Chapter 5

Cohomological properties of almost coherent sheaves

The main goal of this chapter is to establish that almost coherent sheaves share similar cohomological properties to classical coherent sheaves. In particular, we prove almost versions of the proper mapping theorem (both for schemes and nice formal schemes), of the formal GAGA theorem, of the formal function theorem, and of the Grothendieck duality. The formal GAGA theorem is arguably quite surprising in the almost coherent context because almost coherent sheaves are rarely of finite type, so none of the classical proofs of the formal GAGA theorem applies in this situation. We resolve this issue by adapting a new approach to GAGA theorems due to J. Hall (see [31]).

5.1 Almost proper mapping theorem

The main goal of this section is to prove the almost proper mapping theorem which says that derived pushforward along a proper (topologically) finitely presented morphism of nice (formal) schemes preserves almost coherent sheaves.

The idea of the proof is relatively easy: we approximate an almost finitely presented $\mathcal{O}_X$-module by a finitely presented one using Corollary 4.3.5 or Theorem 4.7.6 and then use the usual proper mapping theorem. For this, we will need a version of the proper mapping theorem for a class of non-noetherian rings, which we review below.

Definition 5.1.1. We say that a scheme Y is *universally coherent* if any scheme X that is locally of finite presentation over Y is coherent (i.e. the structure sheaf $\mathcal{O}_X$ is coherent).

Theorem 5.1.2 (Proper mapping theorem [25, Theorem I.8.1.3]). *Let Y be a universally coherent quasi-compact scheme, and let $f : X \to Y$ be a proper morphism of finite presentation. Then $\mathbf{R}f_*$ sends $\mathbf{D}^*_{\mathrm{coh}}(X)$ to $\mathbf{D}^*_{\mathrm{coh}}(Y)$ for any $* \in \{``\ ", +, -, b\}$.*

We want to generalize this theorem to the "almost world". So we pick a ring R and a fixed ideal $\mathfrak{m} \subset R$ such that $\mathfrak{m}^2 = \mathfrak{m}$ and $\widetilde{\mathfrak{m}} = \mathfrak{m} \otimes_R \mathfrak{m}$ is R-flat. In this section, we always consider almost mathematics with respect to this ideal.

Theorem 5.1.3 (Almost proper mapping theorem). *Let Y be a universally coherent quasi-compact R-scheme, and let $f : X \to Y$ be a proper, finitely presented morphism. Then the following statements hold:*

- *The functor $\mathbf{R}f_*$ sends $\mathbf{D}^*_{\mathrm{qc,acoh}}(X)$ to $\mathbf{D}^*_{\mathrm{qc,acoh}}(Y)$ for any $* \in \{``\ ", +, -, b\}$.*
- *The functor $\mathbf{R}f_*$ sends $\mathbf{D}^*_{\mathrm{acoh}}(X)^a$ to $\mathbf{D}^*_{\mathrm{acoh}}(Y)^a$ for any $* \in \{``\ ", +, -, b\}$.*

- *The functor* $\mathbf{R}f_*$ *sends* $\mathbf{D}^+_{\mathrm{acoh}}(X)$ *to* $\mathbf{D}^+_{\mathrm{acoh}}(Y)$.
- *If* Y *has finite Krull dimension, then* $\mathbf{R}f_*$ *sends* $\mathbf{D}^*_{\mathrm{acoh}}(X)$ *to* $\mathbf{D}^*_{\mathrm{acoh}}(Y)$ *for any* $* \in \{\text{“ ”}, +, -, b\}$.

Lemma 5.1.4. *Let* Y *be a quasi-compact scheme of finite Krull dimension, and let* $f: X \to Y$ *be a finite type, quasi-separated morphism. Then* X *has finite Krull dimension, and* f_* *has finite cohomological dimension on* $\mathbf{Mod}_X$.

Proof. First of all, we show that X has finite Krull dimension. Indeed, the morphism $f: X \to Y$ is quasi-compact, therefore X is quasi-compact. So it suffices to show that X locally has finite Krull dimension. Thus, we can assume that $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$ are affine, and the map is given by a finite type morphism $A \to B$. In this case, we have $\dim Y = \dim A$ and $\dim X = \dim B$. Thus, it is enough to show that the Krull dimension of a finite type A-algebra is finite. This readily reduces the question to the case of a polynomial algebra $\dim A[X_1, \dots, X_n]$. Now [3, Chapter 11 Exercise 6] implies that $\dim A[X_1, \dots, X_n] \leq \dim A + 2n$.

Now we prove that f_* has finite cohomological dimension. We note that it suffices to show that there is an integer N such that, for any open affine $U \subset Y$, the cohomology groups $\mathrm{H}^i(X_U, \mathcal{F})$ vanish for $i \geq N$ and any $\mathcal{O}_{X_U}$-module $\mathcal{F}$. We recall that f is quasi-separated, so X_U is quasi-compact, quasi-separated and $\dim X_U \leq \dim X$ for any open $U \subset X$. Therefore, it suffices to show that on any spectral space X, we have $\mathrm{H}^i(X, \mathcal{F}) = 0$ for $i > \dim X$ and $\mathcal{F} \in \mathcal{A}b(X)$. This is proven in [57, Corollary 4.6] (another reference is [68, Tag 0A3G]). Thus, we conclude that $N = \dim X$ does the job. ∎

Proof of Theorem 5.1.3. We divide the proof into several steps.

Step 0: *Reduction to the case of bounded below derived categories.* We note that f_* has a bounded cohomological dimension on $\mathbf{Mod}_X^{\mathrm{qc}}$. Indeed, for any quasi-compact separated scheme X and $\mathcal{F} \in \mathbf{Mod}_X^{\mathrm{qc}}$, we can compute $\mathrm{H}^i(X, \mathcal{F})$ via the *alternating* Čech complex for some finite affine covering of X. Therefore, if X can be covered by N affines, the functor f_* restricted to $\mathbf{Mod}_X^{\mathrm{qc}}$ has cohomological dimension at most N.

Now we use [68, Tag 0D6U] (alternatively, one can use [46, Lemma 3.4]) to reduce the question of proving the claim for any $\mathcal{F} \in \mathbf{D}_{\mathrm{qc,acoh}}(X)$ to the question of proving the claim for all its truncations $\tau^{\geq a}\mathcal{F}$. In particular, we reduce the case of $\mathcal{F} \in \mathbf{D}_{\mathrm{qc,acoh}}(X)$ to the case where $\mathcal{F} \in \mathbf{D}^+{}_{\mathrm{qc,acoh}}(X)$. Similarly, (using Proposition 3.5.23), we reduce the case of $\mathcal{F}^a \in \mathbf{D}_{\mathrm{acoh}}(X)^a$ to the case of $\mathcal{F}^a \in \mathbf{D}^+_{\mathrm{acoh}}(X)^a$.

Using Lemma 5.1.4, a similar argument also allows us to reduce the case of $\mathcal{F} \in \mathbf{D}_{\mathrm{acoh}}(X)$ to the case of $\mathcal{F} \in \mathbf{D}^+_{\mathrm{acoh}}(Y)$ when Y has finite Krull dimension.

Step 1: *Reduction to the case of quasi-coherent almost coherent sheaves.* Using the projection formula Lemma 3.3.5 (resp. Proposition 3.5.23), we see that, in order to show that $\mathbf{R}f_*$ sends $\mathbf{D}^+_{\mathrm{acoh}}(X)$ to $\mathbf{D}^+_{\mathrm{acoh}}(Y)$ (resp. $\mathbf{D}^+_{\mathrm{acoh}}(X)^a$ to $\mathbf{D}^+_{\mathrm{acoh}}(Y)^a$), it suffices to show the analogous result for $\mathbf{D}^+_{\mathrm{qc,acoh}}(X)$. Moreover, we can use the spectral sequence

$$\mathrm{E}_2^{p,q} = \mathrm{R}^p f_* \mathcal{H}^q(\mathcal{F}) \Rightarrow \mathbf{R}^{p+q} f_*(\mathcal{F})$$

to reduce the claim to the fact that higher derived pushforwards of a quasi-coherent, almost coherent sheaf are quasi-coherent and almost coherent.

Step 2: *The case of a quasi-coherent, almost coherent $\mathcal{O}_X$-module $\mathcal{F}$.* We show that $\mathrm{R}^i f_*\mathcal{F}$ is a quasi-coherent, almost coherent $\mathcal{O}_Y$-module for any quasi-coherent, almost coherent $\mathcal{O}_X$-module $\mathcal{F}$ and any i. First, we note that $\mathrm{R}^i f_*\mathcal{F}$ is quasi-coherent, as higher pushforwards along quasi-compact, quasi-separated morphisms preserve quasi-coherence.

Now we show that $\mathrm{R}^i f_*\mathcal{F}$ is almost coherent. Note that it is sufficient to show that $\mathrm{R}^i f_*\mathcal{F}$ is almost finitely presented, as Y is a coherent scheme (this follows from Lemma 4.1.15 and Lemma 4.1.16). We choose some finitely generated ideal $\mathfrak{m}_0 \subset \mathfrak{m}$ and another finitely generated ideal $\mathfrak{m}_1 \subset \mathfrak{m}$ such that $\mathfrak{m}_0 \subset \mathfrak{m}_1^2$. Then we use Corollary 4.3.5 to find a finitely presented $\mathcal{O}_X$-module $\mathcal{G}$ and a morphism

$$\varphi \colon \mathcal{G} \to \mathcal{F}$$

such that $\mathrm{Ker}(\varphi)$ and $\mathrm{Coker}(\varphi)$ are annihilated by $\mathfrak{m}_1$. We define $\mathcal{O}_X$-modules

$$\mathcal{K} := \mathrm{Ker}\,\varphi, \ \mathcal{M} := \mathrm{Im}\,\varphi, \ \mathcal{Q} := \mathrm{Coker}\,\varphi,$$

so we have two short exact sequences

$$0 \to \mathcal{K} \to \mathcal{G} \to \mathcal{M} \to 0,$$
$$0 \to \mathcal{M} \to \mathcal{F} \to \mathcal{Q} \to 0$$

with sheaves $\mathcal{K}$ and $\mathcal{Q}$ killed by $\mathfrak{m}_1$. This easily shows that the natural homomorphisms

$$\mathrm{R}^i f_*(\varphi) \colon \mathrm{R}^i f_*\mathcal{G} \to \mathrm{R}^i f_*\mathcal{F}$$

have kernels and cokernels annihilated by $\mathfrak{m}_1^2$. Since $\mathfrak{m}_0 \subset \mathfrak{m}_1^2$ we conclude that $\mathfrak{m}_0(\mathrm{Ker}\,\mathrm{R}^i f_*(\varphi)) = 0$ and $\mathfrak{m}_0(\mathrm{Coker}\,\mathrm{R}^i f_*(\varphi)) = 0$. Moreover, we know that $\mathrm{R}^i f_*\mathcal{G}$ is a finitely presented $\mathcal{O}_Y$-module by Theorem 5.1.2 ($\mathcal{G}$ is a coherent $\mathcal{O}_X$-module since X is a coherent scheme). Therefore, we use Corollary 4.3.5 to conclude that $\mathrm{R}^i f_*\mathcal{F}$ is an almost finitely presented $\mathcal{O}_Y$-module for any $i \geq 0$. ∎

The next goal is to prove a version of the almost proper mapping theorem for nice formal schemes. But before doing this, we need to establish a slightly more precise version of the usual proper mapping theorem for formal schemes than the one in [25].

