}, \mathcal{G}$.*

Proof. Again, we only show (1) as the other parts follow from this similarly to the proof of Lemma 4.2.4.

The proof of (1) is, in turn, similar to that of Lemma 4.6.5 (1). We consider the completion morphism $c\colon \operatorname{Spf} A \to \operatorname{Spec} A$. Then we have a sequence of isomorphisms

$$\begin{aligned} M^{\Delta} \otimes_{\mathcal{O}_{\mathfrak{X}}} N^{\Delta} &\simeq c^*\big(\widetilde{M}\big) \otimes_{\mathcal{O}_{\mathfrak{X}}} c^*\big(\widetilde{N}\big) \\ &\simeq c^*\big(\widetilde{M} \otimes_{\mathcal{O}_{\operatorname{Spec} A}} \widetilde{N}\big) \simeq c^*\big(\widetilde{M \otimes_A N}\big) \simeq (M \otimes_A N)^{\Delta}. \end{aligned} \qquad \blacksquare$$

Finally, we deal with the functor $\underline{\operatorname{Hom}}_{\mathcal{O}_X^a}(-,-)$. We start with the following preparatory lemma:

Lemma 4.6.7. *Let $\mathfrak{X}$ be a locally topologically finitely presented formal R-scheme for R as in Set-up 4.5.1.*

(1) *Suppose $\mathfrak{X} = \operatorname{Spf} A$ is affine. Then the canonical map*

$$\operatorname{Hom}_A(M,N)^{\Delta} \to \underline{\operatorname{Hom}}_{\mathcal{O}_{\mathfrak{X}}}(M^{\Delta}, N^{\Delta}) \tag{4.6.1}$$

is an almost isomorphism for any almost coherent A-modules M and N.

(2) *Suppose $\mathfrak{X} = \operatorname{Spf} A$ is affine. Then there is a functorial isomorphism*

$$\operatorname{alHom}_{A^a}(M^a, N^a)^{\Delta} \simeq \underline{\operatorname{alHom}}_{\mathcal{O}_{\mathfrak{X}}^a}(M^{a,\Delta}, N^{a,\Delta}) \tag{4.6.2}$$

for any almost coherent A^a-modules M^a and N^a. We also get a functorial almost isomorphism

$$\operatorname{Hom}_{A^a}(M^a, N^a)^{\Delta} \simeq^a \underline{\operatorname{Hom}}_{\mathcal{O}_{\mathfrak{X}}^a}(M^{a,\Delta}, N^{a,\Delta}) \tag{4.6.3}$$

for any almost coherent A^a-modules M^a and N^a.

(3) *Suppose $\mathcal{F}$ and $\mathcal{G}$ are almost coherent $\mathcal{O}_{\mathfrak{X}}$-modules. Then $\underline{\operatorname{Hom}}_{\mathcal{O}_{\mathfrak{X}}}(\mathcal{F}, \mathcal{G})$ is an almost coherent $\mathcal{O}_{\mathfrak{X}}$-module.*

(4) *Suppose $\mathcal{F}^a$ and $\mathcal{G}^a$ are almost coherent $\mathcal{O}_{\mathfrak{X}}^a$-modules. Then*

$$\underline{\operatorname{Hom}}_{\mathcal{O}_{\mathfrak{X}}^a}(\mathcal{F}^a, \mathcal{G}^a) \text{ (resp. } \underline{\operatorname{alHom}}_{\mathcal{O}_{\mathfrak{X}}^a}(\mathcal{F}^a, \mathcal{G}^a))$$

is an almost coherent $\mathcal{O}_{\mathfrak{X}}$-module (resp. $\mathcal{O}_X^a$-module).

Proof. Again, the proof is analogous to that of Lemma 4.2.6 and Corollary 4.2.7 once (1) is proven. So we only give a proof of (1) here.

We note that both M and N are I-adically complete by Lemma 2.12.7. Now we use [25] to say that the natural map $\operatorname{Hom}_A(M,N) \to \operatorname{Hom}_{\mathcal{O}_{\mathfrak{X}}}(M^{\Delta}, N^{\Delta})$ is an isomorphism. This induces a morphism

$$\operatorname{Hom}_A(M,N)^{\Delta} \to \underline{\operatorname{Hom}}_{\mathcal{O}_{\mathfrak{X}}}(M^{\Delta}, N^{\Delta}).$$

In order to prove that it is an almost isomorphism, it suffices to show that the natural map

$$\mathrm{Hom}_A(M,N)\widehat{\otimes}_A A_{\{f\}} \to \mathrm{Hom}_{A_{\{f\}}}\big(M\widehat{\otimes}_A A_{\{f\}}, N\widehat{\otimes}_A A_{\{f\}}\big)$$

is an almost isomorphism for any $f \in A$. Now we note that $\mathrm{Hom}_A(M,N)$ is almost coherent by Corollary 2.6.9. Thus, $\mathrm{Hom}_A(M,N)\otimes_A A_{\{f\}}$ is already complete, so the completed tensor product coincides with the usual one. Similarly, $M\widehat{\otimes}_A A_{\{f\}} \simeq M\otimes_A A_{\{f\}}$ and $N\widehat{\otimes}_A A_{\{f\}} \simeq N\otimes_A A_{\{f\}}$. Therefore, the question boils down to showing that the natural map

$$\mathrm{Hom}_A(M,N)\otimes_A A_{\{f\}} \to \mathrm{Hom}_{A_{\{f\}}}\big(M\otimes_A A_{\{f\}}, N\otimes_A A_{\{f\}}\big)$$

is an almost isomorphism. This, in turn, follows from Lemma 2.9.11. ■

4.7 Formal schemes. Approximation of almost coherent $\mathcal{O}_{\mathfrak{X}}^a$-modules

In this section, we fix a ring R as in Set-up 4.5.1, and a topologically finitely presented formal R-scheme $\mathfrak{X}$.

The main goal of this section is to establish an analogue of Corollary 4.3.5 in the context of formal schemes. More precisely, we show that, for any finitely generated ideal $\mathfrak{m}_0 \subset \mathfrak{m}$, an almost coherent $\mathcal{O}_{\mathfrak{X}}$-module $\mathcal{F}$ can be "approximated" by a coherent $\mathcal{O}_{\mathfrak{X}}$-module $\mathcal{G}_{\mathfrak{m}_0}$ up to $\mathfrak{m}_0 \subset \mathfrak{m}$ torsion. It turns out that this result is more subtle than its algebraic counterpart because, in general, we do not know if we can present an adically quasi-coherent $\mathcal{O}_{\mathfrak{X}}$-module as a filtered colimit of finitely presented $\mathcal{O}_{\mathfrak{X}}$-modules. Also, colimits are much more subtle in the formal set-up due to the presence of topology. It seems unlikely that the method used in the proof Corollary 4.3.5 can be used in the formal set-up. Instead, we take another route and, instead, we first approximate $\mathcal{F}$ up to bounded torsion and then reduce to the algebraic case.

Definition 4.7.1. A map of $\mathcal{O}_{\mathfrak{X}}$-modules $\phi\colon \mathcal{G} \to \mathcal{F}$ is an *FP-approximation* if $\mathcal{G}$ is a finitely presented $\mathcal{O}_{\mathfrak{X}}$-module, and $I^n(\mathrm{Ker}\,\phi) = 0$, $I^n(\mathrm{Coker}\,\phi) = 0$ for some $n > 0$.

If $\mathfrak{m}_0 \subset \mathfrak{m}$ is a finitely generated sub-ideal of $\mathfrak{m}_0$, a map of $\mathcal{O}_{\mathfrak{X}}$-modules $\phi\colon \mathcal{G} \to \mathcal{F}$ is an *FP-$\mathfrak{m}_0$-approximation* if it is an FP-approximation and $\mathfrak{m}_0(\mathrm{Coker}\,\phi) = 0$.

Lemma 4.7.2. *Let $\mathfrak{X} = \mathrm{Spf}\ A$ be an affine topologically finitely presented formal R-scheme, and $\mathcal{F}$ an adically quasi-coherent $\mathcal{O}_{\mathfrak{X}}$-module of almost finite type. Then, for any finitely generated ideal $\mathfrak{m}_0 \subset \mathfrak{m}$, $\mathcal{F}$ admits an FP-$\mathfrak{m}_0$-approximation.*

Proof. Lemma 4.6.2 guarantees that $\mathcal{F} = M^{\Delta}$ for some almost finitely generated A-module M. Then, by definition, there is a submodule $N \subset M$ such that $\mathfrak{m}_0(M/N)$. By assumption, $U := \mathrm{Spec}\, A \setminus \mathrm{V}(I)$ is noetherian, so $\widetilde{N}|_U$ is a finitely presented $\mathcal{O}_U$-module. Then [25, Lemma 0.8.1.6 (2)] guarantees that there is a finitely presented

A-module N' with a surjective map $N' \to N$ such that its kernel K is I^∞-torsion. In particular, $K \subset N'[I^\infty]$. But since A is I-adically complete and noetherian outside I, [24, Theorem 5.1.2 and Definition 4.3.1] guarantee that $N'[I^\infty] = N'[I^n]$ for some $n \geq 0$. In particular, K is an I^n-torsion module.

