Chapter 6

$\mathcal{O}^+/p$-modules

The main goal of this chapter is to discuss the comparison results between $\mathcal{O}^+/p$-modules in the étale, quasi-proétale, and v-topologies. In particular, we show that the categories of O^+/p-vector bundles in all these topologies are canonically equivalent. Furthermore, one can compute cohomology groups with respect to any of these topologies (without passing to almost mathematics). A good understanding of $\mathcal{O}^+/p$-vector bundles in the v-topology will be crucial for our proof of almost coherence of nearby cycles for general $\mathcal{O}^+/p$-vector bundles (see Theorem 7.1.2). We also discuss more general $\mathcal{O}^+/p$-modules in Section 6.7.

In this chapter, we will freely use the notions of perfectoid spaces and their tilts as developed in [58] and [61].

6.1 Recollection: The v-topology

In this section, we discuss the v-topology on adic spaces and show some of its basic properties that seem difficult to find explicitly stated in the literature.

Before we start this discussion, we recall the notion of a diamond and its relation to the notion of an adic space. To motivate this discussion, we remind the reader of the two major problems with the category of adic spaces: the existence of non-sheafy (pre-)adic spaces and the lack of (finite) limits in the category of adic spaces. It turns out that both of these problems go away if we consider a (pre-)adic space over $\mathbf{Q}_p$ as some kind of sheaf $X^\diamond$ on the category of perfectoid spaces of characteristic $p > 0$. It could sound somewhat counter-intuitive to consider a p-adic rigid-analytic variety as a sheaf on characteristic p objects, but it turns out to be quite useful in practice. The main idea is that an $S = \mathrm{Spa}\,(R, R^+)$-point of $X^\diamond$ should be a choice of an untilt $S^\#$ of S (this is a mixed characteristic object) and a morphism $S^\# \to X$. This procedure turns out to remember *a lot* of information about X (e.g., étale cohomology), but not *all* information about X (see Warning 6.1.8).

Definition 6.1.1 ([61, Definitions 8.1, 12.1, and 14.1]). The category Perf is the category of characteristic p perfectoid spaces.

The *v-topology* is the Grothendieck topology on Perf, defined such that a family $\{f_i\colon X_i \to X\}_{i\in I}$ of morphisms in Perf is a covering if, for any quasi-compact open $U \subset X$, there are a finite subset $I_0 \subset I$ and quasi-compact opens $\{U_i \subset X_i\}_{i\in I_0}$ such that $U \subset \bigcup_{i\in I_0} f_i(U_i)$.

A *small v-sheaf* is a v-sheaf Y on Perf such that there is an epimorphism of v-sheaves $Y' \to Y$ for some perfectoid space Y'.

The *v-site* Y_v of a small v-sheaf Y is the site whose objects are all maps $Y' \to Y$ from small v-sheaves Y', with coverings given by families $\{Y_i \to Y\}_{i \in I}$ such that $\bigsqcup_{i \in I} Y_i \to Y$ is an epimorphism of v-sheaves.

Remark 6.1.2. The v-site of a small v-sheaf Y has all finite limits by [61, Proposition 12.10] and [68, Tag 002O].

In what follows, we denote by $\mathbf{Ad}_{\mathbf{Q}_p}$ the category of adic spaces over $\operatorname{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)$ and by $\mathbf{pAd}_{\mathbf{Q}_p}$ the category of pre-adic spaces over $\operatorname{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)$ as defined in [62, Definition 2.1.5] and [41, Definition 8.2.3].[1] The category of pre-adic spaces satisfies the following list of properties (see [62, Proposition 2.1.6] or [41, Section 8.2.3]):

(1) The natural functor $\mathbf{Ad}_{\mathbf{Q}_p} \to \mathbf{pAd}_{\mathbf{Q}_p}$ is fully faithful.

(2) There is a functor $(\text{Tate–Huber}_{(\mathbf{Q}_p,\mathbf{Z}_p)}{}^{\text{comp}})^{\text{op}} \to \mathbf{pAd}_{\mathbf{Q}_p}$ from the opposite category of complete Tate–Huber pairs over $(\mathbf{Q}_p, \mathbf{Z}_p)$ to the category of pre-adic spaces over $\operatorname{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)$. To each such (A, A^+) it assigns the pre-adic affinoid space[2] $\widetilde{\operatorname{Spa}(A, A^+)}$.

(3) For an adic space S and a pre-adic affinoid space $\widetilde{\operatorname{Spa}(A, A^+)}$, the set of morphisms is given by

$$\operatorname{Hom}_{\mathbf{pAd}_{\mathbf{Q}_p}}\big(S, \widetilde{\operatorname{Spa}(A, A^+)}\big) = \operatorname{Hom}_{\text{cont}}\big((A, A^+), (\mathcal{O}_S(S), \mathcal{O}_S^+(S))\big).$$

(4) $\mathbf{pAd}_{\mathbf{Q}_p}$ has all finite limits.

(5) With a pseudo-adic space X, one can functorially associate an underlying topological space $|X|$ such that it coincides with the usual underlying topological space $|X|$ when either $\widetilde{\operatorname{Spa}(A, A^+)}$ is a pre-adic affinoid space or $X = (|X|, \mathcal{O}_X, \mathcal{O}_X^+)$ is an adic space.

(6) With every pre-adic space $X \in \mathbf{pAd}_{\mathbf{Q}_p}$, one can functorially associate an étale site $X_{\text{ét}}$ such that $X_{\text{ét}}$ coincides with the classical étale site when X is a locally strongly noetherian space or a perfectoid space (see [38, Section 2.1] and [58, Section 7]).

Warning 6.1.3. In general, it is not true that $\operatorname{Hom}_{\mathbf{pAd}_{\mathbf{Q}_p}}\big(\widetilde{\operatorname{Spa}(B, B^+)}, \widetilde{\operatorname{Spa}(A, A^+)}\big)$ is equal to $\operatorname{Hom}_{\text{cont}}\big((A, A^+), (B, B^+)\big)$ unless $\operatorname{Spa}(B, B^+)$ is sheafy. In particular, the functor

$$\big(\text{Tate–Huber}^{\text{comp}}_{(\mathbf{Q}_p,\mathbf{Z}_p)}\big)^{\text{op}} \to \mathbf{pAd}_{\mathbf{Q}_p}$$

is *not* fully faithful.

[1] These spaces are called adic in [62], we prefer to call them pre-adic to distinguish them from the usual adic spaces in the sense of Huber.

[2] We follow [41] and use the notation $\widetilde{\operatorname{Spa}(A, A^+)}$ for affinoid *pre*-adic spaces. If A is sheafy, we freely identify it with $\operatorname{Spa}(A, A^+)$.

Definition 6.1.4 ([62, Definition 2.4.1]). Let X_i be a cofiltered inverse system of pre-adic spaces with quasi-compact and quasi-separated transition maps, X a pre-adic space, and $f_i\colon X \to X_i$ a compatible family of morphisms.

We say that X is a *tilde-limit* of X_i, $X \sim \lim_I X_i$ if the map of underlying topological spaces $|X| \to \lim_I |X_i|$ is a homeomorphism and there is an open covering of X by affinoids $\mathrm{Spa}\,\widetilde{(A, A^+)} \subset X$, such that the map

$$\mathrm{colim}_{\mathrm{Spa}\,\widetilde{(A,A^+)} \subset X_i} A_i \to A$$

has dense image, where the filtered colimit runs over all open affinoids

$$\mathrm{Spa}\,\widetilde{(A, A^+)} \subset X_i$$

over which $\mathrm{Spa}\,\widetilde{(A, A^+)} \subset X \to X_i$ factors.

Definition 6.1.5 ([61, Definition 15.5]). The *diamond associated* with $X \in \mathbf{pAd}_{\mathbf{Q}_p}$ is a presheaf

$$X^{\diamond}\colon \mathrm{Perf}^{\mathrm{op}} \to \mathrm{Sets}$$

such that, for any perfectoid space S of characteristic p, we have

$$X^{\diamond}(S) = \big\{\big((S^{\sharp}, \iota), f\colon S^{\sharp} \to X\big)\big\}/\mathrm{isom},$$

where $S^{\sharp}$ is a perfectoid space, $\iota\colon (S^{\sharp})^{\flat} \to S$ is an isomorphism of the tilt of $S^{\sharp}$ with S, and $f\colon S^{\sharp} \to X$ is a morphism of pre-adic spaces.

The *diamantine spectrum* $\mathrm{Spd}\,(A, A^+)$ of a Huber pair (A, A^+) is a presheaf $\mathrm{Spa}\,\widetilde{(A, A^+)}^{\diamond}$.

We list the main properties of this functor:

Proposition 6.1.6. *The diamondification functor factors through the category of v-sheaves. Moreover, the functor $(-)^{\diamond}\colon \mathbf{pAd}_{\mathbf{Q}_p} \to \mathbf{Shv}(\mathrm{Perf}_v)$ satisfies the following list of properties:*

(1) *if X is a perfectoid space, then $X^{\diamond} \simeq X^{\flat}$;*

(2) *$X^{\diamond}$ is a small v-sheaf for any $X \in \mathbf{pAd}_{\mathbf{Q}_p}$;*[3]

(3) *if $\{X_i \to X\}_{i\in I}$ is an open (resp. étale) covering in $\mathbf{pAd}_{\mathbf{Q}_p}$, then the family $\{X_i^{\diamond} \to X^{\diamond}\}_{i\in I}$ is an open (resp. étale) covering of $X^{\diamond}$;*

(4) *there is a functorial homeomorphism $|X| \simeq |X^{\diamond}|$ for any $X \in \mathbf{pAd}_{\mathbf{Q}_p}$;*

(5) *if X is a perfectoid space such that $X \sim \lim_I X_i$ in $\mathbf{pAd}_{\mathbf{Q}_p}$ with quasi-compact quasi-separated transition maps, then $X^{\diamond} \to \lim_I X_i^{\diamond}$ is an isomorphism;*

(6) *the functor $(-)^{\diamond}\colon\colon \mathbf{pAd}_{\mathbf{Q}_p} \to \mathbf{Shv}(\mathrm{Perf}_v)$ commutes with fiber products.*

[3] It is even a locally spatial diamond in the sense of [61, Definition 11.17].

Proof. The first claim follows from [61, Corollary 3.20] and the definition of the diamondification functor. As for the second claim, [61, Proposition 15.6] implies that $X^\diamond$ is a diamond, and so it is a small v-sheaf due to [61, Proposition 11.9] and the definition of a diamond (see [61, Definition 11.1]). The third and fourth claims follow from [61, Lemma 15.6]. The proof of the fifth claim is identical to that of [62, Proposition 2.4.5] (the statement makes the assumption that X and X_i are defined over a perfectoid field, but it is not used in the proof).

Now we give a proof of the sixth claim. Let $U \to V$, $W \to V$ be morphisms in $\mathbf{pAd}_{\mathbf{Q}_p}$ with the fiber product $U \times_V W$. We fix a perfectoid space S of characteristic p. Then we have a sequence of identifications

$$\begin{aligned}
&(U \times_V W)^\diamond(S) \\
&\quad = \big\{\big((S^\sharp, \iota), S^\sharp \to U \times_V W\big)\big\}/\mathrm{isom} \\
&\quad = \big\{\big((S^\sharp, \iota), S^\sharp \to U\big)\big\}/\mathrm{isom} \times_{\{((S^\sharp,\iota),S^\sharp\to V)\}/\mathrm{isom}} \big\{\big((S^\sharp, \iota), S^\sharp \to W\big)\big\}/\mathrm{isom} \\
&\quad = U^\diamond(S) \times_{V^\diamond(S)} W^\diamond(S),
\end{aligned}$$

which is functorial in S. Therefore, this defines an isomorphism

$$(U \times_V W)^\diamond \xrightarrow{\sim} U^\diamond \times_{V^\diamond} W^\diamond. \qquad \blacksquare$$

Warning 6.1.7. The functor $(-)^\diamond$ does not send the final object to the final object. In particular, it does not commute with all finite limits.

Warning 6.1.8. The functor $(-)^\diamond \colon \mathbf{pAd}_{\mathbf{Q}_p} \to \mathbf{Shv}(\mathrm{Perf}_v)$ is not fully faithful. This observation is quite crucial for our proof of Theorem 7.10.3. In that proof, we exploit Theorem 7.10.1 which guarantees that some non-perfectoid affinoid (pre-)adic spaces become perfectoid after diamondification.

The next goal is to discuss some examples of v-covers of $X^\diamond$.

Definition 6.1.9. A family of morphisms $\{f_i \colon X_i \to X\}_{i\in I}$ in $\mathbf{pAd}_{\mathbf{Q}_p}$ is a *naive v-covering* if, for any quasi-compact open $U \subset X$, there are a finite subset $I_0 \subset I$ and quasi-compact opens $\{U_i \subset X_i\}_{i\in I_0}$ such that $|U| \subset \bigcup_{i\in I_0} |f_i|(|U_i|)$.

Remark 6.1.10. Using that the natural morphism $|X \times_Y Z| \to |X| \times_{|Y|} |Z|$ is surjective, it is easy to see that a pullback of a naive v-covering is a naive v-covering.

Lemma 6.1.11. *Let $f \colon X \to Y$ be an étale morphism of pre-adic spaces in $\mathbf{pAd}_{\mathbf{Q}_p}$ (in the sense of* [41, Definition 8.2.19]*). Then f is an open map.*

Proof. By definition, open immersions induce open maps of underlying topological spaces. Therefore, after unraveling the definition of étale morphisms, it suffices to show that a map of pre-adic spaces $|\operatorname{Spa}(\varphi)| \colon |\operatorname{Spa}(B, B^+)| \to |\operatorname{Spa}(A, A^+)|$ is open when $\varphi \colon (A, A^+) \to (B, B^+)$ is a finite étale morphism of Tate–Huber pairs.

In this case, Lemma C.2.9 and Corollary C.3.12 allow us to assume that (A, A^+) is strongly noetherian. Then the result follows from [38, Lemma 1.7.9] (alternatively, one can directly adapt the proof of [38, Lemma 1.7.9] to work in the non-noetherian case). ∎

Example 6.1.12. (1) A quasi-compact surjective morphism $X \to Y$ of pre-adic spaces over Spa $(\mathbf{Q}_p, \mathbf{Z}_p)$ is a naive v-cover;

(2) Lemma 6.1.11 implies that a family of jointly surjective étale morphisms $\{X_i \to X\}$ of pre-adic spaces over Spa $(\mathbf{Q}_p, \mathbf{Z}_p)$ is a naive v-cover.

Our next goal is to show that the diamondification functor $(-)^\diamond$ sends naive v-covers to surjections of small v-sheaves.

Lemma 6.1.13. *Let $f : X \to Y$ be a quasi-compact (resp. quasi-separated) morphism in $\mathbf{pAd}_{\mathbf{Q}_p}$. Then $f^\diamond : X^\diamond \to Y^\diamond$ is quasi-compact (resp. quasi-separated) in the sense of* [61, p. 40].

Proof. We first deal with a quasi-compact f. In order to check that $f^\diamond$ is quasi-compact, it suffices to show that $S \times_{Y^\diamond} X^\diamond$ is quasi-compact for any morphism $S \to Y$ with an affinoid perfectoid S. By definition, this morphism corresponds to a morphism $S^\sharp \to Y$ with an affinoid perfectoid source $S^\sharp$. By Proposition 6.1.6, we have $S \times_{Y^\diamond} X^\diamond \simeq (S^\sharp \times_Y X)^\diamond$, so [61, Lemma 15.6] implies that

$$|S \times_{Y^\diamond} X^\diamond| \simeq |S^\sharp \times_Y X|$$

is quasi-compact by our assumption on f. Now $S \times_{Y^\diamond} X^\diamond$ is quasi-compact due to the combination of [61, Proposition 12.14 (iii) and Lemma 15.6].

The case of a quasi-separated f follows from Proposition 6.1.6 and the quasi-compact case by considering the diagonal morphism $\Delta_f : X \to X \times_Y X$. ∎

Lemma 6.1.14. *Let $\{f_i : X_i \to X\}_{i \in I}$ be a naive v-covering in $\mathbf{pAd}_{\mathbf{Q}_p}$. Then the family $\{f_i^\diamond : X_i^\diamond \to X^\diamond\}_{i \in I}$ is a v-covering as well.*

Proof. We can find a covering $\{U_j \to X\}_{j \in J}$ by open affinoids. Proposition 6.1.6 implies that $\{U_j^\diamond \to X^\diamond\}$ is a v-covering. Therefore, it suffices to show that the family $\{f_{i,j} : X_{i,j} := X_i \times_X U_j \to U_j\}_{i \in I}$ is a v-covering for every $j \in J$. Since naive v-covers are preserved by open base change, we reduce to the case when X is an affinoid.

Moreover, the proof of [61, Proposition 15.4] ensures that there is a v-surjection $f : S \to X^\diamond$ where S is an affinoid perfectoid space. By definition, the map f corresponds to a map $g : S^\sharp \to X$. Proposition 6.1.6 ensures that diamondization commutes with fiber products, so it suffices to show that $\{(X_i \times_X S^\sharp)^\diamond \to (S^\sharp)^\diamond\}_{i \in I}$ is a v-covering. In other words, we can assume that $X = S^\sharp$ is an affinoid perfectoid space.

Now we can find a covering $\{U_{i,j} \to X_i\}_{j\in J_i}$ by open affinoids for each $i \in I$. Then the family $\{U_{i,j} \to X\}_{i\in I, j\in J_i}$ is also a naive v-covering, and so it suffices to show that $\{U_{i,j}^\diamond \to X^\diamond\}_{i\in I, j\in J_i}$ is a v-covering. In other words, we can assume that X is an affinoid perfectoid space and that all X_i are affinoids. A similar argument allows us to assume that each X_i is an affinoid perfectoid space.

Finally, we note that under our assumption that X and X_i are (affinoid) perfectoids, $\{X_i \to X\}_{i\in I}$ is a naive v-covering if and only if $\{X_i^\diamond \to X^\diamond\}_{i\in I}$ is a v-covering since $|X_i^\diamond| \simeq |X_i|$ and $|X^\diamond| = |X|$ by [61, Lemma 15.6]. ■

6.2 Recollection: The quasi-proétale topology

The main goal of this section is to recall the notions of a quasi-proétale morphism and the quasi-proétale topology. This topology will be a crucial intermediate tool to relate the v-topology to the étale topology.

In this section, we will only work with strongly sheafy spaces in the sense of Definition C.4.1. We advise the reader to look at Appendix C for basic definitions involving such spaces. Most likely, this discussion can be generalized to arbitrary affinoid pre-adic spaces, but we do not do this since we will never need this level of generality.

For the purpose of the next definition, we fix a morphism $f: X = \operatorname{Spa}(S, S^+) \to Y = \operatorname{Spa}(R, R^+)$ of strongly sheafy Tate-affinoid adic spaces.

Definition 6.2.1. A morphism $f: \operatorname{Spa}(S, S^+) \to \operatorname{Spa}(R, R^+)$ is an *affinoid strongly pro-étale morphism* if there is a cofiltered system of strongly étale morphisms of strongly sheafy affinoid adic spaces (see Definition C.4.5)

$$\operatorname{Spa}(R_i, R_i^+) \to \operatorname{Spa}(R, R^+)$$

such that $(S, S^+) = \left(\widehat{(\operatorname{colim}_I R_i)_u}, \widehat{(\operatorname{colim}_I R_i)_u^+}\right)$ is the completed uniform filtered colimit of (R_i, R_i^+) (see Definition C.2.4).

We will usually write $\operatorname{Spa}(S, S^+) \approx \lim_I \operatorname{Spa}(R_i, R_i^+) \to \operatorname{Spa}(R, R^+)$ for an affinoid strongly pro-étale presentation of $\operatorname{Spa}(S, S^+) \to \operatorname{Spa}(R, R^+)$.

Remark 6.2.2. Explicitly, Remark C.2.5 implies that $S^+ = (\operatorname{colim}_I R_i^+)^\wedge_\varpi$ is equal to the ϖ-adic completion of $\operatorname{colim}_I R_i^+$ and $S = S^+[\frac{1}{\varpi}]$ for any choice of a pseudo-uniformizer $\varpi \in R^+$.

Remark 6.2.3. We note that Theorem C.3.10 (1) (see also [62, Proposition 2.4.2]) implies that $\operatorname{Spa}(S, S^+) \sim \lim_I \operatorname{Spa}(R_i, R_i^+)$ for an affinoid strongly pro-étale morphism $\operatorname{Spa}(S, S^+) \approx \lim_I \operatorname{Spa}(R_i, R_i^+) \to \operatorname{Spa}(R, R^+)$.

Warning 6.2.4. Definition 6.2.1 is more restrictive than [61, Definition 7.8] when $\operatorname{Spa}(R, R^+)$ is an affinoid perfectoid space.

Definition 6.2.5. A perfectoid space X is *strictly totally disconnected* if X is quasi-compact, quasi-separated, and every étale cover of X splits.