Theorem 5.1.5 (Proper mapping theorem). *Let R be as in Set-up 4.5.1, A a topologically finitely presented R-algebra, $\mathfrak{f}\colon \mathfrak{X} \to \operatorname{Spf} A$ a topologically finitely presented, proper morphism, and $\mathcal{F}$ a coherent $\mathcal{O}_{\mathfrak{X}}$-module. Then $\mathrm{H}^i(\mathfrak{X}, \mathcal{F})$ is a coherent A-module and the natural morphism*

$$\mathrm{H}^i(\mathfrak{X}, \mathcal{F})^{\Delta} \to \mathrm{R}^i \mathfrak{f}_*(\mathcal{F})$$

is an isomorphism for any $i \geq 0$.

Proof. First, we use [25, Theorem I.11.1.2] to conclude that $\mathbf{R}\mathfrak{f}_*\mathcal{F} \in \mathbf{D}^+_{\mathrm{coh}}(\operatorname{Spf} A)$. Therefore, Theorem 4.8.15 implies that $M := \mathbf{R}\Gamma(\operatorname{Spf} A, \mathbf{R}\mathfrak{f}_*\mathcal{F})$ lies in $\mathbf{D}^+_{\mathrm{acoh}}(A)$, and

$$M^{L\Delta} \simeq \mathbf{R}\mathfrak{f}_*\mathcal{F}.$$

Moreover, Lemma 4.8.13 implies that the natural map

$$\mathrm{H}^i(\mathfrak{X}, \mathcal{F})^{\Delta} \simeq \mathrm{H}^i(M)^{\Delta} \to \mathrm{R}^i \mathfrak{f}_*\mathcal{F}$$

is an isomorphism. Finally, we conclude that

$$\mathrm{H}^i(\mathfrak{X}, \mathcal{F}) \simeq \mathrm{H}^0\big(\mathfrak{X}, \mathrm{H}^i(\mathfrak{X}, \mathcal{F})^{\Delta}\big) \simeq \mathrm{H}^0(\mathfrak{X}, \mathrm{R}^i\mathfrak{f}_*\mathcal{F})$$

must be coherent because $\mathrm{R}^i\mathfrak{f}_*\mathcal{F}$ is coherent. ∎

Theorem 5.1.6 (Almost proper mapping theorem). *Let $\mathfrak{Y}$ be a topologically finitely presented formal R-scheme for R as in Set-up 4.5.1, and let $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{Y}$ be a proper, topologically finitely presented morphism. Then the following assertions hold true:*

- *The functor $\mathbf{R}\mathfrak{f}_*$ sends $\mathbf{D}^*_{\mathrm{qc,acoh}}(\mathfrak{X})$ to $\mathbf{D}^*_{\mathrm{qc,acoh}}(\mathfrak{Y})$ for any $* \in \{\text{“ ”}, +, -, b\}$.*
- *The functor $\mathbf{R}\mathfrak{f}_*$ sends $\mathbf{D}^*_{\mathrm{acoh}}(\mathfrak{X})^a$ to $\mathbf{D}^*_{\mathrm{acoh}}(\mathfrak{Y})^a$ for any $* \in \{\text{“ ”}, +, -, b\}$.*
- *The functor $\mathbf{R}\mathfrak{f}_*$ sends $\mathbf{D}^+_{\mathrm{acoh}}(\mathfrak{X})$ to $\mathbf{D}^+_{\mathrm{acoh}}(\mathfrak{Y})$.*
- *If $Y_0 := \mathfrak{Y} \times_{\operatorname{Spf} R} (\operatorname{Spec} R/I)$ has finite Krull dimension, then $\mathbf{R}\mathfrak{f}_*$ sends $\mathbf{D}^*_{\mathrm{acoh}}(\mathfrak{X})$ to $\mathbf{D}^*_{\mathrm{acoh}}(\mathfrak{Y})$ for any $* \in \{\text{“ ”}, +, -, b\}$.*

Moreover, if $\mathfrak{Y} = \operatorname{Spf} A$ is an affine scheme and $\mathcal{F}$ is an adically quasi-coherent, almost coherent $\mathcal{O}_{\mathfrak{X}}$-module, then $\mathrm{H}^n(\mathfrak{X}, \mathcal{F})$ is almost coherent over A, and the natural map $\mathrm{H}^n(\mathfrak{X}, \mathcal{F})^{\Delta} \to \mathrm{R}^n\mathfrak{f}_*\mathcal{F}$ is an isomorphism of $\mathcal{O}_{\mathfrak{Y}}$-modules for $n \geq 0$.

Lemma 5.1.7. *Let $\mathfrak{Y}$ be a quasi-compact adic formal R-scheme, and let $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{Y}$ be a topologically finite type, quasi-separated morphism. Suppose furthermore that the reduction $Y_0 = \mathfrak{Y} \times_{\operatorname{Spf} R} (\operatorname{Spec} R/I)$ (or, equivalently, the “special fiber” $\overline{\mathfrak{Y}} = \mathfrak{Y} \times_{\operatorname{Spf} R} \operatorname{Spec} R/\mathrm{Rad}(I)$) is of finite Krull dimension. Then $\mathfrak{X}$ has finite Krull dimension, and $\mathfrak{f}_*$ is of finite cohomological dimension on $\mathbf{Mod}_{\mathfrak{X}}$.*

Proof. The proof is identical to that of Lemma 5.1.4 once we notice that the underlying topological spaces of $\mathfrak{Y}$, Y_0, and $\overline{\mathfrak{Y}}$ are canonically identified. ∎

Also, before starting the proof of Theorem 5.1.6, we need to establish the following preliminary lemma:

Lemma 5.1.8. *Let* $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{Y} = \operatorname{Spf} A$ *be a morphism as in Theorem* 5.1.6 *with affine* $\mathfrak{Y}$, *and let* $\mathcal{F} \in \mathbf{Mod}_{\mathfrak{X}}$ *be an adically quasi-coherent, almost coherent sheaf. Then* $\mathrm{R}^q\mathfrak{f}_*\mathcal{F}$ *is an adically quasi-coherent, almost coherent* $\mathcal{O}_{\mathfrak{Y}}$*-module if*

(1) *the* A*-module* $\mathrm{H}^q(\mathfrak{X}, \mathcal{F})$ *is almost coherent for any* $q \geq 0$, *and*

(2) *for any* $g \in A$ *with* $\mathfrak{U} = \operatorname{Spf} A_{\{g\}}$, *the canonical map*

$$\mathrm{H}^q(\mathfrak{X}, \mathcal{F}) \otimes_A A_{\{g\}} \to \mathrm{H}^q(\mathfrak{X}_{\mathfrak{U}}, \mathcal{F}),$$

is an isomorphism for any $q \geq 0$.

Proof. Consider an A-module $M := \mathrm{H}^q(\mathfrak{X}, \mathcal{F})$ that is almost coherent by our assumption. So, Lemma 2.12.7 guarantees that M is I-adically complete, and so M^Δ is an adically quasi-coherent, almost coherent $\mathcal{O}_{\mathfrak{X}}$-module. Now note that $\mathrm{R}^q\mathfrak{f}_*\mathcal{F}$ is the sheafification of the presheaf

$$\mathfrak{U} \mapsto \mathrm{H}^q(\mathfrak{X}_{\mathfrak{U}}, \mathcal{F}).$$

Thus, there is a canonical map $M \to \mathrm{H}^0(\mathfrak{Y}, \mathrm{R}^q\mathfrak{f}_*\mathcal{F})$ that induces a morphism

$$M^\Delta \to \mathrm{R}^q\mathfrak{f}_*\mathcal{F}.$$

The second assumption together with Lemma 2.8.1 and Lemma 2.12.7 ensure that this map is an isomorphism on stalks (as the sheafification process preserves stalks). Therefore, $M^\Delta \to \mathrm{R}^q\mathfrak{f}_*\mathcal{F}$ is an isomorphism of $\mathcal{O}_{\mathfrak{X}}$-modules. In particular, $\mathrm{R}^q\mathfrak{f}_*\mathcal{F}$ is adically quasi-coherent and almost coherent. ∎

Proof of Theorem 5.1.6. We use the same reductions as in the proof of Theorem 5.1.3 to reduce to the case of an adically quasi-coherent, almost coherent $\mathcal{O}_{\mathfrak{X}}$-module $\mathcal{F}$. Moreover, the statement is local on $\mathfrak{Y}$, so we can assume that $\mathfrak{Y} = \operatorname{Spf} A$ is affine.

Now we show that both conditions in Lemma 5.1.8 are satisfied in our situation.

Step 1: $\mathrm{H}^q(\mathfrak{X}, \mathcal{F})$ *is almost coherent for every* $q \geq 0$. Fix a finitely generated ideal $\mathfrak{m}_0 \subset \mathfrak{m}$ and another finitely generated ideal $\mathfrak{m}_1 \subset \mathfrak{m}$ such that $\mathfrak{m}_0 \subset \mathfrak{m}_1^2$.

Theorem 4.7.6 guarantees that there are a coherent $\mathcal{O}_{\mathfrak{X}}$-module $\mathcal{G}_{\mathfrak{m}_1}$ and a morphism $\phi_{\mathfrak{m}_1}\colon \mathcal{G}_{\mathfrak{m}_1} \to \mathcal{F}$ such that its kernel and cokernel are annihilated by $\mathfrak{m}_1$. Then it is easy to see that the natural morphism

$$\mathrm{H}^q(\mathfrak{X}, \mathcal{G}_{\mathfrak{m}_1}) \to \mathrm{H}^q(\mathfrak{X}, \mathcal{F})$$

has kernel annihilated by $\mathfrak{m}_1^2$ and cokernel annihilated by $\mathfrak{m}_1$. In particular, both the kernel and cokernel are annihilated by $\mathfrak{m}_0$. Since $\mathfrak{m}_0$ was an arbitrary finitely generated sub-ideal of $\mathfrak{m}$, it suffices to show that $\mathrm{H}^q(\mathfrak{X}, \mathcal{G}_{\mathfrak{m}_1})$ are coherent A-modules for any $q \geq 0$. This follows from Theorem 5.1.5.

Step 2: *The canonical maps* $\mathrm{H}^q(\mathfrak{X}, \mathcal{F}) \otimes_A A_{\{g\}} \to \mathrm{H}^q(\mathfrak{X}_{\mathfrak{U}}, \mathcal{F})$ *are isomorphisms for any* $g \in A$, $q \geq 0$, *and* $\mathfrak{U} = \mathrm{Spf}\ A_{\{g\}}$. Lemma 4.7.5 guarantees that $\mathcal{F}$ admits an FP-approximation $\phi\colon \mathcal{G} \to \mathcal{F}$. Using Lemma 4.5.14, we get the short exact sequences of adically quasi-coherent sheaves

$$0 \to \mathcal{K} \to \mathcal{G} \to \mathcal{M} \to 0,$$
$$0 \to \mathcal{M} \to \mathcal{F} \to \mathcal{Q} \to 0,$$

where $\mathcal{K}$ and $\mathcal{Q}$ are annihilated by I^{n+1} for some $n \geq 0$. So $\mathcal{K}$ and $\mathcal{Q}$ can be identified with quasi-coherent sheaves on $\mathfrak{X}_n := \mathfrak{X} \times_{\mathrm{Spf}\ A} \mathrm{Spec}\ A/I^{n+1}$. Therefore, the natural morphisms

$$\mathrm{H}^q(\mathfrak{X}, \mathcal{K}) \otimes_A A_{\{g\}} \simeq \mathrm{H}^q(\mathfrak{X}_n, \mathcal{K}) \otimes_{A/I^{n+1}} (A/I^{n+1})_g \to \mathrm{H}^q(\mathfrak{X}_{\mathfrak{U},n}, \mathcal{K}),$$
$$\mathrm{H}^q(\mathfrak{X}, \mathcal{Q}) \otimes_A A_{\{g\}} \simeq \mathrm{H}^q(\mathfrak{X}_n, \mathcal{Q}) \otimes_{A/I^{n+1}} (A/I^{n+1})_g \to \mathrm{H}^q(\mathfrak{X}_{\mathfrak{U},n}, \mathcal{Q})$$

are isomorphisms for $q \geq 0$. The morphism

$$\mathrm{H}^q(\mathfrak{X}, \mathcal{G}) \otimes_A A_{\{g\}} \to \mathrm{H}^q(\mathfrak{X}_{\mathfrak{U}}, \mathcal{G}) \tag{5.1.1}$$

is an isomorphism by Theorem 5.1.5. Consequently, the five lemma and A-flatness of $A_{\{g\}}$ imply that the morphism

$$\mathrm{H}^q(\mathfrak{X}, \mathcal{M}) \otimes_A A_{\{g\}} \to \mathrm{H}^q(\mathfrak{X}_{\mathfrak{U}}, \mathcal{M})$$

is an isomorphism for any $q \geq 0$ as well. Applying the five lemma again (and A-flatness of $A_{\{g\}}$), we conclude that the morphism

$$\mathrm{H}^q(\mathfrak{X}, \mathcal{F}) \otimes_A A_{\{g\}} \to \mathrm{H}^q(\mathfrak{X}_{\mathfrak{U}}, \mathcal{F})$$

must be an isomorphism for any $q \geq 0$ as well. ∎

5.2 Characterization of quasi-coherent, almost coherent complexes

The main goal of this section is to show an almost analogue of [68, Tag 0CSI]. This gives a useful characterization of objects in $\mathbf{D}^b_{\mathrm{qc,acoh}}(X)$ on a separated, finitely presented R-scheme for a universally coherent R. This will be crucially used in our proof of the almost version of the formal GAGA theorem (see Theorem 5.3.2).