Therefore, we have an exact sequence

$$0 \to K \to N' \to M \to Q \to 0,$$

whith the properties that N' is finitely presented, M is almost finitely generated, $\mathfrak{m}_0 Q = 0$ and $I^n K = 0$ for some $n \geq 1$. Now Lemma 4.5.14 says that the following sequence is exact:

$$0 \to K^\Delta \to N'^\Delta \to M^\Delta \to Q^\Delta \to 0.$$

In particular, N'^Δ is a finitely presented $\mathcal{O}_{\mathfrak{X}}$-module, $\mathfrak{m}_0(Q^\Delta) = 0$, and $I^n(K^\Delta)$. ■

Lemma 4.7.3. [25, Exercise I.3.4] *Let $\mathfrak{X}$ be a finitely presented formal R-scheme, $\mathcal{F}$ an adically quasi-coherent $\mathcal{O}_{\mathfrak{X}}$-module of finite type, and $\mathcal{G} \subset \mathcal{F}$ an adically quasi-coherent $\mathcal{O}_{\mathfrak{X}}$-submodule. Then $\mathcal{G}$ is a filtered colimit $\mathcal{G} = \operatorname{colim}_{\lambda \in \Lambda} \mathcal{G}_\lambda$ of adically quasi-coherent $\mathcal{O}_{\mathfrak{X}}$-submodules of finite type such that, for all $\lambda \in \Lambda$, $\mathcal{G}/\mathcal{G}_\lambda$ is annihilated by I^n for a fixed $n > 0$.*

Lemma 4.7.4. *Let $\mathfrak{X}$ be a finitely presented formal R-scheme, $\mathcal{F}$ an adically quasi-coherent, almost finitely generated $\mathcal{O}_{\mathfrak{X}}$-module, and $\phi_i \colon \mathcal{G}_i \to \mathcal{F}$ for $i = 1, 2$ two FP-$\mathfrak{m}_0$-approximations of $\mathcal{F}$ for some finitely generated sub-ideal $\mathfrak{m}_0 \subset \mathfrak{m}$. Then there is a commutative diagram*

$$\begin{array}{ccc}
\mathcal{G}_1 & & \\
{\scriptstyle q_1}\downarrow & \searrow^{\phi_1} & \\
\mathcal{H} & \xrightarrow{\phi} & \mathcal{F}, \\
{\scriptstyle q_2}\uparrow & \nearrow_{\phi_2} & \\
\mathcal{G}_2 & &
\end{array}$$

where ϕ and q_i are FP-$\mathfrak{m}_0$-approximations for $i = 1, 2$.

Proof. By assumption, there is an integer $c > 0$ such that $\operatorname{Ker}(\phi_i)$ and $\operatorname{Coker}(\phi_i)$ are annihilated by I^c for $i = 0, 1$. Therefore, we may replace $\mathfrak{m}_0$ by $\mathfrak{m}_0 + I^c$ to assume that $\mathfrak{m}_0$ contains I^c.

Now we define $\mathcal{K}$ to be the kernel of the natural morphism $\mathcal{G}_1 \oplus \mathcal{G}_2 \to \mathcal{F}$. Note that it is an adically quasi-coherent $\mathcal{O}_{\mathfrak{X}}$-submodule of $\mathcal{G}_1 \oplus \mathcal{G}_2$ due to Lemma 4.5.14. Therefore, Lemma 4.7.3 applies to the inclusion $\mathcal{K} \subset \mathcal{G}_1 \oplus \mathcal{G}_2$, so we can write

$\mathcal{K} = \operatorname{colim}_{\lambda \in \Lambda} \mathcal{K}_\lambda$ as a filtered colimit of adically quasi-coherent, finite type $\mathcal{O}_{\mathfrak{X}}$-submodules of $\mathcal{G}_1 \oplus \mathcal{G}_2$ with $I^m(\mathcal{K}/\mathcal{K}_\lambda) = 0$ for some fixed $m > 0$ and every $\lambda \in \Lambda$. We define $\mathcal{H}_\lambda = (\mathcal{G}_1 \oplus \mathcal{G}_2)/\mathcal{K}_\lambda$, it comes with the natural morphisms

$$\phi_\lambda \colon \mathcal{H}_\lambda \to \mathcal{F},$$
$$q_{i,\lambda} \colon \mathcal{G}_i \to \mathcal{H}_\lambda$$

for $i = 1, 2$. We claim that these morphisms satisfy the claim of the lemma for some $\lambda \in \Lambda$, i.e., ϕ_λ, and $q_{i,\lambda}$ are FP-$\mathfrak{m}_0$-approximations.

Since $\mathfrak{X}$ is topologically finitely presented (in particular, it is quasi-compact and quasi-separated), these claims can be checked locally. So we may and do assume that $\mathfrak{X} = \operatorname{Spf} A$ is affine. Then we use Lemma 4.6.2, [25, Theorem I.3.2.8, Proposition I.3.5.4] to reduce to the situation where $\mathfrak{X} = \operatorname{Spf} A$, $\mathcal{F} = M^\Delta$, $\mathcal{G}_1 = N_1^\Delta$, $\mathcal{G}_2 = N_2^\Delta$ for some almost finitely generated A-module M, and finitely presented A-modules N_1, N_2 with maps of sheaves induced by homomorphisms $N_1 \to M$ and $N_2 \to M$. Then Lemma 4.5.14 says that $\mathcal{K} = K^\Delta$ for $K = \operatorname{Ker}(N_1 \oplus N_2 \to M)$, and $K = \operatorname{colim}_{\lambda \in \Lambda} K_\lambda$ for finitely generated A-submodules[8] K_λ with $I^m(K/K_\lambda) = 0$ for some fixed $m > 0$ and all $\lambda \in \Lambda$. So one can use Lemma 4.5.14 once again to conclude that it suffices (due to the assumption that $I^c \subset \mathfrak{m}_0$) to show that, for some $\lambda \in \Lambda$, the natural morphisms $(N_1 \oplus N_2)/K_\lambda \to M$, $N_i \to (N_1 \oplus N_2)/K_\lambda$ have kernels annihilated by some power of I, and cokernels annihilated by $\mathfrak{m}_0$.

The kernels of $N_i \to (N_1 \oplus N_2)/K_\lambda$ (for $i = 1, 2$) embed into the respective kernels for the natural morphisms $N_i \to M$, so they are automatically annihilated by some power of I for any $\lambda \in \Lambda$. Also, clearly, the morphism $(N_1 \oplus N_2)/K_\lambda \to M$ has kernel K/K_λ that is annihilated by I^m by the choice of K_λ.

Therefore, it suffices to show that we can choose $\lambda \in \Lambda$ such that $q_{i,\lambda} \colon N_i \to (N_1 \oplus N_2)/K_\lambda$ (for $i = 1, 2$) and $\phi_\lambda \colon (N_1 \oplus N_2)/K_\lambda \to M$ have cokernels annihilated by $\mathfrak{m}_0$. The latter case is automatic and actually holds for any $\lambda \in \Lambda$. So the only non-trivial thing we need to check is that $\mathfrak{m}_0(\operatorname{Coker} q_{i,\lambda}) = 0$ for some $\lambda \in \Lambda$.