Lemma 6.2.6. *Let each X, Y, Y', and Z be affinoid spaces over* $\mathrm{Spa}\,(\mathbf{Q}_p, \mathbf{Z}_p)$*. We assume that each of them is strongly sheafy.*

(1) *Let $f: X \to Y$ and $g: Y \to Z$ be affinoid strongly pro-étale morphisms. Then the composition $g \circ f: X \to Z$ is also an affinoid strongly pro-étale morphism.*

(2) *Let $f: X \to Y$ be an affinoid strongly pro-étale morphism, let $g: Y' \to Y$ be a morphism of adic spaces with Y' being an affinoid perfectoid space (resp. strictly totally disconnected perfectoid space), and let $X_{Y'} := X \times_Y Y'$ be the fiber product (in pre-adic spaces). Then $X^\diamond_{Y'}$ is an affinoid perfectoid space (resp. strictly totally disconnected perfectoid space) and the morphism $f^\diamond_{Y'}: X^\diamond_{Y'} \to Y'^\diamond$ is an affinoid pro-étale morphism in the sense of* [61, Definition 7.8].

Proof. (1) The proof of [52, Lemma 2.5 (1)] goes through if we use Theorem C.3.10 in place of [61, Proposition 6.4] (and [38, Proposition 1.7.1]).

Now we show (2). We set $X = \mathrm{Spa}\,(S, S^+)$, $Y = \mathrm{Spa}\,(R, R^+)$, $Y' = \mathrm{Spa}\,(R', R'^+)$, and let $X \approx \lim_I \big(X_i = \mathrm{Spa}\,(R_i, R_i^+)\big) \to Y = \mathrm{Spa}\,(R, R^+)$ be an affinoid strongly pro-étale presentation of $X \to Y$. Then Proposition 6.1.6 (5) implies that

$$X^\diamond = \lim_I X_i^\diamond.$$

Therefore, $X^\diamond_{Y'} = \lim_I (X_i \times_Y Y')^\diamond \to Y'^\diamond$. Hence it suffices to show that each $(X_i \times_Y Y')^\diamond$ is represented by an affinoid perfectoid space, and that each morphism $f_i^\diamond: (X_i \times_Y Y')^\diamond \to Y'^\diamond$ is étale. By construction, $f_i^\diamond$ is étale. In particular, $(X_i \times_Y Y')^\diamond$ is represented by a perfectoid space. Furthermore, $f_i^\diamond$ is a composition of finite étale maps and finite disjoint unions of rational subdomains. Therefore, $(X_i \times_Y Y')^\diamond$ is an affinoid perfectoid space due to the combination of [58, Theorem 6.3 and Theorem 7.9].

If Y' is strictly totally disconnected, then [61, Lemma 7.19] implies that $X^\diamond_{Y'}$ is also represented by a strictly totally disconnected perfectoid space. ∎

Warning 6.2.7. [52, Lemma 2.5 (1)] claims a stronger version of Lemma 6.2.6 (2). However, it seems to be false (see Warning 7.6.5).

Now we are ready to show that the issue raised in Warning 6.2.4 disappears when the target is a strictly totally disconnected perfectoid space.

Lemma 6.2.8. *Let $X = \mathrm{Spa}\,(R, R^+)$ be a strictly totally disconnected perfectoid space, and let $f: Y = \mathrm{Spa}\,(S, S^+) \to X = \mathrm{Spa}\,(R, R^+)$ be an affinoid pro-étale* *morphism (in the sense of* [61, Definition 7.8]*). Then f is an affinoid strongly pro-étale morphism.*

Proof. The proof of [61, Lemma 7.19] ensures that f can be realized as a pro-(rational subdomain) inside the pro-(finite étale) morphism $X \times_{\pi_0(X)} \pi_0(Y)$. Each of these morphisms is an affinoid strongly pro-étale morphism. Thus, Lemma 6.2.6 (1) ensures that f is an affinoid strongly pro-étale morphism as well. ∎

For the next definition, we fix a morphism $f: X \to Y$ of adic spaces such that X and Y are strongly sheafy adic spaces over $\mathrm{Spa}\,(\mathbf{Q}_p, \mathbf{Z}_p)$.

Definition 6.2.9. A morphism $f: X \to Y$ is *strongly pro-étale* if, for every point $x \in X$, there are an open affinoid $x \in U \subset X$ and an open affinoid $f(x) \in V \subset Y$ such that $f|_U: U \to V$ is affinoid strongly pro-étale.

Now we are ready to define quasi-proétale morphisms.

Definition 6.2.10 ([61, Definition 10.1 and 14.1]). A morphism of small v-sheaves $f: X \to Y$ is *quasi-proétale* if it is locally separated, and for every morphism $S \to Y$ with a strictly totally disconnected perfectoid S, the fiber product $X_S := X \times_Y S$ is represented by a perfectoid space and $X_S \to S$ is pro-étale.

The *quasi-proétale site* $X_{\text{qproét}}$ of a small v-sheaf is the site whose objects are quasi-proétale morphisms $Y \to X$, with coverings given by families $\{Y_i \to Y\}_{i \in I}$ such that $\bigsqcup_{i \in I} Y_i \to Y$ is a surjection of v-sheaves.

Lemma 6.2.11. *Let $f: X \to Y$ be a strongly pro-étale morphism such that both X and Y are strongly sheafy adic spaces over* $\mathrm{Spa}\,(\mathbf{Q}_p, \mathbf{Z}_p)$*. Then $f^\diamond: X^\diamond \to Y^\diamond$ is quasi-proétale. Furthermore, if f is also a naive v-covering, then $f^\diamond$ is a quasi-proétale covering.*

Proof. The question is local on the source and on the target, so we can assume that f is an affinoid strongly pro-étale morphism. Then it is easy to see that $f^\diamond: X^\diamond \to Y^\diamond$ is a separated morphism (for example, it is quasi-separated due to Lemma 6.1.13 and then the valuative criterion [61, Proposition 10.9] implies that it is separated). Therefore, it suffices to show that, for any strictly totally disconnected perfectoid S and a morphism $S \to Y^\diamond$, the fiber product $S \times_{Y^\diamond} X^\diamond \to S$ is a pro-étale morphism of perfectoid spaces.

Now we recall that a morphism $f: S \to Y^\diamond$ uniquely corresponds to a morphism $g: S^\sharp \to Y$. Proposition 6.1.6 (6) implies that

$$S \times_{Y^\diamond} X^\diamond \simeq (S^\sharp \times_Y X)^\diamond.$$

Therefore, Lemma 6.2.6 (2) implies that $S \times_{Y^\diamond} X^\diamond \to S$ is affinoid pro-étale in the sense of [61, Definition 7.8]. This finishes the proof that $f^\diamond$ is quasi-proétale. If we also assume that f is a naive v-covering, then Lemma 6.1.14 ensures that $f^\diamond$ is a surjection of v-sheaves. Thus, $f^\diamond$ is a quasi-proétale covering in this case. ∎

Finally, we wish to show that strongly sheafy Tate-affinoids $\mathrm{Spa}\,(A, A^+)$ admit affinoid strongly pro-étale covers by strictly totally disconnected perfectoid spaces. For this, we will need some preliminary lemmas:

Lemma 6.2.12. *Let (A, A^+) be an affinoid perfectoid pair. Suppose that every surjective (affinoid) strongly étale morphism $\mathrm{Spa}\,(B, B^+) \to \mathrm{Spa}\,(A, A^+)$ admits a section (see Definition C.4.5). Then $\mathrm{Spa}\,(A, A^+)$ is a strictly totally disconnected perfectoid space.*

Proof. It suffices to show that every étale surjective morphism $X \to \mathrm{Spa}\,(A, A^+)$ admits a section. Any such morphism can be dominated by a surjective morphism of the form $\bigsqcup_{i\in I} X_i \to \mathrm{Spa}\,(A, A^+)$ where $X_i = \mathrm{Spa}\,(B_i, B_i^+) \to \mathrm{Spa}\,(A, A^+)$ is affinoid strongly étale and I is a finite set. Then Remark C.4.7 implies that $\bigsqcup_{i\in I} X_i \to \mathrm{Spa}\,(A_i, A_i^+)$ is itself strongly étale (and affinoid), so it admits a section due to the assumption on X. Therefore, $X \to \mathrm{Spa}\,(A, A^+)$ also admits a section. ∎

Lemma 6.2.13. *Let $\mathrm{Spa}\,(A, A^+)$ denote a strongly sheafy Tate-affinoid space over $\mathrm{Spa}\,(\mathbf{Q}_p, \mathbf{Z}_p)$. Then there is an affinoid strongly pro-étale covering $\mathrm{Spa}\,(A_\infty, A_\infty^+) \to \mathrm{Spa}\,(A, A^+)$ such that the fiber products $\mathrm{Spd}\,(A_\infty, A_\infty^+)^{j/\mathrm{Spd}\,(A,A^+)}$ are represented by strictly totally disconnected (affinoid) perfectoid spaces for $j \geq 1$. In particular,*

$$\mathrm{Spd}\,(A_\infty, A_\infty^+) \to \mathrm{Spd}\,(A, A^+)$$

is a quasi-proétale covering by a strictly totally disconnected perfectoid space.

Proof. For the purpose of the present proof, we say that a strongly étale morphism $f\colon (R, R^+) \to (S, S^+)$ of complete Tate–Huber pairs is a *covering* if the corresponding morphism $|\mathrm{Spa}\,(f)|\colon |\mathrm{Spa}\,(S, S^+)| \to |\mathrm{Spa}\,(R, R^+)|$ is surjective.

To begin with, we fix a set of representatives of all strongly étale coverings $\{(A, A^+) \to (A_i, A_i^+)\}_{i\in I}$. Then, for each finite subset $S \subset I$, we define

$$(A_S, A_S^+) = \widehat{\otimes}_{s\in S}(A_s, A_s^+).$$

Each (A_S, A_S^+) is a strongly étale covering of (A, A^+). For each $S \subset S'$, we put $f_{S,S'}\colon (A_S, A_S^+) \to (A_{S'}, A_{S'}^+)$ to be the natural morphism induced by $S \hookrightarrow S'$. Then we see that $\{(A_S, A_S^+), f_{S,S'}\}_{S\subset I \text{ finite}}$ is a filtered system of strongly étale (A, A^+)-algebras. We put

$$(A(1), A^+(1)) = \Big((\widehat{\mathrm{colim}_S\, A_S^+})\Big[\frac{1}{p}\Big], \widehat{\mathrm{colim}_S\, A_S^+}\Big)$$

to be the completed uniform filtered colimit of (A_S, A_S^+) (see Definition C.2.4). Theorem C.3.10 implies that every strongly étale covering $(A, A^+) \to (B, B^+)$ admits a splitting over $(A(1), A^+(1))$. We repeat the same construction to inductively define

$$\begin{aligned}
\big(A(2), A^+(2)\big) &:= \big(A(1)(1), A(1)^+(1)\big),\\
\big(A(3), A^+(3)\big) &:= \big(A(2)(1), A(2)^+(1)\big), \dots,\\
\big(A(n), A^+(n)\big) &:= \big(A(n-1)(1), A(n-1)^+(1)\big), \dots
\end{aligned}$$

Finally, we let (A_∞, A_∞^+) be the completed uniform filtered colimit of $(A(n), A(n)^+)$. Then Theorem C.3.10 implies that any strongly étale covering of (A_∞, A_∞^+) comes from a covering of some $(A(n), A^+(n))$, hence it admits a splitting over $(A(n+1), A^+(n+1))$. In particular, every strongly étale covering of (A_∞, A_∞^+) admits a splitting. The proof of [61, Lemma 15.3] implies that (A_∞, A_∞^+) is a perfectoid pair. In particular, it is strongly sheafy. Furthermore, Lemma 6.2.12 ensures that it is strictly totally disconnected. We notice that the morphism $\mathrm{Spa}\,(A_\infty, A_\infty^+) \to \mathrm{Spa}\,(A, A^+)$ is an affinoid strongly pro-étale covering. Finally, we conclude that all fiber products $\mathrm{Spd}\,(A_\infty, A_\infty^+)^{j/\mathrm{Spd}\,(A,A^+)}$ are represented by strictly totally disconnected perfectoid spaces due to Lemma 6.2.6 (2). ∎

Lemma 6.2.14. *Let $X = \mathrm{Spa}\,(A, A^+)$ denote a strongly sheafy Tate-affinoid over $\mathrm{Spa}\,(\mathbf{Q}_p, \mathbf{Z}_p)$. Then the set of all morphisms $f^\diamond\colon Y^\diamond \to X^\diamond$ for an affinoid perfectoid Y with an affinoid strongly pro-étale morphism $f\colon Y \to X$ forms a basis of $X^\diamond_{\mathrm{qpro\acute{e}t}}$.*

Proof. Let $Z \to X^\diamond$ be a quasi-proétale morphism. We wish to show that it can be covered (in the quasi-proétale topology) by elements of the form $Y^\diamond \to X^\diamond$ for an affinoid perfectoid Y and an affinoid strongly pro-étale morphism $Y \to X$.

Then Lemma 6.2.13 implies that we can find an affinoid strongly pro-étale covering $X' \to X$ such that X' is a strictly totally disconnected perfectoid space. Since $Z \to X^\diamond$ is quasi-proétale, we conclude that $Z \times_{X^\diamond} X'^\diamond$ is a perfectoid space and $Z \times_{X^\diamond} X'^\diamond$ is pro-étale. Therefore, we can cover it (in the analytic topology) by affinoid perfectoid spaces Z_i such that each Z_i is an affinoid perfectoid space and $Z_i \to X'^\diamond$ is affinoid pro-étale. By construction $\{Z_i \to Z\}_{i\in I}$ is a covering in the quasi-proétale topology.

Now Lemma 6.2.8 implies that each $Z_i \to X'^\diamond$ is affinoid strongly pro-étale. Therefore, when we pass to the corresponding untilts, we get morphisms $Z_i^\sharp \to X'$ that are affinoid strongly pro-étale as well (we use [61, Theorem 3.12 and Theorem 6.1]). Consequently, Lemma 6.2.6 (1) implies that each $Z_i^\sharp \to X$ is an affinoid strongly pro-étale morphism (with an affinoid perfectoid $Z_i^\sharp$). By construction $(Z_i^\sharp)^\diamond = Z_i \to X^\diamond$ cover the morphism $Y \to X^\diamond$. ∎

6.3 Integral structure sheaves

In this section, we define various structure sheaves associated with a (pre-)adic space over $\mathbf{Q}_p$. Then we discuss the relationship between some of these sheaves. We will continue the discussion between these sheaves (and their cohomology) in the next section.

First, we note that the étale, quasi-proétale, and v-sites of a pre-adic space X over $\mathrm{Spa}\,(\mathbf{Q}_p, \mathbf{Z}_p)$ are related via the following sequence of morphisms of sites:

$$X_v^\diamond \xrightarrow{\lambda} X^\diamond_{\text{qproét}} \xrightarrow{\mu} X_{\text{ét}}, \tag{6.3.1}$$

which essentially come from the fact that any étale covering is a quasi-proétale covering, and any quasi-proétale covering is a v-covering.[4] Now we define various structure sheaves on each of these sites:

Definition 6.3.1. Let X be a pre-adic space over $\mathrm{Spa}\,(\mathbf{Q}_p, \mathbf{Z}_p)$.

The *integral "untilted" structure sheaf* $\mathcal{O}^+_{X^\diamond}$ is a sheaf of rings on $X_v^\diamond$ obtained as the sheafification of a pre-sheaf defined by the assignment

$$\{S \to X^\diamond\} \mapsto \mathcal{O}^+_{S^\sharp}(S^\sharp)$$

for any perfectoid space $S \to X^\diamond$ over $X^\diamond$ (the transition maps are defined in the evident way[5]).

The *rational "untilted" structure sheaf* $\mathcal{O}_{X^\diamond}$ is a sheaf of rings on $X_v^\diamond$ given by the formula $\mathcal{O}_{X^\diamond} = \mathcal{O}^+_{X^\diamond}[\frac{1}{p}]$.

The *mod-p structure sheaf* $\mathcal{O}^+_{X^\diamond}/p$ is the quotient of $\mathcal{O}^+_{X^\diamond}$ by p in the category of sheaves of rings on $X_v^\diamond$.

The *quasi-proétale integral "untilted" structure sheaf* $\mathcal{O}^+_{X^\diamond_{\text{qp}}}$ is the restriction of $\mathcal{O}^+_{X^\diamond}$ to the quasi-proétale site of $X^\diamond$, i.e., $\mathcal{O}^+_{X^\diamond_{\text{qp}}} = \lambda_* \mathcal{O}^+_{X^\diamond}$.

The *quasi-proétale mod-p structure sheaf* $\mathcal{O}^+_{X^\diamond_{\text{qp}}}/p$ is the quotient of $\mathcal{O}^+_{X^\diamond_{\text{qp}}}$ by p in the quasi-proétale site $X^\diamond_{\text{qproét}}$.

If X is a strongly sheafy space over $\mathrm{Spa}\,(\mathbf{Q}_p, \mathbf{Z}_p)$, the *étale mod-$p$ structure sheaf* $\mathcal{O}^+_{X_{\text{ét}}}/p$ is the quotient of $\mathcal{O}^+_{X_{\text{ét}}}$ by p in the étale site $X_{\text{ét}}$ (see Definition C.4.9 and Lemma C.4.11).

[4] To show that the natural continuous functors $X_{\text{ét}} \to X^\diamond_{\text{qproét}}$ and $X^\diamond_{\text{qproét}} \to X^\diamond_v$ induce morphisms of sites (in the other direction), one needs to verify that all these sites admit finite limits and these functors commute with all finite limits. We leave this as an exercise to the interested reader.

[5] Recall that a morphism $S \to X^\diamond$ is, by definition, a datum of an untilt $S^\sharp$ with a morphism $S^\sharp \to X$ and an isomorphism $(S^\sharp)^\flat \simeq S$. Thus, a pair of morphisms $T \to S \to X^\diamond$ defines a pair $T^\sharp \to S^\sharp \to X$.

Remark 6.3.2. We note that it is not, a priori, clear whether $\mathcal{O}^+_{X^\diamond_{\mathrm{qp}}}/p \simeq \lambda_*(\mathcal{O}^+_{X^\diamond}/p)$. The problem comes from the fact that λ is not an exact functor, so it is not clear whether it commutes with quotiening by p.

Remark 6.3.3. The relation between $\mathcal{O}^+_{X^\diamond_{\mathrm{qp}}}/p$ and $\mathcal{O}^+_{X_{\text{ét}}}/p$ is even more mysterious. The first sheaf is defined via descent from perfectoid spaces, so it seems subtle to control values of this sheaf on locally noetherian adic spaces. On the contrary, the second sheaf is defined using the étale topology on $X_{\text{ét}}$, so its definition has no direct relation to perfectoid spaces when X is a locally noetherian adic space.

By definition, for a strongly sheafy adic space X over $\mathrm{Spa}\,(\mathbf{Q}_p, \mathbf{Z}_p)$, we can promote Diagram (6.3.1) to a diagram of morphisms of ringed sites:

$$\left(X^\diamond_v, \mathcal{O}^+_{X^\diamond}/p\right) \xrightarrow{\lambda} \left(X^\diamond_{\text{qproét}}, \mathcal{O}^+_{X^\diamond_{\mathrm{qp}}}/p\right) \xrightarrow{\mu} \left(X_{\text{ét}}, \mathcal{O}^+_{X_{\text{ét}}}/p\right). \tag{6.3.2}$$

We also have "tilted" versions of the structure sheaves:

Definition 6.3.4. Let X be a pre-adic space over $\mathrm{Spa}\,(\mathbf{Q}_p, \mathbf{Z}_p)$.

The *integral "tilted" structure sheaf* $\mathcal{O}^{\flat,+}_{X^\diamond}$ is the sheaf of rings on $X^\diamond_v$ obtained as the sheafification of a pre-sheaf defined by the assignment

$$\{S \to X^\diamond\} \mapsto \mathcal{O}^+_S(S)$$

for any perfectoid space $S \to X^\diamond$ over $X^\diamond$.

If X is a pre-adic space over a p-adic perfectoid pair (R, R^+) with a good pseudo-uniformizer $\varpi \in R^+$ (see Definition B.11), the *rational "tilted" structure sheaf* $\mathcal{O}^\flat_{X^\diamond}$ is $\mathcal{O}^{\flat,+}_{X^\diamond}[\frac{1}{\varpi^\flat}]$.

We start with some easy properties of these structure sheaves:

Lemma 6.3.5. *Let $X \in \mathbf{pAd}_{\mathbf{Q}_p}$ be a pre-adic space over* $\mathrm{Spa}\,(\mathbf{Q}_p, \mathbf{Z}_p)$. *Then*

(1) *for any affinoid perfectoid space $Y = \mathrm{Spa}\,(S, S^+) \to X^\diamond$, we have the cohomology groups $\mathrm{H}^0(Y, \mathcal{O}^+_{X^\diamond}) = S^{\sharp,+}$ and $\mathrm{H}^i(Y, \mathcal{O}^+_{X^\diamond}) \simeq^a 0$ for $i \geq 1$;*

(2) *for any affinoid perfectoid space $Y = \mathrm{Spa}\,(S, S^+) \to X^\diamond$, we have the cohomology groups $\mathrm{H}^0(Y, \mathcal{O}^{\flat,+}_{X^\diamond}) = S^+$ and $\mathrm{H}^i(Y, \mathcal{O}^{\flat,+}_{X^\diamond}) \simeq^a 0$ for $i \geq 1$;*

(3) *the sheaf $\mathcal{O}^+_{X^\diamond}$ is derived p-adically complete and p-torsion free;*

(4) *if X is pre-adic space over a perfectoid pair (R, R^+) with a good pseudo-uniformizer $\varpi \in R^+$, the sheaf $\mathcal{O}^{\flat,+}_{X^\diamond}$ is derived $\varpi^\flat$-adically complete and $\varpi^\flat$-torsion free;*

(5) *if X is a pre-adic space over a perfectoid pair (R, R^+) with a good pseudo-uniformizer $\varpi \in R^+$, there is a canonical isomorphism $\mathcal{O}^+_{X^\diamond}/p \simeq \mathcal{O}^{\flat,+}_{X^\diamond}/\varpi^\flat$.*

Proof. (1) and (2) follow directly from [61, Theorem 8.7 and Proposition 8.8].