Our proof follows the proof of [68, Tag 0CSI] quite closely, but we need to make certain adjustments to make the arguments work in the almost coherent setting.

Theorem 5.2.1. *Let* R *be a universally coherent ring with an ideal* $\mathfrak{m}$ *such that* $\mathfrak{m}^2 = \mathfrak{m}$ *and* $\widetilde{\mathfrak{m}} := \mathfrak{m} \otimes_R \mathfrak{m}$ *is flat. Suppose that* $\mathcal{F} \in \mathbf{D}_{\mathrm{qc}}(\mathbf{P}^n_R)$ *is an element such that* $\mathbf{R}\mathrm{Hom}_{\mathbf{P}^n}(\mathcal{P}, \mathcal{F}) \in \mathbf{D}^-_{\mathrm{acoh}}(R)$ *for* $\mathcal{P} = \bigoplus_{i=0}^n \mathcal{O}(i)$. *Then* $\mathcal{F} \in \mathbf{D}^-_{\mathrm{qc,acoh}}(\mathbf{P}^n_R)$.

Proof. We follow the ideas of [68, Tag 0CSG]. Denote the dg algebra $\mathbf{R}\mathrm{Hom}_X(\mathcal{P}, \mathcal{P})$ by S. A computation of cohomology groups of line bundles on $\mathbf{P}^n_R$ implies that S is a "discrete" non-commutative algebra that is finite and flat over R. Now [68, Tag 0BQU][1] guarantees that the functor

$$- \otimes^{\mathbf{L}}_S \mathcal{P} \colon \mathbf{D}(S) \to \mathbf{D}_{\mathrm{qc}}(\mathbf{P}^n_R)$$

is an equivalence of categories, and its quasi-inverse is given by

$$\mathbf{R}\mathrm{Hom}(\mathcal{P}, -) \colon \mathbf{D}_{\mathrm{qc}}(\mathbf{P}^n_R) \to \mathbf{D}(S).$$

So, if we define $M := \mathbf{R}\mathrm{Hom}(\mathcal{P}, \mathcal{F}) \in \mathbf{D}(S)$, our assumption implies that the image of M in $\mathbf{D}(R)$ lies in $\mathbf{D}^-_{\mathrm{acoh}}(R)$.

Therefore, it suffices to show that, for any $N \in \mathbf{D}(S)$ such that its image in $\mathbf{D}(R)$ lies in $\mathbf{D}^-_{\mathrm{acoh}}(R)$, we have that $N \otimes^{\mathbf{L}}_S \mathcal{P}$ lies in $\mathbf{D}^-_{\mathrm{qc,acoh}}(\mathbf{P}^n_R)$.

We use the convergence spectral sequence

$$\mathrm{E}_2^{p,q} = \mathcal{H}^p(\mathrm{H}^q(N) \otimes^{\mathbf{L}}_S \mathcal{P}) \Rightarrow \mathcal{H}^{p+q}(N \otimes^{\mathbf{L}}_S \mathcal{P})$$

to conclude that it suffices to assume that N is just an S-module. Now we fix a finitely generated ideal $\mathfrak{m}_1 \subset \mathfrak{m}$ and a finitely generated ideal $\mathfrak{m}_0 \subset \mathfrak{m}$ such that $\mathfrak{m}_1 \subset \mathfrak{m}_0^2$. Then Lemma 2.8.4 implies that there is a finitely presented right S-module N' with a morphism $f \colon N' \to N$ such that $\mathrm{Ker}\, f$ and $\mathrm{Coker}\, f$ are annihilated by $\mathfrak{m}_0$. Then the universal coherence of R and [68, Tag 0CSF] imply that $N' \otimes^{\mathbf{L}}_S \mathcal{P} \in \mathbf{D}^-_{\mathrm{qc},coh}(\mathbf{P}^n_R)$. Now we note that the functor

$$- \otimes^{\mathbf{L}}_S \mathcal{P} \colon \mathbf{D}(S) \to \mathbf{D}_{\mathrm{qc}}(\mathbf{P}^n_R)$$

is R-linear, so the standard argument shows that the cone of the morphism

$$f \otimes^{\mathbf{L}}_S \mathcal{P} \colon N' \otimes^{\mathbf{L}}_S \mathcal{P} \to N \otimes^{\mathbf{L}}_S \mathcal{P}$$

has cohomology sheaves annihilated by $\mathfrak{m}_1 \subset \mathfrak{m}_0^2$. Since $\mathfrak{m}_1 \subset \mathfrak{m}$ was an arbitrary finitely generated ideal, Lemma 2.5.7 implies that $N \otimes^{\mathbf{L}}_S \mathcal{P}$ is in $\mathbf{D}^-_{\mathrm{qc,acoh}}(\mathbf{P}^n_R)$ and this finishes the proof. ∎

Lemma 5.2.2. *Let R be a universally coherent ring, let X be a finitely presented separated R-scheme, and let $K \in \mathbf{D}_{\mathrm{qc}}(X)$. If $\mathbf{R}\Gamma(X, E \otimes^{\mathbf{L}}_{\mathcal{O}_X} K)$ is in $\mathbf{D}^-_{\mathrm{acoh}}(R)$ for every $E \in \mathbf{D}^-_{\mathrm{coh}}(X)$, then $K \in \mathbf{D}^-_{\mathrm{qc,acoh}}(X)$.*

Proof. We follow the proof of [68, Tag 0CSL]. First, we note that the condition that $K \in \mathbf{D}^-_{\mathrm{qc,acoh}}(X)$ is local on X because X is quasi-compact. Therefore, we can prove it locally around each point x. We use [68, Tag 0CSJ] to find

[1] Note that they have slightly different notations for R and S.

- an open subset $U \subset X$ containing x,
- an open subset $V \subset \mathbf{P}^n_R$,
- a closed subset $Z \subset X \times_R \mathbf{P}^n_R$ with a point $z \in Z$ lying over x,
- an object $E \in \mathbf{D}^-_{\mathrm{coh}}(X \times_R \mathbf{P}^n_R)$,

with a lot of properties listed in the cited lemma. Even though the notation is pretty heavy, the only properties of these objects that we will use are that $x \in U$ and that

$$\mathbf{R}q_*(\mathbf{L}p^*K \otimes^{\mathbf{L}} E)|_V = \mathbf{R}(U \to V)_*(K|_U).$$

The last formula is proven in [68, Tag 0CSK] and we refer to this lemma for a discussion of the morphism $U \to V$ that turns out to be a finitely presented closed immersion.

That being said, it is sufficient to show that $K|_U$ is almost coherent for each such U. Moreover, the formula $\mathbf{R}q_*(\mathbf{L}p^*K \otimes^{\mathbf{L}} E)|_V = \mathbf{R}(U \to V)_*(K|_U)$, the fact that $U \to V$ is a finitely presented closed immersion, and Lemma 2.8.4 imply that it suffices to show that $\mathbf{R}(U \to V)_*(K|_U) = \mathbf{R}q_*(\mathbf{L}p^*K \otimes^{\mathbf{L}} E)|_V$ lies in $\mathbf{D}^-_{\mathrm{qc,acoh}}(V)$. In particular, it is enough to show that $\mathbf{R}q_*(\mathbf{L}p^*K \otimes^{\mathbf{L}} E) \in \mathbf{D}^-_{\mathrm{qc,acoh}}(\mathbf{P}^n_R)$.

Now we check this using Theorem 5.2.1. For doing so, we define a sheaf $\mathcal{P} := \bigoplus_{i=0}^n \mathcal{O}_{\mathbf{P}^n}(i)$ and observe that

$$\begin{aligned}
\mathbf{R}\mathrm{Hom}_{\mathbf{P}^n}\left(\mathcal{P}, \mathbf{R}q_*(\mathbf{L}p^*K \otimes^{\mathbf{L}} E)\right) &= \mathbf{R}\Gamma\left(\mathbf{P}^n, \mathbf{R}q_*(\mathbf{L}p^*K \otimes^{\mathbf{L}} E) \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathbf{P}^n}} \mathcal{P}^\vee\right) \\
&= \mathbf{R}\Gamma\left(\mathbf{P}^n, \mathbf{R}q_*\left(\mathbf{L}p^*K \otimes^{\mathbf{L}} E \otimes^{\mathbf{L}} \mathbf{L}q^*\mathcal{P}^\vee\right)\right) \\
&= \mathbf{R}\Gamma\left(X \times_R \mathbf{P}^n_R, \mathbf{L}p^*K \otimes^{\mathbf{L}} E \otimes^{\mathbf{L}} \mathbf{L}q^*\mathcal{P}^\vee\right) \\
&= \mathbf{R}\Gamma\left(X, \mathbf{R}p_*\left(\mathbf{L}p^*K \otimes^{\mathbf{L}} E \otimes^{\mathbf{L}} \mathbf{L}q^*\mathcal{P}^\vee\right)\right) \\
&= \mathbf{R}\Gamma\left(X, K \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathbf{R}p_*\left(E \otimes^{\mathbf{L}} \mathbf{L}q^*\mathcal{P}^\vee\right)\right),
\end{aligned}$$

where the second equality and the fifth equality come from the projection formula [68, Tag 08EU]. Now we note that the proper mapping theorem (see Theorem 5.1.2) implies that $\mathbf{R}p_*(E \otimes^{\mathbf{L}} \mathbf{L}q^*\mathcal{P}^\vee) \in \mathbf{D}^-_{\mathrm{coh}}(X)$. So our assumption on K implies that

$$\mathbf{R}\mathrm{Hom}_{\mathbf{P}^n}(\mathcal{P}, \mathbf{R}q_*(\mathbf{L}p^*K \otimes^{\mathbf{L}} E)) = \mathbf{R}\Gamma(X, K \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathbf{R}p_*(E \otimes^{\mathbf{L}} \mathbf{L}q^*\mathcal{P}^\vee)) \in \mathbf{D}^-_{\mathrm{acoh}}(R).$$

Now Theorem 5.2.1 finishes the proof. ∎

Theorem 5.2.3. *Let R be a universally coherent ring, let X be a separated, finitely presented R-scheme. If $\mathcal{F} \in \mathbf{D}^-_{\mathrm{qc}}(X)$ is an object such that $\mathbf{R}\mathrm{Hom}_X(\mathcal{P}, \mathcal{F}) \in \mathbf{D}^-_{\mathrm{acoh}}(R)$ for any $\mathcal{P} \in \mathrm{Perf}(X)$, then $\mathcal{F} \in \mathbf{D}^-_{\mathrm{qc,acoh}}(X)$. Analogously, if $\mathbf{R}\mathrm{Hom}_X(\mathcal{P}, \mathcal{F}) \in \mathbf{D}^b_{\mathrm{acoh}}(R)$ for any $\mathcal{P} \in \mathrm{Perf}(X)$, then $\mathcal{F} \in \mathbf{D}^b_{\mathrm{qc,acoh}}(X)$.*

Proof. With Lemma 5.2.2 and the equality $\mathbf{R}\mathrm{Hom}_X(\mathcal{P}, \mathcal{F}) = \mathbf{R}\Gamma(X, \mathcal{P}^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{F})$ at hand, the first part of the theorem is absolutely analogous to [68, Tag 0CSH]. The second part now follows directly from [68, Tag 09IS] and [4, Lemma 3.0.12]. ∎

5.3 The GAGA theorem

The main goal of this section is to prove the formal GAGA theorem for almost coherent sheaves. It roughly says that any adically quasi-coherent, almost coherent sheaf on a completion of a proper, finitely presented scheme admits an essentially unique algebraization, and the same holds for morphisms of those sheaves.