Let $(m_1, \dots, m_d) \in \mathfrak{m}_0$ be a finite set of generators, and $\{y_{i,j}\}_{j \in J_i}$ a finite set of generators of N_i for $i = 1, 2$. Denote by $\overline{y_{i,j}}$ the image of $y_{i,j}$ in M. Define $x_{i,j,k} \in N_{2-i}$ to be a lift of $m_k \overline{y_{i,j}} \in M$ in N_{2-i} for $k = 1, \dots, d$, $i = 1, 2$ and $j \in J_i$. Note that elements $(m_k y_{1,j}, x_{1,j,k}) \in N_1 \oplus N_2$ and $(x_{2,j,k}, m_k y_{2,j}) \in N_1 \oplus N_2$ lie in K. Consequently, for some $\lambda \in \Lambda$, K_λ contains the elements $(m_k y_{1,j}, x_{1,j,k})$ and $(x_{2,j,k}, m_k y_{2,j})$. Then it is easy to see that the cokernels of $N_i \to (N_1 \oplus N_2)/K_\lambda$ are annihilated by $\mathfrak{m}_0$. This finishes the proof. ∎

Lemma 4.7.5. *Let $\mathfrak{X}$ be a finitely presented formal R-scheme, $\mathcal{F}$ an adically quasi-coherent, almost finitely generated $\mathcal{O}_{\mathfrak{X}}$-module. Then, for any finitely generated ideal $\mathfrak{m}_0 \subset \mathfrak{m}$, $\mathcal{F}$ is FP-$\mathfrak{m}_0$-approximated.*

[8] Here, $K_\lambda = \Gamma(\mathfrak{X}, \mathcal{K}_\lambda)$, so the equality follows from [68, Tag 009F].

Proof. First, we note that Lemma 4.7.2 guarantees that the claim holds if $\mathfrak{X}$ is affine. Now choose a covering of $\mathfrak{X}$ by open affines $\mathfrak{X} = \bigcup_{i=1}^n \mathfrak{V}_i$. We know that claim on each $\mathfrak{V}_i$, so it suffices to show that, if $\mathfrak{X} = \mathfrak{U}_1 \cup \mathfrak{U}_2$ is a union of two finitely presented open formal subschemes and $\mathcal{F}$ is FP-$\mathfrak{m}_0$-approximated on $\mathfrak{U}_1$ and $\mathfrak{U}_2$, then $\mathcal{F}$ is FP-$\mathfrak{m}_0$-approximated on $\mathfrak{X}$.

Suppose that $\mathcal{G}_i \to \mathcal{F}|_{\mathfrak{U}_i}$ are FP-$\mathfrak{m}_0$-approximations on $\mathfrak{U}_i$ for $i = 1, 2$. Then the intersection $\mathfrak{U}_{1,2} := \mathfrak{U}_1 \cap \mathfrak{U}_2$ is again a topologically finitely presented formal R-scheme because $\mathfrak{X}$ is so. Therefore, Lemma 4.7.4 guarantees that we can find another FP-$\mathfrak{m}_0$-approximation $\mathcal{H} \to \mathcal{F}|_{\mathfrak{U}_{1,2}}$ that is dominated by both $\mathcal{G}_i|_{\mathfrak{U}_{1,2} \to \mathcal{F}|_{\mathfrak{U}_{1,2}}}$ for $i = 1, 2$. Consider the $\mathcal{O}_{\mathfrak{U}_{1,2}}$-modules

$$\mathcal{K}_i := \mathrm{Ker}(\mathcal{G}_i|_{\mathfrak{U}_{1,2}} \to \mathcal{H}) \text{ for } i = 1, 2.$$

Lemma 4.5.14 guarantees that both $\mathcal{K}_i$ are adically quasi-coherent $\mathcal{O}_{\mathfrak{X}}$-modules of finite type.[9] The fact that $\mathcal{G}_i|_{\mathfrak{U}_{1,2}} \to \mathcal{H}$ are FP-$\mathfrak{m}_0$-approximations ensures that both $\mathcal{K}_i$ are killed by some I^m for $m \geq 1$. In particular, we see that $\mathcal{K}_i \subset \mathcal{G}_i[I^m]|_{\mathfrak{U}_{1,2}}$, so they are naturally quasi-coherent sheaves on $\mathfrak{X}_{m-1} = \mathfrak{X} \times_{\mathrm{Spf}\, R} \mathrm{Spec}\, R/I^m$. Therefore, one can use [68, Tag 01PF] (applied to $\mathfrak{X}_{m-1}$) to extend $\mathcal{K}_i$ to

$$\widetilde{\mathcal{K}}_i \subset \mathcal{G}_i[I^m] \subset \mathcal{G}_i,$$

where $\widetilde{\mathcal{K}}_i$ are adically quasi-coherent $\mathcal{O}_{\mathfrak{X}}$-modules of finite type. Then we see that $\mathcal{G}_i/\widetilde{\mathcal{K}}_i \to \mathcal{F}|_{\mathfrak{U}_i}$ are FP-$\mathfrak{m}_0$-approximations of $\mathcal{F}|_{\mathfrak{U}_i}$ that are isomorphic on the intersection. Therefore, they glue to a global FP-$\mathfrak{m}_0$-approximation $\mathcal{G} \to \mathcal{F}$. ∎

Theorem 4.7.6. *Let $\mathfrak{X}$ be a finitely presented formal R-scheme, $\mathcal{F}$ an almost finitely generated (resp. almost finitely presented) $\mathcal{O}_{\mathfrak{X}}$-module. Then, for any finitely generated ideal $\mathfrak{m}_0 \subset \mathfrak{m}$, there are an adically quasi-coherent, finitely generated (resp. finitely presented) $\mathcal{O}_{\mathfrak{X}}$-module $\mathcal{G}$ and a map $\phi\colon \mathcal{G} \to \mathcal{F}$ such that $\mathfrak{m}_0(\mathrm{Coker}\,\phi) = 0$ and $\mathfrak{m}_0(\mathrm{Ker}\,\phi) = 0$.*

Proof. Without loss of generality, we can replace $\mathcal{F}$ by $\widetilde{\mathfrak{m}} \otimes \mathcal{F}$, so we may and do assume that $\mathcal{F}$ is adically quasi-coherent.

The case of almost adically quasi-coherent, almost finite type $\mathcal{O}_{\mathfrak{X}}$-module $\mathcal{F}$ follows from Lemma 4.7.5. Indeed, there is an FP-$\mathfrak{m}_0$-approximation $\phi'\colon \mathcal{G}' \to \mathcal{F}$, so we define $\phi\colon \mathcal{G} \to \mathcal{F}$ to be the natural inclusion $\mathcal{G} := \mathrm{Im}(\phi') \to \mathcal{F}$. This gives the desired morphism as $\mathcal{G}$ is an adically quasi-coherent $\mathcal{O}_{\mathfrak{X}}$-module of almost finite type by Corollary 4.5.16.

Now suppose $\mathcal{F}$ is an adically quasi-coherent, almost finitely presented $\mathcal{O}_{\mathfrak{X}}$-module. Then we use Lemma 4.7.5 to find an FP-$\mathfrak{m}_0$-approximation $\phi'\colon \mathcal{G}' \to \mathcal{F}$.

[9] Since they are kernels of morphisms between coherent $\mathcal{O}_{\mathfrak{X}}$-modules.

Now we note that any almost finitely presented $\mathcal{O}_{\mathfrak{X}}$-module is almost coherent by Lemma 4.5.22. Therefore, $\operatorname{Ker}\phi$ is again adically quasi-coherent, almost finitely presented. Therefore, we can find an FP-$\mathfrak{m}_0$-approximation $\phi'': \mathcal{G}'' \to \operatorname{Ker}(\phi')$ by Lemma 4.7.5. Denote by $\phi''': \mathcal{G}'' \to G'$ the composition of ϕ'' with the natural inclusion $\operatorname{Ker}(\phi') \to \mathcal{G}'$. Now it is easy to check that $\phi: \operatorname{Coker}(\phi''') \to \mathcal{F}$ gives the desired "approximation". ∎

4.8 Formal schemes. Derived category of almost coherent $\mathcal{O}_{\mathfrak{X}}^a$-modules

We discuss the notion of the derived category of almost coherent sheaves on a formal scheme $\mathfrak{X}$. One major issue is that, in this situation, the derived category of $\mathcal{O}_{\mathfrak{X}}$-modules with adically quasi-coherent cohomology sheaves is not well defined, as adically quasi-coherent sheaves do not form a weak Serre subcategory of $\mathbf{Mod}_{\mathfrak{X}}$.

To overcome this issue, we follow the strategy used in [49] and define another category "$\mathbf{D}_{\mathrm{qc}}(\mathfrak{X})$" completely on the level of derived categories. For the rest of the section, we fix a base ring R as in Set-up 4.5.1.

