Chapter 7

Almost coherence of “p-adic nearby cycles”

7.1 Introduction

The main goal of this chapter is to study the “p-adic nearby cycles” sheaves $\mathbf{R}\nu_*\mathcal{O}^+_{X^\diamond}$ and $\mathbf{R}\nu_*\mathcal{O}^+_{X^\diamond}/p$ for a rigid-analytic variety X. We also study other versions with more general “coefficients” including $\mathcal{O}^+/p$-vector bundles in the v-topology, and sheaves of the form $\mathcal{O}^+_{X^\diamond}/p \otimes \mathcal{F}$ for a Zariski-constructible sheaf $\mathcal{F}$ (see Definition 7.1.7). These complexes turn out to be almost coherent; this makes it possible to study étale cohomology groups of rigid-analytic varieties using (almost) coherent methods on the special fiber.

Before giving precise definitions, let us explain the main motivation to study these sheaves and their relation with étale cohomology of rigid-analytic varieties in the simplest case of the “nearby cycles” of the sheaf $\mathcal{O}^+_{X_{\text{ét}}}/p$. In [59], P. Scholze proved ([59, Theorem 5.1]) that the étale cohomology groups $\mathrm{H}^i(X, \mathbf{F}_p)$ are finite for any smooth, proper rigid-analytic variety X over an algebraically closed p-adic non-archimedean field C. There are two important ingredients: the almost primitive comparison theorem that says that $\mathrm{H}^i(X, \mathcal{O}^+_{X_{\text{ét}}}/p)$ are *almost* isomorphic to $\mathrm{H}^i(X, \mathbf{F}_p) \otimes \mathcal{O}_C/p$, and the almost finiteness of $\mathrm{H}^i(X, \mathcal{O}^+_{X_{\text{ét}}}/p)$.

The proof of the almost finiteness result in [59] uses properness of X in a very elaborate way; first, the proof constructs some specific “good covering” of X by affinoids and then shows that there is enough cancelation in the Čech-to-derived spectral sequence associated with that covering. We note that all terms of the second page of this spectral sequence are not almost finitely generated, but mysteriously there is enough cancelation so that the terms of the ∞-page become almost finitely generated. We refer to [59, Section 5] for details.

Our main goal is to give a more geometric and conceptual way to prove this almost finiteness result. Instead of constructing an explicit “nice” covering of X, we separate the problem into two different problems. We choose an admissible formal $\mathcal{O}_C$-model $\mathfrak{X}$ of X and consider the associated morphism of ringed topoi

$$t\colon \left(X_{\text{ét}}, \mathcal{O}^+_{X_{\text{ét}}}\right) \to \left(\mathfrak{X}_{\text{Zar}}, \mathcal{O}_{\mathfrak{X}}\right)$$

that induces the morphism

$$t\colon \left(X_{\text{ét}}, \mathcal{O}^+_{X_{\text{ét}}}/p\right) \to \left(\mathfrak{X}_{\text{Zar}}, \mathcal{O}_{\mathfrak{X}}/p\right) = \left(\mathfrak{X}_0, \mathcal{O}_{\mathfrak{X}_0}\right),$$

where $\mathfrak{X}_0 := \mathfrak{X} \times_{\operatorname{Spf} \mathcal{O}_C} \operatorname{Spec} \mathcal{O}_C/p$ is the mod-p fiber of $\mathfrak{X}$. Then one can write

$$\mathbf{R}\Gamma\left(X, \mathcal{O}^+_{X_{\text{ét}}}/p\right) \simeq \mathbf{R}\Gamma\left(\mathfrak{X}_0, \mathbf{R}t_*\mathcal{O}^+_{X_{\text{ét}}}/p\right),$$

so one can separately study the “nearby cycles” complex $\mathbf{R}t_*\mathcal{O}^+_{X_{\text{ét}}}/p$ and its derived global sections on $\mathfrak{X}_0$.

The key is that now $\mathfrak{X}$ is proper over Spf $\mathcal{O}_K$ by [51, Lemma 2.6][1] (or [65, Corollary 4.4 and 4.5]). Thus, the almost proper mapping theorem (see Theorem 5.1.3) tells us that, to prove the almost finiteness of $\mathbf{R}\Gamma(X, \mathcal{O}^+_{X_{\text{ét}}}/p)$, it is sufficient only to show that $\mathbf{R}t_*\mathcal{O}^+_{X_{\text{ét}}}/p \in \mathbf{D}^+_{\text{acoh}}(X)$ has almost coherent cohomology sheaves.

The main advantage now is that we can study the “nearby cycles” $\mathbf{R}t_*\mathcal{O}^+_{X_{\text{ét}}}/p$ *locally* on the formal model $\mathfrak{X}$. So this holds for any admissible formal model and not only for proper ones. Moreover, the only place where we use properness of X in our proof is to get properness of the formal model $\mathfrak{X}$ to be able to apply the almost proper mapping theorem (see Theorem 5.1.3). This allows us to avoid all elaborate spectral sequence arguments while at the same time making the essential part of the proof local on $\mathfrak{X}$.

Now we discuss how we prove that $\mathbf{R}t_*\mathcal{O}^+_{X_{\text{ét}}}/p$ is almost coherent. In fact, we will prove a much stronger result that $\mathbf{R}t_*\mathcal{E}$ is almost coherent for any $\mathcal{O}^+/p$-vector bundle $\mathcal{E}$ in the v-topology. However, we find it instructive to discuss the simplest case first.

When $\mathcal{E} = \mathcal{O}^+_{X_{\text{ét}}}/p$, the main idea of the proof is similar to the idea behind the proof [59, Lemma 5.6]: we reduce the general case to the case of an affine $\mathfrak{X}$ with “nice” coordinates, where everything can be reduced to almost coherence of certain continuous group cohomology via perfectoid techniques. In order to make this work, we have to pass to a finer topology that allows towers of finite étale morphisms. There are different possible choices, but we find the v-topology on the associated diamond $X^\diamondsuit$ of X (in the sense of [61]) to be the most convenient for our purposes (see Chapter 6 for the detailed discussion).

The case of a general $\mathcal{O}^+_{X^\diamondsuit}/p$-vector bundle (see Definition 6.5.1) will cause us more trouble; we will use the structure results from Section 6.6 to handle a general $\mathcal{O}^+_{X^\diamondsuit}/p$-vector bundle. The main crucial input that we are going to use is that the category of $\mathcal{O}^+_{X^\diamondsuit}/p$-vector bundles is equivalent to the category of étale $\mathcal{O}^+_{X_{\text{ét}}}/p$-vector bundles and that, locally, any $\mathcal{O}^+_{X^\diamondsuit}/p$-vector bundle can be trivialized by some very particular étale covering (see Corollary 6.6.9).

That being said, we can move to the formulation of the main theorem of this section. We refer to Chapter 6 for the definition of the quasi-proétale and v-topologies on $X^\diamondsuit$ for a rigid-analytic variety over a non-archimedean field K. These sites come with their “integral” structure sheaves $\mathcal{O}^+_{X^\diamondsuit}$, $\mathcal{O}^+_{X^\diamondsuit_{\text{qp}}}$, and $\mathcal{O}^+_{X_{\text{ét}}}$ (see Definition 6.3.1) and a diagram of morphisms of ringed sites (see Diagram (6.3.1) and (6.3.2)):

$$\begin{tikzcd} (X^\diamondsuit_v, \mathcal{O}^+_{X^\diamondsuit}) \arrow[r, "\lambda"] \arrow[rrr, bend right, "\nu"'] & (X^\diamondsuit_{\text{qproét}}, \mathcal{O}^+_{X^\diamondsuit_{\text{qp}}}) \arrow[r, "\mu"] & (X_{\text{ét}}, \mathcal{O}^+_{X_{\text{ét}}}) \arrow[r, "t"] & (\mathfrak{X}_{\text{Zar}}, \mathcal{O}_{\mathfrak{X}}) \end{tikzcd} \qquad (7.1.1)$$

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[1] Strictly speaking, his proof is written under the assumption that $\mathcal{O}_K$ is discretely valued. However, it can be easily generalized to the of a general rank-1 complete valuation ring $\mathcal{O}_K$.

and the mod-p version

$$\begin{tikzcd}
(X_v^\diamond, \mathcal{O}^+_{X^\diamond}/p) \arrow[r, "\lambda"] \arrow[rrr, bend right, "\nu"'] & (X^\diamond_{\text{qproét}}, \mathcal{O}^+_{X^\diamond_{\text{qp}}}/p) \arrow[r, "\mu"] & (X_{\text{ét}}, \mathcal{O}^+_{X_{\text{ét}}}/p) \arrow[r, "t"] & (\mathfrak{X}_{\text{Zar}}, \mathcal{O}_{\mathfrak{X}_0}).
\end{tikzcd}$$

(7.1.2)

If there is any ambiguity in the meaning of ν, we then denote it by $\nu_{\mathfrak{X}}$ to explicitly specify the formal model for these functors.

Recall that for a perfectoid field K, Lemma B.12 ensures that the maximal ideal $\mathfrak{m} \subset \mathcal{O}_K$ is an ideal of almost mathematics with flat $\widetilde{\mathfrak{m}} \simeq \mathfrak{m}^2 = \mathfrak{m}$. For the rest of this section, we fix a p-adic perfectoid field K, and always do almost mathematics with respect to the ideal $\mathfrak{m}$.

We are ready to formulate our first main result. We thank B. Heuer for the suggestion of trying to prove Theorem 7.1.2 for all $\mathcal{O}^+_{X_\diamond}/p$-vector bundles.

Definition 7.1.1. An $\mathcal{O}^+_{X^\diamond}/p$-module $\mathcal{E}$ is a *small $\mathcal{O}^+_{X^\diamond}/p$-vector bundle* if there is a finite étale surjective morphism $V \to U$ such that $\mathcal{E}|_{V_v^\diamond} \simeq (\mathcal{O}^+_{V^\diamond}/p)^r$ for some integer r.

Theorem 7.1.2. *Let $\mathfrak{X}$ be an admissible formal $\mathcal{O}_K$-scheme with adic generic fiber X of dimension d and mod-p fiber $\mathfrak{X}_0$, and let $\mathcal{E}$ be an $\mathcal{O}^+_{X^\diamond}/p$-vector bundle. Then*

(1) $\mathbf{R}\nu_*\mathcal{E} \in \mathbf{D}^+_{\text{qc,acoh}}(\mathfrak{X}_0)$ *and* $(\mathbf{R}\nu_*\mathcal{E})^a \in \mathbf{D}^{[0,2d]}_{\text{acoh}}(\mathfrak{X}_0)^a$*;*

(2) *if* $\mathfrak{X} = \operatorname{Spf} A$ *is affine, then the natural map*

$$\mathrm{H}^i\widetilde{(X_v^\diamond, \mathcal{E})} \to \mathrm{R}^i\nu_*(\mathcal{E})$$

is an isomorphism for every $i \geq 0$;

(3) *the formation of* $\mathrm{R}^i\nu_*(\mathcal{E})$ *commutes with étale base change, i.e., for any étale morphism $\mathfrak{f}\colon \mathfrak{Y} \to \mathfrak{X}$ with adic generic fiber $f\colon Y \to X$, the natural morphism*

$$\mathfrak{f}_0^*\big(\mathrm{R}^i\nu_{\mathfrak{X},*}(\mathcal{E})\big) \to \mathrm{R}^i\nu_{\mathfrak{Y},*}\big(\mathcal{E}|_{Y^\diamond}\big)$$

is an isomorphism for any $i \geq 0$;

(4) *if $\mathfrak{X}$ has an open affine covering $\mathfrak{X} = \bigcup_{i\in I}\mathfrak{U}_i$ such that $\mathcal{E}|_{(\mathfrak{U}_{i,K})^\diamond}$ is small, then*

$$(\mathbf{R}\nu_*\mathcal{E})^a \in \mathbf{D}^{[0,d]}_{\text{acoh}}(\mathfrak{X}_0)^a;$$

(5) *there is an admissible blow-up $\mathfrak{X}' \to \mathfrak{X}$ such that $\mathfrak{X}'$ has an open affine covering $\mathfrak{X}' = \bigcup_{i\in I}\mathfrak{U}_i$ such that $\mathcal{E}|_{(\mathfrak{U}_{i,K})^\diamond}$ is small.*
In particular, there is a cofinal family of admissible formal models $\{\mathfrak{X}'_i\}_{i\in I}$ of X such that

$$\big(\mathbf{R}\nu_{\mathfrak{X}'_i,*}\mathcal{E}\big)^a \in \mathbf{D}^{[0,d]}_{\text{acoh}}(\mathfrak{X}'_{i,0})^a,$$

for each $i \in I$.

Remark 7.1.3. We refer to Definition 4.4.1 and Definition 4.4.2 for the precise definition of all derived categories appearing in Theorem 7.1.2. In order to avoid any confusion, we explicate that the expression $(\mathbf{R}\nu_*\mathcal{E})^a \in \mathbf{D}^{[0,d]}_{\mathrm{acoh}}(\mathfrak{X}_0)^a$ means that the complex $(\mathbf{R}\nu_*\mathcal{E})$ is almost concentrated in degree $[0, d]$ and each of its cohomology sheaves is almost coherent.

Remark 7.1.4. We note that Theorem 7.1.2 (1) implies that the nearby cycles $\mathbf{R}\nu_*\mathcal{E}$ is quasi-coherent on the nose (as opposed to being almost quasi-coherent). This is quite unexpected to the author since all previous results on the cohomology groups of $\mathcal{O}^+/p$ were only available in the almost category.

Remark 7.1.5. If $K = C$ is algebraically closed, the proof gives a non-almost version of cohomological bound. Namely, we see that

$$\mathbf{R}\nu_*\mathcal{E} \in \mathbf{D}^{[0,2d]}_{\mathrm{acoh}}(\mathfrak{X}_0).$$

However, we do not know if $\mathbf{R}\nu_*\mathcal{E}$ is concentrated in degrees $[0, d]$ on the nose (for a cofinal family of formal models).

Remark 7.1.6. Ofer Gabber has informed the author that he knows an example of a smooth affinoid rigid-analytic variety X, a formal model $\mathfrak{X}$, and an $\mathcal{O}^+_{X^\diamond}/p$-vector bundle $\mathcal{E}$ such that $\mathbf{R}\nu_{\mathfrak{X},*}\mathcal{E}$ is *not* almost concentrated in degrees $[0, d]$.

One can prove a slightly more precise version in case $\mathcal{E}$ is equal to the tensor product of a Zariski-constructible étale sheaf of $\mathbf{F}_p$-modules and $\mathcal{O}^+_{X^\diamond}/p$.

Definition 7.1.7 ([32]). An étale sheaf $\mathcal{F}$ of $\mathbf{F}_p$-modules is a *local system* if it is a locally constant sheaf with finite stalks.

An étale sheaf $\mathcal{F}$ of $\mathbf{F}_p$-modules is *Zariski-constructible* if there is a locally finite stratification $X = \bigsqcup_{i\in I} Z_i$ into Zariski locally closed subspaces Z_i such that $\mathcal{F}|_{Z_i}$ is a local system.

The category $\mathbf{D}_{zc}(X; \mathbf{F}_p)$ is a full subcategory of $\mathbf{D}(X_{\mathrm{ét}}; \mathbf{F}_p)$ consisting of objects with Zariski-constructible cohomology sheaves.

Remark 7.1.8. Any Zariski-constructible sheaf $\mathcal{F}$ is overconvergent, i.e., for any morphism $\overline{\eta} \to \overline{s}$ of geometric points in $X_{\mathrm{ét}}$, the specialization map $\mathcal{F}_{\overline{s}} \to \mathcal{F}_{\overline{\eta}}$ is an isomorphism.

Note that any sheaf of $\mathbf{F}_p$-modules on $X_{\mathrm{ét}}$ can be treated as a sheaf on any of the sites $X^\diamond_v$, $X^\diamond_{\mathrm{qproét}}$, or $X_{\mathrm{proét}}$ via the pullback functors along the morphisms in Diagram (7.1.1). In what follows, we abuse the notation and implicitly treat a sheaf $\mathcal{F}$ as a sheaf on any of those sites. We also denote the tensor product $\mathcal{F} \otimes_{\mathbf{F}_p} \mathcal{O}^+_X/p$ simply by $\mathcal{F} \otimes \mathcal{O}^+_X/p$ in what follows.

Now we discuss an integral version of Theorem 7.1.2:

Theorem 7.1.9. *Let $\mathfrak{X}$ be an admissible formal $\mathcal{O}_K$-scheme with adic generic fiber X of dimension d and mod-p fiber $\mathfrak{X}_0$, and $\mathcal{F} \in \mathbf{D}^{[r,s]}_{zc}(X; \mathbf{F}_p)$. Then*

1. *there is an isomorphism $\mathbf{R}t_*\big(\mathcal{F} \otimes \mathcal{O}^+_{X_{\text{ét}}}/p\big) \simeq \mathbf{R}\nu_*\big(\mathcal{F} \otimes \mathcal{O}^+_{X^\diamond}/p\big)$;*
2. *$\mathbf{R}\nu_*\big(\mathcal{F} \otimes \mathcal{O}^+_{X^\diamond}/p\big) \in \mathbf{D}^+_{\text{qc,acoh}}(\mathfrak{X}_0)$, and $\mathbf{R}\nu_*\big(\mathcal{F} \otimes \mathcal{O}^+_{X^\diamond}/p\big)^a \in \mathbf{D}^{[r,s+d]}_{\text{acoh}}(\mathfrak{X}_0)^a$;*
3. *if $\mathfrak{X} = \operatorname{Spf} A$ is affine, then the natural map*
$$\mathrm{H}^i\big(X^\diamond_v, \widehat{\mathcal{F} \otimes \mathcal{O}^+_{X^\diamond}/p}\big) \to \mathrm{R}^i\nu_*\big(\mathcal{F} \otimes \mathcal{O}^+_{X^\diamond}/p\big)$$
is an isomorphism for every $i \geq 0$;
4. *the formation of $\mathrm{R}^i\nu_*\big(\mathcal{F} \otimes \mathcal{O}^+_{X^\diamond}/p\big)$ commutes with étale base change, i.e., for any étale morphism $\mathfrak{f}\colon \mathfrak{Y} \to \mathfrak{X}$ with adic generic fiber $f\colon Y \to X$, the natural morphism*
$$\mathfrak{f}_0^*\big(\mathrm{R}^i\nu_{\mathfrak{X},*}\big(\mathcal{F} \otimes \mathcal{O}^+_{X^\diamond}/p\big)\big) \to \mathrm{R}^i\nu_{\mathfrak{Y},*}\big(f^{-1}\mathcal{F} \otimes \mathcal{O}^+_{Y^\diamond}/p\big)$$
is an isomorphism for any $i \geq 0$.

Definition 7.1.10. An $\mathcal{O}^+_{X^\diamond}$-vector bundle $\mathcal{E}$ is a *small $\mathcal{O}^+_{X^\diamond}$-vector bundle* if $\mathcal{E}/p\mathcal{E}$ is a small $\mathcal{O}^+_{X^\diamond}/p$-vector bundle (see Definition 7.1.1).

Theorem 7.1.11. *Let $\mathfrak{X}$ be an admissible formal $\mathcal{O}_K$-scheme with adic generic fiber X of dimension d, and let $\mathcal{E}$ be an $\mathcal{O}^+_{X^\diamond}$-vector bundle. Then*

1. *$\mathbf{R}\nu_*\mathcal{E} \in \mathbf{D}^+_{\text{qc,acoh}}(\mathfrak{X})$ and $(\mathbf{R}\nu_*\mathcal{E})^a \in \mathbf{D}^{[0,2d]}_{\text{acoh}}(\mathfrak{X})^a$;*
2. *if $\mathfrak{X} = \operatorname{Spf} A$ is affine, then the natural map*
$$\mathrm{H}^i\big(X^\diamond_v, \mathcal{E}\big)^\Delta \to \mathrm{R}^i\nu_*(\mathcal{E})$$
is an isomorphism for every $i \geq 0$;
3. *the formation of $\mathrm{R}^i\nu_*(\mathcal{E})$ commutes with étale base change, i.e., for any étale morphism $\mathfrak{f}\colon \mathfrak{Y} \to \mathfrak{X}$ with adic generic fiber $f\colon Y \to X$, the natural morphism*
$$\mathfrak{f}^*\big(\mathrm{R}^i\nu_{\mathfrak{X},*}(\mathcal{E})\big) \to \mathrm{R}^i\nu_{\mathfrak{Y},*}(\mathcal{E}|_{Y^\diamond})$$
is an isomorphism for any $i \geq 0$;
4. *if $\mathfrak{X}$ has an open affine covering $\mathfrak{X} = \bigcup_{i\in I} \mathfrak{U}_i$ such that $\mathcal{E}|_{(\mathfrak{U}_{i,K})^\diamond}$ is small, then*
$$(\mathbf{R}\nu_*\mathcal{E})^a \in \mathbf{D}^{[0,d]}_{\text{acoh}}(\mathfrak{X})^a;$$
5. *there is an admissible blow-up $\mathfrak{X}' \to \mathfrak{X}$ such that $\mathfrak{X}'$ has an open affine covering $\mathfrak{X}' = \bigcup_{i\in I} \mathfrak{U}_i$ such that $\mathcal{E}|_{(\mathfrak{U}_{i,K})^\diamond}$ is small.*
In particular, there is a cofinal family of admissible formal models $\{\mathfrak{X}'_i\}_{i\in I}$ of X such that
$$(\mathbf{R}\nu_{\mathfrak{X}'_i,*}\mathcal{E})^a \in \mathbf{D}^{[0,d]}_{\text{acoh}}(\mathfrak{X}'_i)^a,$$
for each $i \in I$.

Remark 7.1.12. We refer to Definition 4.8.9 for the precise definition of all derived categories appearing in Theorem 7.1.11.

Remark 7.1.13. One can also prove a version of Theorem 7.1.11 for Zariski-constructible $\mathbf{Z}_p$-sheaves in the sense of [7, Definition 3.32]. However, we prefer not to do this here as it does not require new ideas but instead complicates the notation.

For the version of Theorem 7.1.11 with the pro-étale site $X_{\text{proét}}$ as defined in [59] and [60], see Theorem 7.13.6.

The rest of the memoir is devoted to proving Theorem 7.1.9, Theorem 7.1.2, and Theorem 7.1.11 and discussing their applications. We have decided to work entirely in the v-site of $X^\diamond$ because it is quite flexible for different types of arguments (e.g., proper descent, torsors under pro-finite groups, etc.). However, most of the argument can be done using the more classical pro-étale site defined in [59]. However, it is crucial to use the theory of diamonds to get an almost cohomological bound on $\mathbf{R}\nu_*\mathcal{E}$ for non-smooth X, and it also seems difficult to justify that the sheaves $\mathrm{R}^i\nu_*\mathcal{E}$ are quasi-coherent without using (at least) quasi-proétale topology.

7.2 Digression: Geometric points

In this section, we discuss preliminary results that will be used both in the proof of Theorem 7.1.9 and in deriving applications from it.

We start the section by recalling some definitions.

Definition 7.2.1. [67, Section 2.1.4] An extension of non-archimedean fields[2] $K \subset L$ is *topologically algebraic* if the algebraic closure of K in L is dense in L. Equivalently, $K \subset L$ is topologically algebraic if L is a non-archimedean subfield of $\widehat{\overline{K}}$.

Lemma 7.2.2. (1) *Let $K \subset L$ and $L \subset M$ be two topologically algebraic extensions of non-archimedean fields. Then $K \subset M$ is also topologically algebraic.*

(2) *Let*

$$\begin{tikzcd} N & L \arrow[l] \\ M \arrow[u] & K \arrow[l] \arrow[u] \end{tikzcd}$$

be a commutative diagram of non-archimedean fields such that LM is dense in N and $K \subset L$ is topologically algebraic. Then $M \subset N$ is also a topologically algebraic extension.

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[2] Recall that non-archimedean fields are complete by our convention.

Proof. (1) We know that $L \subset \widehat{\overline{K}}$ and $M \subset \widehat{\overline{L}}$ since both extensions are topologically algebraic. Since $\widehat{\overline{L}}$ is already algebraically closed, we conclude that $M \subset \widehat{\overline{L}} \subset \widehat{\overline{K}}$.

(2) First, we note that

$$LM \subset \widehat{\overline{K}}M \subset \widehat{\overline{K}M} \subset \widehat{\overline{M}},$$

where the composites are taken inside $\widehat{\overline{N}}$. Then we note that $LM \subset N$ is dense, so the inclusion $LM \subset \widehat{\overline{M}}$ uniquely extends to an inclusion $N \subset \widehat{\overline{M}}$. This implies that $M \subset N$ is topologically algebraic. ∎

Definition 7.2.3. A *geometric point* above the point $x \in X$ of an analytic adic space X is a morphism $x\colon \operatorname{Spa}\big(C(x), C(x)^+\big) \to X$ such that $C(x)$ is an algebraically closed non-archimedean field, $C(x)^+$ is an open and bounded valuation subring of $C(x)$, and the corresponding extension of completed residue fields $\widehat{k(x)} \subset C(x)$ is a topologically algebraic extension.