(3) To show that $\mathcal{O}^+_{X^\diamond}$ is p-torsion free, it suffices to show that $\mathcal{O}_{X^\diamond}{}^+(U)$ is p-torsion free on a basis of $X^\diamond_v$. Therefore, it is enough to show that

$$\mathcal{O}^+_{X^\diamond}(Y)$$

is p-torsion free for any affinoid perfectoid space $Y \to X^\diamond$. This follows from (1).

Lemma A.8 ensures that, for the purpose of proving that $\mathcal{O}^+_{X^\diamond}$ is p-adically derived complete, it suffices to show that

$$\mathbf{R}\Gamma(Y, \mathcal{O}^+_{X^\diamond})$$

is derived p-adically complete for any affinoid perfectoid space of the form $Y = \mathrm{Spa}\,(S, S^+) \to X$. Then it suffices to show that each cohomology group $\mathrm{H}^i(Y, \mathcal{O}^+_{X^\diamond})$ is derived p-adically complete. Now (1) implies that

$$\mathrm{H}^0(Y, \mathcal{O}^+_{X^\diamond}) = S^{\sharp,+}$$

is p-adically complete, and so it is derived p-adically complete (see [68, Tag 091R]). Moreover, (1) implies that all higher cohomology groups

$$\mathrm{H}^i(Y, \mathcal{O}^+_{X^\diamond}) \simeq^a 0$$

are almost zero. In particular, they are p-torsion, and so derived p-adically complete. Thus, $\mathbf{R}\Gamma(Y, \mathcal{O}^+_{X^\diamond})$ is derived p-adically complete finishing the proof.

(4) This is completely analogous to the proof of (3) using (2) in place of (1).

(5) Denote by $\mathcal{F}$ the *presheaf* quotient of $\mathcal{O}^+_{X^\diamond}$ by p, and by $\mathcal{G}$ the presheaf quotient of $\mathcal{O}^{\flat,+}_{X^\diamond}$ by $\varpi^\flat$. It suffices to construct a functorial isomorphism

$$\mathcal{F}(U) \simeq \mathcal{G}(U)$$

on a basis of $X^\diamond_v$. Therefore, it suffices to construct such an isomorphism for any affinoid perfectoid space $U \to X^\diamond$. Then (1) and (2) ensure that, for an affinoid perfectoid space $U = \mathrm{Spa}\,(S, S^+) \to X^\diamond$,

$$\mathcal{F}(U) \simeq S^{\sharp,+}/pS^{\sharp,+},$$
$$\mathcal{G}(U) \simeq S^+/\varpi^\flat S^+.$$

Essentially by the definition of a tilt, we have a canonical isomorphism

$$S^{\sharp,+}/pS^{\sharp,+} = S^{\sharp,+}/\varpi S^{\sharp,+} \simeq S^+/\varpi^\flat S^+$$

finishing the proof. ∎

Remark 6.3.6. The conclusion of Lemma 6.3.5 (1), (3) holds for the sheaf $\mathcal{O}^+_{X^\diamond_{\mathrm{qp}}}$ by a similar proof (using [61, Theorem 8.5] in place of [61, Theorem 8.7 and Proposition 8.8]). If X is a perfectoid space, the same conclusions hold for $\mathcal{O}^+_{X_{\text{ét}}}$ with a similar proof (using [61, Theorem 6.3] in place of [61, Theorem 8.7 and Proposition 8.8]).

Our next goal is to discuss the precise relation between $\mathcal{O}^+_{X^\diamond}/p$, $\mathcal{O}^+_{X^\diamond_{\mathrm{qp}}}/p$, and $\mathcal{O}^+_{X_{\text{ét}}}/p$. If one is willing to work in the almost world, then one can quite easily see that each of these sheaves is obtained as the (derived) restriction of the previous one to the smaller site (this essentially boils down to Lemma 6.3.5). However, to understand the relation between the categories of $\mathcal{O}^+/p$-vector bundles in different topologies, it is essential to understand the relation between these sheaves on the nose. This turns out to be quite subtle and will be discussed in the rest of this and the next sections.

Lemma 6.3.7. *Let $X \in \mathbf{pAd}_{\mathbf{Q}_p}$ be a pre-adic space over* $\operatorname{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)$*. Then the natural morphism*

$$\mathcal{O}^+_{X^\diamond_{\mathrm{qp}}}/p \to \lambda_*\big(\mathcal{O}^+_{X^\diamond}/p\big)$$

is an isomorphism. If X is a strongly sheafy adic space over $\operatorname{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)$*, then the natural morphisms*[6]

$$\mu^{-1}\big(\mathcal{O}^+_{X_{\text{ét}}}/p\big) \to \mathcal{O}^+_{X^\diamond_{\mathrm{qp}}}/p,$$
$$\mathcal{O}^+_{X_{\text{ét}}}/p \to \mathbf{R}\mu_*\big(\mathcal{O}^+_{X^\diamond_{\mathrm{qp}}}/p\big)$$

are isomorphisms as well.

Proof. The first result is [52, Proposition 2.13]. For the second result, we note that [52, Lemma 2.7] ensures[7] that, for a strongly sheafy adic space X, the sheaf $\mathcal{O}^+_{X^\diamond_{\mathrm{qp}}}$ is isomorphic to

$$\widehat{\mathcal{O}}^+_{X_{\mathrm{qp}}} := \lim_n \mu^{-1}\big(\mathcal{O}^+_{X_{\text{ét}}}/p^n\big).$$[8]

Now we know that the quasi-proétale site of a diamond is replete (in the sense of [9, Definition 3.1.1]) due to [52, Lemma 1.2]. Therefore, the fact that $\mathcal{O}^+_{X_{\text{ét}}}$ is p-torsion free and [9, Proposition 3.1.10] imply that

$$\widehat{\mathcal{O}}^+_{X_{\mathrm{qp}}} \simeq \mathbf{R}\lim \mu^{-1}\big(\mathcal{O}^+_{X_{\text{ét}}}/p^n\big) \simeq \widehat{\mu^{-1}(\mathcal{O}^+_{X_{\text{ét}}})}$$

[6] The functor $\mu^{-1}\colon \mathcal{Ab}(X_{\text{ét}}) \to \mathcal{Ab}(X^\diamond_{\text{qproét}})$ denotes the pullback of sheaves of abelian groups.

[7] Strictly speaking, the proof of [52, Lemma 2.7] assumes that X is either locally noetherian or perfectoid. However, a similar proof works for any strongly sheafy X if one uses Lemma 6.2.14 in place of [52, Lemma 2.6].

[8] The sheaf $\mathcal{O}^+_{X^\diamond_{\mathrm{qp}}}$ is denoted by $\widehat{\mathcal{O}}^+_{X^\diamond}$ in [52].

is the derived p-adic completion of $\mu^{-1}(\mathcal{O}^+_{X_{\text{ét}}})$. Since $\mathcal{O}^+_{X^\diamond_{\text{qp}}}$ is also p-torsion free by Lemma 6.3.5, the universal property of derived completion implies that

$$\begin{aligned}\mathcal{O}^+_{X^\diamond_{\text{qp}}}/p &\simeq \big[\mathcal{O}^+_{X^\diamond_{\text{qp}}}/p\big] \\ &\simeq \big[\widehat{\mu^{-1}(\mathcal{O}^+_{X_{\text{ét}}})}/p\big] \\ &\simeq \mu^{-1}\big(\mathcal{O}^+_{X_{\text{ét}}}/p\big).\end{aligned}$$

Finally, [61, Proposition 14.8 and Lemma 15.6] imply that

$$\mathcal{O}^+_{X_{\text{ét}}}/p \simeq \mathbf{R}\mu_*\mu^{-1}\big(\mathcal{O}^+_{X_{\text{ét}}}/p\big) \simeq \mathbf{R}\mu_*\big(\mathcal{O}^+_{X^\diamond_{\text{qp}}}/p\big). \qquad \blacksquare$$

Our next goal is to compare $\mathbf{R}\lambda_*\big(\mathcal{O}^+_{X^\diamond}/p\big)$ with $\mathcal{O}^+_{X^\diamond_{\text{qp}}}/p$. To do this, we need a number of preliminary results. This will be done in the next section.

6.4 v-descent for étale cohomology of $\mathcal{O}^+/p$

The main goal of this section is to show that the natural morphism

$$\mathcal{O}^+_{X^\diamond_{\text{qp}}}/p \to \mathbf{R}\lambda_*\big(\mathcal{O}^+_{X^\diamond}/p\big)$$

is an isomorphism. However, our argument is a bit roundabout, and we first show that the étale cohomology complex $\mathbf{R}\Gamma(X_{\text{ét}}, \mathcal{O}^+_{X_{\text{ét}}}/p)$ satisfies v-descent on affinoid perfectoid spaces. Even to formulate this precisely, we will need to use ∞-categories as developed in [48]. In what follows, we denote by $\mathcal{D}(\mathbf{Z})$ the ∞-enhancement of the triangulated derived category of abelian groups $\mathbf{D}(\mathbf{Z})$. We are also going to slightly abuse the notation and identify a (usual) category $\mathcal{C}$ with its nerve $\mathrm{N}(\mathcal{C})$ (see [50, Tag 002M]) considered as an ∞-category.

We fix a category $\mathrm{PerfAff}_{\mathbf{Q}_p}$ of affinoid perfectoid spaces over $\mathrm{Spa}\,(\mathbf{Q}_p, \mathbf{Z}_p)$. For any morphism $Z \to Y$, we can consider its Čech nerve $\check{\mathrm{C}}(Z/Y)$ as a simplicial object in $\mathrm{PerfAff}_{\mathbf{Q}_p}$, i.e., a functor

$$\check{\mathrm{C}}(Z/Y)\colon \Delta^{\text{op}} \to \mathrm{PerfAff}_{\mathbf{Q}_p}.$$

More explicitly, the n-th term

$$\check{\mathrm{C}}(Z/Y)_n = Z^{n/Y}$$

is the n-th fiber product of Z over Y. Face and degeneracy maps are defined in an evident way.

For any functor (in the ∞-categorical sense) $\mathcal{F}\colon \mathrm{PerfAff}^{\text{op}}_{\mathbf{Q}_p} \to \mathcal{D}(\mathbf{Z})$, we can compose $\mathcal{F}$ with $\check{\mathrm{C}}(Z/Y)^{\text{op}}$ to get a cosimplicial object $\check{\mathrm{C}}(Z/Y, \mathcal{F})$ in $\mathcal{D}(\mathbf{Z})$, whose n-th term is given by

$$\check{\mathrm{C}}(Z/Y, \mathcal{F})_n = \mathcal{F}(Z^{n/Y}).$$

Now it makes sense to talk about (derived) limits over this cosimplicial object (see [50, Tag 02VY] for more detail).

Definition 6.4.1. Let $\mathcal{F}\colon \mathrm{PerfAff}_{\mathbf{Q}_p}^{\mathrm{op}} \to \mathcal{D}(\mathbf{Z})$ be a functor (understood in the ∞-categorical sense).

- A morphism $Z \to Y$ is *of $\mathcal{F}$-descent* if the natural morphism

$$\mathcal{F}(Y) \to \mathrm{R}\lim_{n\in\Delta} \check{\mathrm{C}}(Z/Y, \mathcal{F})_n$$

  is an equivalence;
- a morphism $Z \to Y$ is *of universal $\mathcal{F}$-descent* if, for every morphism $Y' \to Y$, the base change $Z \times_Y Y' \to Y'$ is of $\mathcal{F}$-descent;
- $\mathcal{F}$ *satisfies v-descent* (resp. *quasi-proétale descent*) if every v-covering[9] (resp. quasi-proétale covering) $X \to Y$ is of (universal[10]) $\mathcal{F}$-descent;
- $\mathcal{F}$ is a *(derived) v-sheaf* if $\mathcal{F}$ satisfies v-descent and for any $Y_1, Y_2 \in \mathrm{PerfAff}_{\mathbf{Q}_p}$, the natural morphism $\mathcal{F}(Y_1 \sqcup Y_2) \to \mathcal{F}(Y_1) \times \mathcal{F}(Y_2)$ is an equivalence.

Remark 6.4.2. A functor $\mathcal{F}\colon \mathrm{PerfAff}_{\mathbf{Q}_p}^{\mathrm{op}} \to \mathcal{D}(\mathbf{Z})$ is a (derived) v-sheaf in the sense of Definition 6.4.1 if and only if it is a $\mathcal{D}(\mathbf{Z})$-valued sheaf on the (big) v-site $\mathrm{PerfAff}_{\mathbf{Q}_p}$ (see [49, Section A.3.3] for the precise definition). See [49, Proposition A.3.3.1] for a detailed proof of this fact.

Our current goal is to give an explicit condition that ensures that a functor $\mathcal{F}$ satisfies v-descent. Later on, we will show that the étale cohomology of the $\mathcal{O}^+/p$-sheaf satisfies this condition. This will be the crucial input to relate $\mathbf{R}\lambda_*\big(\mathcal{O}^+_{X^\diamond}/p\big)$ to $\mathcal{O}^+_{X^\diamond_{\mathrm{qp}}}/p$.

Lemma 6.4.3 ([47, Lemma 3.1.2]). *Let $\mathcal{F}\colon \mathrm{PerfAff}_{\mathbf{Q}_p}^{\mathrm{op}} \to \mathcal{D}(\mathbf{Z})$ be a functor (in the ∞-categorical sense), and $f\colon Z \to Y$, $g\colon Z' \to Z$ be morphisms in* $\mathrm{PerfAff}_{\mathbf{Q}_p}$. *Then*

(1) *if f has a section, then it is of universal $\mathcal{F}$-descent;*

(2) *if f and g are of universal $\mathcal{F}$-descent, then $f \circ g\colon Z' \to Y$ is of universal $\mathcal{F}$-descent;*

(3) *if $f \circ g$ is of universal $\mathcal{F}$-descent, then f is so.*

Lemma 6.4.4. *Let Y be a strictly totally disconnected perfectoid space, and let $Z \to Y$ be a v-cover by an affinoid perfectoid space. Then there is a presentation*

[9] A morphism $f\colon Z \to Y$ in $\mathrm{PerfAff}_{\mathbf{Q}_p}$ is a v-covering (resp. a quasi-proétale covering) if $f^\diamond\colon X^\diamond \to Y^\diamond$ is so.

[10] We note that if every v-covering (resp. quasi-proétale covering) is of $\mathcal{F}$-descent, then they are automatically of universal $\mathcal{F}$-descent because v-coverings (resp. quasi-proétale coverings) are closed under pullbacks in $\mathrm{PerfAff}_{\mathbf{Q}_p}$.

$Z = \lim_I Z_i \to Y$ as a cofiltered limit of affinoid perfectoid spaces over Y such that each $Z_i \to Y$ admits a section.

Proof. The proof of [52, Lemma 2.11] carries over to this case if one uses [34, Lemma 2.23] in place of [61, Lemma 9.5]. ∎

Definition 6.4.5. A v-covering $Z \to Y$ of affinoid perfectoid spaces is *nice* if it can be written as a cofiltered limit $Z = \lim_I Z_i \to Y$ of affinoid perfectoid spaces over Y such that each $Z_i \to Y$ admits a section.

Remark 6.4.6. ([61, Proposition 6.5]) We recall that the category of affinoid perfectoid spaces PerfAff admits cofiltered limits. Namely, the limit of the cofiltered system $\{\mathrm{Spa}\,(S_i, S_i^+)\}$ is given by $\mathrm{Spa}\,(S, S^+)$ where S^+ is the ϖ-adic completion of $\mathrm{colim}_I\, S_i^+$ (for some compatible choice of pseudo-uniformizers ϖ) and $S = S^+[\frac{1}{\varpi}]$. In particular, $\mathrm{PerfAff}_{\mathbf{Q}_p}$ also admits all cofiltered limits. Moreover, one can choose $\varpi = p$ in this case.

Lemma 6.4.7. *Let $\mathcal{F}\colon \mathrm{PerfAff}_{\mathbf{Q}_p}^{\mathrm{op}} \to \mathcal{D}(\mathbf{Z})$ be a functor (in the ∞-categorical sense) such that*

(1) *$\mathcal{F}$ is universally bounded below, i.e., there is an integer N such that $\mathcal{F}(Y) \in \mathcal{D}^{\geq -N}(\mathbf{Z})$ for any $Y \in \mathrm{PerfAff}_{\mathbf{Q}_p}$;*

(2) *$\mathcal{F}$ satisfies quasi-proétale descent;*

(3) *for an affinoid perfectoid space $Z = \lim_I Z_i$ that is a cofiltered limit of affinoid perfectoid spaces Z_i over $\mathrm{Spa}\,(\mathbf{Q}_p, \mathbf{Z}_p)$, the natural morphism*

$$\mathrm{hocolim}_I\, \mathcal{F}(Z_i) \to \mathcal{F}(Z)$$

is an equivalence.

Then $\mathcal{F}$ satisfies v-descent.

Proof. By shifting, we can assume that $\mathcal{F}(Y) \in \mathcal{D}^{\geq 0}(\mathbf{Z})$ for any $Y \in \mathrm{PerfAff}_{\mathbf{Q}_p}$. We pick a v-covering $f\colon Z \to Y$ in $\mathrm{PerfAff}_{\mathbf{Q}_p}$, and wish to show that it is of universal $\mathcal{F}$-descent. We use [61, Lemma 7.18] to find a quasi-proétale covering $g\colon Y' \to X$ such that Y' is strictly totally disconnected. Then we consider the fiber product

$$\begin{array}{ccc} Z \times_Y Y' & \xrightarrow{g'} & Z \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[g]{} & Y. \end{array}$$

Lemma 6.4.3 implies that f is of universal $\mathcal{F}$-descent if g and f' are so. By assumption, $\mathcal{F}$ satisfies quasi-proétale descent, so g is of universal $\mathcal{F}$-descent. Therefore, it suffices to show that f' is of universal $\mathcal{F}$-descent.

We rename f' by f to reduce the question to showing that any v-cover $f: Z \to Y$ with a strictly totally disconnected Y is of universal $\mathcal{F}$-descent. Further, Lemma 6.4.4 implies that f is nice, so it suffices to show that any nice v-cover (with an arbitrary affinoid perfectoid target space) is of universal $\mathcal{F}$-descent. The property of being nice is preserved by arbitrary pullbacks, so it suffices to show that a nice v-cover is of $\mathcal{F}$-descent.

After all these reductions, we are in the situation of a v-cover $f: Z \to Y$ that can be written as a cofiltered limit $Z = \lim_I Z_i \to Y$ of affinoid perfectoid spaces over Y admitting a Y-section. Lemma 6.4.3 ensures that each $f_i: Z_i \to Y$ is of $\mathcal{F}$-descent since it has a section. We wish to show that

$$\mathcal{F}(Y) \to \mathrm{R}\lim_{n\in\Delta} \check{\mathrm{C}}(Z/Y, \mathcal{F})_n$$

is an equivalence. By assumption, we know that the natural morphism

$$\mathrm{hocolim}_I\, \check{\mathrm{C}}(Z_i/Y, \mathcal{F})_n \to \check{\mathrm{C}}(Z/Y, \mathcal{F})_n$$

is an equivalence for any $n \geq 0$. Now the claim follows from the fact that totalization of a coconnective cosimplisial object commutes with filtered (homotopy) colimits (for example, this follows from [44, Corollary 3.1.13] applied to $\mathcal{C} = \mathrm{Fun}(\Delta, \mathcal{D}(\mathbf{Z}))$, $\mathcal{D} = \mathcal{D}(\mathbf{Z})$, and $F = \mathrm{hocolim}$). ∎

The next goal is to show that the functor (in the ∞-categorical sense)

$$\mathbf{R}\Gamma_{\text{ét}}(-, \mathcal{O}^+/p)\colon \mathrm{PerfAff}^{\mathrm{op}}_{\mathbf{Q}_p} \to \mathcal{D}(\mathbf{Z})$$
$$Y \in \mathrm{PerAff}_{\mathbf{Q}_p} \mapsto \mathbf{R}\Gamma(Y_{\text{ét}}, \mathcal{O}^+_{Y_{\text{ét}}}/p)$$

is a (derived) v-sheaf.