Definition 4.8.1. Let $\mathfrak{X}$ be a locally topologically finitely presented R-scheme. Then we define the *derived category of adically quasi-coherent sheaves* "$\mathbf{D}_{\mathrm{qc}}(\mathfrak{X})$" as a full subcategory of $\mathbf{D}(\mathfrak{X})$ consisting of objects $\mathcal{F}$ such that the following conditions are met:

- For every open affine $\mathfrak{U} \subset \mathfrak{X}$, $\mathbf{R}\Gamma(\mathfrak{U}, \mathcal{F}) \in \mathbf{D}(\mathcal{O}_{\mathfrak{X}}(\mathfrak{U}))$ is derived I-adically complete.
- For every inclusion $\mathfrak{U} \subset \mathfrak{V}$ of affine formal subschemes of $\mathfrak{X}$, the natural morphism

$$\mathbf{R}\Gamma(\mathfrak{V}, \mathcal{F}) \widehat{\otimes}^L_{\mathcal{O}_{\mathfrak{X}}(\mathfrak{V})} \mathcal{O}_{\mathfrak{X}(\mathfrak{U})} \to \mathbf{R}\Gamma(\mathfrak{U}, \mathcal{F})$$

  is an isomorphism, where the completion is understood in the derived sense.

Remark 4.8.2. We refer to [68, Tag 091N] and [68, Tag 0995] for a self-contained discussion on derived completions of modules and sheaves of modules respectively.

We now want to give an interpretation of "$\mathbf{D}_{\mathrm{qc}}(\mathfrak{X})$" in terms of A-modules for an affine formal scheme $\mathfrak{X} = \operatorname{Spf} A$. We recall that in the case of schemes, we have a natural equivalence $\mathbf{D}_{\mathrm{qc}}(\operatorname{Spec} A) \simeq \mathbf{D}(A)$ and the map is induced by $\mathbf{R}\Gamma(\operatorname{Spec} A, -)$. In the case of formal schemes, it is not literally true. We need to impose certain completeness conditions.

Definition 4.8.3. Let A be a ring with a finitely generated ideal I. We define the *complete derived category* $\mathbf{D}_{\mathrm{comp}}(A, I) \subset \mathbf{D}(A)$ as a full triangulated subcategory consisting of the I-adically derived complete objects.

Suppose that $\mathfrak{X} = \operatorname{Spf} A$ is a topologically finitely presented affine formal R-scheme. We note that the natural functor $\mathbf{R}\Gamma(\mathfrak{X}, -) \colon \mathbf{D}(\mathfrak{X}) \to \mathbf{D}(A)$ induces a functor

$$\mathbf{R}\Gamma(\mathfrak{X}, -) \colon \text{“}\mathbf{D}_{\mathrm{qc}}(\mathfrak{X})\text{”} \to \mathbf{D}_{\mathrm{comp}}(A, I).$$

We wish to show that this functor is an equivalence. For this, we need some preliminary lemmas:

Lemma 4.8.4. *Let A be a topologically finitely presented R-algebra for R as in Set-up 4.5.1, let $f \in A$ be any element, and let $(x_1, \dots, x_d) = I$ be a choice of generators for the ideal of definition of R. Denote by $K(A_f; x_1^n, \dots, x_d^n)$ the Koszul complexes for the sequence $(x_1^n, \dots, x_d^n)$. Then the pro-systems $\{K(A_f; x_1^n, \dots, x_d^n)\}$ and $\{A_f/I^n\}$ are isomorphic in* $\operatorname{Pro}(\mathbf{D}(A_f))$.

Proof. The proof is the same as [68, Tag 0921]. The only difference is that one needs to use [24, Theorem 4.2.2 (2) (b)] in place of the usual Artin–Rees lemma. ∎

Lemma 4.8.5. *Let A be a topologically finitely presented R-algebra for R as in Set-up 4.5.1, let $f \in A$ be any element. Then the completed localization $A_{\{f\}}$ coincides with the I-adic derived completion of A_f.*

Proof. Choose some generators $I = (x_1, \dots, x_d)$. Then we know that the derived completion of A_f is given by $\mathbf{R}\lim_n K(A_f; x_1^n, \dots, x_d^n)$, where $K(A_f; x_1^n, \dots, x_d^n)$ is the Koszul complex for the sequence $(x_1^n, \dots, x_d^n)$. Lemma 4.8.4 implies that the pro-systems $\{K(A_f; x_1^n, \dots, x_d^n)\}$ and $\{A_f/I^n\}$ are naturally pro-isomorphic. Thus we have an isomorphism

$$\mathbf{R}\lim_n K(A_f; x_1^n, \dots, x_d^n) \cong \mathbf{R}\lim_n A_f/I^n \simeq A_{\{f\}}.$$

The last isomorphism uses the Mittag-Leffler criterion to ensure vanishing of $\lim^1$. ∎

Theorem 4.8.6 ([49, Corollary 8.2.4.15]). *Let $\mathfrak{X} = \operatorname{Spf} A$ be an affine, finitely presented formal scheme over R as in Set-up 4.5.1. Then the corresponding functor $\mathbf{R}\Gamma(\mathfrak{X}, -) \colon \text{“}\mathbf{D}_{\mathrm{qc}}(\mathfrak{X})\text{”} \to \mathbf{D}_{\mathrm{comp}}(A, I)$ is an equivalence of categories.*

Proof. Lemma 4.8.5 implies that the definition of $\operatorname{Spf} A$ in [49] is compatible with the classical one. Now [49, Proposition 8.2.4.18] ensures that our definition of “$\mathbf{D}_{\mathrm{qc}}(\mathfrak{X})$” is equivalent to the homotopy category of $\operatorname{Qcoh}(\mathfrak{X})$ in the sense of [49]. Therefore, the result follows from [49, Corollary 8.2.4.15] by passing to the homotopy categories.

The proof of [49, Corollary 8.2.4.15] can also be rephrased in our situation without using any derived geometry. However, it would require quite a long digression. ∎

Definition 4.8.7. We denote by

$$(-)^{L\Delta}\colon \mathbf{D}_{\mathrm{comp}}(A, I) \to \text{“}\mathbf{D}_{\mathrm{qc}}(\mathfrak{X})\text{”}$$

the pseudo-inverse to $\mathbf{R}\Gamma(\mathfrak{X}, -)\colon \text{“}\mathbf{D}_{\mathrm{qc}}(\mathfrak{X})\text{”} \to \mathbf{D}_{\mathrm{comp}}(A, I)$. We note that

$$\mathbf{R}\Gamma\big(\mathrm{Spf}\ A_{\{f\}}, M^{L\Delta}\big) \simeq M \widehat{\otimes}_A A_{\{f\}}$$

for any $M \in \mathbf{D}_{\mathrm{comp}}(A, I)$.

Remark 4.8.8. The functor $(-)^{L\Delta}$ *is not* compatible with the “abelian” functor $(-)^{\Delta}$ used the previous sections.

Now we define a category $\mathbf{D}_{\mathrm{qc,acoh}}(\mathfrak{X})$ and show that it is equivalent to $\mathbf{D}_{\mathrm{acoh}}(A)$. Theorem 4.8.6 will be an important technical tool for establishing this equivalence.

Definition 4.8.9. We define $\mathbf{D}_{\mathrm{qc,acoh}}(\mathfrak{X})$ (resp. $\mathbf{D}_{\mathrm{acoh}}(\mathfrak{X})^a$) to be the full triangulated subcategory of $\mathbf{D}(\mathfrak{X})$ (resp. $\mathbf{D}(\mathfrak{X})^a$) consisting of the complexes with adically quasi-coherent, almost coherent (resp. almost coherent) cohomology sheaves (resp. almost sheaves).

Remark 4.8.10. An argument similar to the one in the proof of Lemma 4.4.5 shows that $\mathbf{D}_{\mathrm{acoh}}(\mathfrak{X})^a$ is equivalent to the Verdier quotient $\mathbf{D}_{\mathrm{qc,acoh}}(\mathfrak{X})/\mathbf{D}_{\mathrm{qc},\Sigma_{\mathfrak{X}}}(\mathfrak{X})$.

In order to show an equivalence $\mathbf{D}_{\mathrm{qc,acoh}}(\mathfrak{X}) \simeq \mathbf{D}_{\mathrm{acoh}}(A)$, our first goal is to show that $\mathbf{D}_{\mathrm{qc,acoh}}$ lies inside “$\mathbf{D}_{\mathrm{qc}}(\mathfrak{X})$”. This is not entirely obvious because the definition of $\mathbf{D}_{\mathrm{qc,acoh}}(\mathfrak{X})$ imposes some restrictions on individual cohomology sheaves while the definition of “$\mathbf{D}_{\mathrm{qc}}(\mathfrak{X})$” on the whole complex itself.