We start by recalling the statement of the classical formal GAGA theorem. We fix an I-adically complete noetherian ring A and a proper A-scheme X. Then we consider the I-adic completion $\mathfrak{X}$ as a formal scheme over Spf A. It comes equipped with the natural morphism $c: \mathfrak{X} \to X$ of locally ringed spaces that induces a functor

$$c^*: \mathbf{Coh}_X \to \mathbf{Coh}_{\mathfrak{X}}.$$

The GAGA theorem says that it is an equivalence of categories. Let us say a few words about the classical proof of this theorem. It consists of three essentially independent steps: the first is to show that the morphism c is flat; the second is to show that the functor c^* induces an isomorphism

$$c^*: \mathrm{H}^i(X, \mathrm{F}) \to \mathrm{H}^i(\mathfrak{X}, c^*\mathrm{F})$$

for any $\mathrm{F} \in \mathbf{Coh}_X$ and any integer i. The last is to prove that any coherent sheaf $\mathcal{G} \in \mathbf{Coh}_{\mathbf{P}^N}$ admits a surjection of the form $\bigoplus_i \mathcal{O}(n_i)^{m_i} \to \mathcal{G}$. Though the first two steps generalize to our set-up, there is no chance of having an analogue of the last statement. The reason is easy: existence of such a surjection would automatically imply that the sheaf $\mathcal{G}$ is of finite type, however, almost coherent sheaves are usually not of finite type.

This issue suggests that we should take another approach to GAGA theorems recently developed by J. Hall in his paper [31]. The main advantage of this approach is that it first *constructs* a candidate for algebraization, and only then proves that this candidate indeed provides an algebraization. We adapt this strategy to our almost context.

We start with the discussion of the GAGA functor in the almost world. In what follows, we assume that R is a ring from Set-up 4.5.1. We fix a finitely presented R-scheme X, and we consider its I-adic completion $\mathfrak{X}$ that is a topologically finitely presented formal R-scheme. The formal scheme $\mathfrak{X}$ comes equipped with the canonical morphism of locally ringed spaces

$$c: (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}) \to (X, \mathcal{O}_X)$$

that induces the pullback functor

$$\mathbf{L}c^*: \mathbf{D}(X) \to \mathbf{D}(\mathfrak{X}).$$

We now want to check that this functor preserves quasi-coherent, almost coherent objects. This verification will be necessary even to formulate the GAGA statement.

Lemma 5.3.1. *Let R be a ring as in Set-up 4.5.1, A a topologically finitely presented R-algebra, and X a finitely presented A-scheme. Then the morphism c is flat, and the funtor $c^*\colon \mathbf{Mod}_X \to \mathbf{Mod}_{\mathfrak{X}}$ sends (quasi-coherent and) almost coherent sheaves to (adically quasi-coherent and) almost coherent sheaves. In particular, it induces functors*

$$\mathbf{L}c^*\colon \mathbf{D}^*_{\mathrm{qc,acoh}}(X) \to \mathbf{D}^*_{\mathrm{qc,acoh}}(\mathfrak{X})$$

for any $* \in \{\text{“ ”}, +, -, b\}$.

Proof. The flatness assertion follows from [25, Proposition I.1.4.7 (2)]. Flatness of c implies that it suffices to show that $c^*(\mathcal{G})$ is an adically quasi-coherent, almost coherent $\mathcal{O}_{\mathfrak{X}}$-module for a quasi-coherent, almost coherent $\mathcal{O}_X$-module $\mathcal{G}$. This claim is Zariski-local on X. Thus we can assume that $X = \operatorname{Spec} A$ is affine, so $\mathcal{G} \simeq \widetilde{M}$ for some almost finitely presented A-module M. This case is done in Lemma 4.6.3. ∎

Theorem 5.3.2. *Let R be a ring as in Set-up 4.5.1, A a topologically finitely presented R-algebra, and X a finitely presented, proper A-scheme. Then the functor*

$$\mathbf{L}c^*\colon \mathbf{D}^*_{\mathrm{qc,acoh}}(X) \to \mathbf{D}^*_{\mathrm{qc,acoh}}(\mathfrak{X})$$

induces an equivalence of categories for $* \in \{\text{“ ”}, +, -, b\}$.

Corollary 5.3.3. *Let R, A and X be as in Theorem 5.3.2. Then the functor*

$$\mathbf{L}c^*\colon \mathbf{D}^*_{\mathrm{acoh}}(X)^a \to \mathbf{D}^*_{\mathrm{acoh}}(\mathfrak{X})^a$$

induces an equivalence of categories for $* \in \{\text{“ ”}, +, -, b\}$.

Corollary 5.3.4. *Let R, A, and X be as in Theorem 5.3.2, and let $K \in \mathbf{D}_{\mathrm{qc,acoh}}(X)$. Then the natural map*

$$\beta_K\colon \mathbf{R}\Gamma(X, K) \to \mathbf{R}\Gamma(\mathfrak{X}, \mathbf{L}c^*K)$$

is an isomorphism. Moreover, β_K is an almost isomorphism for $K \in \mathbf{D}_{\mathrm{acoh}}(X)$.

Proof. Note that the case of $K \in \mathbf{D}_{\mathrm{acoh}}(X)$ follows from the case of $K \in \mathbf{D}_{\mathrm{qc,acoh}}(X)$ due to Lemma 3.2.17 and Proposition 3.5.23. So, it suffices to prove the claim for $K \in \mathbf{D}_{\mathrm{qc,acoh}}(X)$.

Now since we are allowed to replace K with $K[i]$ for any integer i, it suffices to show that the map

$$\mathrm{H}^0(\mathbf{R}\Gamma(X, K)) \simeq \mathrm{Hom}_X(\mathcal{O}_X, K) \to \mathrm{Hom}_{\mathfrak{X}}(\mathcal{O}_{\mathfrak{X}}, \mathbf{L}c^*K) \simeq \mathrm{H}^0(\mathbf{R}\Gamma(\mathfrak{X}, \mathbf{L}c^*K))$$

is an isomorphism. This follows from Theorem 5.3.2 together with the observation that $\mathcal{O}_{\mathfrak{X}} \simeq \mathbf{L}c^*\mathcal{O}_X$. ∎

Our proof of Theorem 5.3.2 will follow Jack Hall's proof of the GAGA theorem very closely with some simplifications due to the flatness of the functor c^*. As he works entirely in the setting of pseudo-coherent objects, and almost coherent sheaves may not be pseudo-coherent, we have to repeat some arguments in our setting.

Before we embark on the proof, we need to define the functor in the other direction. Recall that the morphism of locally ringed spaces c defines the derived pushforward functor

$$\mathbf{R}c_*\colon \mathbf{D}(\mathfrak{X}) \to \mathbf{D}(X).$$

This functor is t-exact as $c\colon \mathfrak{X} \to X$ is topologically just a closed immersion. In particular, it preserves boundedness of complexes (in any direction). However, that functor usually does not preserve (almost) coherent objects as can be seen in the example of $\mathbf{R}c_*\mathcal{O}_{\mathfrak{X}} = c_*\mathcal{O}_{\mathfrak{X}}$. A way to fix it is to use the quasi-coherator functor

$$\mathbf{R}Q_X\colon \mathbf{D}(X) \to \mathbf{D}_{\mathrm{qc}}(X)$$

that is defined as the right adjoint to the inclusion $\iota\colon \mathbf{D}_{\mathrm{qc}}(X) \to \mathbf{D}(X)$. It exists by [68, Tag 0CR0]. We define the functor

$$\mathbf{R}c_{\mathrm{qc}}\colon \mathbf{D}(\mathfrak{X}) \to \mathbf{D}_{\mathrm{qc}}(X)$$

as the composition $\mathbf{R}c_{\mathrm{qc}} := \mathbf{R}Q_X \circ \mathbf{R}c_*$.

Combining the adjunctions $(\mathbf{L}c^*, \mathbf{R}c_*)$ and $(\iota, \mathbf{R}Q_X)$, we conclude that we have a pair of the adjoint functors:

$$\mathbf{L}c^*\colon \mathbf{D}_{\mathrm{qc}}(X) \rightleftarrows \mathbf{D}(\mathfrak{X}) :\mathbf{R}c_{\mathrm{qc}}.$$

That gives us the unit and counit morphisms

$$\eta\colon \mathrm{id} \to \mathbf{R}c_{\mathrm{qc}}\mathbf{L}c^* \text{ and } \varepsilon\colon \mathbf{L}c^*\mathbf{R}c_{\mathrm{qc}} \to \mathrm{id}.$$

For future reference, we also note that the above adjunction and the monoidal property of the functor $\mathbf{L}c^*$ define a projection morphism

$$\pi_{\mathrm{G},\mathcal{F}}\colon \mathrm{G} \otimes^{\mathbf{L}}_{\mathcal{O}_X} (\mathbf{R}c_{\mathrm{qc}}\mathcal{F}) \to \mathbf{R}c_{\mathrm{qc}}\big(\mathbf{L}c^*\mathrm{G} \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{F}\big)$$

for any $\mathrm{G} \in \mathbf{D}_{\mathrm{qc}}(X)$ and any $\mathcal{F} \in \mathbf{D}(\mathfrak{X})$. Before discussing the proof of Theorem 5.3.2, we need to establish some properties of these functors.

Lemma 5.3.5. *Let R be a ring as in Set-up 4.5.1, A a topologically finitely presented R-algebra, and X a finitely presented A-scheme. Then there is an integer $N = N(X)$ such that $\mathbf{R}c_{\mathrm{qc}}$ carries $\mathbf{D}^{\leq n}_{\mathrm{qc,acoh}}(\mathfrak{X})$ to $\mathbf{D}^{\leq n+N}_{\mathrm{qc}}(X)$ (resp. $\mathbf{D}^{[a,n]}_{\mathrm{qc,acoh}}(\mathfrak{X})$ to $\mathbf{D}^{[a,n+N]}_{\mathrm{qc}}(X)$) for any integer n. In particular, the natural map*

$$\tau^{\geq a}\mathbf{R}c_{\mathrm{qc}}\mathcal{F} \to \tau^{\geq a}\big(\mathbf{R}c_{\mathrm{qc}}\tau^{\geq a-N}\mathcal{F}\big)$$

is an isomorphism for any $\mathcal{F} \in \mathbf{D}_{\mathrm{qc,acoh}}(\mathfrak{X})$ and any integer a.