Lemma 6.4.8. *The functor* $\mathbf{R}\Gamma_{\text{ét}}(-, \mathcal{O}^+/p)\colon \mathrm{PerfAff}^{\mathrm{op}}_{\mathbf{Q}_p} \to \mathcal{D}(\mathbf{Z})$ *satisfies quasi-pro-étale descent.*

Proof. By Lemma 6.3.7, we have a functorial isomorphism

$$\mathbf{R}\Gamma(Y_{\text{ét}}, \mathcal{O}^+_{Y_{\text{ét}}}/p) \simeq \mathbf{R}\Gamma(Y^{\diamondsuit}_{\mathrm{qp}}, \mathcal{O}^+_{Y^{\diamondsuit}_{\mathrm{qp}}}/p).$$

Now the quasi-proétale cohomology satisfies quasi-proétale descent by definition. ∎

Lemma 6.4.9. *Let* $\{Z_i = \mathrm{Spa}\,(S_i, S_i^+)\}_{i\in I}$ *be a cofiltered system of affinoid perfectoid spaces over* $(\mathbf{Q}_p, \mathbf{Z}_p)$*, and let* $Z_\infty = \lim Z_i$ *with morphisms* $f_i: Z_\infty \to Z_i$*. Then the natural morphism*

$$\mathrm{colim}_I\, f_i^{-1}\mathcal{O}^+_{Z_{i,\text{ét}}}/p \to \mathcal{O}^+_{Z_{\infty,\text{ét}}}/p$$

is an isomorphism, where $f_i: X_\infty \to X_i$ *are the natural projection morphisms and* f_i^{-1} *is the pullback functor on small étale topoi.*

Proof. Note that [61, Proposition 6.5] implies that $Z_\infty = \mathrm{Spa}\,(S_\infty, S_\infty^+)$, where S_∞^+ is the p-adic completion of $\mathrm{colim}_I\, S_i^+$ and $S_\infty = S_\infty^+[\frac{1}{p}]$.

Now we put $\mathcal{F}$ to be the sheaf $\mathrm{colim}_I\, f_i^{-1}\mathcal{O}^+_{Z_i,\text{ét}}$. Since filtered colimits are exact, we conclude that $\mathcal{F}/p = \mathrm{colim}_I\, f_i^{-1}\mathcal{O}^+_{Z_i,\text{ét}}/p$. Because affinoid perfectoid spaces $U_\infty \to Z_\infty$, étale over Z_∞, form a basis of the étale site $Z_{\infty,\text{ét}}$, it thus suffices to show that the natural morphism

$$\mathcal{F}(U_\infty)/p \to \mathcal{O}^+_{Z_{\infty,\text{ét}}}(U_\infty)/p$$

is an isomorphism for any such $U_\infty \to Z_\infty$. Then [61, Proposition 6.4 (iv)] implies that, for some $i_0 \in I$, there is an affinoid perfectoid space U_{i_0} with an étale morphism $U_{i_0} \to Z_{i_0}$ such that

$$U_{i_0} \times_{Z_{i_0}} Z_\infty \simeq U_\infty.$$

For any $j \geq i_0$, we put $U_j := U_{i_0} \times_{Z_{i_0}} Z_j$. Since fiber products commute with limits, we see that

$$U_\infty \simeq \lim_I U_i$$

in the category of affinoid perfectoid spaces. From this it follows that $\mathcal{O}^+_{Z_{\infty,\text{ét}}}(U_\infty) = \left(\mathrm{colim}_{i\geq i_0}\, \mathcal{O}^+_{Z_{i,\text{ét}}}(U_i)\right)^\wedge_p$. Arguing as in [61, Proposition 14.9] (or as in [23, Proposition 5.9.2]), we conclude that $\mathcal{F}(U_\infty) = \mathrm{colim}_{i\geq i_0}\, \mathcal{O}^+_{Z_i,\text{ét}}(U_i)$. Thus, [68, Tag 05GG] ensures that the natural morphism

$$\mathcal{F}(U_\infty)/p \to \mathcal{O}^+_{Z_{\infty,\text{ét}}}(U_\infty)$$

is an isomorphism. ■

Corollary 6.4.10. *Let Z be an affinoid perfectoid space over* $\mathrm{Spa}\,(\mathbf{Q}_p, \mathbf{Z}_p)$*, and let $Z = \lim_I Z_i$ be a cofiltered limit of affinoid perfectoid spaces Z_i over* $\mathrm{Spa}\,(\mathbf{Q}_p, \mathbf{Z}_p)$*. Then the natural morphism*

$$\mathrm{hocolim}_I\, \mathbf{R}\Gamma\left(Z_{i,\text{ét}}, \mathcal{O}^+_{Z_{i,\text{ét}}}/p\right) \to \mathbf{R}\Gamma\left(Z_{\text{ét}}, \mathcal{O}^+_{Z_{\text{ét}}}/p\right)$$

is an equivalence.

Proof. The result is a formal consequence of Lemma 6.4.9 and [61, Proposition 6.4] (for example, argue as in [23, Proposition 5.9.2]). ■

Corollary 6.4.11. *The functor* $\mathbf{R}\Gamma_{\text{ét}}(-, \mathcal{O}^+/p)\colon \mathrm{PerfAff}^{\mathrm{op}}_{\mathbf{Q}_p} \to \mathcal{D}(\mathbf{Z})$ *is a (derived) v-sheaf.*

Proof. Clearly, $\mathbf{R}\Gamma_{\text{ét}}(-, \mathcal{O}^+/p)$ transforms disjoint unions into direct products, so it suffices to show that $\mathbf{R}\Gamma_{\text{ét}}(-, \mathcal{O}^+/p)$ satisfies v-descent. Then it suffices to show that $\mathbf{R}\Gamma_{\text{ét}}(-, \mathcal{O}^+/p)$ satisfies the conditions of Lemma 6.4.7.

By definition, $\mathbf{R}\Gamma(Y_{\text{ét}}, \mathcal{O}^+_{Y_{\text{ét}}}/p) \in \mathcal{D}^{\geq 0}(\mathbf{Z})$ for any $Y \in \text{AffPerf}_{\mathbf{Q}_p}$. Lemma 6.4.8 implies that $\mathbf{R}\Gamma_{\text{ét}}(-, \mathcal{O}^+/p)$ satisfies quasi-proétale descent, and Corollary 6.4.10 ensures that it satisfies the third condition of Lemma 6.4.7. Thus, Lemma 6.4.7 guarantees that $\mathbf{R}\Gamma_{\text{ét}}(-, \mathcal{O}^+/p)$ satisfies v-descent. ■

Lemma 6.4.12. *Let $Y \in \text{PerfAff}_{\mathbf{Q}_p}$, and let $K \to Y^\diamond$ be a v-hypercover in $Y^\diamond_v$ (in the sense of* [68, Tag 01G5]*). Then there is a split (in the sense of* [68, Tag 017P]*) v-hypercover $K' \to Y$ such that each term K'_n is a strictly totally disconnected perfectoid space, and there is a morphism $K'^\diamond \to K$ of augmented (over Y) simplicial objects.*

Proof. This is a standard consequence of the fact that any v-small sheaf X admits a v-covering $f : X' \to X$ with a strictly totally disconnected affinoid perfectoid space X'. Since this reduction is standard, we only indicate that one should argue as in [68, Tag 0DAV] or [21, Theorem 4.16] by inductively constructing a split n-truncated hypercover K' with a morphism $K' \to K_{\leq n}$. For this inductive step, the crucial input is [21, Theorem 4.12] that allows us to construct morphisms from a split (truncated) hypercovering. ■

Lemma 6.4.13. *For an affinoid perfectoid space $Y = \text{Spa}(S, S^+)$ over $\text{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)$, the natural morphism*

$$\mathbf{R}\Gamma(Y_{\text{ét}}, \mathcal{O}^+_{Y_{\text{ét}}}/p) \to \mathbf{R}\Gamma(Y^\diamond_v, \mathcal{O}^+_{Y^\diamond}/p)$$

is an isomorphism.

Proof. We divide the proof into several steps.

Step 1: *Compute $\mathbf{R}\Gamma(Y^\diamond_v, \mathcal{O}^+_{Y^\diamond}/p)$ "explicitly" in terms of hypercovers (see* [68, Tag 01G5] *for a definition of a hypercovering).* Let us denote by $\text{HC}(Y^\diamond)$ the category of all v-hypercovers of $Y^\diamond$ up to homotopy.[11] Likewise, we denote by $\text{HC}(Y)$ the category of all v-hypercovers of Y in $\text{PerfAff}_{\mathbf{Q}_p}$ up to homotopy, and by $\text{HC}_{\text{std}}(Y)$ the full subcategory of hypercovers $K \to Y$ such that each K_n is strictly totally disconnected.

Then the diamondification functor naturally extends to a fully faithful functor $(-)^\diamond : \text{HC}_{\text{std}}(Y) \to \text{HC}(Y^\diamond)$. Lemma 6.4.12 ensures that this functor is cofinal, and so [68, Tag 01H0] implies that, for every integer $i \geq 0$, we have a canonical isomorphism

$$\text{H}^i\left(Y^\diamond_v, \mathcal{O}^+_{Y^\diamond}/p\right) \simeq \text{colim}_{K \in \text{HC}_{\text{std}}(Y)} \check{\text{H}}^i\left(K^\diamond_v, \mathcal{O}^+_{Y^\diamond}/p\right), \tag{6.4.1}$$

where $\check{\text{H}}^i\left(K^\diamond_v, \mathcal{O}^+_{Y^\diamond}/p\right)$ are the Čech cohomology groups associated with a hypercover $K^\diamond \to Y^\diamond$ (see [68, Tag 01GU]).

[11] See [68, Tag 01GZ] for the precise definition.

Moreover, for any affinoid perfectoid space Z with a map $Z \to Y$, we have a natural isomorphism $\mathcal{O}^+_{Y^\diamond}/p|_{Z^\diamond} \simeq \mathcal{O}^+_{Z^\diamond}/p$. Furthermore, Lemma 6.3.7 ensures that

$$\mathrm{H}^0\big(Z_v^\diamond, \mathcal{O}^+_{Y^\diamond}/p\big) \simeq \mathrm{H}^0\big(Z_v^\diamond, \mathcal{O}^+_{Z^\diamond}/p\big) \simeq \mathrm{H}^0\big(Z_{\text{ét}}, \mathcal{O}^+_{Z_{\text{ét}}}/p\big).$$

If $Z = \mathrm{Spa}\,(S, S^+)$ is strictly totally disconnected, we can simplify it even further by noting that all étale sheaves on $Z_{\text{ét}}$ have trivial higher cohomology groups, so

$$\mathrm{H}^0\big(Z_{\text{ét}}, \mathcal{O}^+_{Z_{\text{ét}}}/p\big) \simeq S^+/pS^+.$$

Combining all these observations, we see that Equation (6.4.1) can be simplified to the following form:

$$\mathrm{H}^i\big(Y^\diamond, \mathcal{O}^+_{Y^\diamond}/p\big) \simeq \mathrm{colim}_{K \in \mathrm{HC}_{\mathrm{std}}(Y)}\, \mathrm{H}^i\big(S^+_{0,K}/p \to S^+_{1,K}/p \to \cdots S^+_{n,K}/p \to \cdots\big), \tag{6.4.2}$$

where $K_n = \mathrm{Spa}\,(S_{n,K}, S^+_{n,K})$ is a strictly totally disconnected perfectoid space, and the differentials are given by the usual Čech-type differentials.

Step 2: $\mathbf{R}\Gamma_{\text{ét}}(-, \mathcal{O}^+/p)$ *satisfies* v-hyperdescent. First, we note that Corollary 6.4.11 and [49, Proposition A.3.3.1] ensure that $\mathbf{R}\Gamma_{\text{ét}}\big(-, \mathcal{O}^+_{Y_{\text{ét}}}/p\big)$ is a $\mathcal{D}(\mathbf{Z})$-valued v-sheaf on $\mathrm{PerfAff}_{\mathbf{Q}_p}$. Moreover, for any $Y \in \mathrm{PerfAff}_{\mathbf{Q}_p}$, we know that

$$\mathbf{R}\Gamma\big(Y_{\text{ét}}, \mathcal{O}^+_{Y_{\text{ét}}}/p\big) \in \mathcal{D}^{\geq 0}(\mathbf{Z}).$$

Therefore, [48, Lemma 6.5.2.9] implies that $\mathbf{R}\Gamma_{\text{ét}}(-, \mathcal{O}^+/p)$ is a hypercomplete (derived) v-sheaf. Furthermore, [48, Corollary 6.5.3.13] implies that any hypercomplete (derived) v-sheaf $\mathcal{F}$ (in particular, $\mathbf{R}\Gamma_{\text{ét}}(-, \mathcal{O}^+/p)$) satisfies hyperdescent, i.e., for any v-hypercovering $K \to X$, the natural morphism

$$\mathcal{F}(X) \to \mathbf{R}\lim_{n \in \Delta} \mathcal{F}(K_n)$$

is an equivalence.

Step 3: *Compute* $\mathbf{R}\Gamma(Y_{\text{ét}}, \mathcal{O}^+_{Y_{\text{ét}}}/p)$ *"explicitly" in terms of hypercovers.* By Step 2, we know that, for any v-hypercovering $K \to Y$ in $\mathrm{PerfAff}_{\mathbf{Q}_p}$, the natural morphism

$$\mathbf{R}\Gamma\big(Y_{\text{ét}}, \mathcal{O}^+_{Y_{\text{ét}}}/p\big) \to \mathbf{R}\lim_{n \in \Delta} \mathbf{R}\Gamma\big(K_{n,\text{ét}}, \mathcal{O}^+_{K_{n,\text{ét}}}/p\big)$$

is an isomorphism. Now we assume that each term $K_n = \mathrm{Spa}\,\big(S_{n,K}, S^+_{n,K}\big)$ is strictly totally disconnected, so higher étale cohomology of any étale sheaf on K_n vanishes. Thus, we have

$$\mathbf{R}\Gamma\big(K_{n,\text{ét}}, \mathcal{O}^+_{K_{n,\text{ét}}}/p\big) \simeq \mathrm{H}^0\big(K_{n,\text{ét}}, \mathcal{O}^+_{K_{n,\text{ét}}}/p\big) \simeq S^+_{n,K}/pS^+_{n,K}.$$

Therefore, in this case, the totalization $\mathbf{R}\lim_{n\in\Delta}\mathbf{R}\Gamma\big(K_{n,\text{ét}}, \mathcal{O}^+_{K_{n,\text{ét}}}/p\big)$ can be explicitly computed as the Čech cohomology associated with the hypercovering $K \to Y$. More explicitly, we see that, for every integer $i \geq 0$, we have

$$\mathrm{H}^i\big(Y_{\text{ét}}, \mathcal{O}^+_{Y_{\text{ét}}}/p\big) \simeq \mathrm{H}^i\big(S^+_{0,K}/p \to S^+_{1,K}/p \to \cdots S^+_{n,K}/p \to \cdots\big)$$

with standard Čech-type differentials. Since this formula holds for any v-hypercover $K \to Y$ with strictly totally disconnected terms K_n, we can pass to the filtered colimit[12] over $\mathrm{HC}_{\text{std}}(Y)$ to see that, for every integer $i \geq 0$,

$$\mathrm{H}^i\big(Y_{\text{ét}}, \mathcal{O}^+_{Y_{\text{ét}}}/p\big) \simeq \operatorname{colim}_{K\in \mathrm{HC}_{\text{std}}(Y)} \mathrm{H}^i\big(S^+_{0,K}/p \to S^+_{1,K}/p \to \cdots S^+_{n,K}/p \to \cdots\big), \tag{6.4.3}$$

where $K_n = \operatorname{Spa}\big(S_{n,K}, S^+_{n,K}\big)$ is a strictly totally disconnected perfectoid space, and the differentials are given by the usual Čech-type differentials.

Step 4: *Finish the proof.* Now Equations (6.4.2) and (6.4.3) imply that the natural morphism

$$\mathrm{H}^i\big(Y_{\text{ét}}, \mathcal{O}^+_{Y_{\text{ét}}}/p\big) \to \mathrm{H}^i\big(Y^\diamond_v, \mathcal{O}^+_{Y^\diamond}/p\big)$$

is an isomorphism for every $i \geq 0$. In other words, the morphism

$$\mathbf{R}\Gamma\big(Y_{\text{ét}}, \mathcal{O}^+_{Y_{\text{ét}}}/p\big) \to \mathbf{R}\Gamma\big(Y^\diamond_v, \mathcal{O}^+_{Y^\diamond}/p\big)$$

is an isomorphism. ■

Corollary 6.4.14. *Let* $X \in \mathbf{pAd}_{\mathbf{Q}_p}$ *be a pre-adic space over* $\operatorname{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)$. *Then the natural morphism*

$$\mathcal{O}^+_{X^\diamond_{\text{qp}}}/p \to \mathbf{R}\lambda_*\big(\mathcal{O}^+_{X^\diamond}/p\big)$$

is an isomorphism.

Proof. Lemma 6.3.7 ensures that $\mathcal{O}^+_{X^\diamond_{\text{qp}}}/p \to \lambda_*\big(\mathcal{O}^+_{X^\diamond}/p\big)$ is an isomorphism. Thus, it suffices to show that

$$\mathrm{R}^j\lambda_*\big(\mathcal{O}^+_{X^\diamond}/p\big) \simeq 0$$

for $j \geq 1$. Since strictly totally disconnected spaces form a basis for the quasi-proétale topology of any diamond, it suffices to show that

$$\mathrm{H}^j(Y^\diamond_v, \mathcal{O}^+_{Y^\diamond}/p) = 0$$

for a totally strictly disconnected perfectoid $Y \to X$ and $j \geq 1$. Lemma 6.4.13 implies that

$$\mathrm{H}^j\big(Y^\diamond_v, \mathcal{O}^+_{Y^\diamond}/p\big) \simeq \mathrm{H}^j\big(Y_{\text{ét}}, \mathcal{O}^+_{Y_{\text{ét}}}/p\big).$$

[12] The category $\mathrm{HC}_{\text{std}}(Y)$ is cofiltered because it is a cofinal category in the filtered category $\mathrm{HC}(Y^\diamond)$. See Step 1 and [68, Tag 01GZ] for more detail.

Now the latter group vanishes because any étale sheaf on a strictly totally disconnected perfectoid space has trivial higher cohomology groups. ■

Corollary 6.4.15. *Let X be a strongly sheafy adic space over* $\mathrm{Spa}\,(\mathbf{Q}_p, \mathbf{Z}_p)$. *Then the natural morphisms*

$$\mathbf{R}\Gamma\big(X, \mathcal{O}^+_{X_{\text{ét}}}/p\big) \to \mathbf{R}\Gamma\big(X^{\diamondsuit}_{\text{qp}}, \mathcal{O}^+_{X^{\diamondsuit}_{\text{qp}}}/p\big) \to \mathbf{R}\Gamma\big(X^{\diamondsuit}_{v}, \mathcal{O}^+_{X^{\diamond}}/p\big)$$

are isomorphisms.

Proof. It follows directly from Lemma 6.3.7 and Corollary 6.4.14. ■

Corollary 6.4.16. *Let $X = \mathrm{Spa}\,(R, R^+)$ be a strictly totally disconnected perfectoid space over* $\mathrm{Spa}\,(\mathbf{Q}_p, \mathbf{Z}_p)$. *Then we have $\mathrm{H}^i\big(X^{\diamondsuit}_v, \mathcal{O}^+_{X^{\diamond}}/p\big) \simeq 0$ for every $i \geq 1$, and $\mathrm{H}^0\big(X^{\diamondsuit}_v, \mathcal{O}^+_{X^{\diamond}}/p\big) \simeq R^+/pR^+$.*

Remark 6.4.17. We emphasize that Corollary 6.4.16 guarantees the actual vanishing of higher v-cohomology groups of $\mathcal{O}^+_{X^{\diamond}}/p$ on a strictly totally disconnected perfectoid space X. This is quite surprising for two reasons: this vanishing holds on the nose (without passing to the almost category), the definition of strictly totally disconnected perfectoid spaces, a priori, guarantees vanishing only of étale cohomology groups (as opposed to the v-cohomology groups).

Proof. Corollary 6.4.15 implies that

$$\mathbf{R}\Gamma\big(X^{\diamondsuit}_v, \mathcal{O}^+_{X^{\diamond}}/p\big) \simeq \mathbf{R}\Gamma\big(X, \mathcal{O}^+_{X_{\text{ét}}}/p\big).$$

Since X is a strictly totally disconnected space, so any étale sheaf has no higher cohomology groups. This implies that $\mathrm{H}^i\big(X^{\diamondsuit}_v, \mathcal{O}^+_{X^{\diamond}}/p\big) \simeq 0$ for $i \geq 1$, and

$$\mathrm{H}^0\big(X^{\diamondsuit}_v, \mathcal{O}^+_{X^{\diamond}}/p\big) \simeq \mathrm{H}^0\big(X, \mathcal{O}^+_{X_{\text{ét}}}\big)/p \simeq R^+/pR^+.$$ ■

As an application, we get the following result:

Corollary 6.4.18. *Let K be a p-adic non-archimedean field, let $K^+ \subset K$ be an open and bounded valuation subring, and let X be a locally noetherian adic space over* $\mathrm{Spa}\,(K, K^+)$. *Put $X^\circ := X \times_{\mathrm{Spa}\,(K,K^+)} \mathrm{Spa}\,(K, \mathcal{O}_K)$. Then the natural morphism*

$$\mathbf{R}\Gamma\big(X^{\diamondsuit}_v, \mathcal{O}^+_{X^{\diamond}}/p\big) \otimes_{K^+/p} \mathcal{O}_K/p \to \mathbf{R}\Gamma\big(X^{\circ,\diamondsuit}_v, \mathcal{O}^+_{X^{\circ,\diamond}}/p\big)$$

is an isomorphism. In particular, if (K, K^+) is a perfectoid field pair, then the natural morphism

$$\mathbf{R}\Gamma\big(X^{\diamondsuit}_v, \mathcal{O}^+_{X^{\diamond}}/p\big) \to \mathbf{R}\Gamma\big(X^{\circ,\diamondsuit}_v, \mathcal{O}^+_{X^{\circ,\diamond}}/p\big)$$

is an almost isomorphism.