Lemma 4.8.11. *Let $\mathfrak{X} = \mathrm{Spf}\ A$ be an affine topologically finitely presented formal R-scheme for R as in Set-up 4.5.1. Then the functor $\mathbf{R}\Gamma(\mathfrak{X}, -)\colon \mathbf{D}_{\mathrm{qc,acoh}}(\mathfrak{X}) \to \mathbf{D}(A)$ is t-exact (with respect to the evident t-structures on both sides) and factors through $\mathbf{D}_{\mathrm{acoh}}(A)$. More precisely, there is an isomorphism*

$$\mathrm{H}^i\big(\mathbf{R}\Gamma(\mathfrak{X}, \mathcal{F})\big) \simeq \mathrm{H}^0\big(\mathfrak{X}, \mathcal{H}^i(\mathcal{F})\big) \in \mathbf{Mod}_A^{\mathrm{acoh}}$$

for any object $\mathcal{F} \in \mathbf{D}_{\mathrm{qc,acoh}}(\mathfrak{X})$.

Proof. We note that the vanishing theorem [25, Theorem I.7.1.1] implies that we can use [68, Tag 0D6U] with $N = 0$. Thus, we see that the map $\mathrm{H}^i(\mathbf{R}\Gamma(\mathfrak{X}, \mathcal{F})) \to \mathrm{H}^i(\mathbf{R}\Gamma(\mathfrak{X}, \tau^{\geq i}\mathcal{F}))$ is an isomorphism for any integer i, and that $\mathbf{R}\Gamma(\mathfrak{X}, \mathcal{F}) \in \mathbf{D}_{\mathrm{acoh}}(A)$ for any $\mathcal{F} \in \mathbf{D}_{\mathrm{qc,acoh}}(\mathfrak{X})$. Applying it together with the canonical isomorphism $\mathrm{H}^i(\mathbf{R}\Gamma(\mathfrak{X}, \tau^{\geq i}\mathcal{F})) \simeq \mathrm{H}^0(\mathfrak{X}, \mathcal{H}^i(\mathcal{F}))$, we get the desired result. ∎

Lemma 4.8.12. *Let $\mathfrak{X}$ be a locally topologically finitely presented formal R-scheme for R as in Set-up 4.5.1. Then $\mathbf{D}_{\mathrm{qc,acoh}}(\mathfrak{X})$ is naturally a full triangulated subcategory of “$\mathbf{D}_{\mathrm{qc}}(\mathfrak{X})$”.*

Proof. Both $\mathbf{D}_{\mathrm{qc,acoh}}(\mathfrak{X})$ and "$\mathbf{D}_{\mathrm{qc}}(\mathfrak{X})$" are full triangulated subcategories of $\mathbf{D}(\mathfrak{X})$. Thus, it suffices to show that any $\mathcal{F} \in \mathbf{D}_{\mathrm{qc,acoh}}(\mathfrak{X})$ lies in "$\mathbf{D}_{\mathrm{qc}}(\mathfrak{X})$".

Lemma 4.8.11 and Corollary 2.12.8 imply that $\mathbf{R}\Gamma(\mathfrak{U}, \mathcal{F}) \in \mathbf{D}_{\mathrm{comp}}(A, I)$ for any open affine $\mathfrak{U} \subset \mathfrak{X}$. Now suppose $\mathfrak{U} \subset \mathfrak{V}$ is an inclusion of open affine formal subschemes in $\mathfrak{X}$. We consider the natural morphism

$$\mathbf{R}\Gamma(\mathfrak{V}, \mathcal{F})\widehat{\otimes}^L_{\mathcal{O}_{\mathfrak{X}}(\mathfrak{V})}\mathcal{O}_{\mathfrak{X}}(\mathfrak{U}) \to \mathbf{R}\Gamma(\mathfrak{U}, \mathcal{F}).$$

We note that $\mathcal{O}_{\mathfrak{X}}(\mathfrak{U})$ is flat over $\mathcal{O}_{\mathfrak{X}}(\mathfrak{V})$ by [25, Proposition I.4.8.1]. Thus, the complex

$$\mathbf{R}\Gamma(\mathfrak{V}, \mathcal{F}) \otimes^L_{\mathcal{O}_{\mathfrak{X}}(\mathfrak{V})} \mathcal{O}_{\mathfrak{X}(\mathfrak{U})}$$

lies in $\mathbf{D}_{\mathrm{acoh}}(\mathcal{O}_{\mathfrak{X}}(\mathfrak{U}))$ by Lemma 2.8.1. Therefore, it also lies in $\mathbf{D}_{\mathrm{comp}}(A, I)$ by Corollary 2.12.8. So we conclude that

$$\mathbf{R}\Gamma(\mathfrak{V}, \mathcal{F})\widehat{\otimes}^L_{\mathcal{O}_{\mathfrak{X}}(\mathfrak{V})}\mathcal{O}_{\mathfrak{X}}(\mathfrak{U}) \simeq \mathbf{R}\Gamma(\mathfrak{V}, \mathcal{F}) \otimes^L_{\mathcal{O}_{\mathfrak{X}}(\mathfrak{V})} \mathcal{O}_{\mathfrak{X}}(\mathfrak{U}).$$

Using $\mathcal{O}_{\mathfrak{X}}(\mathfrak{V})$-flatness of $\mathcal{O}_{\mathfrak{X}}(\mathfrak{U})$, we conclude that the question boils down to showing that

$$\mathrm{H}^i(\mathfrak{V}, \mathcal{F}) \otimes_{\mathcal{O}_{\mathfrak{X}}(\mathfrak{V})} \mathcal{O}_{\mathfrak{X}(\mathfrak{U})} \to \mathrm{H}^i(\mathfrak{U}, \mathcal{F})$$

is an isomorphism for all i. Now Lemma 4.8.11 implies that this, in turn, reduces to showing that the natural map

$$\Gamma(\mathfrak{V}, \mathcal{H}^i(\mathcal{F})) \otimes_{\mathcal{O}_{\mathfrak{X}}(\mathfrak{V})} \mathcal{O}_{\mathfrak{X}(\mathfrak{U})} \to \Gamma(\mathfrak{U}, \mathcal{H}^i(\mathcal{F}))$$

is an isomorphism. Without loss of generality, we may assume $\mathfrak{X} = \mathfrak{V} = \mathrm{Spf}\ A$. Then $\mathcal{H}^i(\mathcal{F})$ is an adically quasi-coherent, almost coherent $\mathcal{O}_{\mathfrak{X}}$-module, so Lemma 4.6.1 ensures that it is isomorphic to M^{Δ} for some $M \in \mathbf{Mod}_A^{\mathrm{acoh}}$. Thus, the desired claim follows from [25, Lemma 3.6.4] and the observation that $M \otimes_{\mathcal{O}_{\mathfrak{X}}(\mathfrak{V})} \mathcal{O}_{\mathfrak{X}}(\mathfrak{U})$ is already I-adically complete due to Lemma 2.12.7. ∎

Now we show that the functor $(-)^{L\Delta}$ sends $\mathbf{D}_{\mathrm{acoh}}(A)$ to $\mathbf{D}_{\mathrm{qc,acoh}}(\mathrm{Spf}\ A)$. This is also not entirely obvious as $(-)^{L\Delta}$ is a priori different from the classical version of the $(-)^{\Delta}$-functor studied in previous sections. However, we show that these functors coincide on $\mathbf{Mod}_A^{\mathrm{acoh}}$.

Lemma 4.8.13. *Let $\mathfrak{X} = \mathrm{Spf}\ A$ be an affine topologically finitely presented formal R-scheme for R as in Set-up 4.5.1. Then the functor $(-)^{L\Delta}\colon \mathbf{D}_{\mathrm{acoh}}(A) \to$ "$\mathbf{D}_{\mathrm{qc}}(\mathfrak{X})$" factors through $\mathbf{D}_{\mathrm{qc,acoh}}(\mathfrak{X})$. Moreover, for any $M \in \mathbf{D}_{\mathrm{acoh}}(A)$ and an integer i, there is a functorial isomorphism*

$$\mathrm{H}^i(M)^{\Delta} \simeq \mathcal{H}^i(M^{L\Delta}).$$

Proof. We note that $\mathrm{H}^i(\mathfrak{X}, M^{L\Delta}) \simeq \mathrm{H}^i(M)$ due to its construction. Since $\mathcal{H}^i(M^{L\Delta})$ is canonically isomorphic to the sheafification of the presheaf

$$\mathfrak{U} \mapsto \mathrm{H}^i(\mathfrak{U}, M^{L\Delta}),$$

we get that there is a canonical map $\mathrm{H}^i(M) \to \Gamma(\mathfrak{X}, \mathcal{H}^i(M^{\Delta}))$. By the universal property of the classical $(-)^{\Delta}$ functor, we get a functorial morphism