Proof. We explain the proof that $\mathbf{R}c_{\mathrm{qc}}$ carries $\mathbf{D}^{\leq n}_{\mathrm{qc,acoh}}(\mathfrak{X})$ to $\mathbf{D}^{\leq n+N}_{\mathrm{qc}}(X)$; the case of $\mathbf{D}^{[a,n]}_{\mathrm{qc,acoh}}(\mathfrak{X})$ is similar. We fix an object $\mathcal{F} \in \mathbf{D}^{\leq n}_{\mathrm{qc,acoh}}(\mathfrak{X})$ and note that $\mathbf{R}c_*\mathcal{F} = c_*\mathcal{F}$ since c is topologically a closed immersion. Thus, [68, Tag 0CSA] implies[2] that it suffices to show that

$$\mathrm{H}^i\left(\mathbf{R}\Gamma(U, \mathbf{R}c_*\mathcal{F})\right) = \mathrm{H}^i\left(\mathbf{R}\Gamma(U, c_*\mathcal{F})\right) = 0$$

for any open affine $U \subset X$ and any $i \geq n$. Therefore, we see that

$$\mathrm{H}^i\left(\mathbf{R}\Gamma(U, \mathbf{R}c_*\mathcal{F})\right) = \mathrm{H}^i\left(\mathbf{R}\Gamma(\widehat{U}, \mathcal{F})\right) = \mathrm{H}^i\left(\widehat{U}, \mathcal{F}|_{\widehat{U}}\right),$$

and thus Lemma 4.8.11 implies that $\mathrm{H}^i\left(\widehat{U}, \mathcal{F}|_{\widehat{U}}\right) = 0$ for any $i \geq n$. This finishes the proof of the first claim in the lemma.

The second claim of the lemma follows from the first claim and the distinguished triangle

$$\tau^{\leq a-N-1}\mathcal{F} \to \mathcal{F} \to \tau^{\geq a-N}\mathcal{F} \to \tau^{\leq a-N-1}\mathcal{F}.$$

Namely, we apply the exact functor $\mathbf{R}c_{\mathrm{qc}}$ to this distinguished triangle to get that

$$\mathbf{R}c_{\mathrm{qc}}(\tau^{\leq a-N-1}\mathcal{F}) \to \mathbf{R}c_{\mathrm{qc}}\mathcal{F} \to \mathbf{R}c_{\mathrm{qc}}(\tau^{\geq a-N}\mathcal{F}) \to \mathbf{R}c_{\mathrm{qc}}(\tau^{\leq a-N-1}\mathcal{F}[1])$$

is a distinguished triangle in $\mathbf{D}_{\mathrm{qc}}(X)$ and that $\mathbf{R}c_{\mathrm{qc}}(\tau^{\leq a-N-1}\mathcal{F}) \in \mathbf{D}^{\leq a-1}_{\mathrm{qc}}(X)$. This implies that the map

$$\tau^{\geq a}\mathbf{R}c_{\mathrm{qc}}\mathcal{F} \to \tau^{\geq a}\mathbf{R}c_{\mathrm{qc}}(\tau^{\geq a-N}\mathcal{F})$$

is an isomorphism. ■

Lemma 5.3.6. *Let X be as in Theorem 5.3.2, $\mathcal{F} \in \mathbf{D}^-_{\mathrm{qc,acoh}}(\mathfrak{X})$ and $\mathrm{G} \in \mathbf{D}^-_{\mathrm{qc}}(X)$. Suppose that for each i there is some n_i such that $I^{n_i}\mathcal{H}^i(\mathcal{F}) = 0$ and $I^{n_i}\mathcal{H}^i(\mathrm{G}) = 0$. Then the natural morphisms η_{G} and $\varepsilon_{\mathcal{F}}$ are isomorphisms.*

Proof. We prove the claim only for $\mathcal{F}$ as the other claim is similar.

Reduction to the case when $\mathcal{F} \in \mathbf{D}^b_{\mathrm{qc,acoh}}(\mathfrak{X})$: First, we note that it suffices to show that the natural map

$$\tau^{\geq a}\mathcal{F} \to \tau^{\geq a}\mathbf{L}c^*\mathbf{R}c_{\mathrm{qc}}\mathcal{F}$$

is an isomorphism for any integer a. Moreover, we also note that t-exactness of $\mathbf{L}c^*$ and Lemma 5.3.5 imply that there is an integer N such that the natural map $\tau^{\geq a}\mathbf{L}c^*\mathbf{R}c_{\mathrm{qc}}\mathcal{F} \to \tau^{\geq a}\mathbf{L}c^*\mathbf{R}c_{\mathrm{qc}}\tau^{\geq a-N}\mathcal{F}$ is an isomorphism for any integer a. In particular, we have a commutative diagram

$$\begin{array}{ccc} \tau^{\geq a-N}\mathcal{F} & \longrightarrow & \mathbf{L}c^*\mathbf{R}c_{\mathrm{qc}}(\tau^{\geq a-N}\mathcal{F}) \\ \downarrow & & \downarrow \\ \tau^{\geq a}\mathcal{F} & \longrightarrow & \tau^{\geq a}\mathbf{L}c^*\mathbf{R}c_{\mathrm{qc}}\mathcal{F} \simeq \tau^{\geq a}\mathbf{L}c^*\mathbf{R}c_{\mathrm{qc}}\tau^{\geq a-N}\mathcal{F}, \end{array}$$

[2] We note that the proof of [68, Tag 0CSA] works well with $a = -\infty$ as well.

where the vertical maps induce isomorphisms in degree $\geq a$. Therefore, it suffices to prove the claim for $\tau^{\geq a-N}\mathcal{F}$. So we may and do assume that $\mathcal{F}$ is bounded.

Proof for a bounded $\mathcal{F}$: The case of a bounded $\mathcal{F}$ easily reduces to the case of an adically quasi-coherent, almost coherent $\mathcal{O}_{\mathfrak{X}}$-module $\mathcal{F}$ concentrated in degree 0. In that situation, we have $I^{k+1}\mathcal{F} = 0$ for some k. This implies that $\mathcal{F} = i_{k,*}\mathcal{F}_k = \mathbf{R}i_{k,*}\mathcal{F}_k$ for the closed immersion $i_k \colon X_k \to \mathfrak{X}$. Now it is straightforward to see that the canonical map

$$\mathbf{R}i_{k,*}\mathcal{F}_k \to \mathbf{L}c^*\mathbf{R}c_{\mathrm{qc}}(\mathbf{R}i_{k,*}\mathcal{F}_k)$$

is an isomorphism. The key is flatness of c and the observation that $\mathbf{R}c_*(\mathbf{R}i_{k,*}\mathcal{F}_k)$ is already quasi-coherent, so the quasi-coherator does nothing in this case. ■

Lemma 5.3.7. *If* $\mathrm{G} \in \mathbf{D}_{\mathrm{qc}}(X)$ *and* $\mathcal{F} \in \mathbf{D}(\mathfrak{X})$, *then the natural projection morphism*

$$\pi_{\mathrm{G},\mathcal{F}} \colon \mathrm{G} \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathbf{R}c_{\mathrm{qc}}\mathcal{F} \to \mathbf{R}c_{\mathrm{qc}}(\mathbf{L}c^*\mathrm{G} \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{F})$$

is an isomorphism if G *is perfect.*

Proof. [31, Lemma 4.3]. ■

Now we come to the key input ingredient. Although $\mathbf{R}c_{\mathrm{qc}}$ is quite abstract and difficult to compute in practice, it turns out that the almost proper mapping theorem allows us to check that this functor sends $\mathbf{D}^-_{\mathrm{qc,acoh}}(\mathfrak{X})$ to $\mathbf{D}^-_{\mathrm{qc,acoh}}(X)$. This would give us a candidate for an algebraization.

Lemma 5.3.8. *Let R be a ring as in Set-up 4.5.1, A a topologically finitely presented R-algebra, and X a finitely presented, proper A-scheme. Then* $\mathbf{R}c_{\mathrm{qc}}$ *sends* $\mathbf{D}^*_{\mathrm{qc,acoh}}(\mathfrak{X})$ *to* $\mathbf{D}^*_{\mathrm{qc,acoh}}(X)$ *for* $* \in \{-,\ b\}$.

Proof. We prove only the bounded above case as the other one follows from this using Lemma 5.3.5. We pick any $\mathcal{F} \in \mathbf{D}^-_{\mathrm{qc,acoh}}(\mathfrak{X})$ and use Theorem 5.2.3 to say that it is sufficient to show that $\mathbf{R}\mathrm{Hom}_X(\mathrm{P}, \mathbf{R}c_*\mathcal{F}) \in \mathbf{D}^-_{\mathrm{acoh}}(R)$ for any perfect complex $\mathrm{P} \in \mathrm{Perf}(X)$. For this, we consider the following sequence of isomorphisms:

$$\begin{aligned}\mathbf{R}\mathrm{Hom}_X(\mathrm{P}, \mathbf{R}c_{\mathrm{qc}}\mathcal{F}) &= \mathbf{R}\mathrm{Hom}_{\mathfrak{X}}(\mathbf{L}c^*\mathrm{P}, \mathcal{F}) \\ &= \mathbf{R}\mathrm{Hom}_{\mathfrak{X}}\big(\mathcal{O}_{\mathfrak{X}}, \big(\mathbf{L}c^*\mathrm{P}\big)^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{F}\big) \\ &= \mathbf{R}\Gamma\big(\mathfrak{X}, (\mathbf{L}c^*\mathrm{P})^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{F}\big).\end{aligned}$$

Then we note that $\mathcal{P} := (\mathbf{L}c^*\mathrm{P})^\vee$ is a perfect complex of $\mathcal{O}_{\mathfrak{X}}$-modules, and therefore $\mathcal{P} \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{F}$ lies in $\mathbf{D}^-_{\mathrm{qc,acoh}}(\mathfrak{X})$. Thus, $\mathbf{R}\Gamma(\mathfrak{X}, \mathcal{P} \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{F})$ lies in $\mathbf{D}^-_{\mathrm{acoh}}(R)$ due to the almost proper mapping theorem (see Theorem 5.1.6). ■

Finally, we are ready to give a proof of the GAGA theorem.

Proof of Theorem 5.3.2. For clarity, we divide the proof into the verification of several claims.

Claim 0: *It suffices to show the theorem for* $* = -$, *that is, for bounded above derived categories.* Indeed, flatness of c^* implies that $\mathbf{L}c^*$ preserves boundedness (resp. boundedness above, resp. boundedness below), so it suffices to show that the natural morphisms

$$\eta_{\mathrm{G}}\colon \mathrm{G} \to \mathbf{R}c_{\mathrm{qc}}\mathbf{L}c^*\mathrm{G},$$
$$\varepsilon_{\mathcal{F}}\colon \mathbf{L}c^*\mathbf{R}c_{\mathrm{qc}}\mathcal{F} \to \mathcal{F}$$

are isomorphisms for any $\mathrm{G} \in \mathbf{D}_{\mathrm{qc,acoh}}(X)$ and $\mathcal{F} \in \mathbf{D}_{\mathrm{qc,acoh}}(\mathfrak{X})$.