Remark 7.2.4. If $\operatorname{Spa}\big(C(x), C(x)^+\big) \to X$ is a geometric point, then $C(x)$ can be identified with the completed algebraic closure of $\widehat{k(x)}$ (or, equivalently, of $k(x)$) and $C(x)^+$ with a valuation ring extending $\widehat{k(x)}^+$ (or, equivalently, $k(x)^+$). Therefore, Definition 7.2.3 is more restrictive than [38, Definition 2.5.1], but coincides with the subclass of geometric points constructed in [38, eq. (2.5.2)].

Lemma 7.2.5. *Let K be a non-archimedean field with an open and bounded valuation subring $K^+ \subset K$ and a pseudo-uniformizer ϖ. Let $f\colon X \to Y$ be a morphism of locally of finite type (K, K^+)-adic spaces, and $\overline{y}\colon \operatorname{Spa}\big(C(y), C(y)^+\big) \to Y$ be a geometric point above $y \in Y$. Then the natural morphism*

$$a\colon i^{-1}(\mathcal{O}^+_{X_{\text{ét}}}/\varpi) \to \mathcal{O}^+_{X_{\overline{y},\text{ét}}}/\varpi$$

is an isomorphism, where $i\colon X_{\overline{y}} \to X$ is the "projection" of the geometric fiber $X_{\overline{y}} := X \times_Y \operatorname{Spa}\big(C(y), C(y)^+\big)$ back to X.

Proof. [38, Proposition 2.5.5] ensures that it suffices to show that a is an isomorphism on stalks at geometric points of $X_{\overline{y}}$. Now note that Lemma 7.2.2 implies that any geometric point $\overline{x}\colon \operatorname{Spa}\big(C(x), C(x)^+\big) \to X_{\overline{y}}$ defines a geometric point $\overline{x}'\colon \operatorname{Spa}\big(C(x), C(x)^+\big) \to X$ of X by taking the composition of $\overline{x}$ with i. So it is enough to show that the natural map

$$\big(\mathcal{O}^+_{X_{\text{ét}}}/\varpi\big)_{\overline{x}'} \simeq \big(i^{-1}(\mathcal{O}^+_{X_{\text{ét}}}/\varpi)\big)_{\overline{x}} \to \big(\mathcal{O}^+_{X_{\overline{y},\text{ét}}}/\varpi\big)_{\overline{x}} \tag{7.2.1}$$

is an isomorphism. But [38, Proposition 2.6.1] naturally identifies both sides of (7.2.1) with $C(x)^+/\varpi C(x)^+$ finishing the proof. ∎

Remark 7.2.6. Lemma 7.2.5 is very specific to the adic geometry (and quite counter-intuitive from the algebraic point of view). Its scheme-theoretic version with $\mathcal{O}^+/\varpi$

replaced by $\mathcal{O}$ is false. The main feature of analytic adic geometry (implicitly) used in the proof is that the morphism $\mathcal{O}^+_{X,x} \to k(x)^+$ becomes an isomorphism after the ϖ-adic completion.

Lemma 7.2.7. *Let C be an algebraically closed non-archimedean field, let $C^+ \subset C$ be an open and bounded valuation subring with a pseudo-uniformizer $\varpi \in C^+$, and let (C, C^+) be the corresponding Huber pair. Let $(C, C^+) \to (D, D^+)$ be a finite morphism of complete Huber pairs with a local ring D. Then the natural morphism*

$$C^+/\varpi C^+ \to D^+/\varpi D^+$$

is an isomorphism.

Proof. First, we show that $C^+/\varpi C^+ \to D^+/\varpi D^+$ is injective. For that, suppose that $\overline{c} \in C^+/\varpi C^+$ is an element in the kernel and lift it to $c \in C^+$. The assumption on c implies that $c = \varpi d$ for some $d \in D^+$. Then $d = c/\varpi \in C \cap D^+ = C^+$. Therefore, $\overline{c} = 0$ in $C^+/\varpi C^+$.

Now we check surjectivity. Since D is a local ring that is finite over an algebraically closed field C, we conclude that D is an Artin local ring and $D/\mathrm{nil}(D) \simeq C$. Therefore, for every $d \in D^+$, we can find $c \in C$ and $d' \in \mathrm{nil}(D)$ such that $d = c + d'$. Since $\mathrm{nil}(D) \subset D^{\circ\circ} \subset D^+$, we conclude that $c = d - d' \in D^+ \cap C = C^+$. Now note that d'/ϖ is still a nilpotent element of D, thus $d'/\varpi \in \mathrm{nil}(D) \subset D^+$. So we conclude that

$$d = c + \varpi(d'/\varpi)$$

proving that $C^+/\varpi C^+ \to D^+/\varpi D^+$ is surjective. ■

Corollary 7.2.8. *Let K be a p-adic non-archimedean field, and K^+ an open and bounded valuation subring of K. Let $f\colon X \to Y$ be a finite morphism of locally finite type (K, K^+)-adic spaces. Then the natural morphism*

$$c\colon \big(f_* \underline{\mathbf{F}}_p\big) \otimes \mathcal{O}^+_{Y_{\text{ét}}}/p \to f_*\big(\mathcal{O}^+_{X_{\text{ét}}}/p\big)$$

is an isomorphim on $Y_{\text{ét}}$.

Proof. We use [38, Proposition 2.5.5] to ensure that it suffices to show that c is an isomorphism on stalks at geometric points. Thus, [38, Proposition 2.6.1] and Lemma 7.2.5 imply that it suffices to show that the natural map

$$\mathrm{H}^0_{\text{ét}}\big(X, \mathbf{F}_p\big) \otimes C^+/p \to \mathrm{H}^0_{\text{ét}}\big(X, \mathcal{O}^+_{X_{\text{ét}}}/p\big)$$

is an isomorphism when $Y = \mathrm{Spa}\,(C, C^+)$ for an algebraically closed p-adic non-archimedean field C and an open and bounded valuation subring $C^+ \subset C$. In this case, $X = \mathrm{Spa}\,(D, D^+)$ for some finite morphism of Huber pairs $(C, C^+) \to (D, D^+)$. In particular, D is a finite C-algebra, so it is a finite direct product of local artinian

C-algebras. By passing to a direct factor of D (or, geometrically, to a connected component of Spa (D, D^+)), we can assume that D is local. In particular, D does not have any idempotents, and therefore Spa (D, D^+) is connected. In this case, we have

$$\mathrm{H}^0_{\text{ét}}(X, \mathbf{F}_p) \otimes C^+/pC^+ \simeq C^+/pC^+,$$

since $\mathrm{H}^0_{\text{ét}}(X, \mathbf{F}_p) \simeq \mathbf{F}_p$ since Spa (D, D^+) is connected.

Now we observe that Spa $(D, D^+)_{\mathrm{red}} \simeq$ Spa (C, C^+), so all étale sheaves on Spa (D, D^+) do not have higher cohomology groups. Thus, we have

$$\mathrm{H}^0_{\text{ét}}\big(X, \mathcal{O}^+_{X_{\text{ét}}}/p\big) \simeq D^+/pD^+.$$

In particular, the question boils down to showing that the natural map

$$C^+/pC^+ \to D^+/pD^+$$

is an isomorphism. This was already done in Lemma 7.2.7. ■

Corollary 7.2.9. *Let K be a p-adic non-archimedean field, $f : X \to Y$ a finite morphism of rigid-analytic varieties over K, and $\mathcal{F} \in \mathbf{D}^b_{zc}(X; \mathbf{F}_p)$. Then the natural morphism*

$$c : (f_*\mathcal{F}) \otimes \mathcal{O}^+_{Y_{\text{ét}}}/p \to f_*(\mathcal{F} \otimes \mathcal{O}^+_{X_{\text{ét}}}/p)$$

is an isomorphim on $Y_{\text{ét}}$.

Proof. We recall that [7, Proposition 3.6] says that $\mathbf{D}^b_{zc}(X; \mathbf{F}_p)$ is a thick triangulated subcategory of $\mathbf{D}(X_{\text{ét}}; \mathbf{F}_p)$ generated by objects of the form $g_*\underline{\mathbf{F}}_p$ for finite morphisms $g : X' \to X$. Since both claims in the question satisfy the 2-out-of-3 property and are preserved by passing to direct summands, it suffices to prove the claim only for $\mathcal{F} = g_*\underline{\mathbf{F}}_p$. In this situation, the claim follows from Corollary 7.2.8 by the sequence of isomorphisms

$$\begin{aligned} f_*\big(g_*(\underline{\mathbf{F}}_p)\big) \otimes \mathcal{O}^+_{Y_{\text{ét}}}/p &\simeq (f \circ g)_*(\underline{\mathbf{F}}_p) \otimes \mathcal{O}^+_{Y_{\text{ét}}}/p \\ &\simeq (f \circ g)_*\big(\mathcal{O}^+_{X'_{\text{ét}}}/p\big) \\ &\simeq f_*\big(g_*\mathcal{O}^+_{X'_{\text{ét}}}/p\big) \\ &\simeq f_*\big(g_*\underline{\mathbf{F}}_p \otimes \mathcal{O}^+_{X_{\text{ét}}}/p\big). \end{aligned}$$ ■

7.3 Applications

The main goal of this section is to discuss some applications of Theorem 7.1.9. In particular, we show that “p-adic nearby cycles” commute with proper pushfowards and prove finiteness of the usual étale cohomology of proper rigid-analytic varieties.

For the rest of the section, we fix a p-adic algebraically closed field C with its rank-1 valuation ring $\mathcal{O}_C$, maximal ideal $\mathfrak{m} \subset \mathcal{O}_C$, and a good pseudo-uniformizer $\varpi \in \mathcal{O}_C$ (see Definition B.11). We always do almost mathematics with respect to the ideal $\mathfrak{m}$ in this section. If we need to consider a more general non-archimedean field, we denote it by K.

The first non-trivial consequence of Theorem 7.1.11 is that the v-cohomology groups of $\mathcal{O}^+_{X^\diamond}$-vector bundles have bounded p-torsion.

Lemma 7.3.1. *Let K be a p-adic perfectoid field, let $\mathfrak{X} = \operatorname{Spf} A_0$ be an affine admissible formal $\mathcal{O}_K$-scheme with adic generic fiber X, and let $\mathcal{E}$ be an $\mathcal{O}^+_{X^\diamond}$-vector bundle. Then the cohomology groups $\mathrm{H}^i\left(X^\diamond_v, \mathcal{E}\right)$ are almost finitely presented over A_0. In particular, they are p-adically complete and have bounded torsion p^∞-torsion.*

Proof. This is a straightforward consequence of Theorem 7.1.11, Lemma 2.12.5, and Lemma 2.12.7. ∎

Remark 7.3.2. Lemma 7.3.1 implies that the v-cohomology groups of $\mathcal{O}^+_{X^\diamond}$ behave pretty differently from the analytic cohomology groups of $\mathcal{O}^+_X$. Indeed, we refer to [5, Remark 9.3.4] (that can be easily adapted to the p-adic situation) for an example of an affinoid rigid-analytic variety with unbounded p^∞-torsion in $\mathrm{H}^1_{\mathrm{an}}(X, \mathcal{O}^+_X)$. The same example shows that $\mathrm{H}^1_{\mathrm{et}}(X, \mathcal{O}^+)$ could have unbounded p^∞-torsion.

Theorem 7.3.3. *Let K be a p-adic perfectoid field, let X be a proper rigid-analytic K-variety of dimension d, and let $\mathcal{E}$ be an $\mathcal{O}^+_{X^\diamond}$-vector bundle (resp. $\mathcal{O}^+_{X^\diamond}/p$-vector bundle). Then*

$$\mathbf{R}\Gamma\left(X^\diamond_v, \mathcal{E}\right) \in \mathbf{D}^{[0,2d]}_{\mathrm{acoh}}\left(\mathcal{O}_K\right)^a.$$

Proof. We firstly deal with the case of an $\mathcal{O}^+_{X^\diamond}/p$-vector bundle $\mathcal{E}$. We choose an admissible formal model $\mathfrak{X}$ of X as in Part (5) of Theorem 7.1.2. This formal model is automatically proper by [51, Lemma 2.6] and [65, Corollary 4.4 and 4.5]. Now Theorem 7.1.2 implies that

$$(\mathbf{R}\nu_*\mathcal{E})^a \in \mathbf{D}^{[0,d]}_{\mathrm{acoh}}(\mathfrak{X}_0)^a.$$

Recall that the underlying topological space of $\mathfrak{X}_0$ is equal to the underlying topological space of the special fiber $\overline{\mathfrak{X}} := \mathfrak{X} \times_{\operatorname{Spf} \mathcal{O}_C} \operatorname{Spec} \mathcal{O}_C/\mathfrak{m}$. Thus, [25, Corollary II.10.1.11] implies that $\mathfrak{X}_0$ has Krull dimension d. Therefore, Theorem 5.1.3, [68, Tag 0A3G], and Lemma 3.3.5 imply that

$$\mathbf{R}\Gamma\left(X^\diamond_v, \mathcal{E}\right)^a \simeq \mathbf{R}\Gamma\left(\mathfrak{X}_0, \mathbf{R}\nu_*\left(\mathcal{E}\right)\right)^a \in \mathbf{D}^{[0,2d]}_{\mathrm{acoh}}\left(\mathcal{O}_K/p\right)^a.$$

The case of an $\mathcal{O}^+_{X^\diamond}$-vector bundle follows from the $\mathcal{O}^+_{X^\diamond}/p$-case, Corollary 2.13.3, and Lemma 6.3.5 (3). ∎

Lemma 7.3.4. *Let X be a proper rigid-analytic variety over C of dimension d, and let $\mathcal{F}$ be a Zariski-constructible sheaf of $\mathbf{F}_p$-modules on $X_{\text{ét}}$. Then we have* $\mathbf{R}\Gamma(X^{\diamondsuit}_v, \mathcal{F} \otimes \mathcal{O}^+_{X^\diamond}/p)^a \in \mathbf{D}^{[0,2d]}_{\text{acoh}}(\mathcal{O}_C/p)^a$.

Proof. The proof is analogous to the proof of Theorem 7.3.3 using Theorem 7.1.9 in place of Theorem 7.1.2. ■

Now we discuss finiteness of classical étale cohomology groups. Later, we will generalize it to Zariski-constructible coefficients.

Lemma 7.3.5. *Let X be a proper rigid-analytic variety over C of dimension d. Then*

$$\mathbf{R}\Gamma(X, \mathbf{F}_p) \in \mathbf{D}^{[0,2d]}_{\text{coh}}(\mathbf{F}_p)$$

and the natural morphism

$$\mathbf{R}\Gamma(X, \mathbf{F}_p) \otimes \mathcal{O}_C/p \to \mathbf{R}\Gamma(X^{\diamondsuit}_v, \mathcal{O}^+_{X^\diamond}/p)$$

is an almost isomorphism.

Proof. The proof will be divided into several steps.

Step 1: $\mathbf{R}\Gamma\big(X^{\diamondsuit}_v, \mathcal{O}^{\flat,+}_{X^\diamond}\big)^a \in \mathbf{D}^{[0,2d]}_{\text{acoh}}\big(\mathcal{O}^\flat_C\big)^a$. We consider the tilted integral structure sheaf $\mathcal{O}^{\flat,+}_{X^\diamond}$ (see Definition 6.3.4). Lemma 6.3.5 (4) ensures that $\mathcal{O}^{\flat,+}_{X^\diamond}$ is derived $\varpi^\flat$-adically complete and Lemma 6.3.5 (5) implies that

$$\big[\mathcal{O}^{\flat,+}_{X^\diamond}/\varpi^\flat\big] \simeq \big[\mathcal{O}^+_{X^\diamond}/p\big] \simeq \mathcal{O}^+_{X^\diamond}/p.$$

Therefore, [68, Tag 0BLX] guarantees that $\mathbf{R}\Gamma\big(X^{\diamondsuit}_v, \mathcal{O}^{\flat,+}_{X^\diamond}\big) \in \mathbf{D}\big(\mathcal{O}^\flat_C\big)$ is derived $\varpi^\flat$-adically complete. Moreover, Lemma 7.3.4 implies

$$\big[\mathbf{R}\Gamma\big(X^{\diamondsuit}_v, \mathcal{O}^{\flat,+}_{X^\diamond}\big)^a/\varpi^\flat\big] \simeq \mathbf{R}\Gamma\big(X^{\diamondsuit}_v, \mathcal{O}^+_{X^\diamond}/p\big)^a \in \mathbf{D}^{[0,2d]}_{\text{acoh}}\big(\mathcal{O}_C/p\big)^a.$$

Thus, Corollary 2.13.3 applied to $R = C^+ = \mathcal{O}^\flat_C$ implies that $\mathbf{R}\Gamma\big(X^{\diamondsuit}_v, \mathcal{O}^{\flat,+}_{X^\diamond}\big)^a \in \mathbf{D}^{[0,2d]}_{\text{acoh}}\big(\mathcal{O}^\flat_C\big)^a$.

Step 2: $\mathbf{R}\Gamma(X, \mathbf{F}_p) \in \mathbf{D}^{[0,2d]}_{\text{coh}}(\mathbf{F}_p)$ *and the natural morphism* $\mathbf{R}\Gamma(X, \mathbf{F}_p) \otimes C^\flat \to \mathbf{R}\Gamma\big(X^{\diamondsuit}_v, \mathcal{O}^\flat_{X^\diamond}\big)$ *is an isomorphism.* After inverting $\varpi^\flat$, Step 1 implies that

$$\mathbf{R}\Gamma\big(X^{\diamondsuit}_v, \mathcal{O}^\flat_{X^\diamond}\big) \in \mathbf{D}^{[0,2d]}_{\text{coh}}(C^\flat).$$

Since $\mathcal{O}^\flat_{X^\diamond}$ is a sheaf of $\mathbf{F}_p$-algebras, we have a natural Frobenius morphism

$$F\colon \mathcal{O}^\flat_{X^\diamond} \xrightarrow{f \mapsto f^p} \mathcal{O}^\flat_{X^\diamond}$$

that can be easily seen to be an isomorphism by Lemma 6.3.5 (2) (and Remark B.7). Now we use the Artin–Shreier short exact sequence

$$0 \to \underline{\mathbf{F}}_p \to \mathcal{O}^{\flat}_{X^{\diamond}} \xrightarrow{F-\mathrm{id}} \mathcal{O}^{\flat}_{X^{\diamond}} \to 0$$

on the v-site $X^{\diamondsuit}_v$ to get the associated long exact sequence[3]

$$\cdots \to \mathrm{H}^i(X, \mathbf{F}_p) \to \mathrm{H}^i(X^{\diamondsuit}_v, \mathcal{O}^{\flat}_{X^{\diamond}}) \xrightarrow{\mathrm{H}^i(F)-\mathrm{id}} \mathrm{H}^i(X^{\diamondsuit}_v, \mathcal{O}^{\flat}_{X^{\diamond}}) \to \mathrm{H}^{i+1}(X, \mathbf{F}_p) \to \cdots$$

We already know that each group $\mathrm{H}^i(X^{\diamondsuit}_v, \mathcal{O}^{\flat}_{X^{\diamond}})$ is a finitely generated $C^{\flat}$-vector space, each $\mathrm{H}^i(F)$ is a Frobenius-linear automorphism, and $C^{\flat}$ is an algebraically closed field of characteristic p (see [58, Theorem 3.7]). Thus (the proof of) [68, Tag 0A3L] ensures that $\mathrm{H}^i(F) - \mathrm{id}$ is surjective for each $i \geq 0$ (so $\mathrm{H}^i(X, \mathbf{F}_p) \simeq \mathrm{H}^i(X^{\diamondsuit}_v, \mathcal{O}^{\flat}_{X^{\diamond}})^{F=1}$) and the natural morphism

$$\mathrm{H}^i(X, \mathbf{F}_p) \otimes C^{\flat} \to \mathrm{H}^i(X^{\diamondsuit}_v, \mathcal{O}^{\flat}_{X^{\diamond}})$$

is an isomorphism. In particular, we have $\dim_{\mathbf{F}_p} \mathrm{H}^i(X, \mathbf{F}_p) = \dim_{C^{\flat}} \mathrm{H}^i(X^{\diamondsuit}_v, \mathcal{O}^{\flat}_{X^{\diamond}})$, the natural morphism

$$\mathbf{R}\Gamma(X, \mathbf{F}_p) \otimes C^{\flat} \to \mathbf{R}\Gamma(X^{\diamondsuit}_v, \mathcal{O}^{\flat}_{X^{\diamond}})$$

is an isomorphism, and $\mathbf{R}\Gamma(X, \mathbf{F}_p) \in \mathbf{D}^{[0,2d]}_{\mathrm{coh}}(\mathbf{F}_p)$.

Step 3: *The natural morphism* $\mathbf{R}\Gamma(X, \mathbf{F}_p) \otimes \mathcal{O}_C/p \to \mathbf{R}\Gamma(X^{\diamondsuit}_v, \mathcal{O}^{+}_{X^{\diamond}}/p)$ *is an almost isomorphism.* It suffices to show that

$$\mathbf{R}\Gamma(X, \mathbf{F}_p) \otimes \mathcal{O}^{\flat}_C \to \mathbf{R}\Gamma(X^{\diamondsuit}_v, \mathcal{O}^{\flat,+}_{X^{\diamond}})$$

is an almost isomorphism. The version with $\mathcal{O}^{+}_{X^{\diamond}}/p$ would follow by taking the derived mod-$\varpi^{\flat}$ reduction. Therefore, it suffices to show that

$$\mathrm{H}^i(X, \mathbf{F}_p) \otimes \mathcal{O}^{\flat}_C \to \mathrm{H}^i(X^{\diamondsuit}_v, \mathcal{O}^{\flat,+}_{X^{\diamond}})$$

is an almost isomorphism for each $i \geq 0$. We consider the following commutative diagram:

$$\begin{tikzcd}
\mathrm{H}^i(X, \mathbf{F}_p) \otimes \mathcal{O}^{\flat}_C \arrow[r, "\alpha"] \arrow[d, "i"'] & \mathrm{H}^i(X^{\diamondsuit}_v, \mathcal{O}^{\flat,+}_{X^{\diamond}}) \arrow[d] \\
\mathrm{H}^i(X, \mathbf{F}_p) \otimes C^{\flat} \arrow[r, "\beta"'] & \mathrm{H}^i(X^{\diamondsuit}_v, \mathcal{O}^{\flat}_{X^{\diamond}}).
\end{tikzcd}$$

---

[3] We implicitly use that $\mathrm{H}^i(X, \mathbf{F}_p) \simeq \mathrm{H}^i(X^{\diamondsuit}_v, \mathbf{F}_p)$ by [61, Propositions 14.7, 14.8, and Lemma 15.6].

By Step 2, we know that β is an isomorphism. Since i is injective, we conclude that α is injective as well. So it suffices to show that α is almost surjective.

The actions of Frobenius on $\mathcal{O}_C^\flat$ and on $\mathcal{O}^{\flat,+}_{X^\diamond}$ induce the Frobenius actions on $\mathrm{H}^i(X,\mathbf{F}_p)\otimes\mathcal{O}_C^\flat$ and $\mathrm{H}^i(X_v^\diamondsuit,\mathcal{O}^{\flat,+}_{X^\diamond})$, respectively. Moreover, the map α is Frobenius-equivariant. The action on $\mathrm{H}^i(X,\mathbf{F}_p)\otimes\mathcal{O}_C^\flat$ is an isomorphism because $\mathcal{O}_C^\flat$ is perfect, and the action on $\mathrm{H}^i(X_v^\diamondsuit,\mathcal{O}^{\flat,+}_{X^\diamond})$ is an isomorphism because Frobenius is already an isomorphism on $\mathcal{O}^{\flat,+}_{X^\diamond}$ due to Lemma 6.3.5 (2) (and Remark B.7). Therefore, it makes sense to consider the inverse Frobenius action F^{-1} on both modules and α commutes with this action.