$$\mathrm{H}^i(M)^{\Delta} \to \mathcal{H}^i(M^{L\Delta}).$$

Since $\mathrm{H}^i(M)$ is almost coherent, we only need to show that this map is an isomorphism for any i. This boils down (using almost coherence of $\mathrm{H}^i(M)$) to showing that the natural morphism

$$\mathrm{H}^i(M) \otimes_A A_{\{f\}} \to \mathrm{H}^i\left(\mathrm{Spf}\ A_{\{f\}}, M^{L\Delta}\right)$$

is an isomorphism for all $f \in A$. Now recall that $\mathbf{R}\Gamma(\mathrm{Spf}\ A_{\{f\}}, M^{L\Delta}) \simeq M \widehat{\otimes}^L_A A_{\{f\}}$ for any $f \in A$. Using that $M \in \mathbf{D}_{\mathrm{acoh}}(A)$, $A_{\{f\}}$ is flat over A, and that almost coherent complexes are derived complete by Corollary 2.12.8, we conclude that the natural map

$$\mathrm{H}^i(M) \otimes_A A_{\{f\}} \to \mathrm{H}^i(\mathrm{Spf}\ A_{\{f\}}, M^{L\Delta})$$

is an isomorphism finishing the proof. ∎

Corollary 4.8.14. *Let $\mathfrak{X} = \mathrm{Spf}\ A$ be an affine topologically finitely presented formal R-scheme for R as in Set-up 4.5.1. Suppose that $M \in \mathbf{D}(A)$ has almost zero cohomology modules. Then $\mathcal{H}^i(M^{L\Delta})$ is an almost zero, adically quasi-coherent $\mathcal{O}_{\mathfrak{X}}$-module for each integer i. Therefore, in particular, $(-)^{L\Delta}$ induces a functor $(-)^{L\Delta}\colon \mathbf{D}_{\mathrm{acoh}}(A)^a \to \mathbf{D}_{\mathrm{acoh}}(\mathfrak{X})^a$.*

Proof. We note that any almost zero A-module is almost coherent, thus the result follows directly from the formula $\mathrm{H}^i(M)^{\Delta} \simeq \mathcal{H}^i(M^{L\Delta})$ in Lemma 4.8.13. ∎

Theorem 4.8.15. *Let $\mathfrak{X} = \mathrm{Spf}\ A$ be an affine topologically finitely presented formal R-scheme for R as in Set-up 4.5.1. Then $\mathbf{R}\Gamma(\mathfrak{X}, -)\colon \mathbf{D}_{\mathrm{qc,acoh}}(\mathfrak{X}) \to \mathbf{D}_{\mathrm{acoh}}(A)$ is a t-exact equivalence of triangulated categories with the pseudo-inverse $(-)^{L\Delta}$.*

Proof. Lemma 4.8.11 implies that $\mathbf{R}\Gamma(\mathfrak{X}, -)$ induces the stated functor $\mathbf{D}_{\mathrm{qc,acoh}}(\mathfrak{X}) \to \mathbf{D}_{\mathrm{acoh}}(A)$ and that this functor is t-exact. Lemma 4.8.12 and Theorem 4.8.6 ensure that it is sufficient to show that $(-)^{L\Delta}$ sends $\mathbf{D}_{\mathrm{acoh}}(A)$ to $\mathbf{D}_{\mathrm{qc,acoh}}(\mathfrak{X})$, this follows from Lemma 4.8.13. ∎

Now we can pass to the almost categories using Remark 4.8.10 to get the almost version of Theorem 4.8.15.

Corollary 4.8.16. *Let $\mathfrak{X} = \operatorname{Spf} A$ be an affine topologically finitely presented formal R-scheme for R as in Set-up 4.5.1. Then $\mathbf{R}\Gamma(\mathfrak{X}, -)\colon \mathbf{D}_{\mathrm{acoh}}(\mathfrak{X})^a \to \mathbf{D}_{\mathrm{acoh}}(A)^a$ is a t-exact equivalence of triangulated categories with the pseudo-inverse $(-)^{L\Delta}$.*

4.9 Formal schemes. Basic functors on derived categories of $\mathcal{O}_{\mathfrak{X}}^a$-modules

We discuss the derived analogue of the main results of Section 4.6. We show that the derived completion, derived tensor product, derived pullback, and derived almost Hom functors preserve complexes with almost coherent cohomology sheaves under certain conditions. For the rest of the section, we fix a ring R as in Set-up 4.5.1.

We start with the completion functor. We recall that we have defined the morphism of locally ringed spaces $c\colon \widehat{X} \to X$ for any R-scheme X. If X is locally finitely presented over R or $X = \operatorname{Spec} A$ for a topologically finitely presented R-algebra A, then c is a flat morphism as shown in Lemma 4.6.3 and Corollary 4.6.4.

Lemma 4.9.1. *Let $X = \operatorname{Spec} A$ be an affine R-scheme for R as in Set-up 4.5.1. Suppose that A is either finitely presented or topologically finitely presented over R. Suppose $M \in \mathbf{D}_{\mathrm{acoh}}(A)$. Then $M^{L\Delta} \simeq \mathbf{L}c^*(\widetilde{M})$.*

Proof. First of all, we show that $\mathbf{L}c^*(\widetilde{M}) \in \mathbf{D}_{\mathrm{qc,acoh}}(\widehat{X})$. Indeed, the functor c^* is exact as c is flat. Thus, Lemma 4.6.3 guarantees that we have a sequence of isomorphisms

$$\mathcal{H}^i\big(\mathbf{L}c^*(\widetilde{M})\big) \simeq c^*\big(\widetilde{\mathrm{H}^i(M)}\big) \simeq \big(\mathrm{H}^i(M)\big)^{\Delta}.$$

In particular, Theorem 4.8.6 ensures that the natural morphism

$$M \simeq \mathbf{R}\Gamma(X, \widetilde{M}) \to \mathbf{R}\Gamma\big(\widehat{X}, \mathbf{L}c^*(\widetilde{M})\big)$$

induces the morphism $M^{L\Delta} \to \mathbf{L}c^*(\widetilde{M})$. As c^* is exact, Lemma 4.8.13 implies that it is sufficient to show that the natural map

$$\mathrm{H}^i(M)^{\Delta} \to c^*\big(\widetilde{\mathrm{H}^i(M)}\big)$$

is an isomorphism for all i. This follows from Lemma 4.6.3. ■

Corollary 4.9.2. *Let X be a locally finitely presented R-scheme for a ring R as in Set-up 4.5.1. Then $\mathbf{L}c^*$ induces functors $\mathbf{L}c^*\colon \mathbf{D}^*_{\mathrm{qc,acoh}}(X) \to \mathbf{D}^*_{\mathrm{qc,acoh}}(\widehat{X})$ (resp. $\mathbf{L}c^*\colon \mathbf{D}^*_{\mathrm{acoh}}(X)^a \to \mathbf{D}^*_{\mathrm{acoh}}(\widehat{X})^a$) for any $* \in \{\text{“ ”}, -, b, +\}$.*

Proof. The claim is local, so it suffices to assume that $X = \operatorname{Spec} A$. Then it follows from the exactness of c^* and Lemma 4.9.1. ■

Lemma 4.9.3. *Let $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{Y}$ be a morphism of locally finitely presented formal R-schemes for R as in Set-up 4.5.1.*

(1) *Suppose that $\mathfrak{X} = \operatorname{Spf} B$, $\mathfrak{Y} = \operatorname{Spf} A$ are affine formal R-schemes. Then there is a functorial isomorphism*

$$\mathbf{L}\mathfrak{f}^*(M^{L\Delta}) \simeq (M \otimes_A B)^{L\Delta}$$

for any $M \in \mathbf{D}_{\mathrm{acoh}}(A)$.

(2) *Suppose that $\mathfrak{X} = \operatorname{Spf} B$, $\mathfrak{Y} = \operatorname{Spf} A$ are affine formal R-schemes. Then there is a functorial isomorphism*

$$\mathbf{L}\mathfrak{f}^*(M^{a,L\Delta}) \simeq (M^a \otimes_{A^a} B^a)^{L\Delta}$$

for any $M^a \in \mathbf{D}_{\mathrm{acoh}}(A)$.