Proof. Using the Mayer–Vietoris spectral sequence, we can localize the problem on X. Thus, we can assume that $X = \mathrm{Spa}\,(A, A^+)$ is affinoid. Then we can find a quasi-proétale covering $\mathrm{Spd}\,(A_\infty, A_\infty^+) \to \mathrm{Spd}\,(A, A^+)$ such that all fiber products

$$\mathrm{Spd}\,(A_\infty, A_\infty^+)^{j/\mathrm{Spd}\,(A,A^+)} = \mathrm{Spd}\,(B_j, B_j^+)$$

are strictly totally disconnected (affinoid) perfectoid spaces for $j \geq 1$. Thus, Corollary 6.4.16 implies that

$$\mathrm{H}^i\big(\mathrm{Spd}\,(B_j, B_j^+)_v, \mathcal{O}^+_{X^\diamond}/p\big) \simeq 0$$

for $i, j \geq 1$, and

$$\mathrm{H}^0\big(\mathrm{Spd}\,(B_j, B_j^+)_v, \mathcal{O}^+_{X^\diamond}/p\big) \simeq B_j^+/pB_j^+$$

for $j \geq 1$. Therefore, we can compute $\mathrm{H}^j\big(X_v^\diamondsuit, \mathcal{O}^+_{X^\diamond}/p\big)$ via the Čech cohomology groups of the covering $\mathrm{Spd}\,(A_\infty, A_\infty^+) \to \mathrm{Spa}\,(A, A^+)$. So we get an isomorphism

$$\mathrm{H}^i\big(X_v^\diamondsuit, \mathcal{O}^+_{X^\diamond}/p\big) \simeq \mathrm{H}^i\big(B_1^+/p \to B_2^+/p \to \cdots\big).$$

Now the morphism $\mathrm{Spa}\,(K, \mathcal{O}_K) \to \mathrm{Spa}\,(K, K^+)$ is a pro-open immersion, so the fiber products

$$\mathrm{Spa}\,(B_j, B_j^+) \times_{\mathrm{Spa}\,(K,K^+)} \mathrm{Spa}\,(K, \mathcal{O}_K)$$

are strictly totally disconnected affinoid perfectoid spaces represented by[13]

$$\mathrm{Spa}\,\big(B_j, B_j \widehat{\otimes}_{K^+} \mathcal{O}_K\big).$$

In particular, the same argument as above implies that the $\mathcal{O}^+/p$ cohomology of $X^{\circ,\diamondsuit}$ can be computed as follows:

$$\mathrm{H}^i\big(X_v^{\circ,\diamondsuit}, \mathcal{O}^+_{X^\diamond}/p\big) \simeq \mathrm{H}^i\big(B_1^+/p \otimes_{K^+/p} \mathcal{O}_K/p \to B_2^+/p \otimes_{K^+/p} \mathcal{O}_K/p \to \cdots\big).$$

Now [53, Theorem 10.1] implies that $\mathcal{O}_K$ is an algebraic localization of K^+, so $\mathcal{O}_K$ is K^+-flat. Thus, we get the desired isomorphism

$$\mathbf{R}\Gamma\big(X_v^\diamondsuit, \mathcal{O}^+_{X^\diamond}/p\big) \otimes_{K^+/p} \mathcal{O}_K/p \to \mathbf{R}\Gamma\big(X_v^{\circ,\diamondsuit}, \mathcal{O}^+_{X^{\circ,\diamond}}/p\big).$$

If K is perfectoid, the almost isomorphism

$$\mathbf{R}\Gamma\big(X_v^\diamondsuit, \mathcal{O}^+_{X^\diamond}/p\big) \to \mathbf{R}\Gamma\big(X_v^{\circ,\diamondsuit}, \mathcal{O}^+_{X^{\circ,\diamond}}/p\big)$$

now follows from Lemma B.13. ∎

---

[13] For example, the proof of Lemma 7.4.6 goes through without any changes as $\mathcal{O}_K$ is an algebraic localization of K^+.

6.5 $\mathcal{O}^+/p$-vector bundles in different topologies

The main goal of this section is to show that the categories of v-, quasi-proétale, and étale $\mathcal{O}^+/p$-vector bundles are all equivalent.

The results of this section are mostly due to B. Heuer. A version of these results has also appeared in [35]. We present a slightly different argument that avoids non-abelian cohomology. We heartfully thank B. Heuer for various discussions around these questions and for allowing the author to present a variation of his ideas in this section.

For the next definition, we fix a pre-adic space X over $\mathrm{Spa}\,(\mathbf{Q}_p, \mathbf{Z}_p)$.

Definition 6.5.1. An $\mathcal{O}^+_{X^\diamond}/p$-module $\mathcal{E}$ (in the v-topology on $X^\diamond$) is an *$\mathcal{O}^+_{X^\diamond}/p$-vector bundle* if there is a v-covering $\{X_i \to X^\diamond\}_{i\in I}$ such that $\mathcal{E}|_{X_i} \simeq \big(\mathcal{O}^+_{X^\diamond}/p\big)|_{X_i}^{r_i}$ for some integers r_i. *The category of $\mathcal{O}^+_{X^\diamond}/p$-vector bundles* will be denoted by $\mathrm{Vect}\big(X^\diamond_v, \mathcal{O}^+_{X^\diamond}/p\big)$.

An $\mathcal{O}^+_{X^\diamond_{\mathrm{qp}}}/p$-module $\mathcal{E}$ (in the quasi-proétale topology on $X^\diamond$) is said to be an *$\mathcal{O}^+_{X^\diamond_{\mathrm{qp}}}/p$ -vector bundle* if there is a quasi-proétale covering $\{X_i \to X^\diamond\}_{i\in I}$ such that $\mathcal{E}|_{X_i} \simeq \big(\mathcal{O}^+_{X^\diamond_{\mathrm{qp}}}/p\big)|_{X_i}^{r_i}$ for some integers r_i. We will denote *the category of $\mathcal{O}^+_{X^\diamond_{\mathrm{qp}}}/p$-vector bundles* by $\mathrm{Vect}\big(X^\diamond_{\mathrm{qp}}, \mathcal{O}^+_{X^\diamond_{\mathrm{qp}}}/p\big)$.

Let now X be a strongly noetherian adic space over $\mathrm{Spa}\,(\mathbf{Q}_p, \mathbf{Z}_p)$. An $\mathcal{O}^+_{X_{\text{ét}}}/p$-module $\mathcal{E}$ (in the étale topology on X) is an *$\mathcal{O}^+_{X_{\text{ét}}}/p$-vector bundle* if, there is an étale covering $\{X_i \to X\}_{i\in I}$ such that $\mathcal{E}|_{X_i} \simeq \big(\mathcal{O}^+_{X_{\text{ét}}}/p\big)|_{X_i}^{r_i}$ for some integers r_i. We will denote *the category of $\mathcal{O}^+_{X_{\text{ét}}}/p$-vector bundles* by $\mathrm{Vect}\big(X_{\text{ét}}, \mathcal{O}^+_{X_{\text{ét}}}/p\big)$.

Remark 6.5.2. Note that $\mathcal{O}^+_{X^\diamond}/p$-vector bundles are "big sheaves", i.e., they are defined on the big v-site $X^\diamond_v$. In contrast, $\mathcal{O}^+_{X^\diamond_{\mathrm{qp}}}/p$ and $\mathcal{O}^+_{X_{\text{ét}}}/p$-vector bundles are "small sheaves"; i.e., they are defined on the small quasi-proétale $X^\diamond_{\mathrm{qpro\acute{e}t}}$ or the small étale site $X_{\text{ét}}$ respectively.

The main goal of this section is to show that all these notions of $\mathcal{O}^+/p$-vector bundles are equivalent.

First, we define the functors between these categories of $\mathcal{O}^+/p$-vector bundles which we later show to be equivalences. For this, we note that Lemma 6.3.7 implies that $\mu^{-1}\big(\mathcal{O}^+_{X_{\text{ét}}}/p\big) \simeq \mathcal{O}^+_{X^\diamond_{\mathrm{qp}}}/p$. Consequently, μ^{-1} carries $\mathcal{O}^+_{X_{\text{ét}}}/p$-vector bundles to $\mathcal{O}^+_{X^\diamond_{\mathrm{qp}}}/p$-vector bundles. In particular, it defines the functor

$$\mu^* := \mu^{-1} \colon \mathrm{Vect}\big(X_{\text{ét}}, \mathcal{O}^+_{X_{\text{ét}}}/p\big) \to \mathrm{Vect}\big(X^\diamond_{\mathrm{qp}}, \mathcal{O}^+_{X^\diamond_{\mathrm{qp}}}/p\big).$$

Unfortunately, the natural morphism $\lambda^{-1}\big(\mathcal{O}^+_{X^\diamond_{\mathrm{qp}}}/p\big) \to \mathcal{O}^+_{X^\diamond_v}/p$ is not an isomorphism (see Remark 6.5.2). For this reason, we define λ^* to be the "$\mathcal{O}^+/p$-module pullback" functor

$$\lambda^* \colon \mathrm{Vect}\big(X^\diamond_{\mathrm{qp}}, \mathcal{O}^+_{X^\diamond_{\mathrm{qp}}}/p\big) \to \mathrm{Vect}\big(X^\diamond_v, \mathcal{O}^+_{X^\diamond}/p\big),$$

defined by the formula

$$\lambda^*\mathcal{E} := \lambda^{-1}\mathcal{E} \otimes_{\lambda^{-1}\mathcal{O}^+_{X^\diamond_{\mathrm{qp}}}/p} \mathcal{O}^+_{X^\diamond}/p.$$

Our goal is to show that both λ^* and μ^* are equivalences. Before we do this, we need some preliminary lemmas:

Lemma 6.5.3. *Let X be a pre-adic space over* Spa $(\mathbf{Q}_p, \mathbf{Z}_p)$*, let $\mathcal{E}$ be an $\mathcal{O}^+_{X^\diamond}/p$-vector bundle, and let $Z = \lim_I Z_i$ be a cofiltered limit of affinoid perfectoid spaces over X. Then the natural morphism*

$$\operatorname{colim}_I \mathrm{H}^0\big(Z^\diamond_{i,v}, \mathcal{E}\big) \to \mathrm{H}^0\big(Z^\diamond_v, \mathcal{E}\big)$$

is an isomorphism.

Proof. Without loss of generality, we can assume that I has a final object 0. Then, by the sheaf condition and exactness of filtered colimits, it suffices to show the claim v-locally on Z_0. Therefore, we may assume that $\mathcal{E}|_{Z^\diamond} \simeq \big(\mathcal{O}^+_{Z^\diamond}/p\big)^d$ is a free vector bundle. The claim then follows from Corollary 6.4.10. ■

Corollary 6.5.4. *Let X be a pre-adic space over* Spa $(\mathbf{Q}_p, \mathbf{Z}_p)$*, let $\mathcal{E}$ be an $\mathcal{O}^+_{X^\diamond}/p$-vector bundle, and let $Z \approx \lim_I Z_i \to Z_0$ be an affinoid strongly pro-étale morphism of strongly sheafy Tate-affinoid spaces over X. Then the natural morphism*

$$\operatorname{colim}_I \mathrm{H}^0\big(Z^\diamond_{i,v}, \mathcal{E}\big) \to \mathrm{H}^0\big(Z^\diamond_v, \mathcal{E}\big)$$

is an isomorphism.

Proof. We can prove the claim v-locally on $Z_0^\diamond$. Therefore, we can choose a v-covering $\widetilde{Z}_0 \to Z_0$ with a strictly totally disconnected perfectoid space $\widetilde{Z}_0$. The proof of Lemma 6.2.6 (2) ensures that each $\widetilde{Z}_i := Z_i \times_{Z_0} \widetilde{Z}_0$ is a strictly totally disconnected affinoid space, and the diamond $(Z \times_{Z_0} \widetilde{Z}_0)^\diamond$ is a strictly totally disconnected perfectoid space (of characteristic p). Therefore, we see that the natural morphism

$$\widetilde{Z} := \big((Z \times_{Z_0} \widetilde{Z}_0)^\diamond\big)^\sharp \to Z \times_{Z_0} \widetilde{Z}_0$$

becomes an isomorphism after applying the diamondification functor, and

$$\widetilde{Z} \simeq \lim_I \widetilde{Z}_i$$

in the category of perfectoid spaces over X. Since the question is v-local on $Z_0^\diamond$ and depends only on the associated diamonds of Z_i and Z, we can replace Z_i and Z with $\widetilde{Z}_i$ and $\widetilde{Z}$, respectively, to achieve that each Z_i and Z is an affinoid perfectoid. In this case, the result follows from Lemma 6.5.3. ■

Lemma 6.5.5. *Let X be a pre-adic space over* $\mathrm{Spa}\,(\mathbf{Q}_p, \mathbf{Z}_p)$*, let $\mathcal{E}$ be an $\mathcal{O}^+_{X^\diamond}/p$-vector bundle, and let $Z \approx \lim_I Z_i \to Z_0$ be an affinoid strongly pro-étale morphism of strongly sheafy Tate-affinoid adic spaces over X. If $\mathcal{E}|_{Z^\diamond} \simeq \big(\mathcal{O}^+_{Z^\diamond}/p\big)^d$ for some integer d, then there is an $i \in I$ such that $\mathcal{E}|_{Z_i^\diamond} \simeq \big(\mathcal{O}^+_{Z_i^\diamond}/p\big)^d$.*

Proof. We choose an isomorphism

$$f\colon \big(\mathcal{O}^+_{Z^\diamond}/p\big) \xrightarrow{\sim} \mathcal{E}|_{Z^\diamond}$$

and wish to descend it to a finite level.

Step 1: *We approximate f.* Corollary 6.5.4 ensures that we can find $i \in I$ and a morphism

$$f_i\colon \big(\mathcal{O}^+_{Z_i^\diamond}/p\big)^d \to \mathcal{E}|_{Z_i^\diamond}$$

such that $f_i|_{Z^\diamond} = f$.

Step 2: *Approximate $f^{-1}\colon \mathcal{E}|_{Z^\diamond} \to \big(\mathcal{O}^+_{Z^\diamond}/p\big)^d$.* We note that the dual sheaf

$$\mathcal{E}^\vee = \underline{\mathrm{Hom}}_{\mathcal{O}^+_{X^\diamond}/p}\big(\mathcal{E}, \mathcal{O}^+_{X^\diamond}/p\big)$$

is also an $\mathcal{O}^+_{X^\diamond}/p$-vector bundle. So we can apply the same argument as in Step 1 to

$$(f^{-1})^\vee\colon \big(\mathcal{O}^+_{Z^\diamond}/p\big)^d \to \mathcal{E}^\vee|_{Z^\diamond} = \underline{\mathrm{Hom}}_{\mathcal{O}^+_{X^\diamond}/p}\big(\mathcal{E}, \mathcal{O}^+_{X^\diamond}/p\big)|_{Z^\diamond}$$

to find (after possible enlarging $i \in I$) a morphism

$$g_i'\colon \big(\mathcal{O}^+_{Z_i^\diamond}/p\big)^d \to \mathcal{E}^\vee|_{Z_i^\diamond}$$

such that $g_i'|_{Z^\diamond} = (f^{-1})^\vee$. By dualizing, we get a morphism

$$g_i\colon \mathcal{E}|_{Z_i^\diamond} \to \big(\mathcal{O}^+_{Z_i^\diamond}/p\big)^d$$

such that $g_i|_{Z^\diamond} = f^{-1}$.

Step 3: *Show that $f_i \circ g_i = \mathrm{id}$ and $g_i \circ f_i = \mathrm{id}$ after possibly enlarging $i \in I$.* We show the first claim, the second is proven in the same way (and even easier). We consider $\mathrm{id}_{\mathcal{E}|_{Z_i^\diamond}}$ and $f_i \circ g_i$ as sections of the internal Hom sheaf, i.e.,

$$\mathrm{id}_{\mathcal{E}|_{Z_i^\diamond}}, f_i \circ g_i \in \big(\underline{\mathrm{End}}_{\mathcal{O}^+_{X^\diamond}/p}(\mathcal{E})\big)\big(Z_i^\diamond\big).$$

For brevity, we simply denote $\underline{\mathrm{End}}_{\mathcal{O}^+_{X^\diamond}/p}(\mathcal{E})$ by $\underline{\mathrm{End}}$. Note that $\underline{\mathrm{End}}$ is again an $\mathcal{O}^+_{X^\diamond}/p$-vector bundle, and so Lemma 6.5.3 ensures that

$$\mathrm{colim}_I\, \underline{\mathrm{End}}\,(Z_i^\diamond) = \mathcal{E}(Z^\diamond).$$

Thus if $f_i \circ g_i$ and id are equal in the colimit, they are equal on $Z_i^\diamond$ for some large index i. Similarly, $g_i \circ f_i = \mathrm{id}$ for some $i \in I$. Therefore, $f_i\colon (\mathcal{O}^+_{Z_i^\diamond}/p)^d \xrightarrow{\sim} \mathcal{E}|_{Z_i^\diamond}$ is an isomorphism for $i \gg 0$. ■

Lemma 6.5.6. *Let Y denote a strictly totally disconnected perfectoid space over* $\mathrm{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)$*, and let $\mathcal{E}$ be an $\mathcal{O}^+_{Y^\diamond}/p$-vector bundle. Then there is a finite clopen decomposition $Y = \bigsqcup_{i\in I} Y_i$ such that $\mathcal{E}|_{Y_i^\diamond} \simeq (\mathcal{O}^+_{Y_i^\diamond}/p)^{r_i}$ for some integers r_i.*

Proof. By assumption, there is a v-covering $\{f_j\colon Z_j \to Y^\diamond = Y^\flat\}_{j\in J}$ by affinoid perfectoid spaces. Since Y is quasi-compact, we can assume that J is a finite set.

We put $Y_j''' := f_j(Z_j) \subset Y^\flat$. This subset is pro-constructible by [68, Tag 0A2S] and it is generalizing due to [38, Lemma 1.1.10]. Therefore, [61, Lemma 7.6] implies that there is a canonical structure of an affinoid perfectoid space on Y_j''' such that $\iota_j\colon Y_j''' \to Y^\flat$ is a pro-(rational subdomain). In particular, Y_j''' is strictly totally disconnected for every $j \in J$ (for example, due to [61, Lemma 7.19]).

Lemma 6.4.4 implies that, for each $j \in J$, we can write $Z_j = \lim_{\Lambda_j} Z_{j,\lambda} \to Y_j'''$ as a cofiltered limit of affinoid perfectoid spaces such that $Z_{j,\lambda} \to Y_j'''$ admits a section for each $\lambda \in \Lambda_j$. Therefore, Lemma 6.5.5 ensures that, for each $j \in J$, there is $\lambda_j \in \Lambda_j$ such that $\mathcal{E}|_{Z_{j,\lambda_j}}$ is a free $\mathcal{O}^+_{Y^\diamond}/p$-vector bundle. Since each $Z_{j,\lambda_j} \to Y_j'''$ admits a section, we can pull back this trivialization along the section to conclude that $\mathcal{E}|_{Y_j'''}$ is a free $\mathcal{O}^+_{Y^\diamond}/p$-vector bundle.

Now we use Lemma 6.5.5 and the fact that $\iota_j\colon Y_j''' \to Y^\flat$ is a pro-(rational subdomain) to find a rational open subdomain $Y_j'' \subset Y^\flat$ such that $Y_j''' \subset Y_j''$ and $\mathcal{E}|_{Y_j''}$ is a free $\mathcal{O}^+_{Y^\diamond}/p$-vector bundle of rank $r(j)$. Finally, for each integer i, we put Y_i' to be the union of all Y_j'' such that $r(j) = i$ (in other words, it is the union of Y_j'' such that $\mathcal{E}|_{Y_j''}$ is free of rank i). Then all Y_i' are disjoint and only finitely many of them are non-empty. Finally, we define I to be the (finite) set of integers such that $Y_i' \neq \varnothing$. Then $Y^\diamond = Y^\flat = \bigsqcup_{i\in I} Y_i'$ is a finite clopen decomposition such that $\mathcal{E}|_{Y'}$ is finite free. Then the set of untilts $Y_i := Y_i'^\sharp \subset (Y^\flat)^\sharp = Y$ does the job. ■

Theorem 6.5.7 (see also [35]). *Let X be a pre-adic space over $\mathbf{Q}_p$. Then the functor*

$$\lambda^*\colon \mathrm{Vect}\big(X^\diamond_{\mathrm{qp}}, \mathcal{O}^+_{X^\diamond_{\mathrm{qp}}}/p\big) \to \mathrm{Vect}\big(X^\diamond_v, \mathcal{O}^+_{X^\diamond}/p\big)$$

is an equivalence of categories. Furthermore, for any $\mathcal{O}^+_{X^\diamond_{\mathrm{qp}}}/p$-vector bundle $\mathcal{E}$, the natural morphism

$$\mathcal{E} \to \mathbf{R}\lambda_*\lambda^*\mathcal{E}$$

is an isomorphism.