Next we pick an element $x\in\mathrm{H}^i(X_v^\diamondsuit,\mathcal{O}^{\flat,+}_{X^\diamond})$. Since F is an isomorphism on $\mathrm{H}^i(X_v^\diamondsuit,\mathcal{O}^{\flat,+}_{X^\diamond})$, we conclude that there exists some $x'\in\mathrm{H}^i(X_v^\diamondsuit,\mathcal{O}^{\flat,+}_{X^\diamond})$ such that $F^m(x')=x$ holds. Since $\mathrm{H}^i(X_v^\diamondsuit,\mathcal{O}^{\flat,+}_{X^\diamond})$ is almost coherent, Lemma 2.12.5 implies that it has bounded $(\varpi^\flat)^\infty$-torsion. Combining this with the fact that β is an isomorphism, we conclude that there is an integer N and an element $y'\in\mathrm{H}^i(X_v^\diamondsuit,\mathbf{F}_p)\otimes\mathcal{O}_C^\flat$ such that $\alpha(y')=(\varpi^\flat)^N x'$. Therefore,

$$\left(\varpi^\flat\right)^{N/p^m}x=F^{-m}\left(\left(\varpi^\flat\right)^N x'\right)=F^{-m}\left(\alpha(y')\right)=\alpha\left(F^{-m}(y')\right).$$

Thus $(\varpi^\flat)^{N/p^m}x=\alpha(y)$ where $y=F^{-m}(y')\in\mathrm{H}^i(X,\mathbf{F}_p)\otimes\mathcal{O}_C^\flat$. Since N/p^m can be made arbitrary small by increasing m, we conclude that α is almost surjective. ■

Lemma 7.3.6. *Let X be a proper rigid-analytic variety over C of dimension d, and $\mathcal{F}\in\mathbf{D}^{[r,s]}_{zc}(X;\mathbf{F}_p)$ for some integers $[r,s]$. Then*

$$\mathbf{R}\Gamma(X,\mathcal{F})\in\mathbf{D}^{[r,s+2d]}_{\mathrm{coh}}(\mathbf{F}_p).$$

Proof. First, [38, Corollary 2.8.3] implies that $\mathbf{R}\Gamma(X,\mathcal{F})\in\mathbf{D}^{[r,s+2d]}(\mathbf{F}_p)$. Therefore, it suffices to show that $\mathbf{R}\Gamma(X,\mathcal{F})\in\mathbf{D}_{\mathrm{coh}}(\mathbf{F}_p)$. For this, we recall that [7, Proposition 3.6] says that $\mathbf{D}^b_{zc}(X,\mathbf{F}_p)$ is a thick triangulated subcategory of $\mathbf{D}(X_{\text{ét}};\mathbf{F}_p)$ generated by objects of the form $f_*(\underline{\mathbf{F}}_p)$ for finite morphisms $f\colon X'\to X$. Since Since $\mathbf{D}_{\mathrm{coh}}(\mathbf{F}_p)$ is a thick triangulated subcategory of $\mathbf{D}(\mathbf{F}_p)$, it suffices to prove the claim for $\mathcal{F}=f_*(\underline{\mathbf{F}}_p)$. Then Lemma 7.3.5 and [38, Proposition 2.6.3] imply that

$$\mathbf{R}\Gamma\left(X,f_*(\underline{\mathbf{F}}_p)\right)\simeq\mathbf{R}\Gamma(X',\mathbf{F}_p)\in\mathbf{D}^{[0,2d]}_{\mathrm{coh}}(\mathbf{F}_p).$$ ■

The last thing we discuss is the behavior of the “p-adic nearby cycles” under proper pushforwards. We start with the following lemma:

Lemma 7.3.7. *Let K be a p-adic perfectoid field K, let $f\colon X\to Y$ be a proper morphism of rigid-analytic varieties over K, and let $\mathcal{F}\in\mathbf{D}^b_{zc}(X;\mathbf{F}_p)$. Then the natural morphism*

$$\mathbf{R}f_*\mathcal{F}\otimes\mathcal{O}^+_{Y_{\text{ét}}}/p\to\mathbf{R}f_*\left(\mathcal{F}\otimes\mathcal{O}^+_{X_{\text{ét}}}/p\right)$$

is an almost isomorphism.

Proof. The claim is local on Y, so we can assume that Y is affinoid. Then a similar argument to the proof of Lemma 7.3.6 allows us to reduce to the case when $\mathcal{F} = g_*(\underline{\mathbf{F}}_p)$ for a finite map $g\colon X' \to X$. Therefore, Corollary 7.2.8 implies that it suffices to prove the claim for the morphism $f \circ g\colon X' \to Y$ and $\mathcal{F} = \underline{\mathbf{F}}_p$.

Now [38, Proposition 2.5.5] guarantees that it suffices to show the claim on stalks at geometric points. Therefore, by Lemma 7.2.5 we reduce the question to showing that, for any proper adic space X over a geometric point $\mathrm{Spa}\,(C, C^+)$, the natural morphism

$$\mathbf{R}\Gamma\big(X, \mathbf{F}_p\big) \otimes C^+/p \to \mathbf{R}\Gamma\big(X, \mathcal{O}^+_{X_{\text{ét}}}/p\big)$$

is an almost isomorphism. Denote by $X^\circ := X \times_{\mathrm{Spa}\,(C,C^+)} \mathrm{Spa}\,(C, C^\circ)$. Now [38, Proposition 8.2.3 (ii)] implies that $\mathbf{R}\Gamma(X, \mathbf{F}_p) \simeq \mathbf{R}\Gamma(X^\circ, \mathbf{F}_p)$, Lemma 2.11.2 implies that $C^+/pC^+ \simeq^a \mathcal{O}_C/p\mathcal{O}_C$, and Corollary 6.4.15 and Corollary 6.4.18 imply that

$$\mathbf{R}\Gamma\big(X, \mathcal{O}^+_{X_{\text{ét}}}/p\big) \simeq^a \mathbf{R}\Gamma\big(X^\circ, \mathcal{O}^+_{X^\circ_{\text{ét}}}/p\big).$$

Combining these results, we may replace (C, C^+) with $(C, \mathcal{O}_C)$ and X with X° to achieve that $\mathrm{Spa}\,(C, \mathcal{O}_C)$ is a geometric point of rank-1. In this case, the claim was already proven in Lemma 7.3.6. ■

Now we show that p-adic nearby cycles commute with proper morphisms.

Corollary 7.3.8. *Let K be a p-adic perfectoid field K, let $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{Y}$ be a proper morphism of admissible formal $\mathcal{O}_K$-schemes with adic generic fiber $f\colon X \to Y$, and let $\mathcal{F} \in \mathbf{D}^b_{zc}(X; \mathbf{F}_p)$. Then the natural morphism*

$$\mathbf{R}\nu_{\mathfrak{Y},*}\big(\mathbf{R}f_*\mathcal{F} \otimes \mathcal{O}^+_{Y^\diamond}/p\big) \to \mathbf{R}\mathfrak{f}_{0,*}\big(\mathbf{R}\nu_{\mathfrak{X},*}\big(\mathcal{F} \otimes \mathcal{O}^+_{X^\diamond}/p\big)\big)$$

is an almost isomorphism.

Proof. First, note that $\mathbf{R}f_*\mathcal{F}$ has overconvergent cohomology sheaves by [38, Proposition 8.2.3 (ii)] and Remark 7.1.8. Therefore, Lemma 6.7.10 implies that

$$\mathbf{R}\nu_{\mathfrak{Y},*}\big(\mathbf{R}f_*\mathcal{F} \otimes \mathcal{O}^+_{Y^\diamond}/p\big) \simeq \mathbf{R}t_{\mathfrak{Y},*}\big(\mathbf{R}f_*\mathcal{F} \otimes \mathcal{O}^+_{Y_{\text{ét}}}/p\big),$$

where $t_{\mathfrak{Y}}\colon \big(Y_{\text{ét}}, \mathcal{O}^+_{Y_{\text{ét}}}/p\big) \to \big(\mathfrak{Y}_0, \mathcal{O}_{\mathfrak{Y}_0}\big)$ is the natural morphism of ringed sites. Similarly, we have an isomorphism

$$\mathbf{R}\mathfrak{f}_{0,*}\big(\mathbf{R}\nu_{\mathfrak{X},*}\big(\mathcal{F} \otimes \mathcal{O}^+_{X^\diamond}/p\big)\big) \simeq \mathbf{R}\mathfrak{f}_{0,*}\big(\mathbf{R}t_{\mathfrak{X},*}\big(\mathcal{F} \otimes \mathcal{O}^+_{X_{\text{ét}}}/p\big)\big).$$

Therefore, it suffices to show that the natural morphism

$$\mathbf{R}t_{\mathfrak{Y},*}\big(\mathbf{R}f_*\mathcal{F} \otimes \mathcal{O}^+_{Y_{\text{ét}}}/p\big) \to \mathbf{R}\mathfrak{f}_{0,*}\big(\mathbf{R}t_{\mathfrak{X},*}\big(\mathcal{F} \otimes \mathcal{O}^+_{X_{\text{ét}}}/p\big)\big)$$

is an almost isomorphism.

For this, we observe that the commutative diagram of ringed sites

$$\begin{tikzcd}
\left(X_{\text{ét}}, \mathcal{O}^+_{X_{\text{ét}}}/p\right) \arrow[r, "t_{\mathfrak{X}}"] \arrow[d, "f"'] & \left(\mathfrak{X}_0, \mathcal{O}_{\mathfrak{X}_0}\right) \arrow[d, "\mathfrak{f}_0"] \\
\left(Y_{\text{ét}}, \mathcal{O}^+_{Y_{\text{ét}}}/p\right) \arrow[r, "t_{\mathfrak{Y}}"'] & \left(\mathfrak{Y}_0, \mathcal{O}_{\mathfrak{Y}_0}\right)
\end{tikzcd}$$

implies that

$$\mathbf{R}\mathfrak{f}_{0,*}\left(\mathbf{R}t_{\mathfrak{X},*}\left(\mathcal{F} \otimes \mathcal{O}^+_{X_{\text{ét}}}/p\right)\right) \simeq \mathbf{R}t_{\mathfrak{Y},*}\left(\mathbf{R}f_*\left(\mathcal{F} \otimes \mathcal{O}^+_{X_{\text{ét}}}/p\right)\right).$$

Therefore, the morphism

$$\mathbf{R}t_{\mathfrak{Y},*}\left(\mathbf{R}f_*\mathcal{F}\otimes\mathcal{O}^+_{Y_{\text{ét}}}/p\right) \to \mathbf{R}\mathfrak{f}_{0,*}\left(\mathbf{R}t_{\mathfrak{Y},*}\left(\mathcal{F}\otimes \mathcal{O}^+_{Y_{\text{ét}}}/p\right)\right) \simeq \mathbf{R}t_{\mathfrak{Y},*}\left(\mathbf{R}f_*\left(\mathcal{F}\otimes \mathcal{O}^+_{X_{\text{ét}}}/p\right)\right)$$

is an almost isomorphism due to Lemma 7.3.7 and Proposition 3.5.23. ∎

7.4 Perfectoid covers of affinoids

The main goal of this section is to show almost vanishing of higher v-cohomology groups of a small $\mathcal{O}^+_{X^\diamond}/p$-vector bundle on an affinoid perfectoid space. Later on, we will apply it to certain pro-étale coverings of $\operatorname{Spa}(A, A^+)$ to reduce the computation of v-cohomology groups to the computation of Čech cohomology groups.

Set-up 7.4.1. We fix

(1) a p-adic perfectoid field K together with its rank-1 open and bounded valuation ring denoted by $\mathcal{O}_K$ and a good pseudo-uniformizer $\varpi \in \mathcal{O}_K$ as in Definition B.11 (we always do almost mathematics with respect to the ideal $\mathfrak{m} = \bigcup_n \varpi^{1/p^n}\mathcal{O}_K = K^{\circ\circ}$);

(2) an affine admissible formal $\mathcal{O}_K$-scheme $\mathfrak{X} = \operatorname{Spf} A_0$ with adic generic fiber $X = \operatorname{Spa}(A, A^+)$;

(3) and an affinoid perfectoid pair (A_∞, A^+_∞) (see Definition B.5) with a morphism $(A, A^+) \to (A_\infty, A^+_\infty)$ such that $\operatorname{Spd}(A_\infty, A^+_\infty) \to \operatorname{Spd}(A, A^+)$ is a v-covering (see Definition 6.1.1 and Definition 6.1.5);

(4) a small $\mathcal{O}^+_{X^\diamond}/p$-vector bundle $\mathcal{E}$ (see Definition 7.1.1).

Definition 7.4.2. We say that a p-torsionfree (equivalently, ϖ-torsionfree) $\mathcal{O}_K$-algebra R is *integrally perfectoid* if the Frobenius homomorphism

$$R/\varpi R \xrightarrow{x\mapsto x^p} R/\varpi^p R = R/pR$$

is an isomorphism.

Remark 7.4.3. [8, Lemma 3.10] implies that this definition coincides with [8, Definition 3.5] for p-torsionfree $\mathcal{O}_K$-algebras. In particular, A_∞^+ is an integral perfectoid $\mathcal{O}_K$-algebra by [8, Lemma 3.20].

Lemma 7.4.4. *Under the assumption of Set-up 7.4.1, let $\mathfrak{f}\colon \operatorname{Spf} B_0 \to \operatorname{Spf} A_0$ be an étale morphism of admissible affine formal $\mathcal{O}_K$-schemes. Then $B_\infty^+ := B_0 \widehat{\otimes}_{A_0} A_\infty^+$ is p-torsion free integrally perfectoid $\mathcal{O}_K$-algebra.*

Proof. Firstly, we note that $A_0 \to B_0$ is a flat morphism by [25, Proposition I.4.8.1], so $B_0 \otimes_{A_0} A_\infty^+$ is ϖ-torsion free. Since the ϖ-adic completion of a ϖ-torsionfree algebra is ϖ-torsion free, we conclude that $B_\infty^+ = B_0 \widehat{\otimes}_{A_0} A_\infty^+$ is ϖ-torsion free. We see that the only thing we are left to show is that the Frobenius morphism

$$B_\infty^+/\varpi B_\infty^+ \to B_\infty^+/\varpi^p B_\infty^+$$

is an isomorphism. We consider the commutative diagram

$$\begin{tikzcd}
\operatorname{Spec} B_\infty^+/\varpi \arrow[rd, "F"] \arrow[rrd, bend left, "\Phi_B^*"] \arrow[rdd, bend right, "\mathfrak{f}_\infty/\varpi"'] & & \\
& \operatorname{Spec}\left(B_\infty^+/\varpi^p \otimes_{A_\infty^+/\varpi^p} A_\infty^+/\varpi\right) \arrow[r, "\Phi_A^* \times B_0"'] \arrow[d, "\alpha"] & \operatorname{Spec} B_\infty^+/\varpi^p \arrow[d, "\mathfrak{f}_\infty/\varpi^p"] \\
& \operatorname{Spec} A_\infty^+/\varpi \arrow[r, "\Phi_A^*"] & \operatorname{Spec} A_\infty^+/\varpi^p .
\end{tikzcd}$$

We need to show that Φ_B^* is an isomorphism. We know that $\mathfrak{f}_\infty/\varpi^p$ and $\mathfrak{f}_\infty/\varpi$ are étale morphisms since $\mathfrak{f}$ is so, and moreover the Frobenius Φ_A^* is an isomorphism by Remark 7.4.3. Therefore, the morphism

$$\alpha\colon \operatorname{Spec}\left(B_\infty^+/\varpi^p \otimes_{A_\infty^+/\varpi^p} A_\infty^+/\varpi\right) \to \operatorname{Spec} A_\infty^+/\varpi$$

is étale as a base change of the étale morphism $\mathfrak{f}_\infty/\varpi^p$. Thus, we conclude that F is an étale morphism as a morphism between étale A_∞^+/ϖ-schemes. Now we note that $\Phi_A^* \times B_0$ is an isomorphism since Φ_A^* is an isomorphism. Therefore, Φ_B^* is an étale morphism as a composition of an étale morphism and an isomorphism. However, Φ_B^* is a bijective radiciel morphism since it is the absolute Frobenius morphism. Thus, we conclude that it must be an isomorphism as any étale, bijective radiciel morphism is an isomorphism by [29, Exp. I, Théorème 5.1]. ■

Corollary 7.4.5. *Under the assumption of Set-up 7.4.1, let $\mathfrak{f}\colon \operatorname{Spf} B_0 \to \operatorname{Spf} A_0$ be an étale morphism of admissible affine formal $\mathcal{O}_K$-schemes. Then*

$$(B_\infty, B_\infty^+) := \left((B_0 \widehat{\otimes}_{A_0} A_\infty^+)\left[\tfrac{1}{p}\right], B_0 \widehat{\otimes}_{A_0} A_\infty^+\right)$$

is a perfectoid pair.

Proof. Lemma 7.4.4 states that $B_\infty^+ = B_0 \widehat{\otimes}_{A_0} A_\infty^+$ is a p-torsionfree integral perfectoid. Now $B_0 \otimes_{A_0} A_\infty^+$ is integrally closed in $B_0 \otimes_{A_0} A_\infty^+[\frac{1}{p}]$ because A^+ is integrally closed in A and B_0 is étale over A_0. Therefore, [5, Lemma 5.1.2] ensures that the same holds after completion, i.e., B_∞^+ is integrally closed in B_∞. Thus [8, Lemma 3.20] guarantees that (B_∞, B_∞^+) is a perfectoid pair. ■

Lemma 7.4.6. *Under the assumption of Set-up 7.4.1, let* $\mathfrak{f}\colon \operatorname{Spf} B_0 \to \operatorname{Spf} A_0$ *be an étale morphism of admissible affine formal* $\mathcal{O}_K$*-schemes with adic generic fiber* $\operatorname{Spa}(B, B^+) \to \operatorname{Spa}(A, A^+)$. *Then the natural morphism*

$$\big((B_0 \widehat{\otimes}_{A_0} A_\infty^+)[\tfrac{1}{p}], B_0 \widehat{\otimes}_{A_0} A_\infty^+\big) \to \big(B \widehat{\otimes}_A A_\infty, (B \widehat{\otimes}_A A_\infty)^+\big)$$

is an isomorphism of Tate–Huber pairs.

Proof. By [36, Lemma 1.6], $B \widehat{\otimes}_A A_\infty \simeq (B_0 \widehat{\otimes}_{A_0} A_\infty^+)[\frac{1}{p}]$. Now, $(B \widehat{\otimes}_A A_\infty)^+$ is defined to be the integral closure of the image of the map

$$B^+ \widehat{\otimes}_{A^+} A_\infty^+ \to B \widehat{\otimes}_A A_\infty.$$

By [36, Lemma 1.6], we also have

$$B^+ \widehat{\otimes}_{A^+} A_\infty^+ \simeq (B^+ \otimes_{A^+} A_\infty^+) \otimes_{B_0 \otimes_{A_0} A_\infty^+} (B_0 \widehat{\otimes}_{A_0} A_\infty^+).$$

Since B^+ is integral over B_0, we have that $B^+ \widehat{\otimes}_{A^+} A_\infty^+$ is integral over $B_0 \widehat{\otimes}_{A_0} A_\infty^+$. In particular, we see that $(B \widehat{\otimes}_A A_\infty)^+$ is integral over $B_0 \widehat{\otimes}_{A_0} A_\infty^+$. However, Corollary 7.4.5 implies that $B_0 \widehat{\otimes}_{A_0} A_\infty^+$ is a subalgebra of $B \widehat{\otimes}_A A_\infty$ that is integrally closed in $B \widehat{\otimes}_A A_\infty$. Thus, we have an isomorphism

$$B_0 \widehat{\otimes}_{A_0} A_\infty^+ \simeq (B \widehat{\otimes}_A A_\infty)^+.$$ ■

Remark 7.4.7. It will be crucial for our arguments later that $(B \widehat{\otimes}_A A_\infty)^+$ is *equal* to $B_0 \widehat{\otimes}_{A_0} A_\infty^+$ and not simply to its integral closure.

Lemma 7.4.8. *Under the assumption of Set-up 7.4.1, we put*

$$M_{\mathcal{E}} := \mathrm{H}^0\big(\operatorname{Spd}(A_\infty, A_\infty^+)_v, \mathcal{E}\big).$$

Then $M_{\mathcal{E}}$ *is an almost faithfully flat, almost finitely presented* A_∞^+/p*-module, and for every morphism* $\operatorname{Spa}(D, D^+) \to \operatorname{Spa}(A_\infty, A_\infty^+)$ *of affinoid perfectoid spaces, the natural morphism*

$$M_{\mathcal{E}} \otimes_{A_\infty^+/p} D^+/p \to \mathrm{H}^0\big(\operatorname{Spd}(D, D^+)_v, \mathcal{E}\big)$$

is an almost isomorphism.[4] *Moreover,*

$$\mathrm{H}^i\big(\mathrm{Spd}\,(A_\infty, A_\infty^+)_v, \mathcal{E}\big) \simeq^a 0$$

for $i > 0$.

Proof. We divide the proof into several steps.

Step 1: $\mathrm{H}^0\big(\mathrm{Spd}(A_\infty, A_\infty^+)_v, \mathcal{E}\big)$ *is almost flat and almost finitely presented.* The smallness assumption implies that there is a finite étale surjection $\mathrm{Spa}\,(B, B^+) \to \mathrm{Spa}\,(A_\infty, A_\infty^+)$ such that $\mathcal{E}|_{\mathrm{Spd}\,(B,B^+)} \simeq (\mathcal{O}^+_{X^\diamond}/p)^r$ for some integer $r \geq 0$. The adic space $\mathrm{Spa}\,(B, B^+)$ is affinoid perfectoid by [59, Theorem 7.9].

The natural morphism $A^+ \to B^+$ is almost finitely presented and almost faithfully flat by [59, Theorem 7.9] (see also [5, Theorem 10.0.9] for the almost *faithfully* flat part). Since $\mathcal{E}|_{\mathrm{Spd}\,(B,B^+)}$ is trivial, Lemma 6.3.5 (1) implies that

$$\mathrm{H}^0\big(\mathrm{Spd}\,(B, B^+)_v, \mathcal{E}\big) \simeq^a (B^+/pB^+)^r.$$

In particular, it is almost flat and almost finitely presented. We now want to descend these properties to $\mathrm{H}^0\big(\mathrm{Spd}\,(A_\infty, A_\infty^+)_v, \mathcal{E}\big)$. For this, we use Proposition 6.1.6 to recall that diamondification commutes with fiber products, and so

$$\begin{aligned}
&\mathrm{Spd}\,(B, B^+) \times_{\mathrm{Spd}\,(A_\infty, A_\infty^+)} \mathrm{Spd}\,(B, B^+) \\
&\quad \simeq \big(\mathrm{Spa}\,(B, B^+) \times_{\mathrm{Spa}\,(A_\infty, A_\infty^+)} \mathrm{Spa}\,(B, B^+)\big)^\diamond \\
&\quad \simeq \mathrm{Spd}\,\big(B \widehat{\otimes}_{A_\infty} B, (B \widehat{\otimes}_{A_\infty} B)^+\big).
\end{aligned}$$

By the proof of [58, Proposition 6.18] (and Lemma B.13), we see that $B^+ \widehat{\otimes}_{A_\infty^+} B^+ \to (B \widehat{\otimes}_{A_\infty} B)^+$ is an almost isomorphism (while, a priori, the latter group is the integral closure of the former one inside $B \widehat{\otimes}_{A_\infty} B$). In particular,

$$B^+/p \otimes_{A_\infty^+/p} B^+/p \simeq^a (B \widehat{\otimes}_{A_\infty} B)^+/p(B \widehat{\otimes}_{A_\infty} B)^+.$$

Thus

$$\mathrm{H}^0\big(\mathrm{Spd}\,(B \widehat{\otimes}_{A_\infty} B, (B \widehat{\otimes}_{A_\infty} B)^+)_v, \mathcal{E}\big) \simeq^a \big((B^+/p)^{\otimes^2_{A_\infty^+/p}}\big)^r$$

and the two natural morphisms

$$\begin{aligned}
&\mathrm{H}^0\big(\mathrm{Spd}\,(B, B^+)_v, \mathcal{E}\big) \otimes_{B^+/p} (B^+/p)^{\otimes^2_{A_\infty^+/p}} \\
&\qquad \to \mathrm{H}^0\big(\mathrm{Spd}\,\big(B \widehat{\otimes}_{A_\infty} B, (B \widehat{\otimes}_{A_\infty} B)^+\big)_v, \mathcal{E}\big)
\end{aligned}$$

---

[4] We note that $\mathcal{E}$ is a sheaf on a (big) v-site of $\mathrm{Spd}\,(A, A^+)$, so it makes sense to evaluate $\mathcal{E}$ on $\mathrm{Spd}\,(D, D^+)$.

are almost isomorphisms. We use the sheaf condition and the previous discussion to get the following almost exact sequence:

$$\begin{aligned}0 \to \mathrm{H}^0\big(\mathrm{Spd}\,\big(A_\infty, A_\infty^+\big)_v, \mathcal{E}\big) \to &\mathrm{H}^0\big(\mathrm{Spd}\,(B, B^+)_v, \mathcal{E}\big) \\ \to &\mathrm{H}^0\big(\mathrm{Spd}\,(B, B^+)_v, \mathcal{E}\big) \otimes_{B^+/p} \big((B^+/p)^{\otimes 2}\big).\end{aligned}$$

Theorem 2.10.3 applied to the almost faithfully flat morphism $A_\infty^+/pA_\infty^+ \to B^+/pB^+$ implies that the natural morphism

$$\mathrm{H}^0\big(\mathrm{Spd}\,\big(A_\infty, A_\infty^+\big)_v, \mathcal{E}\big) \otimes_{A_\infty^+/p} B^+/p \to \mathrm{H}^0\big(\mathrm{Spd}\,(B, B^+)_v, \mathcal{E}\big) \qquad (7.4.1)$$

is an almost isomorphism. The above computation tells us that $\mathrm{H}^0\big(\mathrm{Spd}\,(B, B^+)_v, \mathcal{E}\big)$ is almost faithfully flat and almost finitely presented over B^+/pB^+. Thus, the faithfully flat descent for flatness and almost finitely presented modules (see Lemma 2.10.5 and Lemma 2.10.7) implies that $\mathrm{H}^0\big(\mathrm{Spd}\,(A_\infty, A_\infty^+)_v, \mathcal{E}\big)$ is almost faithfully flat and almost finitely presented over A_∞^+/pA_∞^+.