We fix N as in Lemma 5.3.5. Then flatness of c^* and Lemma 5.3.5 guarantee that

$$\mathbf{R}c_{\mathrm{qc}}\mathbf{L}c^*\tau^{\geq a}\mathrm{G} \in \mathbf{D}^{[a,\infty]}(X),$$
$$\mathbf{L}c^*\mathbf{R}c_{\mathrm{qc}}\tau^{\geq a}\mathcal{F} \in \mathbf{D}^{[a,\infty]}(\mathfrak{X}).$$

Therefore, we see that η_{G} is an isomorphism on $\mathcal{H}^i$ for $i < a$ if and only if the same holds for $\eta_{\tau^{\leq a-1}\mathrm{G}}$. Since a was arbitrary, we conclude that it suffices to show that η_{G} is an isomorphism for $\mathrm{G} \in \mathbf{D}^-_{\mathrm{qc,acoh}}(X)$. Similar argument shows that it suffices to show that $\varepsilon_{\mathcal{F}}$ is an isomorphism for $\mathcal{F} \in \mathbf{D}^-_{\mathrm{qc,acoh}}(\mathfrak{X})$. So it suffices to prove the theorem for $* = -$.

Before we formulate the next claim, we need to use the so-called "approximation by perfect complexes" [68, Tag 08EL] to find some $\mathrm{P} \in \mathrm{Perf}(X)$ such that $\tau^{\geq 0}\mathrm{P} \simeq \mathcal{O}_X/I \simeq \mathcal{O}_{X_0}$ and whose support is equal to X_0. We note that it implies that all cohomology sheaves $\mathcal{H}^i(\mathrm{P})$ are killed by some power of I. We also denote its (derived) pullback by $\mathcal{P} := \mathbf{L}c^*\mathrm{P}$.

Claim 1: *If* $\mathrm{G} \in \mathbf{D}^-_{\mathrm{qc,acoh}}(X)$ *such that* $\mathrm{G} \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathrm{P} \simeq 0$, *then we have* $\mathrm{G} \simeq 0$. *Similarly, if* $\mathcal{F} \in \mathbf{D}^-_{\mathrm{qc,acoh}}(\mathfrak{X})$ *such that* $\mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathcal{P} \simeq 0$, *then* $\mathcal{F} \simeq 0$. We choose a maximal m (assuming that $\mathrm{G} \not\simeq 0$) such that $\mathcal{H}^m(\mathrm{G}) \neq 0$. Then we see that $\mathcal{H}^m(\mathrm{G} \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathrm{P}) \simeq \mathcal{H}^m(\mathrm{G}) \otimes_{\mathcal{O}_X} \mathcal{O}_{X_0} = \mathcal{H}^m(\mathrm{G})/I$. Also, $(\mathcal{H}^m(\mathrm{G})/I)(U) = \mathcal{H}^m(\mathrm{G})(U)/I \simeq 0$ on any open affine U. So Nakayama's lemma (see Lemma 2.5.19) implies that $\mathcal{H}^m(\mathrm{G})(U) \simeq 0$ for any such U. This contradicts the choice of m. The proof in the formal set-up is the same once we notice that $\mathcal{H}^0(\mathcal{P}) = \mathcal{O}_{\mathfrak{X}}/I$.

Claim 2: *The map* $\eta_{\mathrm{G}}\colon \mathrm{G} \to \mathbf{R}c_{\mathrm{qc}}\mathbf{L}c^*\mathrm{G}$ *is an isomorphism for any* $\mathrm{G} \in \mathbf{D}^-_{\mathrm{qc,acoh}}(X)$. Claim 1 implies that it is sufficient to show that the map

$$\varepsilon_{\mathrm{G}} \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathrm{P}\colon \mathrm{G} \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathrm{P} \to \mathbf{R}c_{\mathrm{qc}}\mathbf{L}c^*\mathrm{G} \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathrm{P} \tag{5.3.1}$$

is an isomorphism. Recall that the cohomology sheaves of P are killed by some power of I. This property passes to $\mathrm{G} \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathrm{P}$, so we can use Lemma 5.3.6 to get that the map

$$\varepsilon_{\mathrm{G}\otimes^{\mathbf{L}}_{\mathcal{O}_X}\mathrm{P}}\colon \mathrm{G} \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathrm{P} \to \mathbf{R}c_{\mathrm{qc}}\big(\mathbf{L}c^*(\mathrm{G} \otimes^{\mathbf{L}}_{\mathcal{O}_X} P)\big)$$

is an isomorphism. Now comes the key: we fit the morphism $\varepsilon_{\mathrm{G}\otimes^{\mathbf{L}}_{\mathcal{O}_X}\mathrm{P}}$ into the following commutative triangle:

$$\begin{array}{ccc}
\mathrm{G} \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathrm{P} & \xrightarrow{\varepsilon_{\mathrm{G}\otimes^{\mathbf{L}}_{\mathcal{O}_X}\mathrm{P}}} & \mathbf{R}c_{\mathrm{qc}}\mathbf{L}c^*\mathrm{G} \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathrm{P} \\
{\scriptstyle \varepsilon_{\mathrm{G}\otimes^{\mathbf{L}}_{\mathcal{O}_X}\mathrm{P}}}\downarrow & & \downarrow{\scriptstyle \pi_{\mathrm{P},\mathbf{L}c^*\mathrm{G}}} \\
\mathbf{R}c_{\mathrm{qc}}(\mathbf{L}c^*(\mathrm{G} \otimes^{\mathbf{L}}_{\mathcal{O}_X} P)) & \xrightarrow[\sim]{} & \mathbf{R}c_{\mathrm{qc}}(\mathbf{L}c^*\mathrm{G} \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathbf{L}c^*\mathrm{P}),
\end{array}$$

where the bottom horizontal arrow is the isomorphism map induced by the monoidal structure on $\mathbf{L}c^*$. Moreover, we have already established that the left vertical arrow is an isomorphism, and the right vertical arrow is an isomorphism due to Lemma 5.3.7. That shows that the top horizontal must also be an isomorphism.

Claim 3: *The map* $\varepsilon_{\mathcal{F}}\colon \mathbf{L}c^*\mathbf{R}c_{\mathrm{qc}}\mathcal{F} \to \mathcal{F}$ *is an isomorphism for any* $\mathcal{F} \in \mathbf{D}^-_{\mathrm{qc,acoh}}(\mathfrak{X})$. We use Claim 1 again to say that it is sufficient to show that the map

$$\varepsilon_{\mathcal{F}} \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathbf{L}c^*\mathrm{P}\colon \mathbf{L}c^*\mathbf{R}c_{\mathrm{qc}}\mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathbf{L}c^*\mathrm{P} \to \mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathbf{L}c^*\mathrm{P}$$

is an isomorphism. But that map fits into the commutative diagram:

$$\begin{array}{ccc}
\mathbf{L}c^*\mathbf{R}c_{\mathrm{qc}}\mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathbf{L}c^*\mathrm{P} & \xrightarrow{\varepsilon_{\mathcal{F}}\otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}}\mathbf{L}c^*\mathrm{P}} & \mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathbf{L}c^*\mathrm{P} \\
{\scriptstyle \wr}\downarrow & & \uparrow{\scriptstyle \varepsilon_{\mathcal{F}\otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}}\mathbf{L}c^*\mathrm{P}}} \\
\mathbf{L}c^*(\mathbf{R}c_{\mathrm{qc}}\mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathrm{P}) & \xrightarrow[\mathbf{L}c^*(\pi_{\mathrm{P},\mathcal{F}})]{} & \mathbf{L}c^*\mathbf{R}c_{\mathrm{qc}}(\mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}_{\mathfrak{X}}} \mathbf{L}c^*\mathrm{P})
\end{array}$$

where the vertical morphism on the left is the canonical isomorphism induced by the monoidal structure on $\mathbf{L}c^*$, the bottom morphism is an isomorphism by Lemma 5.3.7, and the right vertical morphism is an isomorphism by Lemma 5.3.6. This implies that the top horizontal morphism is an isomorphism as well. This finishes the proof. ■

5.4 The formal function theorem

In this section, we prove the formal function theorem for almost coherent sheaves as a consequence of the formal GAGA theorem established in the previous section.

For the rest of the section, we fix a ring R as in Set-up 4.5.1 and a finitely presented or a topologically finitely presented R-algebra A.

Remark 5.4.1. Both A and $\widehat{A}$ are topologically universally adhesive by [25, Proposition 0.8.5.19], and they are (topologically universally) coherent by [25, Proposition 0.8.5.23].

For the next definition, we fix a finitely presented A-scheme X and an $\mathcal{O}_X$-module $\mathcal{F}$.

Definition 5.4.2. The *natural I-filtration* $\mathrm{F}^\bullet\mathrm{H}^i(X,\mathcal{F})$ on $\mathrm{H}^i(X,\mathcal{F})$ is defined via the formula

$$\mathrm{F}^n\mathrm{H}^i(X,\mathcal{F}) := \mathrm{Im}\big(\mathrm{H}^i(X, I^n\mathcal{F}) \to \mathrm{H}^i(X,\mathcal{F})\big).$$

The *natural I-topology* on $\mathrm{H}^i(X,\mathcal{F})$ is the topology induced by the natural I-filtration.

Lemma 5.4.3. *Let X be a finitely presented A-scheme, $\mathcal{F}$ a quasi-coherent almost finitely generated $\mathcal{O}_X$-module, and $\mathcal{G} \subset \mathcal{F}$ a quasi-coherent $\mathcal{O}_X$-submodule of $\mathcal{F}$. Then, for any n, there is an m such that $I^m\mathcal{F} \cap \mathcal{G} \subset I^n\mathcal{G}$.*

Proof. It suffices to assume that X is affine, in which case the claim follows from Lemma 2.12.6. ∎

Lemma 5.4.4. *Let X be a finitely presented A-scheme, $\mathcal{F}$ and $\mathcal{G}$ quasi-coherent almost finitely generated $\mathcal{O}_X$-modules, and $\varphi\colon \mathcal{G} \to \mathcal{F}$ an $\mathcal{O}_X$-linear homomorphism such that* $\mathrm{Ker}(\varphi)$ *and* $\mathrm{Coker}(\varphi)$ *are annihilated by I^c for some integer c. Then, for every $i \geq 0$, the natural I-topology on $\mathrm{H}^i(X,\mathcal{F})$ coincides with the topology induced by the filtration*

$$\mathrm{Fil}^n_{\mathcal{G}}\mathrm{H}^i(X,\mathcal{F}) = \mathrm{Im}\big(\mathrm{H}^i(X, I^n\mathcal{G}) \to \mathrm{H}^i(X,\mathcal{F})\big).$$

Proof. Consider the short exact sequences

$$0 \to \mathcal{K} \to \mathcal{G} \to \mathcal{H} \to 0,$$
$$0 \to \mathcal{H} \to \mathcal{F} \to \mathcal{Q} \to 0,$$

where $\mathcal{K}$ and $\mathcal{Q}$ are annihilated by I^c. The first short exact sequence induces the following short exact sequence:

$$0 \to \mathcal{K} \cap I^m\mathcal{G} \to I^m\mathcal{G} \to I^m\mathcal{H} \to 0$$

for any $m \geq 0$. Lemma 5.4.3 implies that $\mathcal{K} \cap I^m\mathcal{G} \subset I^c\mathcal{K} = 0$ for $m \gg 0$. Therefore, the natural map $I^m\mathcal{G} \to I^m\mathcal{H}$ is an isomorphism for $m \gg 0$. Note that $\mathcal{H}$ is almost finitely generated and quasi-coherent, so we can replace $\mathcal{G}$ with $\mathcal{H}$ to assume that φ is injective.