Proof. We start the proof by showing that the natural morphism

$$\mathcal{E} \to \mathbf{R}\lambda_*\lambda^*\mathcal{E}$$

is an isomorphism. The claim is quasi-proétale local, hence we can assume that $\mathcal{E}$ is a trivial $\mathcal{O}^+_{X^\diamond_{\mathrm{qp}}}/p$-vector bundle. In this case, the claim follows from Corollary 6.4.14.

This already implies full faithfulness of λ^*. Indeed, it follows from a sequence of isomorphisms:

$$\mathrm{Hom}_{\mathcal{O}^+_{X^\diamond}/p}(\lambda^*\mathcal{E}_1, \lambda^*\mathcal{E}_2) \simeq \mathrm{Hom}_{\mathcal{O}^+_{X^\diamond_{\mathrm{qp}}}/p}(\mathcal{E}_1, \lambda_*\lambda^*\mathcal{E}_2) \simeq \mathrm{Hom}_{\mathcal{O}^+_{X^\diamond_{\mathrm{qp}}}/p}(\mathcal{E}_1, \mathcal{E}_2).$$

To show that λ^* is essentially surjective, it is enough to show that, for an $\mathcal{O}^+_{X^\diamond}/p$-vector bundle $\mathcal{E}$, $\lambda_*\mathcal{E}$ is an $\mathcal{O}^+_{X^\diamond_{\mathrm{qp}}}/p$-vector bundle and the natural morphism

$$\mathcal{E} \to \lambda^*\lambda_*\mathcal{E}$$

is an isomorphism. Both claims are quasi-proétale local on $X^\diamond$, so we can assume that X is a strictly totally disconnected perfectoid space. Then we can assume that $\mathcal{E}$ is a free vector bundle due to Lemma 6.5.6. Then $\lambda_*\mathcal{E}$ is a free $\mathcal{O}^+_{X^\diamond_{\mathrm{qp}}}/p$-vector bundle by Lemma 6.3.7. Thus, the natural morphism

$$\mathcal{E} \to \lambda^*\lambda_*\mathcal{E}$$

is evidently an isomorphism. ■

Lemma 6.5.8. *Let X be a strongly sheafy adic space over* Spa $(\mathbf{Q}_p, \mathbf{Z}_p)$*, and let $\mathcal{E}$ be an $\mathcal{O}^+_{X^\diamond_{\mathrm{qp}}}/p$-vector bundle (equivalently, an $\mathcal{O}^+_{X^\diamond}/p$-vector bundle). Then there is an étale covering $\{X'_i \to X\}_{i\in I}$ such that $\mathcal{E}|_{X'^\diamond_i} \simeq (\mathcal{O}^+_{X'^\diamond_i}/p)^{r_i}$ for some integers r_i.*

Proof. The question is local on X. So we can assume that $X = \mathrm{Spa}\,(A, A^+)$ for a complete strongly sheafy Tate–Huber pair (A, A^+). Then the result follows directly from Lemma 6.2.13, Theorem C.3.10, Lemma 6.5.6, and Lemma 6.5.3. ■

Theorem 6.5.9 (See also [35]). *Let X denote a strongly sheafy adic space over* Spa $(\mathbf{Q}_p, \mathbf{Z}_p)$*. Then the functor*

$$\mu^*\colon \mathrm{Vect}(X_{\text{ét}}, \mathcal{O}^+_{X_{\text{ét}}}/p) \to \mathrm{Vect}(X^\diamond_{\mathrm{qp}}, \mathcal{O}^+_{X^\diamond_{\mathrm{qp}}}/p)$$

is an equivalence of categories. Furthermore, for any $\mathcal{O}^+_{X^\diamond_{\text{ét}}}/p$-vector bundle $\mathcal{E}$, the natural morphism

$$\mathcal{E} \to \mathbf{R}\mu_*\mu^*\mathcal{E}$$

is an isomorphism.

Proof. The proof is completely analogous to the proof of Theorem 6.5.7 making use of Lemma 6.5.8 in place of Lemma 6.5.6. ■

6.6 Trivializing $\mathcal{O}^+/p$-vector bundles

We recall that Theorem 6.5.7 and Theorem 6.5.9 ensure that the categories of $\mathcal{O}^+/p$-vector bundles in the v, quasi-proétale, and étale topologies are equivalent. In particular, any $\mathcal{O}^+/p$-vector bundle in the v-topology can be trivialized étale locally. The main goal of this section is to show that it suffices to consider some very specific étale covers.

To do this, we need to start with the discussion of $\mathcal{O}^+/p$-vector bundles on some very specific adic spaces.

Lemma 6.6.1. *Let $X = \mathrm{Spa}(A, A^+)$ be a Tate affinoid pre-adic space such that A^+ is a Prüfer domain (in the sense of* [28, Theorem 22.1 and the discussion before it]*). Then the specialization map $\mathrm{sp}_X \colon |X| \to |\mathrm{Spf}\, A^+| = |\mathrm{Spec}\, A^+/A^{\circ\circ}|$ is a homeomorphism.*

Proof. First, (the proof of) [5, Theorem 8.1.2] implies that it suffices to show that $\mathrm{Spec}\, A^+$ does not admit any non-trivial admissible blow-ups. For this, it suffices to show that any finitely generated ideal $I \subset A^+$ is invertible. This is, in turn, one of the defining properties of Prüfer domains (see [28, Theorem 22.1]). ■

Lemma 6.6.2. *Let $X = \mathrm{Spa}(K, K^+)$ be a Tate affinoid adic space such that K is a non-archimedean field and K^+ is a Prüfer domain.*[14] *Then the morphism of locally ringed spaces*

$$\mathrm{sp}_X \colon (X_{\mathrm{an}}, \mathcal{O}_X^+) \to (\mathrm{Spf}\, K^+, \mathcal{O}_{\mathrm{Spf}\, K^+})$$

is an isomorphism.

Proof. Lemma 6.6.1 implies that sp_X is a homeomorphism. Therefore, it suffices to show that $\mathrm{sp}_X^\# \colon \mathcal{O}_{\mathrm{Spf}\, K^+} \to \mathrm{sp}_{X,*}(\mathcal{O}_X^+)$ is an isomorphism. It suffices to show that $\mathrm{sp}_X^\#(\mathrm{D}_f)$ is an isomorphism for any $f \in K^+$.

Since K is a non-archimedean field, we conclude that $K^\circ = \mathcal{O}_K$ is a rank-1 valuation ring. Then we consider the inclusions $K^{\circ\circ} \subset K^+ \subset \mathcal{O}_K$ and fix a pseudo-uniformizer $\varpi \in K^+$. Since $\mathcal{O}_K$ is a rank-1 valuation ring, we conclude that the induced topologies on $\mathcal{O}_K$ and K^+ coincide with the ϖ-adic topologies.

Now pick $f \in K^+$. If $f \in K^{\circ\circ}$, then $K^+\left[\frac{1}{f}\right] = K$ and so the principal open $\mathrm{D}(f)$ is empty. In particular, $\mathrm{sp}_X^\#(\mathrm{D}_f)$ is clearly an isomorphism. Therefore, we can assume that $f \in K^+ \smallsetminus K^{\circ\circ}$. Then $K^+\left[\frac{1}{f}\right] \subset \mathcal{O}_K$ is an open subring, so $K^+\left[\frac{1}{f}\right]$ is already complete in the ϖ-adic topology. In particular, we conclude that $\mathcal{O}_{\mathrm{Spf}\, K^+}(\mathrm{D}(f)) = K^+\left[\frac{1}{f}\right]$. Likewise, since $K^+\left[\frac{1}{f}\right] \subset K$ is already complete and integrally closed, we conclude that

$$(\mathrm{sp}_{X,*}\mathcal{O}_X^+)(\mathrm{D}(f)) = \mathcal{O}_X^+\left(X\left(\frac{1}{f}\right)\right) = K^+\left[\frac{1}{f}\right].$$

[14] We do not assume that K^+ is a valuation ring.

In particular, we conclude that $\mathrm{sp}_X^\#(\mathrm{D}_f)$ is an isomorphism, finishing the proof. ∎

Now we recall that any locally noetherian analytic adic space X comes with the natural morphism of ringed sites $i_X\colon (X_{\text{ét}}, \mathcal{O}^+_{X_{\text{ét}}}) \to (X_{\text{an}}, \mathcal{O}^+_X)$. We show that this is an equivalence for some special X.

Lemma 6.6.3. *Let $X = \mathrm{Spa}\,(K, K^+)$ be a Tate affinoid adic space such that K is a non-archimedean field,*[15] *and let $U \subset X$ be a non-empty rational subdomain. Then $U = \mathrm{Spa}\,(K, K'^+)$ for some Tate–Huber pair (K, K'^+).*

Proof. Since U is an affinoid space, we only need to show that $\mathcal{O}_X(U) = K$. First, we choose a pseudo-uniformizer $\varpi \in K^+$. Then we note that $K^\circ = \mathcal{O}_K$ is a rank-1 valuation ring since K is a non-archimedean field. In particular, we conclude that the induced topologies on both K° and K^+ coincide with the ϖ-adic topology (and both are complete with respect to this topology).

Now we consider the case $U = X\left(\frac{f}{g}\right)$ for some $f, g \in K^\times$. Since K is a field, we can assume that $U = X\left(\frac{1}{f}\right)$ for some $f \in K^\times$. If $f \in K^{\circ\circ}$, then $X\left(\frac{1}{f}\right) = \varnothing$, so we can assume that $f \notin K^{\circ\circ}$. Then we recall that the induced topology on K^+ is equal to the ϖ-adic topology to conclude that (see [37, Section 1])

$$\mathcal{O}_X(U) = \left(K^+\left[\frac{1}{f}\right]^\wedge_{(\varpi)}\right)\left[\frac{1}{\varpi}\right],$$

where $K^+\left[\frac{1}{f}\right]$ is the K^+-subalgebra of $K\left[\frac{1}{f}\right] = K$ generated by $\frac{1}{f}$. Since $f \notin K^{\circ\circ}$, we conclude that $K^+\left[\frac{1}{f}\right] \subset \mathcal{O}_K$ is an open subring of $\mathcal{O}_K$. Thus, it is already complete in the ϖ-adic topology. So we conclude that

$$\mathcal{O}_X(U) = \left(K^+\left[\frac{1}{f}\right]\right)\left[\frac{1}{\varpi}\right] = K.$$

A rational subdomain U is equal to $X\left(\frac{f_1,\dots,f_n}{g}\right)$ for some $f_1, \dots, f_n, g \in K^\times$. Denote by U_i the rational subdomain $X\left(\frac{f_i}{g}\right)$. Then

$$U = U_1 \cap U_2 \cap \dots \cap U_n.$$

Therefore, we see that

$$\mathcal{O}_X(U) \simeq \mathcal{O}_X(U_1)\widehat{\otimes}_K \mathcal{O}_X(U_2)\widehat{\otimes}_K \cdots \widehat{\otimes}_K \mathcal{O}_X(U_n) \simeq K\widehat{\otimes}_K K\widehat{\otimes}_K \cdots \widehat{\otimes}_K K \simeq K. \;∎$$

Lemma 6.6.4. *Let $X = \mathrm{Spa}\,(C, C^+)$ be a Tate affinoid adic space such that C is an algebraically closed non-archimedean field.*[16] *Let $\varpi \in C^+$ be a pseudo-uniformizer. Then the morphism of ringed topoi*

$$i_X\colon (X_{\text{ét}}, \mathcal{O}^+_{X_{\text{ét}}}) \to (X_{\text{an}}, \mathcal{O}^+_X)$$

[15] We do not assume that K^+ is a valuation ring.

[16] We do not assume that C^+ is a valuation ring.

is an equivalence. In particular, the functor i_X^{-1} induces an equivalence of categories

$$i_X^{-1}\colon \mathrm{Vect}\big(X_{\mathrm{an}}, \mathcal{O}_X^+/\varpi\big) \xrightarrow{\sim} \mathrm{Vect}\big(X_{\mathrm{\acute{e}t}}, \mathcal{O}_{X_{\mathrm{\acute{e}t}}}^+/\varpi\big).$$

Proof. We verify conditions (a)–(d) of [38, Corollary A.5]. Conditions (a) and (c) are clear. Condition (d) follows from the fact that étale maps are open. Indeed, in the notation of [38, Corollary A.5], we can take $I = \{0\}$, $X_0 = X$, $Y_0 = \varphi(X)$, and $X_0 \to Y_0$ the map induced by φ.

Therefore, we are only left to check condition (c) of loc. cit. That is, we need to show that any étale morphism $f\colon Y \to X$ admits an étale covering $\{Y_i \to X\}_{i\in I}$ such that $Y_i \to X$ is an open immersion.

Without loss of generality, we can assume that $Y = \mathrm{Spa}\,(A, A^+)$ is affinoid. We can construct Y_i analytically locally on X. Lemma 6.6.2 implies that we can freely replace X with any non-empty open affinoid without changing the assumptions on X. Therefore, [38, Lemma 2.2.8] implies that we can assume that $f\colon Y \to X$ factors as a composition of an open immersion $j\colon Y \to \overline{Y}^{/X}$ followed by a finite étale morphism $\overline{f}^{/X}\colon \overline{Y}^{/X} \to X$. Since X is strongly noetherian, we conclude that the category X_{fet} of finite étale adic spaces over X is equivalent to the category C_{fet} of finite étale C-algebras. Since C is algebraically closed, we conclude that $\overline{Y}^{/X} = \bigsqcup_{i\in I} X_i$ is a disjoint union of a finite number of copies of X ($X_i \simeq X$). Therefore,

$$\big\{j_i\colon Y_i := X_i \cap Y \to Y\big\}_{i\in I}$$

gives the desired covering of Y.

Now to conclude that $i_X^{-1}\colon \mathrm{Vect}\big(X_{\mathrm{an}}, \mathcal{O}_X^+/\varpi\big) \to \mathrm{Vect}\big(X_{\mathrm{\acute{e}t}}, \mathcal{O}_{X_{\mathrm{\acute{e}t}}}^+/\varpi\big)$ is an equivalence, it suffices to show that $i_X^{-1}\mathcal{O}_X^+/\varpi = \mathcal{O}_{X_{\mathrm{\acute{e}t}}}^+/\varpi$. Since i_X^{-1} is exact, it suffices to show that $i_X^{-1}\mathcal{O}_X^+ = \mathcal{O}_{X_{\mathrm{\acute{e}t}}}^+$. For this, it suffices to show that $i_{X,*}\mathcal{O}_{X_{\mathrm{\acute{e}t}}}^+ = \mathcal{O}_X^+$, but this is evident from the definition. ∎

Lemma 6.6.5. *Let K be a non-archimedean field with an open and bounded valuation subring $K^+ \subset K$ and a pseudo-uniformizer $\varpi \in K^+$. Let K^{sep} be a separable closure of K, and let $\{K_i\}_{i\in I}$ be a filtered system of finite subextensions $K \subset K_i \subset K^{\mathrm{sep}}$. For each $i \in I$, we put K_i^+ to be the integral closure of K^+ in K_i. Then the completed colimit*

$$C^+ := \big(\mathrm{colim}_I\, K_i^+\big)^\wedge_{(\varpi)}$$

is a Prüfer domain and $C := C^+\big[\frac{1}{\varpi}\big]$ is an algebraically closed non-archimedean field.

Proof. First, we note that [36, Lemma 1.6] implies that C is the usual completion of the topological field K^{sep}. Therefore, [12, Proposition 3.4.1/3 and Proposition 3.4.1/6] imply that C is algebraically closed. So we only need to show that C^+ is a Prüfer domain.

First, we note that [28, Theorem 22.1] ensures that K^+ is a Prüfer domain. Then [28, Theorem 22.3] implies that each K_i^+ is a Prüfer domain. Now we note that $\mathrm{colim}_I\, K_i^+$ is a domain, so [28, Proposition 22.6] ensures that it is a Prüfer domain. Then [56, Theorem 4] implies that it suffices to show that every torsionfree C^+-module M is flat. Clearly, $M[\frac{1}{\varpi}]$ is a flat $C = C^+[\frac{1}{\varpi}]$-module because C is a field. Furthermore, [19, Chapter VII, Proposition 4.5] applied to $A = M$ and $\Lambda = \mathrm{colim}_I\, K_i^+$ implies that M is flat over $\mathrm{colim}_I\, K_i^+$. In particular, $M/\varpi M$ is flat over $C^+/\varpi \simeq \left(\mathrm{colim}_I\, K_i^+\right)/\varpi$. Therefore, [11, Lemma 8.2/1] concludes that M is flat over C^+ and finishes the proof. ∎

Lemma 6.6.6. *In the notation of Lemma 6.6.5, any finite projective C^+/ϖ-module is free.*

Proof. First, we note that $C^+/\varpi \simeq \mathrm{colim}_I (K_i^+/\varpi)$. Therefore, a standard approximation argument reduces the question of showing that every finite projective K_i^+/ϖ-module is finite free. Let us denote the residue field of (the rank-1 valuation ring) $K_i^\circ = \mathcal{O}_{K_i}$ by k_i. Then we observe that $K_i^{\circ\circ} = \mathrm{rad}(\varpi)$, and thus $K_i^+/\mathrm{rad}(\varpi) = K_i^+/K_i^{\circ\circ} \subset \mathcal{O}_{K_i}/K_i^{\circ\circ} = k_i$ is a domain. In particular,

$$|\mathrm{Spec}\, K_i^+/\varpi| = |\mathrm{Spec}\, K_i^+/K_i^{\circ\circ}|$$

is irreducible. Furthermore, [17, Ch. VI Section 8.3, Thm. 1 and Ch.VI, Section 8.6, Prop. 6] imply that each K_i^+ is semi-local. In particular, the ring K_i^+/ϖ is semi-local as well. Therefore, [68, Tag 02M9] and the above observation, that $|\mathrm{Spec}\, K_i^+/\varpi|$ is irreducible, guarantee that any finite projective K_i^+/ϖ-module is free. ∎

Corollary 6.6.7. *In the notation of Lemma 6.6.5, put $X = \mathrm{Spa}\,(C, C^+)$. Then any $\mathcal{O}^+_{X_{\text{ét}}}/\varpi$-vector bundle is free.*

Proof. Lemma 6.6.2, Lemma 6.6.4, and Lemma 6.6.5 imply that the category of $\mathcal{O}^+_{X_{\text{ét}}}/\varpi$-vector bundles is equivalent to the category of usual vector bundles on the scheme $\mathrm{Spec}\, C^+/\varpi$. Any such vector bundle is free due to Lemma 6.6.6. ∎

Now we can prove the main result of this section:

Theorem 6.6.8. *Let X be a strongly sheafy adic space over $\mathrm{Spa}\,(\mathbf{Q}_p, \mathbf{Z}_p)$, let $x \in X$ be a point, and let $\mathcal{E}$ be an $\mathcal{O}^+_{X^\diamond}/p$-vector bundle. Then there are an affinoid open subset $x \in U_x \subset X$ and a finite étale surjective morphism $\widetilde{U}_x \to U_x$ such that $\mathcal{E}|_{\widetilde{U}_x} \simeq \left(\mathcal{O}^+_{\widetilde{U}_x^\diamond}/p\right)^d$ for some integer d.*

Proof. For clarity, we divide the proof into several steps.

Step 1: *The space* $X = \operatorname{Spa}(K, K^+)$ *for a non-archimedean field* K *and an open and bounded valuation subring* $K^+ \subset K$. In this case, we first fix a separable closure K^{sep} of K. We put $\{K_i\}_{i\in I}$ to be a filtered system of all finite sub-extensions $K \subset K_i \subset K^{\mathrm{sep}}$, we also put K_i^+ to be the integral closure of K^+ in K_i. Then Lemma 6.6.5 ensures that $C^+ := \left(\operatorname{colim}_I K_i^+\right)^{\wedge}_{(p)}$ is a Prüfer domain and $C := C^+\left[\frac{1}{p}\right]$ is an algebraically closed non-archimedean field. Therefore, Corollary 6.6.7 implies that $\mathcal{E}|_{\operatorname{Spd}(C,C^+)}$ is free. By construction,

$$\operatorname{Spa}(C, C^+) \approx \lim_I \operatorname{Spa}(K_i, K_i^+) \to \operatorname{Spa}(K, K^+)$$

is an affinoid strongly pro-étale morphism. Therefore, Lemma 6.5.5 implies that there is an index $i \in I$ such that $\mathcal{E}|_{\operatorname{Spd}(K_i,K_i^+)}$ is free. Now the result follows from the evident observation that $\operatorname{Spa}(K_i, K_i^+) \to \operatorname{Spa}(K, K^+)$ is a surjective finite étale morphism.