Step 2: $\mathrm{H}^0\big(\mathrm{Spd}(A_\infty, A_\infty^+)_v, \mathcal{E}\big)$ *almost commutes with base change.* By the proof of [58, Proposition 6.18] (and by virtue of Lemma B.13), we can conclude that $\mathrm{Spa}\,(B, B^+) \times_{\mathrm{Spa}\,(A_\infty, A_\infty^+)} \mathrm{Spa}\,(D, D^+)$ exists as an adic space and is represented by $\mathrm{Spa}\,(R, R^+)$ for a perfectoid pair (R, R^+) such that

$$B^+/p \otimes_{A_\infty^+/p} D^+/p \to R^+/p \qquad (7.4.2)$$

is an almost isomorphism. Thus, the proof of Step 1 and (7.4.2) imply that

$$\mathrm{H}^0\big(\mathrm{Spd}\,(D, D^+)_v, \mathcal{E}\big) \otimes_{A_\infty^+/p} B^+/p \to \mathrm{H}^0\big(\mathrm{Spd}\,(R, R^+)_v, \mathcal{E}\big)$$

is an almost isomorphism. Now we wish to show that the natural morphism

$$\mathrm{H}^0\big(\mathrm{Spd}\,(A_\infty, A_\infty^+)_v, \mathcal{E}\big) \otimes_{A_\infty^+/p} D^+/p \to \mathrm{H}^0\big(\mathrm{Spd}\,(D, D^+)_v, \mathcal{E}\big)$$

is an almost isomorphism. By the faithfully flat descent, it suffices to check after tensoring against B^+/p over A_∞^+/p. Therefore, we use (7.4.1) and (7.4.2) to see that it suffices to show that

$$\mathrm{H}^0\big(\mathrm{Spd}\,(B, B^+)_v, \mathcal{E}\big) \otimes_{B^+/p} R^+/p \to \mathrm{H}^0\big(\mathrm{Spd}\,(R, R^+)_v, \mathcal{E}\big)$$

is an almost isomorphism. Now Lemma 6.3.5 (1) almost identifies (in the technical sense) this morphism with the identity morphism

$$(B^+/pB^+)^r \otimes_{B^+/p} R^+/p \to (R^+/pR^+)^r$$

since $\mathcal{E}|_{\mathrm{Spd}\,(B,B^+)}$ is a trivial $\mathcal{O}^+/p$-vector bundle of rank r. This map is clearly an isomorphism.

Step 3: $\mathrm{H}^i\big(\mathrm{Spd}(A_\infty, A_\infty^+)_v, \mathcal{E}\big)$ *is almost zero for* $i > 0$. As in Step 1, we use that

$$\mathrm{Spa}\,(B, B^+) \to \mathrm{Spa}\,(A_\infty, A_\infty^+)$$

is a finite étale morphism of affinoid perfectoid spaces to conclude that all fiber products

$$\mathrm{Spa}\,(B, B^+)^{j/\mathrm{Spd}\,(A_\infty, A_\infty^+)}$$

are represented by affinoid perfectoid spaces $\mathrm{Spa}\,(B_j, B_j^+)$ and the natural morphisms

$$(B^+/pB^+)^{\otimes^j_{A_\infty^+/pA_\infty^+}} \to B_j^+/pB_j^+$$

are almost isomorphisms. Since each restriction $\mathcal{E}|_{\mathrm{Spd}\,(B_j, B_j^+)}$ is trivial, it is ensured by Lemma 6.3.5 (1) that the higher cohomology of $\mathcal{E}$ on $\mathrm{Spd}\,(B_j, B_j^+)$ almost vanishes. Thus, $\mathbf{R}\Gamma\big(\mathrm{Spd}\,(A_\infty, A_\infty^+)_v, \mathcal{E}\big)$ is almost isomorphic to the Čech complex associated with the covering $\mathrm{Spd}\,(B, B^+) \to \mathrm{Spd}\,(A_\infty, A_\infty^+)$. Step 2 implies that this complex is almost isomorphic to the standard Amitsur complex

$$0 \to M_{\mathcal{E}} \to M_{\mathcal{E}} \otimes_{A_\infty^+/p} B^+/p \to M_{\mathcal{E}} \otimes_{A_\infty^+/p} B^+/p \otimes_{A_\infty^+/p} B^+/p \to \cdots$$

Almost exactness of this complex follows from Lemma 2.10.4. ∎

7.5 Strictly totally disconnected covers of affinoids

The main goal of this section is to eliminate almost mathematics in Lemma 7.4.8 under some stronger assumptions on A_∞.

Set-up 7.5.1. We fix

1. a p-adic perfectoid field K with its rank-1 open and bounded valuation ring denoted by $\mathcal{O}_K$ and a good pseudo-uniformizer $\varpi \in \mathcal{O}_K$ (we always do almost mathematics with respect to the ideal $\mathfrak{m} = \bigcup_n \varpi^{1/p^n}\mathcal{O}_K = K^{\circ\circ}$);
2. an affine admissible formal $\mathcal{O}_K$-scheme $\mathfrak{X} = \mathrm{Spf}\ A_0$ with adic generic fiber $X = \mathrm{Spa}\,(A, A^+)$;
3. a strictly totally disconnected affinoid perfectoid space $\mathrm{Spa}\,(A_\infty, A_\infty^+)$ (see Definition 6.2.5) with a morphism

$$\mathrm{Spa}\,(A_\infty, A_\infty^+) \to \mathrm{Spa}\,(A, A^+)$$

such that $\mathrm{Spd}\,(A_\infty, A_\infty^+) \to \mathrm{Spd}\,(A, A^+)$ is a v-covering and all fiber products

$$\mathrm{Spd}\,(A_\infty, A_\infty^+)^{j/\mathrm{Spd}\,(A, A^+)}$$

are strictly totally disconnected affinoid perfectoid spaces.

Corollary 7.5.2. *Under the assumption of Set-up 7.5.1, let* $\mathfrak{f}\colon \operatorname{Spf} B_0 \to \operatorname{Spf} A_0$ *be an étale morphism of admissible affine formal* $\mathcal{O}_K$*-schemes. Then*

$$(B_\infty, B_\infty^+) := \big((B_0 \widehat{\otimes}_{A_0} A_\infty^+)[\tfrac{1}{p}], B_0 \widehat{\otimes}_{A_0} A_\infty^+\big)$$

is a perfectoid pair and $\operatorname{Spa}(B_\infty, B_\infty^+)$ *is a strictly totally disconnected (affinoid) perfectoid space.*

Proof. Corollary 7.4.5 already implies that $\operatorname{Spa}(B_\infty, B_\infty^+)$ is an affinoid perfectoid space. Moreover, Lemma 7.4.6 implies that

$$\operatorname{Spa}(B_\infty, B_\infty^+) \simeq \operatorname{Spa}(B, B^+) \times_{\operatorname{Spa}(A,A^+)} \operatorname{Spa}(A_\infty, A_\infty^+),$$

where $\operatorname{Spa}(B, B^+)$ is the generic fiber of $\operatorname{Spf} B_0$. So $\operatorname{Spa}(B_\infty, B_\infty^+) \to \operatorname{Spa}(A_\infty, A_\infty^+)$ is an étale morphism, thus the claim follows from [61, Lemma 7.19]. ■

Lemma 7.5.3. *Under the assumption of Set-up 7.5.1, let* $M_{\mathcal{E}}$ *be the* A_∞^+/pA_∞^+*-module*

$$M_{\mathcal{E}} := \mathrm{H}^0\big(\operatorname{Spd}(A_\infty, A_\infty^+)_v, \mathcal{E}\big).$$

Then $M_{\mathcal{E}}$ *is a finite projective* $(A_\infty^+/p)^r$*-module. Moreover, for every morphism* $\operatorname{Spa}(D, D^+) \to \operatorname{Spa}(A_\infty, A_\infty^+)$ *of strictly totally disconnected affinoid perfectoid spaces, the natural morphism*

$$M_{\mathcal{E}} \otimes_{A_\infty^+/p} D^+/p \to \mathrm{H}^0\big(\operatorname{Spd}(D, D^+)_v, \mathcal{E}\big)$$

is an isomorphism. Furthermore,

$$\mathrm{H}^i\big(\operatorname{Spd}(A_\infty, A_\infty^+)_v^{j/\operatorname{Spd}(A,A^+)}, \mathcal{E}\big) \simeq 0$$

for $i, j \geq 1$.

Proof. Lemma 6.5.6 implies that we can replace $\operatorname{Spa}(A_\infty, A_\infty^+)$ by a finite clopen decomposition to assume[5] that $\mathcal{E}|_{\operatorname{Spd}(A_\infty,A_\infty^+)} \simeq \big(\mathcal{O}^+_{\operatorname{Spd}(A_\infty,A_\infty^+)}/p\big)^r$ for some integer r. Then Corollary 6.4.16 implies that $M_{\mathcal{E}} \simeq (A_\infty^+/p)^r$. The same applies to $\mathcal{E}|_{\operatorname{Spd}(D,D^+)}$, therefore the natural morphism

$$M_{\mathcal{E}} \otimes_{A_\infty^+/p} D^+/p = (A_\infty^+/p)^r \otimes_{A_\infty^+/p} D^+/p \to (D^+/p)^r$$

is clearly an isomorphism. Furthermore, Corollary 6.4.16 implies that

$$\mathrm{H}^i\big(\operatorname{Spd}(A_\infty, A_\infty^+)_v^{j/\operatorname{Spd}(A,A^+)}, \mathcal{E}\big) \simeq 0$$

for $i, j \geq 1$ because we assume that all fiber products $\operatorname{Spd}\big(A_\infty, A_\infty^+\big)^{j/\operatorname{Spd}(A,A^+)}$ are representable by strictly totally disconnected (affinoid) perfectoid spaces. ■

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[5] At this step, the map $\operatorname{Spd}(A_\infty, A_\infty^+) \to \operatorname{Spd}(A, A^+)$ might not be a v-covering anymore. But this will not matter for the rest of the proof.

Corollary 7.5.4. *Under the assumption of Set-up 7.5.1, let* $\mathfrak{f}\colon \operatorname{Spf} B_0 \to \operatorname{Spf} A_0$ *be an étale morphism, and let* (B_∞, B_∞^+) *be the perfectoid pair from Corollary 7.4.5. Then the natural morphism*

$$\Gamma\big(\operatorname{Spd}(A_\infty, A_\infty^+)_v^{j/\operatorname{Spd}(A,A^+)}, \mathcal{E}\big) \otimes_{A_0/pA_0} B_0/pB_0 \\ \to \Gamma\big(\operatorname{Spd}(B_\infty, B_\infty^+)_v^{j/\operatorname{Spd}(B,B^+)}, \mathcal{E}\big)$$

is an isomorphism for $j \geq 1$.

Proof. For $j = 1$, the result follows from Lemma 7.5.3 and Corollary 7.5.2. For $j > 1$, we know that $X_j := \operatorname{Spd}(A_\infty, A_\infty^+)^{j/\operatorname{Spd}(A,A^+)}$ is represented by a strictly totally disconnected perfectoid space. The morphism $X_j \to \operatorname{Spd}(A, A^+)$ defines a strictly totally disconnected perfectoid space $X_j^\sharp$ with a morphism $X_j^\sharp \to \operatorname{Spa}(A, A^+)$. One checks that $X_j^\sharp \to \operatorname{Spa}(A, A^+)$ satisfies the assumptions of Set-up 7.5.1, so we can replace $\operatorname{Spa}(A_\infty, A_\infty^+)$ with $X_j^\sharp$ to reduce the case of $j > 1$ to the case $j = 1$. ■

Corollary 7.5.5. *Under the assumption of Set-up 7.5.1, let* $\mathfrak{f}\colon \operatorname{Spf} B_0 \to \operatorname{Spf} A_0$ *be an étale morphism, and let* $\operatorname{Spa}(B, B^+)$ *be the adic generic fiber of* $\operatorname{Spf}(B_0)$*. Then the natural morphism*

$$\mathrm{H}^i\big(\operatorname{Spd}(A, A^+)_v, \mathcal{E}\big) \otimes_{A_0/pA_0} B_0/pB_0 \to \mathrm{H}^i\big(\operatorname{Spd}(B, B^+)_v, \mathcal{E}\big)$$

is an isomorphism for $i \geq 0$.

Proof. Arguing as in the proof of Corollary 7.5.4, we see that Lemma 7.5.3 implies that

$$\mathrm{H}^i\big(\operatorname{Spd}(A_\infty, A_\infty^+)_v^{j/\operatorname{Spd}(A,A^+)}, \mathcal{E}\big) \simeq 0$$

for $i, j \geq 1$. Consequently, the cohomology groups $\mathrm{H}^i(\operatorname{Spd}(A, A^+)_v, \mathcal{E})$ can be computed via the cohomology of the Čech complex associated with the covering $\operatorname{Spd}(A_\infty, A_\infty^+) \to \operatorname{Spd}(A, A^+)$. By Corollary 7.5.2, the same applies to $\operatorname{Spa}(B, B^+)$ and the Čech complex associated with the covering $\operatorname{Spd}(B_\infty, B_\infty^+) \to \operatorname{Spd}(B, B^+)$. Therefore, the claim follows from Corollary 7.5.4. ■

Corollary 7.5.6. *Under the assumption of Set-up 7.4.1, let* $K \subset C$ *be a completed algebraic closure of* K*, and* $\operatorname{Spa}(A_C, A_C^+) = \operatorname{Spa}(A, A^+) \times_{\operatorname{Spa}(K,\mathcal{O}_K)} \operatorname{Spa}(C, \mathcal{O}_C)$. *Then the natural morphism*

$$\mathrm{H}^i\big(\operatorname{Spd}(A, A^+)_v, \mathcal{E}\big) \otimes_{\mathcal{O}_K/p} \mathcal{O}_C/p \to \mathrm{H}^i\big(\operatorname{Spd}(A_C, A_C^+)_v, \mathcal{E}\big)$$

is an almost isomorphism.

Proof. The proof is similar to that of Corollary 7.5.4 and Corollary 7.5.5. The only change we need to make is that the fiber product

$$\mathrm{Spa}\,(A_\infty, A_\infty^+) \times_{\mathrm{Spa}\,(K,\mathcal{O}_K)} \mathrm{Spa}\,(L, \mathcal{O}_L)$$

is a strictly totally disconnected affinoid perfectoid space with the $+$-ring *almost* isomorphic to $A_\infty^+ \widehat{\otimes}_{\mathcal{O}_K} \mathcal{O}_L$. The strictly totally disconnected claim follows from [61, Lemma 7.19] and the almost computation of the $+$-ring follows from the proof of [59, Proposition 6.18]. ■

7.6 Perfectoid torsors

We apply the results of Section 7.4 to certain pro-étale covers of $\mathrm{Spa}\,(A, A^+)$ to see that the computation of v-cohomology groups can often be reduced to the computation of certain continuous cohomology groups. To make this precise, we need to define the notion of a G-torsor under a pro-finite group G.

Definition 7.6.1. A *v-sheaf $\underline{G}$ associated with a pro-finite group* G is a v-sheaf $\underline{G}\colon \mathrm{Perf}^{\mathrm{op}} \to \mathrm{Sets}$ such that $\underline{G}(S) = \mathrm{Hom}_{\mathrm{cont}}(|S|, G)$.

A morphism of v-sheaves $X \to Y$ is a *$\underline{G}$-torsor* if it is a v-surjection and there is an action $a\colon \underline{G} \times X \to X$ over Y such that the morphism $a \times_Y p_2\colon \underline{G} \times X \to X \times_Y X$ is an isomorphism, where $p_2\colon \underline{G} \times X \to X$ is the canonical projection.

Remark 7.6.2. If a pro-finite group G is a cofiltered limit of finite groups, that is, $G \simeq \lim_I G_i$, then $\underline{G} \simeq \lim_I \underline{G_i}$.

Now we can formulate the precise set-up we are going to work in.

Set-up 7.6.3. We fix

(1) a p-adic perfectoid field K with its rank-1 open and bounded valuation ring denoted by $\mathcal{O}_K$ and a good pseudo-uniformizer $\varpi \in \mathcal{O}_K$ (we always do almost mathematics with respect to the ideal $\mathfrak{m} = \bigcup_n \varpi^{1/p^n}\mathcal{O}_K = K^{\circ\circ}$);

(2) an admissible formal $\mathcal{O}_K$-scheme $\mathfrak{X} = \mathrm{Spf}\ A_0$ with adic generic fiber $X = \mathrm{Spa}\,(A, A^+)$;

(3) a morphism $(A, A^+) \to (A_\infty, A_\infty^+)$ such (A_∞, A_∞^+) is a perfectoid pair and $\mathrm{Spd}\,(A_\infty, A_\infty^+) \to \mathrm{Spd}\,(A, A^+)$ is a $\underline{\Delta_\infty}$-torsor under a pro-finite group Δ_∞;

(4) a small $\mathcal{O}_{X^\diamond}^+/p$-vector bundle $\mathcal{E}$.

At the beginning of this section, we analyze the structure of the fiber products $\mathrm{Spd}\,(A_\infty, A_\infty^+)^{j/\mathrm{Spd}\,(A,A^+)}$ for $j \geq 1$. For a general v-cover, we cannot say much about these fiber products. However, we have much more control in the case of $\underline{G}$-torsors.

Lemma 7.6.4. *With the notation and under the assumption of Set-up 7.6.3, the fiber product* $\mathrm{Spd}\,(A_\infty, A_\infty^+)^{j/\mathrm{Spd}\,(A,A^+)}$ *is represented by an affinoid perfectoid space*[6] $\mathrm{Spa}\,(T_j, T_j^+)$*, for every* $j \geq 0$*. Moreover, for every* $j \geq 0$*,*

$$\left(T_j, T_j^+\right) \simeq \left(\mathrm{Map}_{\mathrm{cont}}(\Delta_\infty^{j-1}, A_\infty^\flat), \mathrm{Map}_{\mathrm{cont}}(\Delta_\infty^{j-1}, A_\infty^{\flat,+})\right)$$

and $T_j^{\sharp,+}/pT_j^{\sharp,+} \simeq T_j^+/\varpi^\flat T_j^+ \simeq \mathrm{Map}_{\mathrm{cont}}\left(\Delta_\infty^{j-1}, A_\infty^+/pA_\infty^+\right)$.

Proof. We first show that $\mathrm{Spd}\,(A_\infty, A_\infty^+)^{j/\mathrm{Spd}\,(A,A^+)}$ are representable by affinoid perfectoid spaces. We write a presentation of $\Delta_\infty = \lim_I \Delta_i$ as a cofiltered limit of finite groups. Since $\mathrm{Spd}\,(A_\infty, A_\infty^+) \to \mathrm{Spd}\,(A, A^+)$ is a $\underline{\Delta}_\infty$-torsor, we get that

$$\begin{aligned}
\mathrm{Spd}\,(A_\infty, A_\infty^+)^{j/\mathrm{Spd}\,(A,A^+)} &\simeq \mathrm{Spd}\,(A, A^+) \times \underline{\Delta}_\infty^{j-1} \\
&\simeq \lim_I \left(\mathrm{Spa}\,(A_\infty^\flat, A_\infty^{\flat,+}) \times \underline{\Delta}_i^{j-1}\right) \\
&\simeq \lim_I \left(\mathrm{Spa}\,\left(\mathrm{Map}(\Delta_i^{j-1}, A_\infty^\flat), \mathrm{Map}(\Delta_i^{j-1}, A_\infty^{\flat,+})\right)\right)
\end{aligned}$$

is a cofiltered limit of affinoid perfectoid spaces, so it is an affinoid perfectoid space $\mathrm{Spa}\,(T_j, T_j^+)$ by [61, Proposition 6.5]. Moreover, loc. cit. implies that T_j^+ is equal to the $\varpi^\flat$-adic completion of the filtered colimit $\mathrm{colim}_I\, \mathrm{Map}(\Delta_i^{j-1}, A_\infty^{\flat,+})$ and $T_j = T_j^+[\frac{1}{\varpi^\flat}]$. In particular, we already see that

$$\begin{aligned}
T_j^{\sharp,+}/pT_j^{\sharp,+} \simeq T_j^+/(\varpi)^\flat T_j^+ &\simeq \left(\mathrm{colim}_I\, \mathrm{Map}(\Delta_i^{j-1}, A_\infty^{\flat,+})\right)/(\varpi)^\flat \\
&\simeq \mathrm{colim}_I\, \mathrm{Map}\left(\Delta_i^{j-1}, A_\infty^{\flat,+}/(\varpi)^\flat A_\infty^{\flat,+}\right) \\
&\simeq \mathrm{colim}_I\, \mathrm{Map}\left(\Delta_i^{j-1}, A_\infty^+/\varpi A_\infty^+\right) \\
&\simeq \mathrm{colim}_I\, \mathrm{Map}\left(\Delta_i^{j-1}, A_\infty^+/pA_\infty^+\right) \\
&\simeq \mathrm{Map}_{\mathrm{cont}}\left(\Delta_\infty^{j-1}, A_\infty^+/pA_\infty^+\right).
\end{aligned}$$

Now we compute T_j^+ and T_j. We start with T_j^+:

$$\begin{aligned}
T_j^+ &\simeq \lim_n \left(\mathrm{colim}_I\, \mathrm{Map}(\Delta_i^{j-1}, A_\infty^{\flat,+})/(\varpi^\flat)^n\right) \\
&\simeq \lim_n \left(\mathrm{colim}_I\, \mathrm{Map}(\Delta_i^{j-1}, A_\infty^{\flat,+}/(\varpi^\flat)^n A_\infty^{\flat,+})\right) \\
&\simeq \lim_n \mathrm{Map}\left(\Delta_\infty^{j-1}, A_\infty^{\flat,+}/(\varpi^\flat)^n A_\infty^{\flat,+}\right) \\
&\simeq \mathrm{Map}_{\mathrm{cont}}\left(\Delta_\infty^{j-1}, \lim_n A_\infty^{\flat,+}/(\varpi^\flat)^n A_\infty^{\flat,+}\right) \\
&\simeq \mathrm{Map}_{\mathrm{cont}}\left(\Delta_\infty^{j-1}, A_\infty^{\flat,+}\right).
\end{aligned}$$

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[6] Recall that $\mathrm{Spd}\,(A_\infty, A_\infty^+)$ is itself represented by an affinoid perfectoid $\mathrm{Spa}\,(A_\infty^\flat, A_\infty^{\flat,+})$.