Clearly, $\mathrm{Fil}^k_{\mathcal{G}}\mathrm{H}^i(X,\mathcal{F}) \subset \mathrm{F}^k\mathrm{H}^i(X,\mathcal{F})$ for every k. So it suffices to show that, for any k, there is m such that $\mathrm{F}^m\mathrm{H}^i(X,\mathcal{F}) \subset \mathrm{Fil}^k_{\mathcal{G}}\mathrm{H}^i(X,\mathcal{F})$. We consider the short exact sequence

$$0 \to \mathcal{G} \cap I^m\mathcal{F} \to I^m\mathcal{F} \to I^m\mathcal{Q} \to 0.$$

If $m \geq c$, we get that $\mathcal{G} \cap I^m \mathcal{F} = I^m \mathcal{F}$ because $I^c \mathcal{Q} \simeq 0$. Now we use Lemma 5.4.3 to conclude there is $m \geq c$ such that

$$I^m \mathcal{F} = \mathcal{G} \cap I^m \mathcal{F} \subset I^k \mathcal{G}.$$

Therefore, $\mathrm{F}^m \mathrm{H}^i(X, \mathcal{F}) \subset \mathrm{Fil}^k_{\mathcal{G}} \mathrm{H}^i(X, \mathcal{F})$. ∎

Lemma 5.4.5. *Let X be a finitely presented A-scheme, $\mathcal{F}$ and $\mathcal{G}$ quasi-coherent almost finitely generated $\mathcal{O}_X$-modules, and $\varphi\colon \mathcal{G} \to \mathcal{F}$ an $\mathcal{O}_X$-linear homomorphism such that* $\mathrm{Ker}(\varphi)$ *and* $\mathrm{Coker}(\varphi)$ *are annihilated by I^c for some integer c. Suppose that the natural I-topology on $\mathrm{H}^i(X, \mathcal{G})$ is the I-adic topology. Then the same holds for $\mathrm{H}^i(X, \mathcal{F})$.*

Proof. Clearly, $I^n \mathrm{H}^i(X, \mathcal{F}) \subset \mathrm{F}^n \mathrm{H}^i(X, \mathcal{F})$. So it suffices to show that, for every n, there is an m such that $\mathrm{F}^m \mathrm{H}^i(X, \mathcal{F}) \subset I^n \mathrm{H}^i(X, \mathcal{F})$.

The assumption that the natural I-topology on $\mathrm{H}^i(X, \mathcal{G})$ coincides with the I-adic topology guarantees that $\mathrm{F}^k \mathrm{H}^i(X, \mathcal{G}) \subset I^n \mathrm{H}^i(X, \mathcal{G})$ for large enough k. Pick such k. Lemma 5.4.4 implies that

$$\mathrm{F}^m \mathrm{H}^i(X, \mathcal{F}) \subset \mathrm{Im}(\mathrm{H}^i(X, I^k \mathcal{G}) \to \mathrm{H}^i(X, \mathcal{F}))$$

for large enough m. So we get that

$$\begin{aligned} \mathrm{F}^m \mathrm{H}^i(X, \mathcal{F}) &\subset \mathrm{Im}\big(\mathrm{H}^i(X, I^k \mathcal{G}) \to \mathrm{H}^i(X, \mathcal{F})\big) \\ &\subset \mathrm{Im}\big(I^n \mathrm{H}^i(X, \mathcal{G}) \to \mathrm{H}^i(X, \mathcal{F})\big) \subset I^n \mathrm{H}^i(X, \mathcal{F}) \end{aligned}$$

for a large enough m. ∎

Theorem 5.4.6. *Let X be a proper, finitely presented A-scheme, and $\mathcal{F}$ a quasi-coherent, almost coherent $\mathcal{O}_X$-module. Then the natural I-topology on $\mathrm{H}^i(X, \mathcal{F})$ coincides with the I-adic topology for any $i \geq 0$.*

Proof. Lemma 4.7.3 guarantees that there are a finitely presented $\mathcal{O}_X$-module $\mathcal{G}$ and a morphism $\varphi\colon \mathcal{G} \to \mathcal{F}$ such that $I^n(\mathrm{Ker}\,\varphi) = 0$ and $I^n(\mathrm{Coker}\,\varphi) = 0$ for some integer $n > 0$. Lemma 5.4.5 then ensures that it suffices to prove the claim for $\mathcal{G}$. In this case, the claim follows [25, Proposition I.8.5.2 and Lemma 0.7.4.3] and Remark 5.4.1. ∎

Now we consider a proper, finitely presented A-scheme X, and an almost coherent $\mathcal{O}_X$-module $\mathcal{F}$. We denote the I-adic completion of X by $\mathfrak{X}$, so we have the following commutative diagram:

$$\begin{array}{ccc} \mathfrak{X} & \xrightarrow{\ c\ } & X \\ {\scriptstyle \widehat{f}}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ \mathrm{Spf}\,(\widehat{A}) & \longrightarrow & \mathrm{Spec}\, A. \end{array} \qquad (5.4.1)$$

Then we consider four different cohomology groups

$$\mathrm{H}^i(\mathfrak{X}, c^*\mathcal{F}),\ \widehat{\mathrm{H}^i(X,\mathcal{F})},\ \mathrm{H}^i(X,\mathcal{F})\otimes_A \widehat{A},\ \text{and}\ \lim_n \mathrm{H}^i(X_n,\mathcal{F}_n),$$

and note that they are related via the following A-linear homomorphisms:

$$\begin{array}{ccc} \mathrm{H}^i(X,\mathcal{F})\otimes_A \widehat{A} & \xrightarrow{\alpha^i_{\mathcal{F}}} & \widehat{\mathrm{H}^i(X,\mathcal{F})} \\ \beta^i_{\mathcal{F}}\big\downarrow & & \big\downarrow \phi^i_{\mathcal{F}} \\ \mathrm{H}^i(\mathfrak{X}, c^*\mathcal{F}) & \xrightarrow[\gamma^i_{\mathcal{F}}]{} & \lim_n \mathrm{H}^i(X_n,\mathcal{F}_n). \end{array} \tag{5.4.2}$$

We show that all these morphisms are (almost) isomorphisms:

Theorem 5.4.7. *In the notation as above, all maps $\alpha^i_{\mathcal{F}}$, $\beta^i_{\mathcal{F}}$, $\gamma^i_{\mathcal{F}}$, $\phi^i_{\mathcal{F}}$ are almost isomorphisms for any almost coherent $\mathcal{O}_X$-module $\mathcal{F}$. If $\mathcal{F}$ is quasi-coherent, almost coherent, then these maps are isomorphisms.*

Proof. Once again, we divide the proof into several (numbered) steps.

Step 0: *Reduction to the case of a quasi-coherent, almost coherent sheaf $\mathcal{F}$.* We observe that Lemma 3.3.1, Lemma 3.2.17 and the fact that limits of two almost isomorphic direct systems are almost the same, allow us to replace $\mathcal{F}$ with $\widetilde{\mathfrak{m}}\otimes\mathcal{F}$ to assume that $\mathcal{F}$ is quasi-coherent and almost coherent.

Step 1: *$\alpha^i_{\mathcal{F}}$ is an isomorphism.* This is just a consequence of Lemma 2.12.7, as we established in Theorem 5.1.3 that $\mathrm{H}^i(X,\mathcal{F})$ is an almost coherent A-module.

Step 2: *$\beta^i_{\mathcal{F}}$ is an isomorphism.* We note that the assumptions on A imply that the map $A\to\widehat{A}$ is flat by [25, Proposition 0.8.218]. Thus, flat base change for quasi-coherent cohomology groups implies that $\mathrm{H}^i(X,\mathcal{F})\otimes_A\widehat{A}\simeq \mathrm{H}^i(X_{\widehat{A}},\mathcal{F}_{\widehat{A}})$. Therefore, we may and do assume that A is I-adically complete. Then the map $\mathrm{H}^i(X,\mathcal{F})\to\mathrm{H}^i(\mathfrak{X},c^*\mathcal{F})$ is an isomorphism by Theorem 5.3.2.

Step 3: *$\alpha^i_{\mathcal{F}}$ is injective.* Theorem 5.4.6 and Corollary 5.3.4 imply that the I-adic topology of $\mathrm{H}^i(X,\mathcal{F})$ coincides with the natural I-topology. Therefore,

$$\widehat{\mathrm{H}^i(X,\mathcal{F})}\simeq \lim_n \frac{\mathrm{H}^i(X,\mathcal{F})}{\mathrm{Im}\big(\mathrm{H}^i(X,I^{n+1}\mathcal{F})\to\mathrm{H}^i(X,\mathcal{F})\big)}.$$

Clearly, we have an inclusion

$$\frac{\mathrm{H}^i(X,\mathcal{F})}{\mathrm{Im}\big(\mathrm{H}^i(X,I^{n+1}\mathcal{F})\to\mathrm{H}^i(X,\mathcal{F})\big)}\hookrightarrow \mathrm{H}^i(X_n,\mathcal{F}_n).$$

Therefore, we conclude that $\alpha^i_{\mathcal{F}}$ is injective by left exactness of the limit functor.

Step 4: $\gamma^i_{\mathcal{F}}$ *is surjective.* Recall that $\mathcal{F} \simeq \lim_k \mathcal{F}_k$ because $\mathcal{F}$ is adically quasi-coherent. Therefore, [25, Corollary 0.3.2.16] implies that it is sufficient to show that there is a basis of opens $\mathcal{B}$ such that, for every $\mathfrak{U} \in \mathcal{B}$,

$$\mathrm{H}^i(\mathfrak{U}, \mathcal{F}) = 0 \text{ for } i \geq 1,$$

and

$$\mathrm{H}^0(\mathfrak{U}, \mathcal{F}_{k+1}) \to \mathrm{H}^0(\mathfrak{U}, \mathcal{F}_k) \text{ is surjective for any } k \geq 0.$$

Vanishing of the higher cohomology groups of adically quasi-coherent sheaves on affine formal schemes (see [25, Theorem I.7.1.1]) implies that one can take $\mathcal{B}$ to be the basis consisting of open affine formal subschemes of $\mathfrak{X}$. Therefore, we get that $\gamma^i_{\mathcal{F}}$ is indeed surjective for any $i \geq 0$.

Step 5: $\alpha^i_{\mathcal{F}}$ *and* $\gamma^i_{\mathcal{F}}$ *are isomorphisms.* This follows formally from commutativity of Diagram (5.4.1) and the previous steps. ∎

5.5 Almost version of Grothendieck duality

For this section, we fix a universally coherent ring R with an ideal $\mathfrak{m}$ such that $\widetilde{\mathfrak{m}} := \mathfrak{m} \otimes_R \mathfrak{m}$ is R-flat and $\mathfrak{m}^2 = \mathfrak{m}$. Since R is universally coherent, there is a good theory of a functor $f^!$ for morphisms f between finitely presented, separated R-schemes.[3]

Proposition 5.5.1. *Let* $f: X \to Y$ *be a morphism between separated, finitely presented* R*-schemes. Then* $f^!$ *sends* $\mathbf{D}^+_{\mathrm{qc,acoh}}(Y)$ *to* $\mathbf{D}^+_{\mathrm{qc,acoh}}(X)$.

Proof. The only thing that we need to check here is that $f^!$ preserves almost coherence of cohomology sheaves. This statement is local, so we can assume that both X and Y are affine. Then we can choose a closed embedding $X \to \mathbf{A}^n_Y \to Y$. So, it suffices to prove the claim for a finitely presented closed immersion and for the morphism $\mathbf{A}^n_Y \to Y$.

In the case $f: X \to Y$ a finitely presented closed immersion, we know that for any $\mathcal{F} \in \mathbf{D}^+_{\mathrm{qc}}(Y)$,

$$f^!\mathcal{F} \simeq \mathbf{R}\underline{\mathrm{Hom}}_Y(f_*\mathcal{O}_X, \mathcal{F}).$$

Since Y is a coherent scheme and f is finitely presented, we conclude that $f_*\mathcal{O}_X$ is an almost coherent $\mathcal{O}_Y$-module. So, $f^!\mathcal{F} = \mathbf{R}\underline{\mathrm{Hom}}_Y(f_*\mathcal{O}_X, \mathcal{F}) \in \mathbf{D}_{\mathrm{qc,acoh}}(X)$ by Corollary 4.4.11.