Step 2: *General* X. Step 1 constructs a finite separable extension $\widehat{k(x)} \subset K$ such that $\mathcal{E}|_{\operatorname{Spd}(K,K^+)}$ is free, where K^+ is the integral closure of $\widehat{k(x)}^+$ in K.

Now, [5, Proposition 7.5.5 (5)] implies that $\mathcal{O}^+_{X,x}$ is p-adically henselian, and there is a natural isomorphism $\left(\mathcal{O}^+_{X,x}\right)^{\wedge}_{(p)} \simeq \widehat{k(x)}^+$. So, [26, Proposition 5.4.54] says that we can find a finite étale morphism $\mathcal{O}_{X,x} \to A$ such that $A \otimes_{\mathcal{O}_{X,x}} \widehat{k(x)} = K$. Since $\mathcal{O}_{X,x}$ is a local ring with residue field $k(x)$, we easily conclude that $\mathcal{O}_{X,x} \to A$ is also faithfully flat. Now we recall that $\mathcal{O}_{X,x} = \operatorname{colim}_{x\in V\subset X} \mathcal{O}_X(V)$, so a standard approximation argument implies that we can find an affinoid open $x \in V \subset X$ and a faithfully flat finite étale morphism $\mathcal{O}_X(V) \to A_V$ such that $A_V \otimes_{\mathcal{O}_X(V)} \mathcal{O}_{X,x} \simeq A$.

For each affinoid open subset $x \subset W \subset V$, we put $A_W := A_V \otimes_{\mathcal{O}_X(V)} \mathcal{O}_X(W)$ and put A_W^+ to be the integral closure of $\mathcal{O}_X(W)$ in A_W. Then Lemma C.1.1 ensures that (A_W, A_W^+) is a complete Tate–Huber pair for each open affinoid $x \subset W \subset V$. Furthermore, the corresponding morphism $f_W\colon \operatorname{Spa}(A_W, A_W^+) \to W$ is a finite étale surjection due to Lemma C.1.2. By construction, we have that

$$\operatorname{Spa}(K, K^+) \approx \lim_{x\in W\subset V} \operatorname{Spa}(A_W, A_W^+) \to \operatorname{Spa}(A_V, A_V^+)$$

is an affinoid strongly pro-étale morphism, and $\mathcal{E}|_{\operatorname{Spd}(K,K^+)}$ is free. So, Lemma 6.5.5 implies that there is an open affinoid subspace $x \in U_x \subset V_x$ such that $\mathcal{E}|_{\operatorname{Spd}(A_U,A_U^+)}$ is free. Then $\widetilde{U}_x = \operatorname{Spa}(A_U, A_U^+)$ does the job. ■

Now we summarize all results about various $\mathcal{O}^+/p$-vector bundles below:

Corollary 6.6.9. *Let* X *be a strongly sheafy adic space over* $\operatorname{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)$. *Then*

(1) *the categories* $\operatorname{Vect}(X_{\text{ét}}; \mathcal{O}^+_{X_{\text{ét}}}/p)$, $\operatorname{Vect}(X^{\diamond}_{\mathrm{qp}}; \mathcal{O}^+_{X^{\diamond}_{\mathrm{qp}}}/p)$, *and* $\operatorname{Vect}(X^{\diamond}_v; \mathcal{O}^+_{X^{\diamond}}/p)$ *are equivalent;*

(2) *these equivalences preserve cohomology groups;*

(3) *for any $\mathcal{O}^+_{X^\diamond}/p$-vector bundle $\mathcal{E}$ and a point $x \in X$, there exists an open affinoid subspace $x \in U_x \subset X$ and a finite étale surjective morphism $\widetilde{U}_x \to U_x$ such that $\mathcal{E}|_{\widetilde{U}_x^\diamond}$ is a free vector bundle.*

6.7 Étale coefficients

The main goal of this section is to relate the étale and v-cohomology groups of étale "overconvergent" $\mathcal{O}^+/p$-modules.

We fix a strongly sheafy adic space X over Spa $(\mathbf{Q}_p, \mathbf{Z}_p)$. Then we note that any étale sheaf of $\mathbf{F}_p$-modules $\mathcal{F}$ on X defines sheaves $\mu^{-1}\mathcal{F}$ and $\lambda^{-1}\mu^{-1}\mathcal{F}$ of $\mathbf{F}_p$-modules on $X^\diamond_{\mathrm{qp}}$ and $X^\diamond_v$ respectively, see Diagram (6.3.1). In what follows, we abuse the notation and denote $(\lambda^{-1}\mu^{-1}\mathcal{F}) \otimes_{\mathbf{F}_p} \mathcal{O}^+_{X^\diamond}/p$ simply by $\mathcal{F} \otimes \mathcal{O}^+_{X^\diamond}/p$ for any $\mathcal{F} \in \mathbf{Shv}(X_{\text{ét}}; \mathbf{F}_p)$. Similarly, we denote by $(\mu^{-1}\mathcal{F}) \otimes_{\mathbf{F}_p} \mathcal{O}^+_{X^\diamond_{\mathrm{qp}}}/p$ simply by $\mathcal{F} \otimes \mathcal{O}^+_{X^\diamond_{\mathrm{qp}}}/p$.

Before we go to the comparison results, we need to discuss some preliminary results on sheaves on pro-finite sets. They turn out to be tied up with overconvergent étale sheaves on strictly totally disconnected spaces.

Definition 6.7.1. For S a pro-finite set, a sheaf of $\mathbf{F}_p$-modules $\mathcal{F}$ is *constructible* if there exists a finite decomposition of S into a disjoint union of clopen subsets $S = \bigsqcup_{i=1}^n S_i$ such that $\mathcal{F}|_{S_i}$ is a constant sheaf of finite rank.

Lemma 6.7.2. *Let S be a pro-finite set, and let $f : \mathcal{F} \to \mathcal{G}$ be a morphism of constructible sheaves of $\mathbf{F}_p$-modules. Then* Ker f *and* Coker f *are constructible.*

Proof. Since S is pro-finite, each point $s \in S$ admits a clopen subset $s \in U_s \subset S$ such that both $\mathcal{F}|_{U_s}$ and $\mathcal{G}|_{U_s}$ are constant. Since S is quasi-compact, we can find a finite disjoint union decomposition $S = \bigsqcup_{i=1}^n U_i$ such that both $\mathcal{F}|_{U_i}$ and $\mathcal{G}|_{U_i}$ are constant. So we can assume that both $\mathcal{F}$ and $\mathcal{G}$ are constant. Then it is easy to see that the kernel and the cokernel are constant as well. ∎

Lemma 6.7.3. *Let S be a pro-finite set, and let $\mathcal{F}$ be a sheaf of $\mathbf{F}_p$-vector spaces. Then $\mathcal{F} \simeq \mathrm{colim}_I\, \mathcal{F}_i$ for a filtered system of constructible sheaves $\mathcal{F}_i$.*

Proof. We use [68, Tag 093C], with $\mathcal{B}$ being the collection of clopen subsets of S, to write $\mathcal{F}$ as a filtered colimit of the form

$$\mathcal{F} \simeq \mathrm{colim}_I\, \mathrm{Coker}\Big(\bigoplus_{j=1}^m \underline{\mathbf{F}}_{p,V_j} \to \bigoplus_{i=1}^n \underline{\mathbf{F}}_{p,U_i}\Big).$$

Now Lemma 6.7.2 implies that each cokernel is constructible finishing the proof. ∎

Definition 6.7.4. A sheaf of $\mathbf{F}_p$-modules $\mathcal{F}$ on $X_{\text{ét}}$ is *overconvergent* if, for every specialization $\overline{\eta} \to \overline{s}$ of geometric points of X, the specialization map $\mathcal{F}_{\overline{s}} \to \mathcal{F}_{\overline{\eta}}$ is an isomorphism.

Definition 6.7.5. An étale sheaf of $\mathbf{F}_p$-modules $\mathcal{F}$ on a strictly totally disconnected perfectoid space X is *special* if there exists a finite decomposition of X into a disjoint union of clopen subsets $X = \bigsqcup_{i=1}^n X_i$ such that $\mathcal{F}|_{X_i}$ is a constant sheaf of finite rank.

Lemma 6.7.6. *Let X be a strictly totally disconnected perfectoid space, and $\mathcal{F}$ an overconvergent étale sheaf of $\mathbf{F}_p$-modules. Then $\mathcal{F} \simeq \operatorname{colim}_I \mathcal{F}_i$ for a filtered system of special sheaves $\mathcal{F}_i$ of $\mathbf{F}_p$-modules.*

Proof. Since X is strictly totally disconnected, the étale and analytic sites of X are equivalent. So we can argue on the analytic site of X. By [61, Lemma 7.3], there is a continuous surjection $\pi\colon X \to \pi_0(X)$ onto a pro-finite set $\pi_0(X)$ of connected components.

Step 1: *The natural map $\pi^*\pi_*\mathcal{F} \to \mathcal{F}$ is an isomorphism.* It suffices to check that it is an isomorphism on stalks. Pick a point $x \in X$, then [61, Lemma 7.3] implies that the connected component of x has a unique closed point s. Then after unraveling all definitions, one gets that the map $(\pi^*\pi_*\mathcal{F})_x \to \mathcal{F}_x$ is naturally identified with the specialization map $\mathcal{F}_s \to \mathcal{F}_x$ that is an isomorphism by the overconvergent assumption.

Step 2: *Finish the proof.* Lemma 6.7.3 ensures that $\pi_*\mathcal{F} \simeq \operatorname{colim}_I \mathcal{G}'_i$ is a filtered colimit of constructible sheaves. Since pullback commutes with all colimits, we get isomorphisms $\mathcal{F} \simeq \pi^*\pi_*\mathcal{F} \simeq \operatorname{colim}_I \pi^*\mathcal{G}'_i$. This finishes the proof since each $\mathcal{G}_i :=$ $\pi^*\mathcal{G}'_i$ is special. ∎

Lemma 6.7.7. *Let X be a strongly sheafy adic space over* $\operatorname{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)$*, and let $\mathcal{F}$ be an overconvergent étale sheaf of $\mathbf{F}_p$-modules. Then the natural morphism*

$$\mathcal{O}^+_{X^\diamond_{\mathrm{qp}}}/p \otimes \mathcal{F} \to \mathbf{R}\lambda_*\big(\mathcal{O}^+_{X^\diamond}/p \otimes \mathcal{F}\big)$$

is an isomorphism.

Proof. Since strictly totally disconnected spaces form a basis for the quasi-proétale topology on $X^\diamond$, it suffices to show that a is an isomorphism on such spaces. Then we can write $\mathcal{F} \simeq \operatorname{colim}_I \mathcal{F}_i$ as a filtered colimit of special sheaves by Lemma 6.7.6. One easily checks that α is a coherent morphism of algebraic topoi, and thus each $\mathrm{R}^i\lambda_*(\mathcal{O}^+_{X^\diamond}/p \otimes -)$ commutes with filtered colimits by [2, Exp. VI Theoreme 5.1]. Thus, it suffices to prove the claim for a special $\mathcal{F}$. By the definition of a special sheaf, there exists a disjoint decomposition $X = \bigsqcup_{i=1}^n X_i$ into clopen subsets such that $\mathcal{F}|_{X_i}$ is constant of finite rank. Since the question is local on $X^\diamond_{\text{qproét}}$, we can replace X with each X_i to assume that $\mathcal{F}$ is constant. In this case, the claim follows from Corollary 6.4.14. ■

Remark 6.7.8. We do not know if Lemma 6.7.7 holds for non-overconvergent étale sheaves $\mathcal{F}$.

Now we discuss the relation between the étale and quasi-proétale topologies.

Lemma 6.7.9. *Let X be a strongly sheafy adic space over* Spa $(\mathbf{Q}_p, \mathbf{Z}_p)$*, and let $\mathcal{F}$ be an overconvergent étale sheaf of $\mathbf{F}_p$-modules. Then the natural morphism*

$$\mathcal{O}^+_{X_{\text{ét}}}/p \otimes \mathcal{F} \to \mathbf{R}\mu_*\big(\mathcal{O}^+_{X^\diamond_{\text{qp}}}/p \otimes \mathcal{F}\big)$$

is an isomorphism.

Proof. As a consequence of Lemma 6.3.7, the right-hand side is canonically isomorphic to $\mathbf{R}\mu_*\mu^{-1}\big(\mathcal{O}^+_{X_{\text{ét}}}/p \otimes \mathcal{F}\big)$. So the result follows from [61, Proposition 14.8]. ■

Now we combine all these results together:

Lemma 6.7.10. *Let X be a strongly sheafy adic space over* Spa $(\mathbf{Q}_p, \mathbf{Z}_p)$*, and $\mathcal{F}$ an overconvergent étale sheaf of $\mathbf{F}_p$-modules on X. Then the natural morphisms*

$$\mathcal{O}^+_{X_{\text{ét}}}/p \otimes \mathcal{F} \to \mathbf{R}\mu_*\big(\mathcal{O}^+_{X^\diamond_{\text{qp}}}/p \otimes \mathcal{F}\big),$$
$$\mathcal{O}^+_{X^\diamond_{\text{qp}}}/p \otimes \mathcal{F} \to \mathbf{R}\lambda_*\big(\mathcal{O}^+_{X^\diamond}/p \otimes \mathcal{F}\big)$$

are isomorphisms.

6.8 Application: $\mathcal{O}^+$ and $\mathcal{O}$ vector bundles

In this section, we discuss the relation between $\mathcal{O}^+_{X^\diamond}$ and $\mathcal{O}_{X^\diamond}$ vector bundles in different topologies. As an application of the methods developed in this section, we reprove a theorem of Kedlaya–Liu saying that, for a perfectoid space X, the categories of $\mathcal{O}_{X^\diamond}$-vector bundles in the analytic, étale, quasi-proétale, and v-topologies are all equivalent. To achieve this result, we prove an intermediate claim that the categories of $\mathcal{O}^+_{X^\diamond}$-vector bundles in the étale, quasi-proétale, and v-topologies are equivalent. The results of this section will not be used in the rest of the memoir.

We define the categories of v, quasi-proétale, and étale $\mathcal{O}^+$-vector bundles on X (resp. $\mathcal{O}$-vector bundles on X) similarly to Definition 6.5.1.

We start by understanding the category of $\mathcal{O}^+_{X^\diamond}$-torsors on an affinoid perfectoid space X.

Lemma 6.8.1. *Let (R, R^+) be a perfectoid pair, and let $f\colon (R^+)^d \to (R^+)^d$ be an R^+-linear homomorphism such that $\overline{f}\colon (R^+/R^{\circ\circ})^d \to (R^+/R^{\circ\circ})^d$ is an isomorphism. Then f is an isomorphism.*

Proof. Lemma B.9 (2) together with a standard approximation argument imply that $f \bmod \varpi\colon (R^+/\varpi)^d \to (R^+/\varpi)^d$ is an isomorphism. Then [68, Tag 0315] implies that f is surjective, put $K = \operatorname{Ker} f$. We note that K is derived ϖ-adically complete due to [68, Tag 091U]. Furthermore, our assumption implies that $K/\varpi K = 0$, so [68, Tag 09B9] ensures that $K = 0$. In particular, f is an isomorphism. ∎

Lemma 6.8.2. *Let $X = \operatorname{Spa}(R, R^+)$ be an affinoid perfectoid space over $\operatorname{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)$, and let $\mathcal{E}$ be an $\mathcal{O}^+_{X^\diamond}$-vector bundle. If $\mathcal{E}/p$ is a free $\mathcal{O}^+_{X^\diamond}/p$-vector bundle, then $\mathcal{E}$ is a free $\mathcal{O}^+_{X^\diamond}$-vector bundle.*

Proof. In this proof, we put $\mathfrak{m} = R^{\circ\circ}$ and always do almost mathematics with respect to this ideal (see Lemma B.12).

Lemma 6.3.5 (1) implies that $\mathbf{R}\Gamma(X^\diamond_v, \mathcal{E}/p)$ is almost concentrated in degree 0. Then Lemma 6.3.5 (3), [68, Tag 0A0G], and Lemma A.5 imply that $\mathbf{R}\Gamma(X^\diamond_v, \mathcal{E})$ is almost concentrated in degree 0. This implies that

$$\mathfrak{m} \otimes_{R^+} \mathbf{R}\Gamma(X^\diamond_v, \mathcal{E}) = \mathbf{R}\Gamma(X^\diamond_v, \mathfrak{m} \otimes_{R^+} \mathcal{E}) = \mathbf{R}\Gamma(X^\diamond_v, \mathfrak{m}\mathcal{E}) \tag{6.8.1}$$

and

$$\mathfrak{m} \otimes_{R^+} \mathbf{R}\Gamma(X^\diamond_v, \mathcal{E}/p) = \mathbf{R}\Gamma(X^\diamond_v, \mathfrak{m} \otimes_{R^+} \mathcal{E}/p) = \mathbf{R}\Gamma(X^\diamond_v, \mathfrak{m}\mathcal{E}/p\mathfrak{m}\mathcal{E}) \tag{6.8.2}$$

are concentrated in degree 0. Since $\mathcal{E}/p$ is trivial, we conclude that $\mathcal{E}/\mathfrak{m}\mathcal{E}$ is a trivial $\mathcal{O}^+_{X^\diamond}/\mathfrak{m}$-vector bundle. We choose an isomorphism $\mathcal{E}/\mathfrak{m}\mathcal{E} \simeq \left(\mathcal{O}^+_{X^\diamond}/\mathfrak{m}\right)^d$ and define a basis

$$e''_1, \dots, e''_r \in \mathrm{H}^0\left(X^\diamond_v, \mathcal{E}/\mathfrak{m}\mathcal{E}\right).$$

Then we consider the short exact sequence

$$0 \to \mathfrak{m} \otimes_{R^+} (\mathcal{E}/p) \to \mathcal{E}/p\mathfrak{m}\mathcal{E} \to \mathcal{E}/\mathfrak{m}\mathcal{E} \to 0.$$

Now (6.8.1) implies that we can lift e''_i to elements $e'_i \in \mathrm{H}^0\left(X^\diamond_v, \mathcal{E}/p\mathfrak{m}\mathcal{E}\right)$. Then we use the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathfrak{m} \otimes_{R^+} p\mathcal{E} & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{E}/p\mathfrak{m}\mathcal{E} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{E} & \xrightarrow{\cdot p} & \mathcal{E} & \longrightarrow & \mathcal{E}/p & \longrightarrow & 0
\end{array}$$

and (6.8.2) to conclude that the natural morphism $\mathrm{H}^0\left(X^\diamond_v, \mathcal{E}/p\mathfrak{m}\mathcal{E}\right) \to \mathrm{H}^0\left(X^\diamond_v, \mathcal{E}/p\right)$ factors through $\mathrm{H}^0\left(X^\diamond_v, \mathcal{E}\right)/p \subset \mathrm{H}^0\left(X^\diamond_v, \mathcal{E}/p\right)$. This implies that we can lift e'_i to elements $e_i \in \mathrm{H}^0\left(X^\diamond_v, \mathcal{E}\right)$. This defines a morphism

$$\varphi\colon \left(\mathcal{O}^+_{X^\diamond}\right)^d \to \mathcal{E}.$$

By construction, φ mod $\mathfrak{m}$ becomes an isomorphism. We wish to show that this implies that φ is an isomorphism. This can be checked v-locally on X, so we can assume that $\mathcal{E} \simeq (\mathcal{O}^+_{X^\diamond})^d$, and we need to check that $\varphi(X')$ is an isomorphism for any affinoid perfectoid $X' \to X$. Then the result follows directly from Lemma 6.3.5 and Lemma 6.8.1. ■

Corollary 6.8.3. *Let X be a perfectoid space over* Spa $(\mathbf{Q}_p, \mathbf{Z}_p)$*, and let $\mathcal{E}$ be an $\mathcal{O}^+_{X^\diamond}$-vector bundle. Then, for each $x \in X$, there are an open subspace $x \in U_x \subset X$ and a finite étale surjective morphism $\widetilde{U}_x \to U_x$ such that $\mathcal{E}|_{\widetilde{U}_x}$ is trivial.*

Proof. This formally follows from Corollary 6.6.9 and Lemma 6.8.2. ■

Now we denote by

$$\mu^* = \mu^{-1} \otimes_{\mu^{-1}\mathcal{O}^+_{X_{\text{ét}}}} \mathcal{O}^+_{X^\diamond_{\text{qp}}} \colon \mathrm{Vect}\big(X_{\text{ét}}; \mathcal{O}^+_{X_{\text{ét}}}\big) \to \mathrm{Vect}\big(X_{\text{qp}}; \mathcal{O}^+_{X^\diamond_{\text{qp}}}\big)$$

and

$$\lambda^* = \lambda^{-1} \otimes_{\lambda^{-1}\mathcal{O}^+_{X^\diamond_{\text{qp}}}} \mathcal{O}^+_{X^\diamond} \colon \mathrm{Vect}\big(X^\diamond_{\text{qp}}; \mathcal{O}^+_{X^\diamond_{\text{qp}}}\big) \to \mathrm{Vect}\big(X^\diamond_v; \mathcal{O}^+_{X^\diamond}\big)$$

the pullback functors.

Now we can show that the categories of $\mathcal{O}^+$-vector bundles in the étale, quasi-proétale, and v topologies are all equivalent:

Theorem 6.8.4. *Let X be a pre-adic space over* Spa $(\mathbf{Q}_p, \mathbf{Z}_p)$.