Since Δ_∞ is compact and $A^{\flat}_{\infty} \simeq A^{\flat,+}_{\infty}[\frac{1}{\varpi^\flat}]$, we also have

$$\begin{aligned} T_j &\simeq T_j^+\left[\frac{1}{\varpi^\flat}\right] \\ &\simeq \operatorname{colim}_{\times\varpi^\flat, n} \operatorname{Map}_{\mathrm{cont}}\left(\Delta_\infty^{j-1}, A_\infty^{\flat,+}\right) \\ &\simeq \operatorname{Map}_{\mathrm{cont}}\left(\Delta_\infty^{j-1}, \operatorname{colim}_{\times\varpi^\flat} A_\infty^{\flat,+}\right) \\ &\simeq \operatorname{Map}_{\mathrm{cont}}\left(\Delta_\infty^{j-1}, A_\infty^{\flat}\right) \end{aligned}$$

finishing the proof. ■

Warning 7.6.5. The fiber product $\operatorname{Spa}(A_\infty, A_\infty^+) \times_{\operatorname{Spa}(A,A^+)} \operatorname{Spa}(A_\infty, A_\infty^+)$ in the category of adic spaces is often not a perfectoid space. This already happens for $\operatorname{Spa}(A, A^+) = \operatorname{Spa}(\mathbf{Q}_p, \mathbf{Z}_p)$ and $\operatorname{Spa}(A_\infty, A_\infty^+) = \operatorname{Spa}(\mathbf{Q}_p(\mu_{p^\infty})^\wedge, \mathbf{Z}_p[\mu_{p^\infty}]^\wedge)$. However, the diamond $\operatorname{Spd}(A_\infty, A_\infty^+) \times_{\operatorname{Spd}(A,A^+)} \operatorname{Spd}(A_\infty, A_\infty^+)$ is always represented by an affinoid perfectoid space as guaranteed by Lemma 7.6.4.

Note that since $\operatorname{Spd}(A_\infty, A_\infty^+) \to \operatorname{Spd}(A, A^+)$ is a $\underline{\Delta}_\infty$-torsor, there is a canonical continuous A^+-linear action of Δ_∞ on A_∞^+. Now we want to relate v-cohomology groups of $\mathcal{E}$ to the continuous group cohomology of Δ_∞. This is done in the following lemmas:

Lemma 7.6.6. *Under the assumption of Set-up 7.6.3, we define $M_{\mathcal{E}}$ to be the A_∞^+/p-module $\mathrm{H}^0\left(\operatorname{Spd}(A_\infty, A_\infty^+)_v, \mathcal{E}\right)$. Then $M_{\mathcal{E}}$ is an almost faithfully flat, almost finitely presented A_∞^+/p-module, and for every $i, j \geq 1$,*

$$\mathrm{H}^0\left(\operatorname{Spd}(A_\infty,A_\infty^+)_v^{j/\operatorname{Spd}(A,A^+)},\mathcal{E}\right) \simeq^a \operatorname{Map}_{\mathrm{cont}}\left(\Delta_\infty^{j-1},M_{\mathcal{E}}\right) \simeq^a \operatorname{Map}_{\mathrm{cont}}\left(\Delta_\infty^{j-1},(M_{\mathcal{E}}^a)_!\right),$$
$$\mathrm{H}^i\left(\operatorname{Spd}(A_\infty,A_\infty^+)_v^{j/\operatorname{Spd}(A,A^+)}, \mathcal{E}\right) \simeq^a 0.$$

Proof. Lemma 7.6.4 implies that all fiber products $\operatorname{Spd}(A_\infty, A_\infty^+)^{j/\operatorname{Spd}(A,A^+)}$ satisfy the assumptions of Lemma 7.4.8. Thus, Lemma 7.4.8 and the computation of fiber products in Lemma 7.6.4 imply that

$$\mathrm{H}^i\left(\operatorname{Spd}(A_\infty, A_\infty^+)_v^{j/\operatorname{Spd}(A,A^+)}, \mathcal{E}\right) \simeq^a 0$$

for every $i, j \geq 1$, and the natural morphism

$$M_{\mathcal{E}} \otimes_{A_\infty^+/p} \operatorname{Map}_{\mathrm{cont}}\left(\Delta_\infty^{j-1}, A_\infty^+/p\right) \to \mathrm{H}^0\left(\operatorname{Spd}(A_\infty, A_\infty^+)_v^{j/\operatorname{Spd}(A,A^+)}, \mathcal{E}\right)$$

is an almost isomorphism for every $j \geq 1$. Thus, it suffices to show that the natural morphism

$$M_{\mathcal{E}} \otimes_{A_\infty^+/p} \operatorname{Map}_{\mathrm{cont}}\left(\Delta_\infty^{j-1}, A_\infty^+/p\right) \to \operatorname{Map}_{\mathrm{cont}}\left(\Delta_\infty^{j-1}, M_{\mathcal{E}}\right)$$

is an isomorphism. This can be done by writing $\Delta_\infty = \lim_I \Delta_i$ and reducing to the case of a finite group similarly to the proof of Lemma 7.6.4. The almost isomorphism

$$\operatorname{Map}_{\mathrm{cont}}(\Delta_\infty^{j-1}, M_{\mathcal{E}}) \simeq^a \operatorname{Map}_{\mathrm{cont}}(\Delta_\infty^{j-1}, (M_{\mathcal{E}}^a)_!)$$

is achieved similarly using that $(-)_!$ commutes with colimits being a left adjoint functor. ■

Lemma 7.6.7. *Under the assumption of Set-up 7.6.3, let* $M_{\mathcal{E}}$ *be the* A^+_∞/pA^+_∞*-module* $\mathrm{H}^0\big(\mathrm{Spd}\,(A_\infty, A^+_\infty)_v, \mathcal{E}\big)$*. Then there is a canonical continuous action of* Δ_∞ *on* $(M^a_{\mathcal{E}})_!$ *compatible with the action of* Δ_∞ *on* A^+_∞/p*, i.e.,* $g(am) = g(a)g(m)$ *for any* $a \in A^+_\infty/p$ *and* $m \in M_{\mathcal{E}}$*.*

Proof. By Lemma 7.6.4, the fiber product $\mathrm{Spd}\,(A_\infty, A^+_\infty) \times_{\mathrm{Spd}\,(A,A^+)} \mathrm{Spd}\,(A_\infty, A^+_\infty)$ is represented by an affinoid perfectoid space $\mathrm{Spa}\,(T_2, T_2^+)$ of characteristic p. Therefore, we can uniquely write it as $\mathrm{Spd}\,(S, S^+)$ for an untilt of (T_2, T_2^+) corresponding to the morphism $\mathrm{Spa}\,(T_2, T_2^+) \to \mathrm{Spd}\,(A, A^+) \to \mathrm{Spd}\,(\mathbf{Q}_p, \mathbf{Z}_p)$.

Lemma 7.4.8 implies that the descent data for the sheaf $\mathcal{E}$ provide us with an $(S^+/pS^+)^a$-isomorphism

$$(S^+/p)^a \otimes_{(A^+_\infty/p)^a} (M_{\mathcal{E}})^a \to (M_{\mathcal{E}})^a \otimes_{(A^+_\infty/p)^a} (S^+/p)^a$$

satisfying the cocycle condition. By Corollary 2.2.4 (2), this defines an $(A^+_\infty/p)^a$-linear morphism

$$\big(M_{\mathcal{E}}\big)^a \to \big(M_{\mathcal{E}}\big)^a \otimes_{(A^+_\infty/p)^a} \big(S^+/p\big)^a.$$

By Lemma 7.4.8 and Lemma 7.6.6, this is equivalent to an $(A^+_\infty/p)^a$-linear morphism

$$\big(M_{\mathcal{E}}\big)^a \to \mathrm{Map}_{\mathrm{cont}}\big(\Delta_\infty, (M^a_{\mathcal{E}})_!\big)^a.$$

By Lemma 2.1.9 (3), this is the same as an (A^+_∞/pA^+_∞)-linear morphism

$$\phi\colon \big(M^a_{\mathcal{E}}\big)_! \to \mathrm{Map}_{\mathrm{cont}}\big(\Delta_\infty, \big(M^a_{\mathcal{E}}\big)_!\big).$$

This defines a morphism

$$\gamma\colon \Delta_\infty \to \mathrm{Hom}_{A^+_\infty/p}\big(\big(M_{\mathcal{E}}\big)_!, \big(M_{\mathcal{E}}\big)_!\big)$$

by the rule

$$\gamma(g)(m) = (\phi(m))(g).$$

One checks that the cocycle condition translates to the statement that γ is a group homomorphism, i.e., it defines an action of Δ_∞. Similarly, one checks that A^+_∞/p-linearity of ϕ translates into the fact that this action is compatible with the action on A^+_∞/p. And continuity of ϕ translates to the fact that γ defines a continuous action, i.e., the natural morphism

$$\mathrm{colim}_{U_i \triangleleft \Delta_\infty, \mathrm{open}}\big(M^a_{\mathcal{E}}\big)_!^{U_i} \to \big(M^a_{\mathcal{E}}\big)_!^{\Delta_\infty}$$

is an isomorphism. ■

Corollary 7.6.8. *Under the assumption of Set-up 7.6.3, let $M_{\mathcal{E}}$ be the A_∞^+/p-module* $\mathrm{H}^0\big(\mathrm{Spd}\,(A_\infty, A_\infty^+)_v, \mathcal{E}\big)$. *Then*

$$\mathrm{H}^i\big(\mathrm{Spd}\,(A, A^+)_v, \mathcal{E}\big) \simeq^a \mathrm{H}^i_{\mathrm{cont}}\big(\Delta_\infty, (M_{\mathcal{E}}^a)_!\big).$$

Proof. Lemma 7.6.6 implies that

$$\mathrm{H}^i\big(\mathrm{Spd}\,(A_\infty, A_\infty^+)_v^{j/\mathrm{Spd}\,(A,A^+)}, \mathcal{E}\big) \simeq^a 0$$

for $i, j \geq 1$. Consequently, the cohomology groups $\mathrm{H}^i\big(\mathrm{Spd}\,(A, A^+)_v, \mathcal{E}\big)$ can be almost computed via cohomology of the Čech complex associated with the covering $\mathrm{Spd}\,(A_\infty, A_\infty^+) \to \mathrm{Spd}\,(A, A^+)$. Moreover, Lemma 7.6.6 also implies that the terms of this complex can be almost identified with the bar complex computing the continuous cohomology of the pro-finite group Δ_∞ with coefficients in the discrete module $(M_{\mathcal{E}}^a)_!$. We leave it to the reader to verify that the differentials in the Čech complex coincide with the differentials in the bar complex computing the continuous cohomology. ■

For future reference, we also discuss the following base change result:

Lemma 7.6.9. *Let G be a pro-finite group, and let M be a discrete R-module that has a continuous R-linear action of G. Suppose that $R \to A$ is a flat homomorphism of rings. Then the canonical morphism $\mathrm{H}^i_{\mathrm{cont}}(G, M) \otimes_R A \to \mathrm{H}^i_{\mathrm{cont}}(G, M \otimes_R A)$ is an isomorphism for $i \geq 0$.*

Proof. We first prove the claim for H^0. Since G acts on M continuously, we can write $M = \mathrm{colim}_I\, M_i$ as a filtered colimit of G-stable R-submodules of M such that the action of G on M_i factors through a finite group G_i. Since both $\mathrm{H}^0_{\mathrm{cont}}(G, -) \otimes_R A$ and $\mathrm{H}^0_{\mathrm{cont}}(G, - \otimes_R A)$ commute with filtered colimits, we can reduce to the case when the action of G factors through a finite group quotient. In this case, the result is classical (see, for example, [29, Exp. V, Proposition 1.9]).

In general, the result follows from the following sequence of isomorphisms:

$$\begin{aligned}
\mathrm{H}^i_{\mathrm{cont}}(G, M) \otimes_R A &\cong \big(\mathrm{colim}_{H \triangleleft G, \mathrm{open}}\, \mathrm{H}^i(G/H, M^H)\big) \otimes_R A \\
&\simeq \mathrm{colim}_{H \triangleleft G, \mathrm{open}} \big(\mathrm{H}^i(G/H, M^H) \otimes_R A\big) \\
&\simeq \mathrm{colim}_{H \triangleleft G, \mathrm{open}}\, \mathrm{H}^i\big(G/H, M^H \otimes_R A\big) \\
&\simeq \mathrm{colim}_{H \triangleleft G, \mathrm{open}}\, \mathrm{H}^i\big(G/H, (M \otimes_R A)^H\big) \\
&\simeq \mathrm{H}^i_{\mathrm{cont}}\big(G, M \otimes_R A\big).
\end{aligned}$$

■

7.7 Nearby cycles are quasi-coherent

We start the proof of Theorem 7.1.9 and Theorem 7.1.2 in this section. Namely, we show that the complex $\mathbf{R}\nu_*\mathcal{E}$ is quasi-coherent and commutes with étale base change for an $\mathcal{O}^+_{X^\diamond}/p$-vector bundle $\mathcal{E}$. The main idea is to apply the results of Section 7.4 to a particular perfectoid covering of X.

For the rest of this section, we fix a perfectoid p-adic field K with a good pseudo-uniformizer $\varpi \in \mathcal{O}_K$ (see Definition B.11). We always do almost mathematics with respect to the ideal $\mathfrak{m} = \bigcup_n \varpi^{1/p^n}\mathcal{O}_K$.

Lemma 7.7.1. *Let $\mathfrak{X} = \operatorname{Spf} A_0$ be an admissible affine formal $\mathcal{O}_K$-scheme with an affinoid generic fiber $X = \operatorname{Spa}(A, A^+)$, and let $\mathcal{E}$ be an $\mathcal{O}^+_{X^\diamond}/p$-vector bundle. Then $\mathrm{R}^i\nu_*\mathcal{E}$ is quasi-coherent for $i \geq 0$. More precisely, the natural morphism*

$$\widetilde{\mathrm{H}^i\left(X_v^\diamond, \mathcal{E}\right)} \to \mathrm{R}^i\nu_*\mathcal{E}$$

is an isomorphism for any $i \geq 0$.

Proof. The universal property of the tilde-construction implies that we do have a natural morphism

$$c : \widetilde{\mathrm{H}^i\left(X_v^\diamond, \mathcal{E}\right)} \to \mathrm{R}^i\nu_*\mathcal{E}.$$

Recall that $\mathrm{R}^i\nu_*\mathcal{E}$ is the sheafification of the presheaf defined by the rule

$$\mathfrak{U} \mapsto \mathrm{H}^i\left(\mathfrak{U}^\diamond_{K,v}, \mathcal{E}\right).$$

Thus, in order to show that c is an isomorphism, it suffices to show that the natural morphism

$$\mathrm{H}^i\left(X_v^\diamond, \mathcal{E}\right) \otimes_{A_0/p} \left(A_0/p\right)_f \to \mathrm{H}^i\left(\mathfrak{U}^\diamond_{K,v}, \mathcal{E}\right)$$

is an isomorphism for any open formal subscheme $\operatorname{Spf}(A_0)_{\{f\}} \subset \operatorname{Spf} A_0$. We choose a covering $\operatorname{Spa}(A_\infty, A_\infty) \to \operatorname{Spa}(A, A^+)$ from Lemma 6.2.13. Then the result follows from Corollary 7.5.5 since $(A, A^+) \to (A_\infty, A^+_\infty)$ fits into Set-up 7.5.1. ∎

Theorem 7.7.2. *Let $\mathfrak{X}$ be an admissible formal $\mathcal{O}_K$-scheme with adic generic fiber $X = \mathfrak{X}_K$, and let $\mathcal{E}$ be an $\mathcal{O}^+_{X^\diamond}/p$-vector bundle. Then $\mathrm{R}^i\nu_*\mathcal{E}$ is quasi-coherent for $i \geq 0$. Furthermore, if $\mathfrak{f} : \mathfrak{Y} \to \mathfrak{X}$ is an étale morphism with generic fiber $f : Y \to X$, then the natural morphism*

$$\mathfrak{f}_0^*\left(\mathrm{R}^i\nu_{\mathfrak{X},*}\mathcal{E}\right) \to \mathrm{R}^i\nu_{\mathfrak{Y},*}\left(\mathcal{E}|_{Y_v^\diamond}\right)$$

is an isomorphism for any $i \geq 0$.

Proof. Both claims are local on $\mathfrak{X}$ and $\mathfrak{Y}$, so we can assume that $\mathfrak{X} = \operatorname{Spf} A_0$ and $\mathfrak{Y} = \operatorname{Spf} B_0$ are affine. Then quasi-coherence of $\mathrm{R}^i\nu_*(\mathcal{E})$ directly follows from Lemma 7.7.1. In order to show that $\mathfrak{f}_0^*\left(\mathrm{R}^i\nu_{\mathfrak{X},*}\mathcal{E}\right) \to \mathrm{R}^i\nu_{\mathfrak{Y},*}\left(\mathcal{E}|_{Y_v^\diamond}\right)$ is an isomorphism, it suffices to show that the natural morphism

$$\mathrm{H}^i\left(X_v^\diamond, \mathcal{E}\right) \otimes_{A_0/p} B_0/p \to \mathrm{H}^i\left(Y_v^\diamond, \mathcal{E}\right)$$

is an isomorphism. This follows by the application of Corollary 7.5.5 to the covering $\operatorname{Spa}(A_\infty, A_\infty^+) \to \operatorname{Spa}(A, A^+)$ from Lemma 6.2.13. ■

For future reference, we also prove the following result:

Lemma 7.7.3. *Let* $X = \operatorname{Spa}(A, A^+)$ *be an affinoid rigid-analytic space over* K*, let* $\mathcal{E}$ *be an* $\mathcal{O}_{X^\diamond}^+/p$*-vector bundle, and let* $K \subset C$ *be a completed algebraic closure of* K*. Then*

$$\mathrm{H}^i\left(X_v^\diamond, \mathcal{E}\right) \otimes_{\mathcal{O}_K/p} \mathcal{O}_C/p \to \mathrm{H}^i\left(X_{C,v}^\diamond, \mathcal{E}\right)$$

is an almost isomorphism.

Proof. Similarly to the reasoning above, this follows directly from Corollary 7.5.6 using the covering $\operatorname{Spa}(A_\infty, A_\infty^+) \to \operatorname{Spa}(A, A^+)$ from Lemma 6.2.13. ■

7.8 Nearby cycles are almost coherent for smooth X and small $\mathcal{E}$

The main goal of this section is to show that the complex $\mathbf{R}\nu_*\mathcal{E}$ has almost coherent cohomology sheaves for an admissible formal $\mathcal{O}_K$-scheme with *smooth* generic fiber. The main idea is to apply the results of Section 7.6 to a particular "small" perfectoid torsor cover of X, where one has good control over the structure group Δ_∞.

For the rest of the section, we fix a p-adic perfectoid field K with a good pseudo-uniformizer $\varpi \in \mathcal{O}_K$. We always do almost mathematics with respect to the ideal $\mathfrak{m} = \bigcup_n \varpi^{1/p^n}\mathcal{O}_K$.

Before we embark on the proof, we discuss the overall strategy of the proof. We proceed in four steps: first, we show the result for $\widehat{\mathbf{G}}_m^n$ and $\mathcal{E} = \mathcal{O}_{X^\diamond}^+/p$; then we deduce the result for affine formal schemes such that the adic generic fiber admits a map to a torus $\mathbf{T}_C^n$ that is a composition of finite étale maps and rational embeddings. After that, we finish the proof for $\mathcal{E} = \mathcal{O}_{X^\diamond}^+/p$ and a general smooth X by choosing a "good" covering of $\mathfrak{X}$, possibly after an admissible blow-up of $\mathfrak{X}$. We reduce the general case to the case $\mathcal{E} = \mathcal{O}_{X^\diamond}^+/p$ via Corollary 6.6.9.

The main ingredient for the third step is Achinger's result ([1, Proposition 6.6.1]) that any étale morphism $g\colon \operatorname{Spa}(A, A^+) \to \mathbf{D}_K^n$ can be replaced with a finite étale morphism

$$g'\colon \operatorname{Spa}(A, A^+) \to \mathbf{D}_K^n.$$

The proof of this result in [1] is given only for rigid-analytic varieties over discretely valued non-archimedean fields, but we need to apply it in the perfectoid situation that is never discretely valued. So Appendix D provides the reader with a detailed proof of this result without any discreteness assumptions.

To realize the above sketched strategy, we consider $\mathfrak{X} = \operatorname{Spf} \mathcal{O}_K\langle T_1^{\pm 1}, \dots, T_n^{\pm 1}\rangle$, and set $R^+ := \mathcal{O}_K\langle T_1^{\pm 1}, \dots, T_n^{\pm 1}\rangle$ and $R_m^+ := \mathcal{O}_K\langle T_1^{\pm 1/p^m}, \dots, T_n^{\pm 1/p^m}\rangle$. We note that the map Spf $R^+_m \to$ Spf R^+ defines a $\mu^n_{p^m}$-torsor, thus $\mu^n_{p^m}$ continuously acts on R^+_m by R^+-linear automorphisms.

Now we consider the R^+-algebra

$$R^+_\infty = \mathcal{O}_K\langle T_1^{\pm 1/p^\infty}, \dots, T_n^{\pm 1/p^\infty}\rangle = \left(\operatorname{colim}_n\ R^+_m\right)^\wedge$$

where $^\wedge$ stands for the p-adic completion. It comes with a continuous R^+-linear action of the profinite group $\Delta_\infty := \mathbf{Z}_p(1)^n = T_p(\mu_{p^\infty})$ on R^+_∞. We trivialize $\mathbf{Z}_p(1)$ by choosing some compatible system of p^i-th roots of unity $(\zeta_p, \zeta_{p^2}, \zeta_{p^3}, \dots)$. To describe the action of Δ_∞ on R^+_∞ we need the following definition:

Definition 7.8.1. For any $a \in \mathbf{Z}[\frac{1}{p}]$, we define ζ^a as $\zeta_{p^l}^{ap^l}$ whenever $ap^l \in \mathbf{Z}$. It is easy to see that this definition does not depend on the choice of l.

Essentially by definition, the k-th basis vector $\gamma_k \in \Delta_\infty \simeq \mathbf{Z}_p^n$ acts on R^+_∞ as

$$\gamma_k\left(T_1^{a_1} \cdots T_n^{a_n}\right) = \zeta^{a_k} T_1^{a_1} \cdots T_n^{a_n}.$$

Lemma 7.8.2 ([59, Lemma 5.5]). *Let R^+, R^+_∞ and Δ_∞ be as above. Then the cohomology groups $\mathrm{H}^i_{\mathrm{cont}}(\Delta_\infty, R^+_\infty/p)$ are almost coherent R^+/p-modules. Moreover, the natural map*

$$\mathrm{H}^i_{\mathrm{cont}}\left(\Delta_\infty, R^+_\infty/p\right) \otimes_{R^+/p} A^+/p \to \mathrm{H}^i_{\mathrm{cont}}\left(\Delta_\infty, R^+_\infty/p \otimes_{R^+/p} A^+/p\right)$$

is an isomorphism for a p-torsionfree R^+-algebra A^+ and $i \geq 0$.

Proof. We note that R^+/p is an almost noetherian ring due Theorem 2.11.5. Thus, Corollary 2.7.8 implies that $\mathrm{H}^i_{\mathrm{cont}}(\Delta_\infty, R^+_\infty/pR^+_\infty)$ is almost coherent if it is almost finitely generated.

Now [8, Lemma 7.3] says that $\mathbf{R}\Gamma_{\mathrm{cont}}(\Delta_\infty, R^+_\infty/p)$ is computed via the Koszul complex $K\left(R^+_\infty/p; \gamma_1 - 1, \dots, \gamma_n - 1\right)$. Then, similarly to [6, Lemma 4.6], we can write

$$\begin{aligned} &K\left(R^+_\infty/p; \gamma_1 - 1, \dots, \gamma_n - 1\right) \\ &\quad = K\left(R^+/p; 0, 0, \dots, 0\right) \oplus \bigoplus_{\substack{(a_1,\dots,a_n) \\ \in (\mathbf{Z}[1/p]\cap(0,1))^n}} K\left(R^+/p; \zeta^{a_1} - 1, \dots, \zeta^{a_n} - 1\right). \end{aligned}$$

We observe that

$$\mathrm{H}^i\left(K(R^+/p; 0, 0, \dots, 0)\right) = \wedge^i\left(R^+/p\right)$$

is a free finitely presented R^+/p-module. For each $(a_1, \dots, a_n) \in \left(\mathbf{Z}[\frac{1}{p}] \cap (0,1)\right)^n$, we can assume that a_1 has the minimal p-adic valuation for the purpose of proving that

$$K\left(R^+_\infty/p; \gamma_1 - 1, \dots, \gamma_n - 1\right)$$

has almost finitely generated cohomology groups. Then [8, Lemma 7.10] implies that $\mathrm{H}^i\big(K(R^+/p;\zeta^{a_1}-1,\dots,\zeta^{a_n}-1)\big)$ is finitely presented over R^+/p and a $\zeta^{a_1}-1$-torsion module. Note that

$$v_p(\zeta^{a_1}-1)=v_p(\zeta_{p^l}-1)=\frac{v(p)}{p^l-p^{l-1}}\to 0$$

where $a_1=b/p^l$ with $\gcd(b,p)=1$. Furthermore, for any $h\in\mathbf{Z}$, there are only finitely many indices $(a_1,\dots,a_n)\in(\mathbf{Z}[\frac{1}{p}]\cap(0,1))^n$ with $v_p(a_j)\geq h$. This implies that

$$\mathrm{H}^i_{\mathrm{cont}}\big(\Delta_\infty,R^+_\infty/p\big)=\mathrm{H}^i\big(K(R^+_\infty/p;\gamma_1-1,\dots,\gamma_n-1)\big)$$

is a finitely presented R^+/p-module up to any ϖ^{1/p^n}-torsion. In particular, this module is almost finitely presented.