[3] This theory does not seem to be addressed in the literature in this generality, however, all arguments from [68, Tag 0DWE] can be adapted to this level of generality with little or no extra work. See [71, Section 2.1–2.2] for more detail.

Now we consider the case of a relative affine space $f\colon X = \mathbf{A}^n_Y \to Y$. In this case, we have $f^!\mathcal{F} \simeq \mathbf{L}f^*\mathcal{F} \otimes^L_{\mathcal{O}_X} \Omega^n_{X/Y}[n]$. Then $\mathbf{L}f^*(\mathcal{F}) \in \mathbf{D}^+_{\mathrm{qc,acoh}}(X)$ in view of Lemma 4.4.7 (4), and so $\mathbf{L}f^*\mathcal{F} \otimes^L_{\mathcal{O}_X} \Omega^n_{X/Y}[n] \in \mathbf{D}^+_{\mathrm{qc,acoh}}(X)$ because $\Omega^n_{X/Y}$ is (non-canonically) isomorphic to $\mathcal{O}_X$. ■

Now we use Proposition 5.5.1 to define the almost version of the upper shriek functor:

Definition 5.5.2. Let $f\colon X \to Y$ be a morphism of separated, finitely presented R-schemes. We define the *almost upper shriek functor* $f^!_a\colon \mathbf{D}^+_{\mathrm{aqc}}(Y)^a \to \mathbf{D}^+_{\mathrm{aqc}}(X)^a$ as $f^!_a(\mathcal{F}) := (f^!(\mathcal{F}_!))^a$.

Remark 5.5.3. In what follows, we will usually denote the functor $f^!_a$ simply by $f^!$ as it will not cause any confusion.

Lemma 5.5.4. *Let $f\colon X \to Y$ be a morphism between separated, finitely presented R-schemes. Then $f^!$ carries $\mathbf{D}^+_{\mathrm{acoh}}(Y)^a$ to $\mathbf{D}^+_{\mathrm{acoh}}(X)^a$.*

Proof. This follows from Proposition 5.5.1. ■

Theorem 5.5.5. *Let $f\colon X \to Y$ be as above. Suppose that f is proper. Then the functor $f^!\colon \mathbf{D}^+_{\mathrm{aqc}}(Y)^a \to \mathbf{D}^+_{\mathrm{aqc}}(X)^a$ is a right adjoint to $\mathbf{R}f_*\colon \mathbf{D}^+_{\mathrm{aqc}}(Y)^a \to \mathbf{D}^+_{\mathrm{aqc}}(X)^a$.*

We note that the theorem makes sense as $\mathbf{R}f_*$ carries $\mathbf{D}^+_{\mathrm{aqc}}(X)^a$ into $\mathbf{D}^+_{\mathrm{aqc}}(Y)$ due to Lemma 4.4.9.

Proof. This follows from a sequence of canonical isomorphisms:

$$\begin{aligned}
\mathrm{Hom}_{\mathbf{D}(Y)^a}\big(\mathbf{R}f_*\mathcal{F}^a, \mathcal{G}^a\big) &\simeq \mathrm{Hom}_{\mathbf{D}(Y)}\big(\widetilde{\mathfrak{m}} \otimes \mathbf{R}f_*\mathcal{F}, \mathcal{G}\big) && \text{Lemma 3.1.13}\\
&\simeq \mathrm{Hom}_{\mathbf{D}(Y)}\big(\mathbf{R}f_*(\widetilde{\mathfrak{m}} \otimes \mathcal{F}), \mathcal{G}\big) && \text{Lemma 3.3.5}\\
&\simeq \mathrm{Hom}_{\mathbf{D}(X)}\big(\widetilde{\mathfrak{m}} \otimes \mathcal{F}, f^!(\mathcal{G})\big) && \text{Grothendieck duality}\\
&\simeq \mathrm{Hom}_{\mathbf{D}(X)^a}\big(\mathcal{F}^a, f^!(\mathcal{G})^a\big). && \text{Lemma 3.1.13.}
\end{aligned}$$
■

Now suppose that $f\colon X \to Y$ is a proper morphism of separated, finitely presented R-schemes, $\mathcal{F}^a \in \mathbf{D}^+_{\mathrm{aqc}}(X)^a$, and $\mathcal{G}^a \in \mathbf{D}^+_{\mathrm{aqc}}(Y)^a$. Then we want to construct a canonical morphism

$$\mathbf{R}f_*\mathbf{R}\underline{\mathrm{alHom}}_X(\mathcal{F}^a, f^!(\mathcal{G}^a)) \to \mathbf{R}\underline{\mathrm{alHom}}_Y(\mathbf{R}f_*(\mathcal{F}^a), \mathcal{G}^a).$$

Lemma 3.5.16 says that such a map is equivalent to a map

$$\mathbf{R}f_*\mathbf{R}\underline{\mathrm{alHom}}_X\big(\mathcal{F}^a, f^!(\mathcal{G}^a)\big) \otimes^L_{\mathcal{O}_X} \mathbf{R}f_*(\mathcal{F}^a) \to \mathcal{G}^a.$$

We construct the latter map as the composition

$$\mathbf{R}f_*\mathbf{R}\underline{\mathrm{alHom}}_X\big(\mathcal{F}^a, f^!(\mathcal{G}^a)\big) \otimes^L_{\mathcal{O}_X} \mathbf{R}f_*(\mathcal{F}^a)$$

$$\to \mathbf{R}f_*\big(\mathbf{R}\underline{\mathrm{alHom}}_X(\mathcal{F}^a, f^!(\mathcal{G}^a)) \otimes^L_{\mathcal{O}_X} \mathcal{F}^a\big) \to \mathbf{R}f_* f^! \mathcal{G}^a \to \mathcal{G}^a,$$

where the first map is induced by the relative cup product (see [68, Tag 0B68]), the second map comes from Remark 3.5.15, and the last map is the counit of the $(\mathbf{R}f_*, f^!)$-adjunction.

Lemma 5.5.6. *Let $f: X \to Y$ be a proper morphism of separated, finitely presented R-schemes, $\mathcal{F}^a \in \mathbf{D}^-_{\mathrm{acoh}}(X)^a$, and $\mathcal{G}^a \in \mathbf{D}^+_{\mathrm{aqc}}(Y)^a$. Then the map*

$$\mathbf{R}f_*\mathbf{R}\underline{\mathrm{alHom}}_X(\mathcal{F}^a, f^!(\mathcal{G}^a)) \to \mathbf{R}\underline{\mathrm{alHom}}_Y(\mathbf{R}f_*(\mathcal{F}^a), \mathcal{G}^a)$$

is an (almost) isomorphism in $\mathbf{D}^+_{\mathrm{aqc}}(X)^a$.

Proof. We note that $\mathbf{R}f_*\mathbf{R}\underline{\mathrm{alHom}}_X(\mathcal{F}^a, f^!(\mathcal{G}^a))$ lies in $\mathbf{D}^+_{\mathrm{aqc}}(Y)^a$ as a consequence of Lemma 4.4.10 (4) and Lemma 4.4.9. Likewise, $\mathbf{R}\underline{\mathrm{alHom}}_Y(\mathbf{R}f_*(\mathcal{F}^a), \mathcal{G}^a)$ lies in $\mathbf{D}^+_{\mathrm{aqc}}(Y)^a$ by Theorem 5.1.3 and Lemma 4.4.10 (4). Therefore, it suffices to show that

$$\begin{aligned}&\mathbf{R}\mathrm{Hom}_Y\big(\mathcal{H}^a, \mathbf{R}f_*\mathbf{R}\underline{\mathrm{alHom}}_X\big(\mathcal{F}^a, f^!(\mathcal{G}^a)\big)\big)\\ &\qquad \to \mathbf{R}\mathrm{Hom}_Y\big(\mathcal{H}^a, \mathbf{R}\underline{\mathrm{alHom}}_Y\big(\mathbf{R}f_*(\mathcal{F}^a), \mathcal{G}^a\big)\big)\end{aligned}$$

is an isomorphism for any $\mathcal{H}^a \in \mathbf{D}^+_{\mathrm{aqc}}(Y)^a$. This follows from the following sequence of isomorphisms:

$$\begin{aligned}&\mathbf{R}\mathrm{Hom}_Y\big(\mathcal{H}^a, \mathbf{R}f_*\mathbf{R}\underline{\mathrm{alHom}}_X\big(\mathcal{F}^a, f^!(\mathcal{G}^a)\big)\big)\\ &\quad \simeq \mathbf{R}\mathrm{Hom}_X\big(\mathbf{L}f^*\mathcal{H}^a, \mathbf{R}\underline{\mathrm{alHom}}_X\big(\mathcal{F}^a, f^!(\mathcal{G}^a)\big)\big)\\ &\quad \simeq \mathbf{R}\mathrm{Hom}_X\big(\mathbf{L}f^*\mathcal{H}^a \otimes^L_{\mathcal{O}_X} \mathcal{F}^a, f^!(\mathcal{G}^a)\big)\\ &\quad \simeq \mathbf{R}\mathrm{Hom}_Y\big(\mathbf{R}f_*\big(\mathbf{L}f^*\mathcal{H}^a \otimes^L_{\mathcal{O}_X} \mathcal{F}^a\big), \mathcal{G}^a\big)\\ &\quad \simeq \mathbf{R}\mathrm{Hom}_Y\big(\mathcal{H}^a \otimes \mathbf{R}f_*(\mathcal{F}^a), \mathcal{G}^a\big)\\ &\quad \simeq \mathbf{R}\mathrm{Hom}_Y\big(\mathcal{H}^a, \mathbf{R}\underline{\mathrm{alHom}}_Y\big(\mathbf{R}f_*(\mathcal{F}^a), \mathcal{G}^a\big)\big).\end{aligned}$$

The first isomorphism holds by Corollary 3.5.26. The second isomorphism holds by Lemma 3.5.16. The third isomorphism holds by Theorem 5.5.5. The fourth isomorphism holds by Proposition 4.4.12. The fifth equality holds by Lemma 3.5.16. ∎

Theorem 5.5.7. *Let $f: X \to Y$ be as above. Suppose that f is smooth of pure dimension d. Then $f^!(-) \simeq \mathbf{L}f^*(-) \otimes^L_{\mathcal{O}_X} \Omega^d_{X/Y}[d]$.*

Proof. It follows from the corresponding statement in the classical Grothendieck duality. ∎

We summarize all results of this section in the following theorem:

Theorem 5.5.8. *Let R be a universally coherent ring with an ideal $\mathfrak{m}$ such that $\widetilde{\mathfrak{m}} := \mathfrak{m} \otimes_R \mathfrak{m}$ is R-flat and $\mathfrak{m}^2 = \mathfrak{m}$, and FPS_R be the category of finitely presented, separated R-schemes. Then there is a well-defined pseudo-functor $(-)^!$ from FPS_R into the 2-category of categories such that*

(1) $(X)^! = \mathbf{D}^+_{\mathrm{aqc}}(X)^a$;

(2) *for a smooth morphism $f: X \to Y$ of pure relative dimension d, we have a natural isomorphism $f^! \simeq \mathbf{L}f^*(-) \otimes^L_{\mathcal{O}^a_X} \Omega^d_{X/Y}[d]$;*

(3) *for a proper morphism $f: X \to Y$, the functor $f^!$ is the right adjoint of $\mathbf{R}f_*: \mathbf{D}^+_{\mathrm{aqc}}(X)^a \to \mathbf{D}^+_{\mathrm{aqc}}(Y)^a$.*