(3) *The functor $\mathbf{L}\mathfrak{f}^*$ carries $\mathbf{D}^-_{\mathrm{qc,acoh}}(\mathfrak{Y})$ to $\mathbf{D}^-_{\mathrm{qc,acoh}}(\mathfrak{X})$.*

(4) *The functor $\mathbf{L}\mathfrak{f}^*$ carries $\mathbf{D}^-_{\mathrm{acoh}}(\mathfrak{Y})^a$ to $\mathbf{D}^-_{\mathrm{acoh}}(\mathfrak{X})^a$.*

Proof. The proof is similar to the proof of Lemma 4.6.5. We use Lemma 4.9.1 and Lemma 4.8.13 to reduce to the analogous algebraic facts that were already proven in Lemma 4.2.3. ∎

Lemma 4.9.4. *Let $\mathfrak{X}$ be a locally topologically finitely presented formal R-scheme for R as in Set-up 4.5.1.*

(1) *Suppose that $\mathfrak{X} = \operatorname{Spf} A$ is affine. Then there is a functorial isomorphism*

$$M^{L\Delta} \otimes^L_{\mathcal{O}_{\mathfrak{X}}} N^{L\Delta} \simeq (M \otimes^L_A N)^{L\Delta}$$

for any $M, N \in \mathbf{D}_{\mathrm{acoh}}(A)$.

(2) *Suppose that $\mathfrak{X} = \operatorname{Spf} A$ is affine. Then there is a functorial isomorphism*

$$M^{a,L\Delta} \otimes^L_{\mathcal{O}^a_{\mathfrak{X}}} N^{a,L\Delta} \simeq (M^a \otimes^L_{A^a} N^a)^{L\Delta}$$

for any $M^a, N^a \in \mathbf{D}_{\mathrm{acoh}}(A)^a$.

(3) *Let $\mathcal{F}, \mathcal{G} \in \mathbf{D}^-_{\mathrm{qc,acoh}}(\mathfrak{X})$. Then $\mathcal{F} \otimes^L_{\mathcal{O}_{\mathfrak{X}}} \mathcal{G} \in \mathbf{D}^-_{\mathrm{qc,acoh}}(\mathfrak{X})$.*

(4) *Let $\mathcal{F}^a, \mathcal{G}^a \in \mathbf{D}^-_{\mathrm{acoh}}(\mathfrak{X})^a$. Then $\mathcal{F}^a \otimes^L_{\mathcal{O}^a_{\mathfrak{X}}} \mathcal{G}^a \in \mathbf{D}^-_{\mathrm{acoh}}(\mathfrak{X})^a$.*

Proof. Similarly to Lemma 4.9.3, we use Lemma 4.9.1 and Lemma 4.8.13 to reduce to the analogous algebraic facts that were already proven in Lemma 4.2.4. ∎

Now we discuss the functor $\mathbf{R}\underline{\mathrm{alHom}}_{\mathcal{O}_{\mathfrak{X}}}(-,-)$. Our strategy of showing that $\mathbf{R}\underline{\mathrm{alHom}}(-,-)$ preserves almost coherent complexes will be slightly different from the schematic case. The main technical problem corresponds to defining the map $\mathbf{R}\mathrm{alHom}_{A^a}(M^a, N^a)^{L\Delta} \to \mathbf{R}\underline{\mathrm{alHom}}_{\mathcal{O}^a_{\mathfrak{X}}}(M^{a,L\Delta}, N^{a,L\Delta})$ in the affine case.

The main issue is that we do not know if $(-)^{L\Delta}$ is a left adjoint to the functor of global section on the whole category $\mathbf{D}(\mathfrak{X})$; we only know that it becomes a pseudo-inverse to $\mathbf{R}\Gamma(\mathfrak{X}, -)$ after restriction to "$\mathbf{D}_{\mathrm{qc}}(\mathfrak{X})$". However, the complex $\mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{O}_{\mathfrak{X}}}(M^{L\Delta}, N^{L\Delta})$ itself usually does not lie inside "$\mathbf{D}_{\mathrm{qc}}(\mathfrak{X})$". To overcome this issue, we will show that

$$\widetilde{\mathfrak{m}} \otimes \mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{O}_{\mathfrak{X}}}(M^{L\Delta}, N^{L\Delta})$$

does lie in "$\mathbf{D}_{\mathrm{qc}}(\mathfrak{X})$" for $M \in \mathbf{D}^-_{\mathrm{acoh}}(A)$ and $N \in \mathbf{D}^+_{\mathrm{acoh}}(A)$.

Since "$\mathbf{D}_{\mathrm{qc}}(\mathfrak{X})$" was defined in a bit abstract way, it is probably the easiest way to show that $\widetilde{\mathfrak{m}} \otimes \mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{O}_{\mathfrak{X}}}(M^{L\Delta}, N^{L\Delta})$ actually lies in $\mathbf{D}_{\mathrm{qc,acoh}}(\mathfrak{X})$. That is sufficient by Lemma 4.8.12.

Lemma 4.9.5. *Let $\mathfrak{X} = \mathrm{Spf}\, A$ be a topologically finitely presented formal R-scheme for R as in Set-up 4.5.1. For $M, N \in \mathbf{Mod}_A^{\mathrm{acoh}}$, there is a natural almost isomorphism*

$$\mathrm{Ext}^p_A(M, N)^\Delta \xrightarrow{\sim} \underline{\mathrm{Ext}}^p_{\mathcal{O}_{\mathfrak{X}}}(M^\Delta, N^\Delta)$$

for every integer p.

Proof. We recall that $\underline{\mathrm{Ext}}^p_{\mathcal{O}_{\mathfrak{X}}}(M^\Delta, N^\Delta)$ is canonically isomorphic to the sheafification of the presheaf

$$\mathfrak{U} \mapsto \mathrm{Ext}^p_{\mathcal{O}_{\mathfrak{U}}}(M^\Delta|_{\mathfrak{U}}, N^\Delta|_{\mathfrak{U}}).$$

In particular, there is a canonical map $\mathrm{Ext}^p_{\mathcal{O}_{\mathfrak{X}}}(M^\Delta, N^\Delta) \to \Gamma(\mathfrak{X}, \underline{\mathrm{Ext}}^p_{\mathcal{O}_{\mathfrak{X}}}(M^\Delta, N^\Delta))$. It induces a morphism

$$\mathrm{Ext}^p_{\mathcal{O}_{\mathfrak{X}}}(M^\Delta, N^\Delta)^\Delta \to \underline{\mathrm{Ext}}^p_{\mathcal{O}_{\mathfrak{X}}}(M^\Delta, N^\Delta). \tag{4.9.1}$$

Now we note that the classical $(-)^\Delta$ functor and the derived version coincide on almost coherent modules thanks to Lemma 4.8.13. As a consequence, the equivalence "$\mathbf{D}_{\mathrm{qc}}(\mathfrak{X})$" $\simeq \mathbf{D}_{\mathrm{comp}}(A, I)$ coming from Theorem 4.8.6 and Lemma 4.8.13 ensures that $\mathrm{Ext}^p_{\mathcal{O}_{\mathfrak{X}}}(M^\Delta, N^\Delta) \simeq \mathrm{Ext}^p_A(M, N)$. So the map (4.9.1) becomes a map

$$\mathrm{Ext}^p_A(M, N)^\Delta \to \underline{\mathrm{Ext}}^p_{\mathcal{O}_{\mathfrak{X}}}(M^\Delta, N^\Delta).$$

We note that $\mathrm{Ext}^p_A(M, N)$ is an almost coherent A-module by Proposition 2.6.19. Using that almost coherent modules are complete, we conclude that it suffices to show that

$$\mathrm{Ext}^p_A(M, N) \otimes_A A_{\{f\}} \to \mathrm{Ext}^p_{\mathrm{Spf}\, A_{\{f\}}}\left(M^\Delta|_{\mathrm{Spf}\, A_{\{f\}}}, N^\Delta|_{\mathrm{Spf}\, A_{\{f\}}}\right)$$

is an almost isomorphism. Using Lemma 4.6.5 and the equivalence "$\mathbf{D}_{\mathrm{qc}}(\mathfrak{X})$" $\simeq$ $\mathbf{D}_{\mathrm{comp}}(A, I)$ as above, we see that the map above becomes the canonical map

$$\mathrm{Ext}^p_A(M, N) \otimes_A A_{\{f\}} \to \mathrm{Ext}^p_{A_{\{f\}}}(M \otimes_A A_{\{f\}}, N \otimes_A A_{\{f\}}).$$

Finally, this map is an almost isomorphism by Lemma 2.9.12. ■

Corollary 4.9.6. *Let $\mathfrak{X}$ be a locally topologically finitely presented formal R-scheme for R as in Set-up 4.5.1. Then*

$$\widetilde{\mathfrak{m}} \otimes \mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{O}_{\mathfrak{X}}}(\mathcal{F}, \mathcal{G}) \in \mathbf{D}^{+}_{\mathrm{qc,acoh}}(\mathfrak{X})$$

for $\mathcal{F} \in \mathbf{D}^{-}_{\mathrm{qc,acoh}}(\mathfrak{X})$, and $\mathcal{G} \in \mathbf{D}^{+}_{\mathrm{qc,acoh}}(\mathfrak{X})$.