Now we show that $\mathrm{H}^i_{\mathrm{cont}}(\Delta_\infty,R^+_\infty/p)$ commutes with base change for any $\mathcal{O}_K$-flat algebra A^+. In order to show this, we observe that the $(R^+/p)[\Delta_\infty]$-module R^+_∞/p comes as a tensor product $M\otimes_{\mathcal{O}_K/p}R^+/p$ for the $(\mathcal{O}_K/p)[\Delta_\infty]$-module

$$M:=\bigoplus_{\substack{(a_1,\dots,a_n)\\ \in(\mathbf{Z}[1/p]\cap[0,1))^n}}(\mathcal{O}_K/p\mathcal{O}_K)T_1^{a_1}\cdots T_n^{a_n},$$

where the basis element γ_k acts by

$$\gamma_k(T_1^{a_1}\cdots T_n^{a_n})=\zeta^{a_k}T_1^{a_1}\cdots T_n^{a_n}.$$

Therefore, the desired claim follows from a sequence of isomorphisms

$$\begin{aligned}&\mathrm{H}^i_{\mathrm{cont}}\big(\Delta_\infty,R^+_\infty/p\big)\otimes_{R^+/p}A^+/p\\ &\quad\simeq\big(\mathrm{H}^i_{\mathrm{cont}}(\Delta_\infty,M)\otimes_{\mathcal{O}_K/p}R^+/p\big)\otimes_{R^+/p}A^+/p\\ &\quad\simeq\mathrm{H}^i_{\mathrm{cont}}\big(\Delta_\infty,M\big)\otimes_{\mathcal{O}_K/p}A^+/p\\ &\quad\simeq\mathrm{H}^i_{\mathrm{cont}}\big(\Delta_\infty,M\otimes_{\mathcal{O}_K/p}A^+/p\big)\\ &\quad\simeq\mathrm{H}^i_{\mathrm{cont}}\big(\Delta_\infty,R^+_\infty/p\otimes_{R^+/p}A^+/p\big),\end{aligned}$$

where the third isomorphism uses Lemma 7.6.9. ■

Lemma 7.8.2 combined with Corollary 7.6.8 essentially settles the first step of our strategy. Now we move to the second step. We start with the following preliminary result:

Lemma 7.8.3. *Let A_0 be a topologically finitely presented $\mathcal{O}_K$-algebra, and P a topologically free A_0-module, i.e., $P=\widehat{\bigoplus}_I A_0$ for some set I. Then M is A_0-flat.*

Proof. We start the proof by noting that [68, Tag 00M5] guarantees that it suffices to show that $\mathrm{Tor}_1^{A_0}(P, M) = 0$ for any finitely presented A_0-module M. We choose a presentation

$$0 \to Q \to A_0^n \to M \to 0$$

and observe that Q is finitely presented because A_0 is coherent. So vanishing of Tor_1 is equivalent to showing that

$$P \otimes_{A_0} Q \to P \otimes_{A_0} A_0^n$$

is injective.

Now note that $Q[p^\infty]$, $A_0^n[p^\infty]$, and $M[p^\infty]$ are bounded by [11, Lemma 7.3/7], so the same holds for $\bigoplus_I Q$, $\bigoplus_I A_0^n$, and $\bigoplus_I M$. Therefore, the usual p-adic completions of $\bigoplus_I Q$, $\bigoplus_I A_0^n$ and $\bigoplus_I M$ coincide with their derived p-adic completions. Since derived p-adic completion is exact (in the sense of triangulated categories) and coincides with the usual one on these modules, we get that the sequence

$$0 \to \widehat{\bigoplus_I} Q \to \widehat{\bigoplus_I} A_0^n \to \widehat{\bigoplus_I} M \to 0$$

is exact.

Now we want to show that this short exact sequence is the same as the sequence

$$P \otimes_{A_0} Q \to P \otimes_{A_0} A_0^n \to P \otimes_{A_0} M \to 0.$$

As a consequence, this will show that $P \otimes_{A_0} Q \to P \otimes_{A_0} A_0^n$ is injective.

For each A_0-module N, there is a canonical map

$$P \otimes_{A_0} N \to \widehat{\bigoplus_I} N.$$

So we have a morphism of sequences:

$$\begin{tikzcd}
 & P \otimes_{A_0} Q \arrow[r] \arrow[d] & P \otimes_{A_0} A_0^n \arrow[r] \arrow[d] & P \otimes_{A_0} M \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \widehat{\bigoplus}_I Q \arrow[r] & \widehat{\bigoplus}_I A_0^n \arrow[r] & \widehat{\bigoplus}_I M \arrow[r] & 0.
\end{tikzcd}$$

The map $A_0^n \otimes_{A_0} P \to \widehat{\bigoplus}_I A_0^n$ is an isomorphism because $A_0^n \otimes_{A_0} P = P^n$ is already p-adically complete. This implies that the arrow

$$M \otimes_{A_0} P \to \widehat{\bigoplus_I} M$$

is surjective. But then

$$P \otimes_{A_0} Q \to \widehat{\bigoplus_I} Q$$

is surjective since M was an arbitrary finitely presented A-module. Now a diagram chase implies that

$$M \otimes_{A_0} P \to \widehat{\bigoplus_I} M$$

is also injective. And, therefore, it is an isomorphism. So

$$P \otimes_{A_0} Q \to \widehat{\bigoplus_I} Q$$

is also an isomorphism. Therefore, these two sequences are the same. In particular,

$$P \otimes_{A_0} Q \to P \otimes_{A_0} A_0^n$$

is injective. ■

To establish the second part of our strategy, we will also need a slightly refined version of [59, Lemma 4.5] specific to the situation of an étale morphism to a torus.

We recall that we have defined

$$R^+ := \mathcal{O}_K\langle T_1^{\pm 1}, \dots, T_n^{\pm 1}\rangle,$$
$$R_m^+ := \mathcal{O}_K\langle T_1^{\pm 1/p^m}, \dots, T_n^{\pm 1/p^m}\rangle,$$

and

$$R_\infty^+ = \mathcal{O}_K\big\langle T_1^{\pm 1/p^\infty}, \dots, T_n^{\pm 1/p^\infty}\big\rangle = \big(\operatorname{colim}_n\ R_m^+\big)^\wedge,$$

and the group $\Delta_\infty \simeq \mathbf{Z}_p^m$ continuously acts on R_∞^+. We also define R (resp. R_m and R_∞) as $R^+\big[\frac{1}{p}\big]$ (resp. $R_m^+\big[\frac{1}{p}\big]$ and $R_\infty^+\big[\frac{1}{p}\big]$). Furthermore, for an étale morphism $\operatorname{Spa}(A, A^+) \to \operatorname{Spa}(R, R^+) = \mathbf{T}^n$, we define a Huber pair

$$\big(A_m, A_m^+\big) := \big(R_m \otimes_R A, (R_m \otimes_R A)^+\big) = \big(R_m \widehat{\otimes}_R A, (R_m \widehat{\otimes}_R A)^+\big),$$

where $(R_m \widehat{\otimes}_R A)^+$ is the *integral closure* of the image of $R_m^+ \widehat{\otimes}_{R^+} A^+$ in $R_m \widehat{\otimes}_R A$. Similarly, we define

$$A_\infty^+ := \big(\operatorname{colim}_n\ A_m^+\big)^\wedge$$

and $A_\infty := A_\infty^+\big[\frac{1}{p}\big]$.

Lemma 7.8.4. [59, Lemma 4.5] *Let* $\operatorname{Spa}(A, A^+) \to \operatorname{Spa}(R, R^+) = \mathbf{T}^n$ *be a morphism that is a composition of finite étale maps and rational embeddings. Then the* *pair* (A_∞, A_∞^+) *is an affinoid perfectoid pair,* $\operatorname{Spd}(A_\infty, A_\infty^+) \to \operatorname{Spd}(A, A^+)$ *is a* $\underline{\Delta}_\infty$*-torsor, and, for any* $n \in \mathbf{Z}$*, there exists* m *such that the morphism*

$$A_m^+ \widehat{\otimes}_{R_m^+} R_\infty^+ \to A_\infty^+$$

is injective with cokernel annihilated by ϖ^{1/p^n}.

Proof. We note that [59, Lemma 4.5] proves that (A_∞, A_∞^+) is an affinoid perfectoid space (denoted by (S_∞, S_∞^+) there). By construction (and Proposition 6.1.6 (6)), $\operatorname{Spd}(A_m, A_m^+) \to \operatorname{Spd}(A, A^+)$ is a $\underline{(\mathbf{Z}/p^m\mathbf{Z})^n}$-torsor. Therefore $\operatorname{Spd}(A_\infty, A_\infty^+) \simeq \lim_m \operatorname{Spd}(A_m, A_m^+)$ (see Proposition 6.1.6 (5)) is a $\underline{\Delta}_\infty \simeq \lim_m \underline{(\mathbf{Z}/p^n\mathbf{Z})^n}$-torsor. Thus, we are only left to show that, for any $n \in \mathbf{Z}$, there exists m such that the morphism

$$A_m^+ \widehat{\otimes}_{R_m^+} R_\infty^+ \to A_\infty^+$$

is injective with the cokernel annihilated by ϖ^{1/p^n}.

In the following, we denote by $\widetilde{A}_m$ the p-adic completion of the p-torsionfree quotient of $A_m^+ \otimes_{R_m^+} R_\infty^+$ ($\widetilde{A}_m$ is denoted by A_m in [59, Lemma 4.5]). Then [59, Lemma 4.5] shows that, for any $n \in \mathbf{Z}$, there exists m such that the map $\widetilde{A}_m \to A_\infty^+$ has cokernel annihilated by ϖ^{1/p^n}. Moreover, the map becomes an isomorphism after inverting p. We observe that this implies that $\widetilde{A}_m \to A_\infty^+$ is injective as the kernel should be p^∞-torsion, but the p-adic completion of a p-torsionfree ring is p-torsion free. Thus, the only thing we need to show is that $A_m^+ \otimes_{R_m^+} R_\infty^+$ is already p-torsion free for any m. We note that R_∞^+ is topologically free as an R_m^+-module because

$$\begin{aligned} R_\infty^+ &= \mathcal{O}_K\langle T_1^{\pm 1/p^\infty}, \dots, T_n^{\pm 1/p^\infty}\rangle \\ &= \widehat{\bigoplus_{(b_1,\dots,b_n)\in \mathbf{Z}^n \setminus m\mathbf{Z}^n}} \mathcal{O}_K\langle T_1^{\pm 1/p^m}, \dots, T_n^{\pm 1/p^m}\rangle T_1^{1/p^{b_1}} \cdots T_n^{1/p^{b_n}} \\ &= \widehat{\bigoplus_{(b_1,\dots,b_n)\in \mathbf{Z}^n \setminus m\mathbf{Z}^n}} R_m^+ \cdot T_1^{1/p^{b_1}} \cdots T_n^{1/p^{b_n}}. \end{aligned}$$

Thus, R_∞^+ is R_m^+-flat for any m due to Lemma 7.8.3. Therefore, $A_m^+ \otimes_{R_m^+} R_\infty^+$ is flat over A_m^+, so it is, in particular, $\mathcal{O}_K$-flat. As a consequence, it does not have any nonzero p-torsion. This finishes the proof. ■

Lemma 7.8.5. *Let* $\mathfrak{X} = \operatorname{Spf} A_0$ *be an affine admissible formal* $\mathcal{O}_K$*-scheme with generic fiber* $X = \operatorname{Spa}(A, A^+)$ *that admits a map* $f \colon X \to \mathbf{T}^n = \operatorname{Spa}(R, R^+)$ *that factors as a composition of finite étale morphisms and rational embeddings. Then the cohomology groups*

$$\mathrm{H}^i\left(X_v^\diamond, \mathcal{O}_{X^\diamond}^+/p\right)$$

are almost coherent A_0/p*-modules for* $i \geq 0$.

Proof. We denote the completed algebraic closure of K by C. Then we note that Lemma 7.7.3 implies that

$$\mathrm{H}^i\left(X_v^\diamondsuit, \mathcal{O}^+_{X^\diamond}/p\right) \otimes_{\mathcal{O}_K/p} \mathcal{O}_C/p \to \mathrm{H}^i\left(X^\diamondsuit_{C,v}, \mathcal{O}^+_{X^\diamond_C}/p\right)$$

is an almost isomorphism for any $i \geq 0$. Therefore, faithful flatness of the morphism $\mathcal{O}_K/p \to \mathcal{O}_C/p$ and Lemma 2.10.5 imply that it suffices to prove the claim under the additional assumption that $K = C$ is algebraically closed.

Theorem 2.11.5 ensures that A_0 is an almost noetherian ring, thus it suffices to show that $\mathrm{H}^i(X_v^\diamondsuit, \mathcal{O}^+_{X^\diamond}/p)$ are almost finitely generated A_0/p-modules.

Now the generic fiber X is smooth over C, so [12, Corollary 6.4.1/5] implies that $A^+ = A^\circ$ is a flat, topologically finitely type $\mathcal{O}_C$-algebra that is finite over A_0. Thus Lemma 2.8.3 ensures that it suffices to show that $\mathrm{H}^i\left(X_v^\diamondsuit, \mathcal{O}^+_{X^\diamond}/p\right)$ is an almost finitely generated A^+/pA^+-module for $i \geq 0$. We note that A^+ is almost noetherian as a topologically finitely generated $\mathcal{O}_C$-algebra, so almost coherent and almost finitely generated A^+-modules coincide.

We consider the $\underline{\Delta}_\infty$-torsor $\mathrm{Spd}\,(A_\infty, A_\infty^+) \to \mathrm{Spd}\,(A, A^+)$ that was constructed in Lemma 7.8.4. Thus, Corollary 7.6.8 ensures that

$$\mathbf{R}\Gamma\left(X_v^\diamondsuit, \mathcal{O}^+_{X^\diamond}/p\right) \simeq^a \mathbf{R}\Gamma_{\mathrm{cont}}\left(\Delta_\infty, A^+_\infty/p\right).$$

So we reduce the problem to showing that the complex $\mathbf{R}\Gamma_{\mathrm{cont}}\left(\Delta_\infty, A^+_\infty/p\right)$ has almost finitely generated cohomology modules.

We pick any $\varepsilon \in \mathbf{Q}_{>0}$ and use Lemma 7.8.4 to find m such that the map

$$A^+_m \widehat{\otimes}_{R^+_m} R^+_\infty \to A^+_\infty$$

is injective with cokernel killed by p^ε. Thus, we conclude that the map

$$A^+_m/p \otimes_{R^+_m/p} R^+_\infty/p \to A^+_\infty/p$$

has kernel and cokernel annihilated by p^ε. Then it is clear that the induced map

$$\mathrm{H}^i_{\mathrm{cont}}\left(\Delta_\infty, A^+_m/p \otimes_{R^+_m/p} R^+_\infty/p\right) \to \mathrm{H}^i_{\mathrm{cont}}\left(\Delta_\infty, A^+_\infty/p\right)$$

has kernel and cokernel annihilated by $p^{2\varepsilon}$ for any $i \geq 0$. Therefore, Lemma 2.5.7 implies that it is sufficient to show that $\mathrm{H}^i_{\mathrm{cont}}\left(\Delta_\infty, A^+_m/p \otimes_{R^+_m/p} R^+_\infty/p\right)$ is almost finitely generated over A^+/p for any $m \geq 0$ and any $i \geq 0$.

The trick now is to consider the subgroup $p^m\Delta_\infty$ that acts trivially on A^+_m/p to pull it out of the cohomology group by Lemma 7.8.2. More precisely, we consider the Hochschild–Serre spectral sequence

$$\begin{aligned} \mathrm{E}_2^{i,j} =& \mathrm{H}^i\left(\Delta_\infty/p^m\Delta_\infty, \mathrm{H}^j_{\mathrm{cont}}(p^m\Delta_\infty, A^+_m/p \otimes_{R^+_m/p} R^+_\infty/p)\right) \\ &\Rightarrow \mathrm{H}^{i+j}_{\mathrm{cont}}\left(\Delta_\infty, A^+_m/p \otimes_{R^+_m/p} R^+_\infty/p\right). \end{aligned}$$

We recall that the group cohomology of any finite group G can be computed via an explicit bar complex. Namely, for a G-module M, the complex looks like

$$\mathrm{C}^0(G,M) \xrightarrow{\mathrm{d}^0} \mathrm{C}^1(G,M) \xrightarrow{\mathrm{d}^1} \cdots,$$

where

$$\mathrm{C}^i(G,M) = \{f : G^i \to M\} \simeq M^{\oplus i \cdot \# G}$$

and

$$\begin{aligned} d^i(f)(g_0, g_1, \dots, g_i) = & g_0 \cdot f(g_1, \dots, g_i) \\ & + \sum_{j=1}^{i} (-1)^j f(g_0, \dots, g_{j-2}, g_{j-1}g_j, g_{j+1}, \dots, g_i) \\ & + (-1)^{i+1} f(g_0, \dots, g_{i-1}). \end{aligned}$$

In case M is an A^+/p-module and G acts A^+/p-linearly on M, all terms $\mathrm{C}^i(G,M)$ have a natural structure of an A^+/p-module, and the differentials are A^+/p-linear. Moreover, the terms $\mathrm{C}^i(G,M)$ are finite direct sums of M as an A^+/p-module. In particular, they are almost coherent, if so is M. Thus, Lemma 2.6.8 guarantees that all cohomology groups $\mathrm{H}^i(G,M)$ are almost coherent over A^+/p if M is almost coherent (equivalently, almost finitely generated) over A^+/p.

We now apply this observation (together with Lemma 2.6.8) to

$$G = \Delta_\infty / p^m \Delta_\infty \quad \text{and} \quad M = \mathrm{H}^j_{\mathrm{cont}}\big(p^m \Delta_\infty, A^+_m/p \otimes_{R^+_m/p} R^+_\infty/p\big)$$

to conclude that it suffices to show that $\mathrm{H}^j_{\mathrm{cont}}\big(p^m \Delta_\infty, A^+_m/p \otimes_{R^+_m/p} R^+_\infty/p\big)$ is almost coherent (equivalently, almost finitely generated) over A^+/p for any $j \geq 0$, $m \geq 0$. We note that A^+_m is finite over A^+ by [12, Corollary 6.4.1/5]. Thus, Lemma 2.8.3 implies that it is enough to show that $\mathrm{H}^j_{\mathrm{cont}}\big(p^m \Delta_\infty, A^+_m/p \otimes_{R^+_m/p} R^+_\infty/p\big)$ is almost finitely generated over A^+_m/p for $i \geq 0$ and $m \geq 0$. Now we use Lemma 7.8.2 to write

$$\mathrm{H}^j_{\mathrm{cont}}\big(p^m \Delta_\infty, A^+_m/p \otimes_{R^+_m/p} R^+_\infty/p\big) \simeq \mathrm{H}^j_{\mathrm{cont}}\big(p^m \Delta_\infty, R^+_\infty/p\big) \otimes_{R^+_m/p} A^+_m/p.$$

Moreover, Lemma 7.8.2 guarantees that $\mathrm{H}^j_{\mathrm{cont}}\big(p^m \Delta_\infty, R^+_\infty/p\big)$ is almost finitely generated over R^+_m/p. Thus $\mathrm{H}^j_{\mathrm{cont}}\big(p^m \Delta_\infty, R^+_\infty/p\big) \otimes_{R^+_m/p} A^+_m/p$ is almost finitely generated over A^+_m/p by Lemma 2.8.1. ■

Corollary 7.8.6. *Let $\mathfrak{X} = \operatorname{Spf} A_0$ and $X = \operatorname{Spa}(A, A^+)$ be as in Lemma 7.8.5, and let $\mathcal{E}$ be a small $\mathcal{O}^+_{X^\diamond}/p$-vector bundle. Then the cohomology group $\mathrm{H}^i(X^\diamond_v, \mathcal{E})$ is almost coherent over A_0/pA_0 for any $i \geq 0$.*

Proof. Similarly to the proof of Lemma 7.8.5, we can assume that $K = C$ is algebraically closed and $A_0 = A^\circ = A^+$ is almost noetherian.

By assumption, we can find a finite étale surjection $Y \to X$ that splits $\mathcal{E}$. Since X is noetherian, we can dominate it by a Galois cover to assume that $Y \to X$ is a G-torsor for a finite group G such that $\mathcal{E}|_{Y_v^\diamond} \simeq (\mathcal{O}^+_{Y^\diamond}/p)^r$ for some r. Then we have the Hochschild–Serre spectral sequence

$$\mathrm{E}_2^{i,j} = \mathrm{H}^i\big(G, \mathrm{H}^j\big(Y_v^\diamond, (\mathcal{O}^+_{Y^\diamond}/p)^r\big)\big) \Rightarrow \mathrm{H}^{i+j}\big(X_v^\diamond, \mathcal{E}\big).$$

Now note that [12, Corollary 6.4/5] implies that $\mathcal{O}^+_X(X) \to \mathcal{O}^+_Y(Y)$ is a finite morphism. Therefore, similarly to the proof of Lemma 7.8.5, the argument with the explicit bar complex computing $\mathrm{H}^i(G, -)$ implies that it is sufficient to show that $\mathrm{H}^j\big(Y_v^\diamond, (\mathcal{O}^+_{Y^\diamond}/p)^r\big)$ is almost coherent over $\mathcal{O}^+_{Y^\diamond}(Y^\diamond)/p$ for $j \geq 0$. But this is done in Lemma 7.8.5. ■

Lemma 7.8.7. *Let K be a p-adic perfectoid field, let $\mathfrak{X}$ be an admissible formal $\mathcal{O}_K$-scheme with adic generic fiber $X = \mathfrak{X}_K$, and let $\mathcal{E}$ be an $\mathcal{O}^+_{X^\diamond}/p$-vector bundle on $X_v^\diamond$. Then there is a collection of*

1. *an admissible blow-up $\mathfrak{X}' \to \mathfrak{X}$,*
2. *a finite open affine cover $\mathfrak{X}' = \bigcup_{i \in I} \mathfrak{U}_i$,*

such that, for every $i \in I$, the restriction $\mathcal{E}|_{(\mathfrak{U}_{i,K})_v^\diamond}$ is small.

Proof. Corollary 6.6.9 ensures that there is a finite open cover $X = \bigcup_{i \in I} U_i$ such that $\mathcal{E}|_{(U_{i,K})_v^\diamond}$ can be trivialized by a finite étale surjection. Therefore, [11, Lemma 8.4/5] implies that there is an admissible blow-up $\mathfrak{X}' \to \mathfrak{X}$ with a covering $\mathfrak{X}' = \bigcup_{i \in I} \mathfrak{U}_i$ such that $\mathfrak{U}_{i,K} = U_i$. We can then refine $\mathfrak{U}$ to assume that each $\mathfrak{U}_i = \operatorname{Spf} A_{i,0}$ is affine. ■

Theorem 7.8.8. *Let $\mathfrak{X}$ be an admissible formal $\mathcal{O}_K$-scheme with smooth adic generic fiber X and mod-p fiber $\mathfrak{X}_0$. Then*

$$(\mathbf{R}\nu_* \mathcal{E})^a \in \mathbf{D}^+_{\mathrm{acoh}}(\mathfrak{X}_0)^a$$

for any $\mathcal{O}^+_{X^\diamond}/p$-vector bundle $\mathcal{E}$.

Proof. First, we note that the claim is clearly Zariski-local on $\mathfrak{X}$ and descends through rig-isomorphisms by the almost proper mapping theorem (see Theorem 5.1.3). Thus Lemma 7.8.7 implies that it suffices to prove the theorem for $\mathfrak{X} = \operatorname{Spf} A_0$ an affine formal $\mathcal{O}_K$-scheme and a small $\mathcal{E}$.

Now we note that $\mathfrak{X}$ is rig-smooth in the terminology of [15, Section 3]. Thus, [15, Proposition 3.7] states that there are an admissible blow-up $\pi: \mathfrak{X}' \to \mathfrak{X}$ and a covering of $\mathfrak{X}'$ by open affine formal subschemes $\mathfrak{U}'_i$ with rig-étale morphisms $\mathfrak{f}'_i: \mathfrak{U}'_i \to \widehat{\mathbf{A}}^{n_i}_{\mathcal{O}_K}$, i.e., the adic generic fibers $\mathfrak{f}'_{i,K}: \mathfrak{U}'_{i,K} \to \mathbf{D}^{n_i}_K$ are étale. We apply the almost proper mapping theorem (see Theorem 5.1.3) again to conclude that it suffices to show the theorem for $\mathfrak{X}'$. Moreover, since the claim is Zariski-local on $\mathfrak{X}$, we can even pass to each $\mathfrak{U}'_i$ separately. So we reduce to the case where $\mathfrak{X} = \operatorname{Spf} A_0$ is affine, admits a rig-étale morphism $\mathfrak{f}' : \mathfrak{X} \to \widehat{\mathbf{A}}^d_{\mathcal{O}_K}$, and $\mathcal{E}$ is small.