(1) *Then the functor $\lambda^* \colon \mathrm{Vect}\big(X^\diamond_{\text{qp}}; \mathcal{O}^+_{X^\diamond_{\text{qp}}}\big) \to \mathrm{Vect}\big(X^\diamond_v; \mathcal{O}^+_{X^\diamond}\big)$ is an equivalence. Furthermore, for any $\mathcal{O}^+_{X^\diamond_{\text{qp}}}$-vector bundle $\mathcal{V}$, the natural morphism*

$$\mathcal{V} \to \mathbf{R}\lambda_* \lambda^* \mathcal{V}$$

is an isomorphism.

(2) *If X is perfectoid, then the functor $\mu^* \colon \mathrm{Vect}\big(X_{\text{ét}}; \mathcal{O}^+_{X_{\text{ét}}}\big) \to \mathrm{Vect}\big(X^\diamond_{\text{qp}}; \mathcal{O}^+_{X^\diamond_{\text{qp}}}\big)$ is an equivalence. Furthermore, for any $\mathcal{O}^+_{X_{\text{ét}}}$-vector bundle $\mathcal{E}$, the natural morphism*

$$\mathcal{E} \to \mathbf{R}\mu_* \mu^* \mathcal{E}$$

is an isomorphism.

Proof. First, we note that the second claim is quasi-proétale local on X, so we can assume that X is a perfectoid space. Then the proof is very similar to that of Theorem 6.5.7. We spell out the main steps.

We first show that the natural morphisms

$$\alpha \colon \mathcal{O}^+_{X_{\text{ét}}} \to \mathbf{R}\mu_* \mathcal{O}^+_{X^\diamond_{\text{qp}}},$$
$$\beta \colon \mathcal{O}^+_{X^\diamond_{\text{qp}}} \to \mathbf{R}\lambda_* \mathcal{O}^+_{X^\diamond}$$

are isomorphisms. For this, we note that Remark 6.3.6 implies that $\mathcal{O}^+_{X_{\text{ét}}}$, $\mathcal{O}^+_{X^\diamond_{\text{qp}}}$, and $\mathcal{O}^+_{X^\diamond}$ are derived p-adically complete and p-torsion free. Therefore, we can check that α and β are isomorphisms modulo p (in the derived sense). This follows from Theorem 6.5.9 and Theorem 6.5.7.

This formally implies that the maps $\mathcal{V} \to \mathbf{R}\lambda_*\lambda^*\mathcal{V}$ and $\mathcal{E} \to \mathbf{R}\mu_*\mu^*\mathcal{E}$ are isomorphisms. This, in turn, formally implies that λ^* and μ^* are fully faithful. Essential surjectivity of both functors follows from Corollary 6.8.3. ∎

Remark 6.8.5. We note that [35, Theorem 4.27] gives a much more general version of Theorem 6.8.4. However, Corollary 6.8.3 does not seem to be addressed in [35].

Now we discuss the case of $\mathcal{O}_X$-vector bundles. We first wish to show that any $\mathcal{O}_{X^\diamond}$-vector bundle (in the v-topology) admits an $\mathcal{O}^+_{X^\diamond}$-lattice étale locally on X. This will be our key tool to reduce questions about $\mathcal{O}$-vector bundles to the case of $\mathcal{O}^+$-vector bundles. For this, we will need a number of preliminary lemmas:

Lemma 6.8.6. *Let A be an f-henselian ring for some regular element $f \in A$, and let $\widehat{A}$ be its f-adic completion. Then the natural morphism*

$$\mathrm{GL}_n\left(A\left[\tfrac{1}{f}\right]\right)/\mathrm{GL}_n(A) \to \mathrm{GL}_n\left(\widehat{A}\left[\tfrac{1}{f}\right]\right)/\mathrm{GL}_n(\widehat{A})$$

is a bijection.

Proof. In this proof, we denote by $\underline{\mathrm{Vect}}_n(R)$ the groupoid of finite projective R-modules of rank-n, and by $\mathrm{Vect}_n(R)$ the set of isomorphism classes of finite projective R-modules of rank-n.

Now we start the proof. First, [68, Tag 0BNS] ensures that $(A \to \widehat{A}, f)$ is a gluing data. Second, [68, Tag 0BNW] ensures that any finite projective A-module is glueable. Therefore, [68, Tag 0BP2] and [68, Tag 0BP6] imply that the following diagram of groupoids:

$$\begin{array}{ccc} \underline{\mathrm{Vect}}_n(A) & \xrightarrow{-\otimes_A \widehat{A}} & \underline{\mathrm{Vect}}_n(\widehat{A}) \\ \downarrow{\scriptstyle -\otimes_A A_f} & & \downarrow{\scriptstyle -\otimes_{\widehat{A}} \widehat{A}[\frac{1}{f}]} \\ \underline{\mathrm{Vect}}_n(A_f) & \xrightarrow[-\otimes_{A_f} \widehat{A}[\frac{1}{f}]]{} & \underline{\mathrm{Vect}}_n\left(\widehat{A}\left[\tfrac{1}{f}\right]\right) \end{array}$$

is cartesian. Therefore, we can pass to homotopy groups at the free module A^n to get a long exact sequence of pointed sets:

$$0 \longrightarrow \mathrm{GL}_n(A) \longrightarrow \mathrm{GL}_n\left(A\left[\tfrac{1}{f}\right]\right) \times \mathrm{GL}_n(\widehat{A}) \longrightarrow \mathrm{GL}_n\left(\widehat{A}\left[\tfrac{1}{f}\right]\right)$$

$$\mathrm{Vect}_n(A) \underset{\alpha}{\leftrightarrows} \mathrm{Vect}_n(\widehat{A}) \times \mathrm{Vect}_n\left(A\left[\tfrac{1}{f}\right]\right) \longrightarrow \mathrm{Vect}_n\left(\widehat{A}\left[\tfrac{1}{f}\right]\right) \longrightarrow 0.$$

To prove the claim, it suffices to show that the fiber of α over the pair of trivial rank-n modules is just a point. This follows from [68, Tag 0D4A] which even implies that the map $\mathrm{Vect}_n(A) \to \mathrm{Vect}_n(\widehat{A})$ is a bijection. ∎

Definition 6.8.7. Let X be a pre-adic space over $\mathrm{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)$. The *(pre-)sheaf of invertible matrices* $\mathrm{GL}_{n,X^\diamond}$ on $X_v^\diamond$ is defined via the rule

$$(S \to X^\diamond) \mapsto \mathrm{GL}_n\big(\mathcal{O}_{S^\sharp}(S^\sharp)\big)$$

for any affinoid perfectoid space S over $X^\diamond$.

We define the *(pre-)sheaf of integral invertible matrices* $\mathrm{GL}^+_{n,X^\diamond}$ on $X_v^\diamond$ via the rule

$$(S \to X^\diamond) \mapsto \mathrm{GL}_n\big(\mathcal{O}^+_{S^\sharp}(S^\sharp)\big)$$

for any affinoid perfectoid space S over $X^\diamond$.

One easily checks that a $\mathrm{GL}_{n,X^\diamond}$ is a v-sheaf since it is isomorphic to the diamond associated with the classical (pre)-adic space $\mathrm{GL}_{n,\mathbf{Q}_p} \times_{\mathbf{Q}_p} X$. Similarly, $\mathrm{GL}^+_{n,X^\diamond}$ is a v-sheaf since it is isomorphic to the diamond associated with the (pre-)adic space $\big(\mathrm{GL}_{n,\mathbf{Q}_p} \cap \mathbf{D}^{n^2}_{\mathbf{Q}_p}\big) \times_{\mathbf{Q}_p} X$.

For the next definition, we fix a pre-adic spce X over $\mathrm{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)$ and an $\mathcal{O}_{X^\diamond}$-vector bundle $\mathcal{E}$.

Definition 6.8.8. The *sheaf of lattices* $\mathrm{Latt}_X(\mathcal{E})$ is the v-sheaf defined by the formula

$$(S \to X^\diamond) \mapsto \left\{\mathcal{E}^+ \in \mathrm{Vect}\big(S_v^{\sharp,\diamond}; \mathcal{O}^+_{S^{\sharp,\diamond}}\big), \varphi\colon \mathcal{E}^+\left[\frac{1}{p}\right] \xrightarrow{\sim} \mathcal{E}|_S\right\}/\mathrm{isom}$$

for each affinoid perfectoid $S \to X^\diamond$ over $X^\diamond$.

Lemma 6.8.9. *Let X denote a pre-adic space over* $\mathrm{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)$*, and let $\mathcal{E}$ be an $\mathcal{O}_{X^\diamond}$-vector bundle. Then, v-locally on $X^\diamond$, the sheaf* $\mathrm{Latt}_X(\mathcal{E})$ *is isomorphic to* $\mathrm{GL}_{n,X^\diamond}/\mathrm{GL}^+_{n,X^\diamond}$.

Proof. The claim is v-local on X by design, so we can assume that $\mathcal{E} \simeq \mathcal{O}^d_{X^\diamond}$. Then we note that $\mathrm{GL}_{n,X^\diamond}$ acts $\mathcal{E}$, i.e., for any $g \in \mathrm{GL}_{n,X^\diamond}(S)$ we have an isomorphism $g^*\colon \mathcal{E}_S \to \mathcal{E}_S$. Therefore, it also acts on $\mathrm{Latt}_X(\mathcal{E})$ via the rule

$$g\big(\mathcal{E}^+, \varphi\colon \mathcal{E}^+ \xrightarrow{\sim} \mathcal{E}\big) = \big(\mathcal{E}^+, g^* \circ \varphi\big).$$

Now let $\mathcal{E}^+_0 \subset \mathcal{E}$ be the trivial lattice $\big(\mathcal{O}^+_{X^\diamond}\big)^d \subset \mathcal{O}^d_{X^\diamond} = \mathcal{E}$, this defines a point $\ell_0 \in \mathrm{Latt}_X(\mathcal{E})$. Then the orbit map defines a morphism of sheaves $\alpha\colon \mathrm{GL}_{n,X^\diamond} \to \mathrm{Latt}_X(\mathcal{E})$ via the rule

$$g \mapsto g(\ell_0).$$

The stabilizer of ℓ_0 is equal to $\mathrm{GL}^+_{n,X^\diamond}$, so α factors through an injective morphism

$$\beta\colon \mathrm{GL}_{n,X^\diamond}/\mathrm{GL}^+_{n,X^\diamond} \hookrightarrow \mathrm{Latt}_X(\mathcal{E}).$$

So we are only left to show that it is surjective. Let $S \to \mathrm{Latt}_X(\mathcal{E})$ be a point corresponding to a lattice $(\mathcal{E}^+, \varphi)$. We need to show that this point lies in the image of β locally in the v-topology. By definition, there is a v-covering $S' \to S$ such that $\mathcal{E}^+|_{S'}$ becomes a free $\mathcal{O}^+_{X^\diamond}$-vector bundle. But then there is an element $g \in \mathrm{GL}_{n,X^\diamond}(S')$ such that $g(\mathcal{E}^+) = \mathcal{E}_0^+|_{S'}$. In particular, $(\mathcal{E}^+|_{S'}, \varphi|_{S'})$ lies in the image of $\beta(S')$. ∎

Corollary 6.8.10. *Let X be a pre-adic space over* $\mathrm{Spa}\,(\mathbf{Q}_p, \mathbf{Z}_p)$*, let $\mathcal{E}$ be an $\mathcal{O}_{X^\diamond}$-vector bundle, and let $Z = \lim_I Z_i$ be a cofiltered limit of affinoid perfectoid spaces over X. Then the natural morphism*

$$\mathrm{colim}_I\, \mathrm{Latt}_X(\mathcal{E})(Z_i^\diamond) \to \mathrm{Latt}_X(\mathcal{E})(Z^\diamond)$$

is a bijection.

Proof. Let $Z_i = \mathrm{Spa}\,(R_i, R_i^+)$, we put $R^+_\infty := \mathrm{colim}_I\, R_i^+$ and denote by $\widehat{R}^+_\infty$ its p-adic completion. Then we note that the claim is v-local on Z, so we can assume that $\mathcal{E}$ is a free $\mathcal{O}_{X^\diamond}$-vector bundle. Then Lemma 6.8.9 implies that it suffices to show that

$$\mathrm{GL}_n\big(R^+_\infty[\tfrac{1}{p}]\big)/\mathrm{GL}_n\big(R^+_\infty\big) \to \mathrm{GL}_n\big(\widehat{R}^+_\infty[\tfrac{1}{p}]\big)/\mathrm{GL}_n\big(\widehat{R}^+_\infty\big)$$

is a bijection. This follows directly from Lemma 6.8.6, [68, Tag 0ALJ], and [68, Tag 0FWT]. ∎

Corollary 6.8.11. *Let X be a pre-adic space over* $\mathrm{Spa}\,(\mathbf{Q}_p, \mathbf{Z}_p)$*, let $\mathcal{E}$ be an $\mathcal{O}_{X^\diamond}$-vector bundle, and let $Z \approx \lim_I Z_i \to Z_0$ be an affinoid strongly pro-étale morphism of strongly sheafy Tate-affinoid adic spaces over X. Then the natural morphism*

$$\mathrm{colim}_I\, \mathrm{Latt}_X(\mathcal{E})\big(Z_i^\diamond\big) \to \mathrm{Latt}_X(\mathcal{E})\big(Z^\diamond\big)$$

is a bijection.

Proof. This is a direct consequence of Corollary 6.8.10 (one can argue as in Corollary 6.5.4). ∎

Corollary 6.8.12. *Let X be a strongly sheafy adic space over* $\mathrm{Spa}\,(\mathbf{Q}_p, \mathbf{Z}_p)$*, let $\mathcal{E}$ be an $\mathcal{O}_{X^\diamond}$-vector bundle. Then there are an étale covering $X' \to X$, an $\mathcal{O}^+_{X'^\diamond}$-vector bundle $\mathcal{E}^+$, and an isomorphism $\mathcal{E}^+\big[\frac{1}{p}\big] \simeq \mathcal{E}|_{X'}$.*

Proof. First, we note that we want to show that $\mathrm{Latt}_X(\mathcal{E})$ admits a section for some étale covering $X' \to X$. For this, we can assume that X is an affinoid space.

If X is strictly totally disconnected, the result follows from the observation that such a section exists after a v-surjection, Lemma 6.4.4, and Corollary 6.8.10. Then the result follows from Lemma 6.2.13 and Corollary 6.8.11. ∎

We will later be able to prove a more precise version of Corollary 6.8.12. But, before that, we show that all possible versions of $\mathcal{O}_X$-vector bundles coincide on perfectoid spaces. For this, we denote by $\pi\colon (X_{\text{ét}}, \mathcal{O}_{X_{\text{ét}}}) \to (X_{\text{an}}, \mathcal{O}_X)$ the natural morphism of ringed sites. We also denote by

$$\begin{aligned}
\pi^* &= \pi^{-1} \otimes_{\pi^{-1}\mathcal{O}_{X_{\text{an}}}} \mathcal{O}_{X_{\text{ét}}} : \mathrm{Vect}(X_{\text{an}}; \mathcal{O}_{X_{\text{an}}}) \to \mathrm{Vect}(X_{\text{ét}}; \mathcal{O}_{X_{\text{ét}}}),\\
\mu^* &= \mu^{-1} \otimes_{\mu^{-1}\mathcal{O}_{X_{\text{ét}}}} \mathcal{O}_{X^\diamond_{\text{qp}}} : \mathrm{Vect}(X_{\text{ét}}; \mathcal{O}_{X_{\text{ét}}}) \to \mathrm{Vect}(X^\diamond_{\text{qp}}; \mathcal{O}_{X^\diamond_{\text{qp}}}),\\
\lambda^* &= \lambda^{-1} \otimes_{\lambda^{-1}\mathcal{O}_{X^\diamond_{\text{qp}}}} \mathcal{O}_{X^\diamond} : \mathrm{Vect}(X^\diamond_{\text{qp}}; \mathcal{O}_{X^\diamond_{\text{qp}}}) \to \mathrm{Vect}(X^\diamond_v; \mathcal{O}_{X^\diamond})
\end{aligned}$$

the natural pullback functors.

Theorem 6.8.13 ([42, Theorem 3.5.8], [63, Lemma 17.1.8], [35, Theorem 4.27]). *Let X be a pre-adic space over* $\mathrm{Spa}\,(\mathbf{Q}_p, \mathbf{Z}_p)$.

(1) *If X is strongly sheafy, then $\pi^*\colon \mathrm{Vect}(X_{\text{an}}, \mathcal{O}_X) \to \mathrm{Vect}(X_{\text{ét}}, \mathcal{O}_{X_{\text{ét}}})$ is an equivalence. Moreover, the natural morphism*

$$\mathcal{L} \to \mathbf{R}\pi_*\pi^*\mathcal{L}$$

is an isomorphism for any $\mathcal{O}_X$-vector bundle $\mathcal{L}$. Further, if $X = \mathrm{Spa}\,(A, A^+)$ for a strongly sheafy Tate ring A, then $\mathrm{Vect}(X_{\text{an}}, \mathcal{O}_X)$ is equivalent to the category of finitely generated projective R-modules.

(2) *If X is perfectoid, then $\mu^*\colon \mathrm{Vect}(X_{\text{ét}}; \mathcal{O}_{X_{\text{ét}}}) \to \mathrm{Vect}(X_{\text{qp}}; \mathcal{O}_{X^\diamond_{\text{qp}}})$ is an equivalence. Furthermore, the natural morphism*

$$\mathcal{E} \to \mathbf{R}\mu_*\mu^*\mathcal{E}$$

is an isomorphism for any $\mathcal{O}_{X_{\text{ét}}}$-vector bundle $\mathcal{E}$.

(3) *The functor $\lambda^*\colon \mathrm{Vect}\big(X^\diamond_{\text{qp}}; \mathcal{O}_{X^\diamond_{\text{qp}}}\big) \to \mathrm{Vect}\big(X^\diamond_v; \mathcal{O}_{X^\diamond}\big)$ is an equivalence. Furthermore, the natural morphism*

$$\mathcal{V} \to \mathbf{R}\lambda_*\lambda^*\mathcal{V}$$

is an equivalence for any $\mathcal{O}_{X^\diamond_{\text{qp}}}$-vector bundle $\mathcal{V}$.

Proof. (1) follows from [41, Theorem 8.2.22 (c), (d)].

Part (3) is quasi-proétale local on X, so we can assume that X is an affinoid perfectoid for the purpose of proving (2) and (3).

Then it follows from Theorem 6.8.4 that the natural maps $\mathcal{O}^+_{X_{\text{ét}}} \to \mathbf{R}\mu_*\mathcal{O}_{X^\diamond_{\text{qp}}}$ and $\mathcal{O}_{X^\diamond_{\text{qp}}} \to \mathbf{R}\lambda_*\mathcal{O}_{X^\diamond}$ are isomorphisms. Then this formally implies that the maps $\mathcal{E} \to \mathbf{R}\mu_*\mu^*\mathcal{E}$ and $\mathcal{V} \to \mathbf{R}\lambda_*\lambda^*\mathcal{V}$ are isomorphisms. This, in turn, formally implies that μ^* and λ^* are fully faithful. In order to show essential surjectivity, it suffices to show that any $\mathcal{O}_{X^\diamond}$-vector bundle can be trivialized étale locally on X (for a perfectoid space X). This follows from the combination of Corollary 6.8.12 and Corollary 6.8.3. ∎

Finally, we give a more refined version of Corollary 6.8.12:

Corollary 6.8.14. *Let X be a strongly sheafy adic space over* $\mathrm{Spa}\,(\mathbf{Q}_p, \mathbf{Z}_p)$*, and let $\mathcal{E}$ be an $\mathcal{O}_{X^\diamond}$-vector bundle. Then, for each $x \in X$, there are an open subspace $x \in U_x \subset X$, a finite étale surjective morphism $\widetilde{U}_x \to U_x$, and an $\mathcal{O}^+_{\widetilde{U}_x^\diamond}$-vector bundle $\mathcal{E}^+_x$ such that $\mathcal{E}^+_x\left[\frac{1}{p}\right] \simeq \mathcal{E}|_{\widetilde{U}_x}$.*

Proof. Using Corollary 6.8.11 in place of Lemma 6.5.5, we can repeat the argument of Theorem 6.6.8 once we know that $\mathcal{E}|_{\mathrm{Spd}\,(C,C^+)}$ admits a lattice for any morphism $\mathrm{Spa}\,(C, C^+) \to X$ such that C is an algebraically closed non-archimedean field (and any open, integrally closed, bounded subring $C^+ \subset C$).[17] For this, we note that (C, C^+) is a perfectoid pair, so Theorem 6.8.13 implies that the category of $\mathcal{O}_{\mathrm{Spd}\,(C,C^+)}$-vector bundles is equivalent to the category of finite-dimensional C-vector spaces. In particular, any $\mathcal{E}|_{\mathrm{Spd}\,(C,C^+)}$ is a free bundle, so it clearly admits an $\mathcal{O}^+_{X^\diamond}$-lattice. This finishes the proof. ∎

[17]The proof of Theorem 6.6.8 ensures that it suffices to prove this claim for a very restrictive class of such pairs (C, C^+), but this is irrelevant for the current proof.