Proof. The claim is local, so we can assume that $\mathfrak{X} = \mathrm{Spf}\ A$. Then we use the Ext-spectral sequence and Lemma 4.5.18 to reduce to the case when $\mathcal{F}$ and $\mathcal{G}$ are in $\mathbf{Mod}^{\mathrm{qc,acoh}}_{\mathfrak{X}}$. Thus, Lemma 4.6.2 ensures that $\mathcal{F} = M^{\Delta}$ and $\mathcal{G} = N^{\Delta}$ for some M, $N \in \mathbf{Mod}^{\mathrm{acoh}}_{A}$. So Lemma 4.9.5 guarantees that

$$\mathcal{H}^{p}\big(\mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{O}_{\mathfrak{X}}}(\mathcal{F}, \mathcal{G})\big) \simeq^{a} \mathrm{Ext}^{p}_{A}(M, N)^{\Delta}.$$

In other words,

$$\widetilde{\mathfrak{m}} \otimes \mathcal{H}^{p}\big(\mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{O}_{\mathfrak{X}}}(\mathcal{F}, \mathcal{G})\big) \simeq \widetilde{\mathfrak{m}} \otimes \mathrm{Ext}^{p}_{A}(M, N)^{\Delta}.$$

Now $\mathrm{Ext}^{p}_{A}(M, N)^{\Delta}$ is an adically quasi-coherent, almost coherent $\mathcal{O}_{\mathfrak{X}}$-module as a consequence of Proposition 2.6.19 and Lemma 4.6.1. So Lemma 4.5.10 guarantees that $\widetilde{\mathfrak{m}} \otimes \mathrm{Ext}^{p}_{A}(M, N)^{\Delta}$ is also adically quasi-coherent and almost coherent. Therefore, $\widetilde{\mathfrak{m}} \otimes \mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{O}_{\mathfrak{X}}}(\mathcal{F}, \mathcal{G}) \in \mathbf{D}^{+}_{\mathrm{qc,acoh}}(\mathfrak{X})$. ■

Lemma 4.9.7. *Let $\mathfrak{X}$ be a locally topologically finitely presented formal R-scheme for R as in Set-up 4.5.1.*

(1) *Suppose $\mathfrak{X} = \mathrm{Spf}\ A$ is affine. Then there is a functorial isomorphism*

$$\mathbf{R}\mathrm{alHom}_{A^{a}}(M^{a}, N^{a})^{L\Delta} \to \mathbf{R}\underline{\mathrm{alHom}}_{\mathcal{O}^{a}_{\mathfrak{X}}}\big(M^{a,L\Delta}, N^{a,L\Delta}\big),$$

for $M \in \mathbf{D}^{-}_{\mathrm{acoh}}(A)^{a}$ and $N \in \mathbf{D}^{+}_{\mathrm{acoh}}(A)^{a}$.

(2) *Suppose $\mathcal{F}^{a} \in \mathbf{D}^{+}_{\mathrm{acoh}}(\mathfrak{X})^{a}$ and $\mathcal{G}^{a} \in \mathbf{D}^{-}_{\mathrm{acoh}}(\mathfrak{X})$ are almost coherent $\mathcal{O}^{a}_{\mathfrak{X}}$-modules. Then $\mathbf{R}\underline{\mathrm{alHom}}_{\mathcal{O}^{a}_{\mathfrak{X}}}(\mathcal{F}^{a}, \mathcal{G}^{a}) \in \mathbf{D}^{+}_{\mathrm{acoh}}(\mathfrak{X})^{a}$.*

Proof. We start with (1). Proposition 3.5.8 implies that the map

$$\big(\widetilde{\mathfrak{m}} \otimes \mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{O}_{\mathfrak{X}}}\big(M^{\Delta}, N^{\Delta}\big)\big)^{a} \to \mathbf{R}\underline{\mathrm{alHom}}_{\mathcal{O}^{a}_{\mathfrak{X}}}\big(M^{a,\Delta}, N^{a,\Delta}\big)$$

is an isomorphism in $\mathbf{D}(\mathfrak{X})^{a}$. Similarly, the map

$$\big(\widetilde{\mathfrak{m}} \otimes \mathbf{R}\mathrm{Hom}_{A}(M, N)^{\Delta}\big)^{a} \to \mathbf{R}\mathrm{alHom}_{A^{a}}(M^{a}, N^{a})^{\Delta}$$

is an isomorphism by Lemma 4.9.5. Thus, it suffices to construct a functorial isomorphism

$$\widetilde{\mathfrak{m}} \otimes \mathbf{R}\mathrm{Hom}_{A}(M, N)^{L\Delta} \to \widetilde{\mathfrak{m}} \otimes \mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{O}_{\mathfrak{X}}}\big(M^{L\Delta}, N^{L\Delta}\big).$$

Now Lemma 4.8.13 and Corollary 4.9.6 guarantee that

$$\widetilde{\mathfrak{m}} \otimes \mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{O}_{\mathfrak{X}}}\big(M^{L\Delta}, N^{L\Delta}\big) \in \mathbf{D}_{\mathrm{qc,acoh}}(\mathfrak{X}).$$

Proposition 2.6.19, Lemma 4.6.1, and Lemma 4.5.10 also guarantee that

$$\widetilde{\mathfrak{m}} \otimes \mathbf{R}\mathrm{Hom}_A(M, N)^{\Delta} \in \mathbf{D}_{\mathrm{qc,acoh}}(\mathfrak{X}).$$

Thus, Theorem 4.8.6 ensures that, in order to construct the desired isomorphism, it suffices to do it after applying $\mathbf{R}\Gamma(\mathfrak{X}, -)$. The projection formula (see Lemma 3.3.5) and the definition of the functor $(-)^{L\Delta}$ provide us with functorial isomorphisms

$$\mathbf{R}\Gamma\big(\mathfrak{X}, \widetilde{\mathfrak{m}} \otimes \mathbf{R}\mathrm{Hom}_A(M, N)^{L\Delta}\big) \simeq \widetilde{\mathfrak{m}} \otimes \mathbf{R}\mathrm{Hom}_A(M, N)$$

and

$$\begin{aligned}\mathbf{R}\Gamma\big(\mathfrak{X}, \widetilde{\mathfrak{m}} \otimes \mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{O}_{\mathfrak{X}}}(M^{L\Delta}, N^{L\Delta})\big) &\simeq \widetilde{\mathfrak{m}} \otimes \mathbf{R}\Gamma\big(\mathfrak{X}, \mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{O}_{\mathfrak{X}}}(M^{L\Delta}, N^{L\Delta})\big) \\ &\simeq \widetilde{\mathfrak{m}} \otimes \mathbf{R}\mathrm{Hom}_{\mathcal{O}_X}(M^{L\Delta}, N^{L\Delta}) \\ &\simeq \widetilde{\mathfrak{m}} \otimes \mathbf{R}\mathrm{Hom}_A(M, N)\end{aligned}$$

where the last isomorphism uses the equivalence from Theorem 4.8.6. Thus, we see

$$\mathbf{R}\Gamma\big(\mathfrak{X}, \widetilde{\mathfrak{m}} \otimes \mathbf{R}\mathrm{Hom}_A(M, N)^{L\Delta}\big) \simeq \mathbf{R}\Gamma\big(\mathfrak{X}, \widetilde{\mathfrak{m}} \otimes \mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{O}_{\mathfrak{X}}}(M^{L\Delta}, N^{L\Delta})\big).$$

As a consequence, we have a functorial isomorhism

$$\widetilde{\mathfrak{m}} \otimes \mathbf{R}\mathrm{Hom}_A(M, N)^{L\Delta} \xrightarrow{\sim} \widetilde{\mathfrak{m}} \otimes \mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{O}_{\mathfrak{X}}}(M^{L\Delta}, N^{L\Delta}).$$

This induces the desired isomorphism

$$\mathbf{R}\mathrm{alHom}_{A^a}(M^a, N^a)^{L\Delta} \xrightarrow{\sim} \mathbf{R}\underline{\mathrm{alHom}}_{\mathcal{O}^a_{\mathfrak{X}}}(M^{a,L\Delta}, N^{a,L\Delta}).$$

(2) is an easy consequence of (1), Proposition 2.6.19, and Corollary 4.8.14. ∎