We wish to reduce the question to the situation of Corollary 7.8.6, though we are still not quite there. The key trick now is to use Theorem D.4 to find a finite rig-étale morphism $\mathfrak{f} : \mathfrak{X} \to \widehat{\mathbf{A}}^d_{\mathcal{O}_K}$. In particular, the generic fiber $\mathfrak{f}_K : X \to \mathbf{D}^d_K$ is a finite étale morphism. So the only thing we are left to do is to embed $\mathbf{D}^d_K$ into $\mathbf{T}^d_K$ as a rational subset. This is done by observing that

$$\mathbf{D}^d_K \simeq \mathbf{T}^d_K\left(\frac{T_1 - 1}{p}, \dots, \frac{T_d - 1}{p}\right) \subset \mathbf{T}^d_K.$$

In particular, X admits an étale morphism to a torus that is a composition of a finite étale morphism and a rational embedding. Therefore, Corollary 7.8.6 implies that

$$\mathbf{R}\Gamma\left(X^\diamondsuit_v, \mathcal{E}\right)^a \in \mathbf{D}^+_{\mathrm{acoh}}\left(A_0/pA_0\right)^a.$$

Finally, we note that Theorem 7.7.2 ensures that $\widetilde{\mathbf{R}\Gamma\left(X^\diamondsuit_v, \mathcal{E}\right)} \simeq \mathbf{R}\nu_*\mathcal{E}$, so

$$(\mathbf{R}\nu_*\mathcal{E})^a \in \mathbf{D}^+_{\mathrm{acoh}}(\mathfrak{X}_0)^a$$

by Theorem 4.4.6. ∎

7.9 Nearby cycles are almost coherent for general X and $\mathcal{E}$

The main goal of this section is to generalize Theorem 7.8.8 to the case of a general generic fiber X. The idea is to reduce the general case to the smooth case by means of Lemma 5.4.4, resolution of singularities, and proper hyperdescent.

For the rest of this section, we fix a perfectoid p-adic field K with a good pseudo-uniformizer $\varpi \in \mathcal{O}_K$ (see Definition B.11). We always do almost mathematics with respect to the ideal $\mathfrak{m} = \bigcup_n \varpi^{1/p^n}\mathcal{O}_K$.

Lemma 7.9.1. *Let* $\operatorname{Spf} A_0$ *be an admissible affine formal* $\mathcal{O}_K$*-scheme with adic generic fiber* $\operatorname{Spa}(A, A^+)$. *Let* $f : X \to \operatorname{Spa}(A, A^+)$ *be a proper morphism with smooth* X, *and let* $\mathcal{E}$ *be an* $\mathcal{O}^+_{\operatorname{Spd}(A,A^+)}/p$*-vector bundle. Then* $\mathrm{H}^i\left(X^\diamondsuit_v, \mathcal{E}\right)$ *is an almost coherent* A_0/p*-module for any* $i \geq 0$.

Proof. First, [13, Assertion (c) on p. 307] implies that we can choose an admissible formal $\mathcal{O}_K$-model $\mathfrak{X}$ of X with a morphism $\mathfrak{f} : \mathfrak{X} \to \operatorname{Spa} A_0$ such that $\mathfrak{f}_K = f$. The map f is proper by [51, Lemma 2.6] (or [65, Corollary 4.4 and 4.5]). Now we can compute

$$\mathbf{R}\Gamma\left(X^\diamondsuit_v, \mathcal{E}\right) \simeq \mathbf{R}\Gamma\left(\mathfrak{X}_0, \mathbf{R}\nu_*\mathcal{E}\right).$$

Theorem 7.8.8 implies that $\mathbf{R}\nu_*\mathcal{E} \in \mathbf{D}^+_{\mathrm{acoh}}(\mathfrak{X}_0)$ as X is smooth. Thus, Theorem 5.1.3 implies that

$$\mathbf{R}\Gamma\left(X^\diamondsuit_v, \mathcal{E}\right) \simeq \mathbf{R}\Gamma\left(\mathfrak{X}_0, \mathbf{R}\nu_*\mathcal{E}\right) \in \mathbf{D}^+_{\mathrm{acoh}}\left(A_0/p\right).$$ ∎

Now we recall the notion of a hypercovering that will be crucial for our proof. We refer to [68, Tag 01FX] and [21] for more detail.

Definition 7.9.2. Let $\mathscr{C}$ be a category admitting finite limits. Let $\mathbf{P}$ be a class of morphisms in $\mathscr{C}$ which is stable under base change, preserved under composition (hence under products), and contains all isomorphisms. A simplicial object $X_\bullet$ in $\mathscr{C}$ is said to be a $\mathbf{P}$-*hypercovering* if, for all $n \geq 0$, the natural adjunction map[7]

$$X_\bullet \to \operatorname{cosk}_n(\operatorname{sk}_n(X_\bullet))$$

induces a map $X_{n+1} \to (\operatorname{cosk}_n(\operatorname{sk}_n(X_\bullet)))_{n+1}$ in degree $n+1$ which is in $\mathbf{P}$. If $X_\bullet$ is an augmented simplicial complex, we make a similar definition but also require the case $n = -1$ (and then we say $X_\bullet$ is a $\mathbf{P}$-hypercovering of X_{-1}).

Lemma 7.9.3. *Let X be a quasi-compact, quasi-separated rigid-analytic variety over K. Then there is a proper hypercovering $a\colon X_\bullet \to X$ such that all X_i are smooth over K.*

Proof. First, we note that quasi-compact rigid-analytic varieties over $\operatorname{Spa}(K, \mathcal{O}_K)$ admit resolution of singularities by [66, Theorem 5.2.2]. Thus, the proof of [21, Theorem 4.16] (or [68, Tag 0DAX]) carries over to show that there is a proper hypercovering $a\colon X_\bullet \to X$ such that all X_i are smooth over $\operatorname{Spa}(K, \mathcal{O}_K)$. ■

Lemma 7.9.4. *Let $a\colon X_\bullet \to X$ be a proper hypercovering of a rigid-analytic variety X. Then $a^\diamond\colon X_\bullet^\diamond \to X^\diamond$ is a v-hypercovering of $X^\diamond$.*

Proof. The functor $(-)^\diamond$ commutes with fiber products by Proposition 6.1.6 (6). So

$$\left((\operatorname{cosk}_n(\operatorname{sk}_n X_\bullet))_{n+1}\right)^\diamond \simeq \left(\operatorname{cosk}_n\left(\operatorname{sk}_n X_\bullet^\diamond\right)\right)_{n+1}.$$

Therefore, the only thing we need to show is that $(-)^\diamond$ sends proper coverings to v-coverings. This follows from Lemma 6.1.14 and Example 6.1.12. ■

Theorem 7.9.5. *Let $\mathfrak{X}$ be an admissible formal $\mathcal{O}_K$-scheme with adic generic fiber X and mod-p fiber $\mathfrak{X}_0 := \mathfrak{X} \times_{\operatorname{Spf} \mathcal{O}_K} \operatorname{Spec} \mathcal{O}_K/p$. Then*

$$\mathbf{R}\nu_*\mathcal{E} \in \mathbf{D}^+_{\mathrm{acoh}}(\mathfrak{X}_0)$$

for any $\mathcal{O}^+_{X^\diamond}/p$-vector bundle $\mathcal{E}$.

Proof. The claim is Zariski-local on $\mathfrak{X}$, so we can assume that $\mathfrak{X} = \operatorname{Spf} A_0$ is affine. Thus, Theorem 7.7.2 and Theorem 4.4.6 ensure that it suffices to show that

$$\mathbf{R}\Gamma\left(X_v^\diamond, \mathcal{E}\right) \in \mathbf{D}^+_{\mathrm{acoh}}\left(A_0/p\right).$$

---

[7] See [21, Section 3] (or [68, Tag 0AMA]) for the definition of the coskeleton functor.

Lemma 7.9.3 shows that there is a proper hypercovering $a\colon X_\bullet \to X$ with smooth X_i, and Lemma 7.9.4 implies that $a\colon X_\bullet^\diamond \to X^\diamond$ is then a v-hypercovering.

The proof of [68, Tag 01GY] implies that there is a spectral sequence

$$\mathrm{E}_1^{i,j} = \mathrm{H}^j\big(X_{i,v}^\diamond, \mathcal{E}\big) \Rightarrow \mathrm{H}^{i+j}\big(X_v^\diamond, \mathcal{E}\big).$$

Lemma 7.9.1 guarantees that $\mathrm{H}^j(X_{i,v}^\diamond, \mathcal{E})$ is almost coherent over A_0/p for every $i, j \geq 0$. Therefore, Lemma 2.6.8 guarantees that $\mathrm{H}^{i+j}\big(X_v^\diamond, \mathcal{E}\big)$ is almost coherent over A_0/p for every $i + j \geq 0$. ■

7.10 Cohomological bound on nearby cycles

The main goal of this section is to show that $\mathbf{R}\nu_*\mathcal{E}$ is almost concentrated in degrees $[0, d]$ for a small vector bundle $\mathcal{E}$. This claim turns out to be pretty hard. To achieve this result, we have to use a recent notion of perfectoidization developed in [10] that gives a stronger version of the almost purity theorem in the world of diamonds. Our approach is strongly motivated by the proof of [30, Proposition 7.5.2].

For the rest of this section, we fix a perfectoid p-adic field K with a good pseudo-uniformizer $\varpi \in \mathcal{O}_K$. We always do almost mathematics with respect to the ideal $\mathfrak{m} = \bigcup_n \varpi^{1/p^n}\mathcal{O}_K$.

In this section, it is crucial that we work on the level of diamonds. The main observation is that the functor

$$(-)^\diamond\colon \big\{\text{(Pre-)Adic Analytic Spaces}\big\} \to \big\{\text{Diamonds}\big\}$$

is not fully faithful, so it is possible that a non-perfectoid (pre)-adic space becomes representable by an affinoid perfectoid space after diamondification (we already saw this phenomenon in Warning 7.6.5). An explicit construction of such examples is the crux of our argument in this section. To construct such spaces, we need the following theorem of B. Bhatt and P. Scholze:

Theorem 7.10.1 ([10, Theorem 10.11]). *Let R be an integral perfectoid ring.*[8] *Let $R \to S$ be the p-adic completion of an integral map. Then there exists an integral perfectoid ring S_{perfd} together with a map of R-algebras $S \to S_{\mathrm{perfd}}$, such that it is initial among all R-algebra maps $S \to S'$ for S' being integral perfectoid.*

Now we show how this result can be used to obtain a cohomological bound on $\mathbf{R}\nu_*\mathcal{E}$. We recall that a torus

$$\mathbf{T}^d = \mathrm{Spa}\big(K\langle T_1^{\pm 1}, \dots, T_d^{\pm 1}\rangle, \mathcal{O}_K\langle T_1^{\pm 1}, \dots, T_d^{\pm 1}\rangle\big) = \mathrm{Spa}\big(R, R^+\big)$$

---

[8] We use [8, Definition 3.5] as the definition of integral perfectoid rings here. This definition coincides with Definition 7.4.2 in the p-torsionfree case, but it is less restrictive in general.

admits a map

$$\mathbf{T}^d_\infty = \operatorname{Spa}\left(K\langle T_1^{\pm 1/p^\infty}, \dots, T_d^{\pm 1/p^\infty}\rangle, \mathcal{O}_K\langle T_1^{\pm 1/p^\infty}, \dots, T_d^{\pm 1/p^\infty}\rangle\right) \to \mathbf{T}^d$$

such that $\mathbf{T}^d_\infty$ is an affinoid perfectoid space, and the map becomes a $\underline{\Delta}_\infty = \underline{\mathbf{Z}_p(1)^d}$-torsor after applying the diamondification functor.

Now we can embed a d-dimensional disk $\mathbf{D}^d$ as a rational subdomain

$$\mathbf{D}^d = \mathbf{T}^d\left(\frac{T_1 - 1}{p}, \dots, \frac{T_n - 1}{p}\right) \subset \mathbf{T}^d,$$

so the fiber product

$$\mathbf{D}^d_\infty = \mathbf{D}^d \times_{\mathbf{T}^d} \mathbf{T}^d_\infty \to \mathbf{D}^d$$

is again an affinoid perfectoid covering of $\mathbf{D}^d$ by Lemma 7.8.4.

If $X = \operatorname{Spa}(A, A^+) \to \mathbf{D}^d$ is an arbitrary finite morphism, then the fiber product $X \times_{\mathbf{D}^d} \mathbf{D}^d_\infty$ may not be an affinoid perfectoid space (or even an adic space). However, it turns out that the associated diamond is always representable by an affinoid perfectoid space.

Lemma 7.10.2. *Let $f : X = \operatorname{Spa}(A, A^+) \to \mathbf{D}^d$ be a finite morphism of rigid-analytic K-spaces. Then the fiber product $X^\diamond_\infty := X^\diamond \times_{\mathbf{D}^{d,\diamond}} \mathbf{D}^{d,\diamond}_\infty$ is representable by an affinoid perfectoid space (of characteristic p).*

Proof. Let us say that $\mathbf{D}^d = \operatorname{Spa}(S, S^+)$ and $\widehat{\mathbf{D}}^d_\infty = \operatorname{Spa}(S_\infty, S^+_\infty)$. The map f defines an integral morphism $S^+ \to A^+$, we define

$$A^\dagger_\infty := S^+_\infty \widehat{\otimes}_{S^+} A^+.$$

This is a p-adic completion of an integral morphism over an integral perfectoid ring S^+_∞ (see [8, Lemma 3.20]), so there is a map

$$A^\dagger_\infty \to (A^\dagger_\infty)_{\text{perfd}}$$

initial to an integral perfectoid ring. We define A_∞ as $A^\dagger_\infty[\frac{1}{p}]$ and A^+_∞ as the integral closure of $A^\dagger_\infty$ in A_∞. Then (A_∞, A^+_∞) is an affinoid perfectoid pair by [8, Lemma 3.21]. Therefore, it suffices to show that the natural morphism

$$\operatorname{Spd}\left(A_\infty, A^+_\infty\right) \to \operatorname{Spd}\left(A, A^+\right) \times_{\operatorname{Spd}(S,S^+)} \operatorname{Spd}\left(S_\infty, S^+_\infty\right)$$

is an isomorphism. This can be easily checked on the level of rational points by the universal property of $(A^\dagger_\infty)_{\text{perfd}}$ and the construction of the diamondification functor in Definition 6.1.5 (and [8, Lemma 3.20] that relates affinoid perfectoid pairs and integral affinoid rings). ■

Theorem 7.10.3. *Let* $\mathfrak{X} = \mathrm{Spf}\ A_0$ *be an admissible formal* $\mathcal{O}_K$*-scheme with adic generic fiber* $X = \mathrm{Spa}\,(A, A^+)$ *of dimension* d*, and let* $\mathcal{E}$ *be a small* $\mathcal{O}^+_{X^\diamond}/p$*-vector bundle. Then*

$$\mathbf{R}\Gamma\left(X_v^\diamond, \mathcal{E}\right)^a \in \mathbf{D}^{[0,d]}_{\mathrm{acoh}}\left(A_0/p\right)^a.$$

Proof. Lemma 7.9.1 ensures that $\mathbf{R}\Gamma\left(X_v^\diamond, \mathcal{E}\right) \in \mathbf{D}_{\mathrm{acoh}}(A_0/p)$, so it suffices to show that

$$\mathrm{H}^i\left(X_v^\diamond, \mathcal{E}\right) \simeq^a 0$$

for $i > d$. The Noether normalization theorem (see [11, Proposition 3.1.3]) implies that there is a finite morphism $f\colon X \to \mathbf{D}^d$. We consider the $\underline{\Delta_\infty} \simeq \underline{\mathbf{Z}_p(1)^d}$-torsor

$$X_\infty^\diamond \simeq X^\diamond \times_{\mathbf{D}^{d,\diamond}} \mathbf{D}^{d,\diamond}_\infty \to X^\diamond.$$

As a consequence of Lemma 7.10.2, $X_\infty^\diamond$ is represented by an affinoid perfectoid space $\mathrm{Spd}\left(A_\infty, A_\infty^+\right) = \mathrm{Spa}\left(A_\infty^\flat, A_\infty^{\flat,+}\right)$. Thus, we are in the situation of Set-up 7.6.3. So Corollary 7.6.8 implies that

$$\mathrm{H}^i\left(X_v^\diamond, \mathcal{E}\right) \simeq^a \mathrm{H}^i_{\mathrm{cont}}\left(\Delta_\infty, (M_{\mathcal{E}}^a)_!\right),$$

where $M_{\mathcal{E}} \simeq \mathrm{H}^0\left(X^\diamond_{\infty,v}, \mathcal{E}\right)$. Therefore, the claim follows from the observation that the cohomological dimension of $\Delta_\infty \simeq \mathbf{Z}_p(1)^d \simeq \mathbf{Z}_p^d$ is d due to [8, Lemma 7.3]. ∎

7.11 Proof of Theorem 7.1.2

The main goal of this section is to give a full proof of Theorem 7.1.2. Most of the hard work was already done in the previous sections.

For the rest of this section, we fix a perfectoid p-adic field K with a pseudo-uniformizer $\varpi \in \mathcal{O}_K$ as in Remark B.10. We always do almost mathematics with respect to the ideal $\mathfrak{m} = \bigcup_n \varpi^{1/p^n}\mathcal{O}_K$.

Theorem 7.11.1. *Let* $\mathfrak{X}$ *be an admissible formal* $\mathcal{O}_K$*-scheme with adic generic fiber* X *of dimension* d *and mod-*p *fiber* $\mathfrak{X}_0$*, and let* $\mathcal{E}$ *be an* $\mathcal{O}^+_{X^\diamond}/p$*-vector bundle. Then*

(1) $\mathbf{R}\nu_*\mathcal{E} \in \mathbf{D}^+_{\mathrm{qc,acoh}}(\mathfrak{X}_0)$ *and* $(\mathbf{R}\nu_*\mathcal{E})^a \in \mathbf{D}^{[0,2d]}_{\mathrm{acoh}}(\mathfrak{X}_0)^a$*;*

(2) *if* $\mathfrak{X} = \mathrm{Spf}\ A$ *is affine, then the natural map*

$$\widetilde{\mathrm{H}^i\left(X_v^\diamond, \mathcal{E}\right)} \to \mathrm{R}^i\nu_*\left(\mathcal{E}\right)$$

is an isomorphism for every $i \geq 0$*;*

(3) *the formation of* $\mathrm{R}^i\nu_*(\mathcal{E})$ *commutes with étale base change, i.e., for any étale morphism* $\mathfrak{f}\colon\mathfrak{Y} \to \mathfrak{X}$ *with adic generic fiber* $f\colon Y \to X$*, the natural morphism*

$$\mathfrak{f}_0^*\left(\mathrm{R}^i\nu_{\mathfrak{X},*}(\mathcal{E})\right) \to \mathrm{R}^i\nu_{\mathfrak{Y},*}\left(\mathcal{E}|_{Y^\diamond}\right)$$

is an isomorphism for any $i \geq 0$*;*

(4) *if* $\mathfrak{X}$ *has an open affine covering* $\mathfrak{X} = \bigcup_{i \in I} \mathfrak{U}_i$ *such that* $\mathcal{E}|_{(\mathfrak{U}_{i,K})^\diamond}$ *is small, then*

$$\left(\mathbf{R}\nu_*\mathcal{E}\right)^a \in \mathbf{D}^{[0,d]}_{\mathrm{acoh}}\left(\mathfrak{X}_0\right)^a;$$

(5) *there is an admissible blow-up* $\mathfrak{X}' \to \mathfrak{X}$ *such that* $\mathfrak{X}'$ *has an open affine covering* $\mathfrak{X}' = \bigcup_{i \in I} \mathfrak{U}_i$ *such that* $\mathcal{E}|_{(\mathfrak{U}_{i,K})^\diamond}$ *is small.*
In particular, there is a cofinal family of admissible formal models $\{\mathfrak{X}'_i\}_{i \in I}$ *of* X *such that*

$$\left(\mathbf{R}\nu_{\mathfrak{X}'_i,*}\mathcal{E}\right)^a \in \mathbf{D}^{[0,d]}_{\mathrm{acoh}}\left(\mathfrak{X}'_{i,0}\right)^a,$$

for each $i \in I$.

Proof. The first part of (1), (2), and (3) follow from Theorem 7.7.2 together with Theorem 7.9.5. Now to show that $\mathbf{R}\nu_*\mathcal{E}$ is almost concentrated in degrees $[0, 2d]$, it suffices to show that, for every affine $\mathfrak{U} = \mathrm{Spf}\ A_0 \subset \mathfrak{X}$, the complex $\mathbf{R}\Gamma(\mathfrak{U}^\diamond_{K,v}, \mathcal{E})^a$ (almost) lies in $\mathbf{D}^{[0,2d]}(A_0/p)^a$. By Lemma 7.7.3 and full faithful flatness of $\mathcal{O}_K/p \to \mathcal{O}_C/p$, it is sufficient to prove it under the additional assumption that $K = C$ is algebraically closed. Then Theorem 6.5.7 and Theorem 6.5.9 imply that

$$\mathcal{E}' := \mathbf{R}\mu_*\mathbf{R}\lambda_*\mathcal{E}$$

is an $\mathcal{O}^+_{X_{\text{ét}}}/p$-vector bundle concentrated in degree 0. Therefore,

$$\mathbf{R}\Gamma\left(\mathfrak{U}^\diamond_{C,v}, \mathcal{E}\right) \simeq \mathbf{R}\Gamma\left(\mathfrak{U}_{C,\text{ét}}, \mathcal{E}'\right),$$

and

$$\mathbf{R}\Gamma\left(\mathfrak{U}_{C,\text{ét}}, \mathcal{E}'\right) \in \mathbf{D}^{[0,2d]}(A_0/p)$$

due to [38, Corollary 2.8.3 and Corollary 1.8.8].

In order to show (4), we consider an open affine covering $\mathfrak{X} = \bigcup_{i \in I} \mathfrak{U}_i$ and denote $\mathfrak{U}_i = \mathrm{Spf}\ A_i$. Then Part (2) implies that it suffices to show that

$$\mathbf{R}\Gamma\left((\mathfrak{U}_{i,K})^\diamond_v, \mathcal{E}\right)^a \in \mathbf{D}^{[0,d]}_{\mathrm{acoh}}\left(A_i/p\right)^a$$

for each $i \in I$. This follows from Theorem 7.10.3 and the assumption that $\mathcal{E}|_{(\mathfrak{U}_{i,K})^\diamond}$ is small.

(5) now follows from Lemma 7.8.7. ■

7.12 Proof of Theorem 7.1.9

The main goal of this section is to prove Theorem 7.1.9. The idea is to reduce to the case of a constant Zariski-constructible sheaf through a sequence of reductions; in this case, the result follows directly from Theorem 7.1.2.

For the rest of this section, we fix a perfectoid p-adic field K with a pseudo-uniformizer $\varpi \in \mathcal{O}_K$ as in Remark B.10. We always do almost mathematics with respect to the ideal $\mathfrak{m} = \bigcup_n \varpi^{1/p^n}\mathcal{O}_K$.

We consider the following diagram of morphisms of ringed sites:

$$\begin{tikzcd}
(X^{\diamond}_{v}, \mathcal{O}^+_{X^\diamond}/p) \arrow[r, "\lambda"] \arrow[rrr, bend right, "\nu"'] & (X^{\diamond}_{\text{qproét}}, \mathcal{O}^+_{X^\diamond_{\text{qp}}}/p) \arrow[r, "\mu"] & (X_{\text{ét}}, \mathcal{O}^+_{X_{\text{ét}}}/p) \arrow[r, "t"] & (\mathfrak{X}_{\text{Zar}}, \mathcal{O}_{\mathfrak{X}_0}).
\end{tikzcd}$$

Both ν_* and t_* will play an important role in the proof.

Lemma 7.12.1. *Let $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{Y}$ be a finite morphism of admissible formal $\mathcal{O}_K$-schemes with adic generic fiber $f\colon X \to Y$, and $\mathcal{F} \in \mathbf{D}^b_{zc}(X; \mathbf{F}_p)$. Then the natural morphism*

$$\mathbf{R}\nu_{\mathfrak{Y},*}\big(f_*\mathcal{F} \otimes \mathcal{O}^+_{Y^\diamond}/p\big) \to \mathbf{R}\mathfrak{f}_{0,*}\big(\mathbf{R}\nu_{\mathfrak{X},*}(\mathcal{F} \otimes \mathcal{O}^+_{X^\diamond}/p)\big)$$

is an isomorphism in $\mathbf{D}(\mathfrak{Y}_0)$.

Proof. First, we note that f is finite, and so $f_* \simeq \mathbf{R}f_*$ due to [38, Proposition 2.6.3]. Now the proof of Corollary 7.3.8 just goes through using Corollary 7.2.9 (that does not use Theorem 7.1.9 as an input) in place of Lemma 7.3.7. ■

Lemma 7.12.2. *Let $f\colon X \to Y$ be a finite morphism of quasi-compact, quasi-separated rigid-analytic varieties over K, and $\mathcal{F} \in \mathbf{D}^{[r,s]}_{zc}(X; \mathbf{F}_p)$ such that*

$$\mathbf{R}\nu_{\mathfrak{X},*}\big(\mathcal{F} \otimes \mathcal{O}^+_{X^\diamond}/p\big)^a \in \mathbf{D}^{[r,s+d]}_{\text{acoh}}(\mathfrak{X}_0)^a$$

(resp.

$$\mathbf{R}\nu_{\mathfrak{X},*}\big(\mathcal{F} \otimes \mathcal{O}^+_{X^\diamond}/p\big) \in \mathbf{D}^+_{\text{qc,acoh}}(\mathfrak{X}_0))$$

for any formal $\mathcal{O}_K$-model $\mathfrak{X}$ of X. Then, for any formal $\mathcal{O}_K$-model $\mathfrak{Y}$ of Y,

$$\mathbf{R}\nu_{\mathfrak{Y},*}\big(f_*\mathcal{F} \otimes \mathcal{O}^+_{Y^\diamond}/p\big)^a \in \mathbf{D}^{[r,s+d]}_{\text{acoh}}(\mathfrak{Y}_0)^a$$

(resp.

$$\mathbf{R}\nu_{\mathfrak{Y},*}\big(f_*\mathcal{F} \otimes \mathcal{O}^+_{Y^\diamond}/p\big) \in \mathbf{D}^+_{\text{qc,acoh}}(\mathfrak{Y}_0)).$$

Proof. First, we note that we can choose a finite morphism $\mathfrak{f}\colon \mathfrak{X} \to \mathfrak{Y}$ such that its generic fiber $\mathfrak{f}_K$ is equal to f (for example, this follows from [25, Corollary II.5.3.3, II.5.3.4]).

Now Lemma 7.12.1 ensures that the natural morphism

$$\mathbf{R}\nu_{\mathfrak{Y},*}\big(f_*\mathcal{F} \otimes \mathcal{O}^+_{Y^\diamond}/p\big) \to \mathbf{R}\mathfrak{f}_{0,*}\big(\mathbf{R}\nu_{\mathfrak{X},*}(\mathcal{F} \otimes \mathcal{O}^+_{X^\diamond}/p)\big)$$

is an isomorphism. Therefore, $\mathbf{R}\nu_{\mathfrak{Y},*}\big(f_*\mathcal{F} \otimes \mathcal{O}^+_{Y^\diamond}/p\big)$ already lies in $\mathbf{D}_{\text{acoh}}(\mathfrak{Y}_0)^a$ (resp. in $\mathbf{D}_{\text{qc,acoh}}(\mathfrak{Y}_0)$) by Theorem 5.1.3. The cohomological bound follows from

Proposition 3.5.23 and the fact that the finite morphism $\mathfrak{f}_0$ is (almost) exact on (almost) quasi-coherent sheaves. ■

Lemma 7.12.3. *Let $\mathfrak{X}$ be an admissible formal $\mathcal{O}_K$-scheme with adic generic fiber X of dimension d and mod-p fiber $\mathfrak{X}_0$, and let $\mathcal{F} \in \mathbf{D}^{[r,s]}_{zc}(X; \mathbf{F}_p)$. Then*

$$\mathbf{R}t_*\big(\mathcal{F} \otimes \mathcal{O}^+_{X_{\text{ét}}}/p\big) \simeq \mathbf{R}\nu_*\big(\mathcal{F} \otimes \mathcal{O}^+_{X^\diamond}/p\big) \in \mathbf{D}^+_{\text{qc,acoh}}(\mathfrak{X}_0),$$

and

$$\mathbf{R}\nu_*\big(\mathcal{F} \otimes \mathcal{O}^+_{X^\diamond}/p\big)^a \in \mathbf{D}^{[r,s+d]}_{\text{qc,acoh}}(\mathfrak{X}_0)^a.$$

Proof. Lemma 6.7.10 and Remark 7.1.8 imply that

$$\mathbf{R}t_*\big(\mathcal{F} \otimes \mathcal{O}^+_{X_{\text{ét}}}/p\big) \simeq \mathbf{R}\nu_*\big(\mathcal{F} \otimes \mathcal{O}^+_{X^\diamond}/p\big).$$

In what follows, we will freely identify these sheaves. Also, we can assume that $\mathcal{F}$ is concentrated in degree 0, i.e., $\mathcal{F}$ is a usual Zariski-constructible sheaf.

Step 1: *The case of a local system $\mathcal{F}$.* In this case, $\mathcal{E} := \mathcal{F} \otimes \mathcal{O}^+_{X^\diamond}/p$ fits the assumption of Theorem 7.11.1. Since an $\mathbf{F}_p$-local system on any rigid-analytic variety Y splits by a finite étale cover, so $\mathcal{F} \otimes \mathcal{O}^+_{X^\diamond}/p$ is small for any open affinoid $U \subset X$. Thus, the desired claim follows from Theorem 7.11.1.

Step 2: *Case of a zero-dimensional X.* If X is of dimension 0, then any Zariski-constructible sheaf on X is a local system. So the claim follows from Step 1.

Now we argue by induction on $\dim X$. We suppose the claim is known for every rigid-analytic variety of dimension less than d (and any Zariski-constructible $\mathcal{F}$) and wish to prove the claim for X of dimension d.

Step 3: *Reduction to the case of a reduced X.* Consider the reduction morphism $i\colon X_{\text{red}} \to X$. Then $i_{\text{ét}}$ is an equivalence of étale topoi, we see that

$$i_* i^{-1}\mathcal{F} \to \mathcal{F}$$

is an isomorphism. Thus the claim follows from Lemma 7.12.2.

Step 4: *Reduction to the case of a normal X.* Consider the normalization morphism $f\colon X' \to X$. It is finite by [20, Theorem 2.1.2] and an isomorphism outside of a nowhere dense Zariski-closed subset Z. We use [38, Proposition 2.6.3] and argue on stalks to conclude that the natural morphism $\mathcal{F} \to f_* f^{-1}\mathcal{F}$ is injective. Therefore, there is an exact sequence

$$0 \to \mathcal{F} \to f_* f^{-1}\mathcal{F} \to i_*\mathcal{G} \to 0,$$

where $i\colon Z \to X$ is a Zariski-closed immersion with $\dim Z < \dim X$ and $\mathcal{G}$ is a Zariski-constructible sheaf on Z. Now the induction hypothesis and Lemma 7.12.2

ensure that

$$\mathbf{R}\nu_*\big(i_*\mathcal{G} \otimes \mathcal{O}^+_{X^\diamond}/p\big) \in \mathbf{D}^+_{\text{qc,acoh}}(\mathfrak{X}_0),$$
$$\mathbf{R}\nu_*\big(i_*\mathcal{G} \otimes \mathcal{O}^+_{X^\diamond}/p\big)^a \in \mathbf{D}^{[0,d-1]}_{\text{acoh}}(\mathfrak{X}_0)^a.$$

Therefore, it suffices to show the claim for $f_*f^{-1}\mathcal{F}$. Thus, Lemma 7.12.2 guarantees that it suffices to show that

$$\mathbf{R}\nu_{\mathfrak{X}',*}\big(f^{-1}\mathcal{F} \otimes \mathcal{O}^+_{X'^\diamond}/p\big) \in \mathbf{D}^+_{\text{qc,acoh}}(\mathfrak{X}'_0),$$
$$\mathbf{R}\nu_{\mathfrak{X}',*}\big(f^{-1}\mathcal{F} \otimes \mathcal{O}^+_{X'^\diamond}/p\big)^a \in \mathbf{D}^{[r,s+d]}_{\text{acoh}}(\mathfrak{X}'_0)^a$$

for any admissible formal $\mathcal{O}_K$-model $\mathfrak{X}'$ of X'. So we may and do assume that X is normal.

Step 5: *Reduction to the case* $\mathcal{F} = \underline{\mathbf{F}}_p$. By definition of a Zariski-constructible sheaf, there are a nowhere dense Zariski-closed subset $i: Z \to X$ with the open complement $j: U \to X$ and an $\mathbf{F}_p$-local system $\mathbf{L}$ on U such that $\mathcal{F}|_U \simeq \mathbf{L}$. In particular, there is a short exact sequence

$$0 \to j_!\mathbf{L} \to \mathcal{F} \to i_*\mathcal{F}|_Z \to 0.$$

Similarly to the argument in Step 4, it suffices to prove the claim for $\mathcal{F} = j_!\mathbf{L}$.

Then "méthode de la trace" (see [68, Tag 03SH]) implies that there is a finite étale covering $g: U' \to U$ of degree prime-to-p such that $\mathbf{L}' := \mathbf{L}|_{U'}$ is an iterated extension of constant sheaves $\underline{\mathbf{F}}_p$. Then $\mathbf{L}$ is a direct summand of $g_*(\mathbf{L}')$. Thus, it is enough to prove the claim for

$$\mathcal{F} = j_!(g_*\mathbf{L}').$$

Moreover, it suffices to prove the claim for $\mathcal{F} = j_!(g_*\underline{\mathbf{F}}_p)$ because the claim of Lemma 7.12.3 satisfies the 2-out-of-3 property, and both functors g_* and $j_!$ are exact.

Now we use [32, Theorem 1.6] to extend g to a finite morphism $g': X' \to X$. Then a similar reduction shows that it suffices to prove the claim for $\mathcal{F} = g'_*(\underline{\mathbf{F}}_p)$. This case follows from Step 1 and Lemma 7.12.2. ■

Theorem 7.12.4. *Let $\mathfrak{X}$ be an admissible formal $\mathcal{O}_K$-scheme with adic generic fiber X of dimension d and mod-p fiber $\mathfrak{X}_0$, and $\mathcal{F} \in \mathbf{D}^{[r,s]}_{zc}(X; \mathbf{F}_p)$. Then*

(1) *there is an isomorphism* $\mathbf{R}t_*\big(\mathcal{F} \otimes \mathcal{O}^+_{X_{\text{ét}}}/p\big) \simeq \mathbf{R}\nu_*\big(\mathcal{F} \otimes \mathcal{O}^+_{X^\diamond}/p\big)$;

(2) $\mathbf{R}\nu_*\big(\mathcal{F} \otimes \mathcal{O}^+_{X^\diamond}/p\big) \in \mathbf{D}^+_{\text{qc,acoh}}(\mathfrak{X}_0)$, *and* $\mathbf{R}\nu_*\big(\mathcal{F} \otimes \mathcal{O}^+_{X^\diamond}/p\big)^a \in \mathbf{D}^{[r,s+d]}_{\text{acoh}}(\mathfrak{X}_0)^a$;

(3) *if* $\mathfrak{X} = \operatorname{Spf} A$ *is affine, then the natural map*

$$\mathrm{H}^i\big(X_v^{\diamond}\widetilde{, \mathcal{F} \otimes \mathcal{O}}^+_{X^\diamond}/p\big) \to \mathrm{R}^i\nu_*\big(\mathcal{F} \otimes \mathcal{O}^+_{X^\diamond}/p\big)$$

is an isomorphism for every $i \geq 0$;

(4) *the formation of* $\mathrm{R}^i\nu_*\big(\mathcal{F}\otimes\mathcal{O}^+_{X^\diamond}/p\big)$ *commutes with étale base change, i.e., for any étale morphism* $\mathfrak{f}\colon\mathfrak{Y}\to\mathfrak{X}$ *with adic generic fiber* $f\colon Y\to X$, *the natural morphism*

$$\mathfrak{f}_0^*\big(\mathrm{R}^i\nu_{\mathfrak{X},*}\big(\mathcal{F}\otimes\mathcal{O}^+_{X^\diamond}/p\big)\big)\to\mathrm{R}^i\nu_{\mathfrak{Y},*}\big(f^{-1}\mathcal{F}\otimes\mathcal{O}^+_{Y^\diamond}/p\big)$$

is an isomorphism for any $i\geq 0$.

Proof. (1) and (2) follow from Lemma 7.12.3. Now (3) follows from Lemma 4.4.4 and the isomorphism

$$\mathbf{R}\Gamma\big(\mathfrak{X}_0,\mathbf{R}\nu_*\big(\mathcal{F}\otimes\mathcal{O}^+_{X^\diamond}/p\big)\big)\simeq\mathbf{R}\Gamma\big(X^\diamond_v,\mathcal{F}\otimes\mathcal{O}^+_{X^\diamond}/p\big).$$

We are left to show (4). By (1), it suffices to show that the natural morphism

$$\mathfrak{f}_0^*\big(\mathrm{R}^i t_{\mathfrak{X},*}\big(\mathcal{F}\otimes\mathcal{O}^+_{X_{\text{ét}}}/p\big)\big)\to\mathrm{R}^i t_{\mathfrak{Y},*}\big(f^{-1}\mathcal{F}\otimes\mathcal{O}^+_{Y_{\text{ét}}}/p\big)$$

is an isomorphism. Moreover, [7, Proposition 3.6] ensures that it suffices to prove the claim for $\mathcal{F}=g_*(\underline{\mathbf{F}}_p)$ for some finite morphism $g\colon X'\to X$. Then we can lift it to a finite morphism $\mathfrak{g}\colon\mathfrak{X}'\to\mathfrak{X}$ as in the proof of Lemma 7.12.2. Then we have a commutative diagram

$$\begin{tikzcd}
 & (Y'_{\text{ét}},\mathcal{O}^+_{Y'_{\text{ét}}}/p) \arrow[rr, "t_{\mathfrak{Y}'}"] \arrow[dl, "f'"'] \arrow[dd, "g'"] & & (\mathfrak{Y}'_0,\mathcal{O}_{\mathfrak{Y}'_0}) \arrow[dl, "\mathfrak{f}'_0"'] \arrow[dd, "\mathfrak{g}'_0"] \\
(X'_{\text{ét}},\mathcal{O}^+_{X'_{\text{ét}}}/p) \arrow[rr, "t_{\mathfrak{X}'}"] \arrow[dd, "g"] & & (\mathfrak{X}'_0,\mathcal{O}_{\mathfrak{X}'_0}) \arrow[dd, "\mathfrak{g}_0"] & \\
 & (Y_{\text{ét}},\mathcal{O}^+_{Y_{\text{ét}}}/p) \arrow[rr, "t_{\mathfrak{Y}}"] \arrow[dl, "f"'] & & (\mathfrak{Y}_0,\mathcal{O}_{\mathfrak{Y}_0}) \arrow[dl, "\mathfrak{f}_0"'] \\
(X,\mathcal{O}^+_{X_{\text{ét}}}/p) \arrow[rr, "t_{\mathfrak{X}}"] & & (\mathfrak{X}_0,\mathcal{O}_{\mathfrak{X}_0}) &
\end{tikzcd}$$

(7.12.1)

with $\mathfrak{Y}'=\mathfrak{Y}\times_{\mathfrak{X}}\mathfrak{X}'$ and Y' being its adic generic fiber. Then we have a sequence of isomorphisms:

$$\begin{aligned}
\mathfrak{f}_0^*\big(\mathbf{R}t_{\mathfrak{X},*}\big(g_*(\underline{\mathbf{F}}_p)\otimes\mathcal{O}^+_{X_{\text{ét}}}/p\big)\big) &\simeq \mathfrak{f}_0^*\big(\mathbf{R}t_{\mathfrak{X},*}\big(\mathbf{R}g_*\mathcal{O}^+_{X'_{\text{ét}}}/p\big)\big)\\
&\simeq \mathfrak{f}_0^*\big(\mathbf{R}\mathfrak{g}_{0,*}\big(\mathbf{R}t_{\mathfrak{X}',*}\mathcal{O}^+_{X'_{\text{ét}}}/p\big)\big)\\
&\simeq \mathbf{R}\mathfrak{g}'_{0,*}\big(\mathfrak{f}_0'^{\,*}\big(\mathbf{R}t_{\mathfrak{X}',*}\mathcal{O}^+_{X'_{\text{ét}}}/p\big)\big)\\
&\simeq \mathbf{R}\mathfrak{g}'_{0,*}\big(\mathbf{R}t_{\mathfrak{Y}',*}\big(\mathcal{O}^+_{Y'_{\text{ét}}}/p\big)\big)\\
&\simeq \mathbf{R}t_{\mathfrak{Y},*}\big(\mathbf{R}g'_*\mathcal{O}^+_{Y'_{\text{ét}}}/p\big)
\end{aligned}$$

$$\begin{aligned} &\simeq \mathbf{R}t_{\mathfrak{Y},*}\big(g'_*(\underline{\mathbf{F}}_p) \otimes \mathcal{O}^+_{Y_{\text{ét}}}/p\big) \\ &\simeq \mathbf{R}t_{\mathfrak{Y},*}\big(f^{-1}(g_*\underline{\mathbf{F}}_p) \otimes \mathcal{O}^+_{Y_{\text{ét}}}/p\big). \end{aligned}$$

The first isomorphism holds by (the proof of) Corollary 7.2.9. The second isomorphism is formal and follows from Diagram (7.12.1). The third isomorphism holds by flat base change applied to $\mathfrak{f}_0$. The fourth isomorphism follows from Theorem 7.11.1 applied to $\mathcal{E} = \mathcal{O}^+_{X'^{\diamond}}/p$ and the étale morphism $\mathfrak{Y}' \to \mathfrak{X}'$. The fifth isomorphism is formal again. The sixth isomorphism follows from (the proof of) Corollary 7.2.9. Finally, the last isomorphism follows from [38, Theorem 4.3.1]. ■

7.13 Proof of Theorem 7.1.11

The main goal of this section is to prove Theorem 7.1.11. The proof is a formal reduction to the case of $\mathcal{O}^+_{X^{\diamond}}/p$-vector bundles. After that, we also discuss a version of this theorem for the classical pro-étale site from [59].

For the rest of this section, we fix a perfectoid p-adic field K with a good pseudo-uniformizer $\varpi \in \mathcal{O}_K$. We always do almost mathematics with respect to the ideal $\mathfrak{m} = \bigcup_n \varpi^{1/p^n}\mathcal{O}_K$.

Lemma 7.13.1. *Let X be a rigid-analytic variety over K, and let $\mathcal{E}$ be an $\mathcal{O}^+_{X^{\diamond}}$-vector bundle on X. Then $\mathcal{E}$ is derived p-adically complete.*

Proof. It suffices to prove the claim v-locally on $X^{\diamond}_v$. Therefore, we may and do assume that $\mathcal{E} = \big(\mathcal{O}^+_{X^{\diamond}}\big)^r$ for some integer r. Then the claim follows immediately from Lemma 6.3.5 (3). ■

Lemma 7.13.2. *Let $\mathfrak{X} = \operatorname{Spf} A_0$ be an affine admissible formal $\mathcal{O}_K$-scheme with adic generic fiber $X = \operatorname{Spa}(A, A^+)$ of dimension d, and let $\mathcal{E}$ be an $\mathcal{O}^+_{X^{\diamond}}$-vector bundle. Then*

$$\mathbf{R}\Gamma\big(X^{\diamond}_v, \mathcal{E}\big)^a \in \mathbf{D}^{[0,2d]}_{\text{acoh}}(A_0).$$

Moreover,

$$\mathbf{R}\Gamma\big(X^{\diamond}_v, \mathcal{E}\big)^a \in \mathbf{D}^{[0,d]}_{\text{acoh}}(A_0)$$

if $\mathcal{E}$ is small (see Definition 7.1.10).

Proof. Lemma 7.13.1 implies that $\mathcal{E}$ is derived p-adically complete. Thus, the result follows from Theorem 7.1.2, [68, Tag 0A0G], and Corollary 2.13.3. ■

Lemma 7.13.3. *Let $\mathfrak{X} = \operatorname{Spf} A_0$ be an admissible affine formal $\mathcal{O}_K$-scheme with adic generic fiber $X = \operatorname{Spa}(A, A^+)$, and $\mathfrak{f} \colon \operatorname{Spf} B_0 \to \operatorname{Spf} A_0$ an étale morphism* *with adic generic fiber* $f\colon Y \to X$*, and* $\mathcal{E}$ *an* $\mathcal{O}^+_{X^\diamond}$*-vector bundle on* X*. Then the natural morphism*

$$r\colon \mathbf{R}\Gamma\big(X_v^\diamond, \mathcal{E}\big) \otimes_{A_0} B_0 \to \mathbf{R}\Gamma\big(Y_v^\diamond, \mathcal{E}\big)$$

is an isomorphism.

Proof. The morphism $A_0 \to B_0$ is flat since $\mathfrak{f}$ is étale. Now Lemma 7.13.2 and Lemma 2.12.7 ensure that the cohomology groups of both $\mathbf{R}\Gamma\big(X_v^\diamond, \mathcal{E}\big) \otimes_{A_0} B_0$ and $\mathbf{R}\Gamma\big(Y_v^\diamond, \mathcal{E}\big)$ are (classically) p-adically complete. In particular, both complexes are derived p-adically complete. So it suffices to show that r is an isomorphism after taking derived mod-p fiber (see [68, Tag 0G1U]). Then the claim follows from Theorem 7.11.1 (3) (4). ■

Theorem 7.13.4. *Let* $\mathfrak{X}$ *be an admissible formal* $\mathcal{O}_K$*-scheme with adic generic fiber* X *of dimension* d*, and let* $\mathcal{E}$ *be an* $\mathcal{O}^+_{X^\diamond}$*-vector bundle. Then*

(1) $\mathbf{R}\nu_*\mathcal{E} \in \mathbf{D}^+_{\mathrm{qc,acoh}}(\mathfrak{X})$ *and* $(\mathbf{R}\nu_*\mathcal{E})^a \in \mathbf{D}^{[0,2d]}_{\mathrm{acoh}}(\mathfrak{X})^a$*;*

(2) *if* $\mathfrak{X} = \mathrm{Spf}\ A$ *is affine, then the natural map*

$$\mathrm{H}^i\big(X_v^\diamond, \mathcal{E}\big)^\Delta \to \mathrm{R}^i\nu_*(\mathcal{E})$$

is an isomorphism for every $i \geq 0$*;*

(3) *the formation of* $\mathrm{R}^i\nu_*(\mathcal{E})$ *commutes with étale base change, i.e., for any étale morphism* $\mathfrak{f}\colon \mathfrak{Y} \to \mathfrak{X}$ *with adic generic fiber* $f\colon Y \to X$*, the natural morphism*

$$\mathfrak{f}^*\big(\mathrm{R}^i\nu_{\mathfrak{X},*}(\mathcal{E})\big) \to \mathrm{R}^i\nu_{\mathfrak{Y},*}\big(\mathcal{E}|_{Y^\diamond}\big)$$

is an isomorphism for any $i \geq 0$*;*

(4) *if* $\mathfrak{X}$ *has an open affine covering* $\mathfrak{X} = \bigcup_{i\in I}\mathfrak{U}_i$ *such that* $\mathcal{E}|_{(\mathfrak{U}_{i,K})^\diamond}$ *is small, then*

$$\big(\mathbf{R}\nu_*\mathcal{E}\big)^a \in \mathbf{D}^{[0,d]}_{\mathrm{acoh}}\big(\mathfrak{X}\big)^a;$$

(5) *there is an admissible blow-up* $\mathfrak{X}' \to \mathfrak{X}$ *such that* $\mathfrak{X}'$ *has an open affine covering* $\mathfrak{X}' = \bigcup_{i\in I}\mathfrak{U}_i$ *such that* $\mathcal{E}|_{(\mathfrak{U}_{i,K})^\diamond}$ *is small.*
In particular, there is a cofinal family of admissible formal models $\{\mathfrak{X}'_i\}_{i\in I}$ *of* X *such that*

$$\big(\mathbf{R}\nu_{\mathfrak{X}'_i,*}\mathcal{E}\big)^a \in \mathbf{D}^{[0,d]}_{\mathrm{acoh}}\big(\mathfrak{X}'_i\big)^a,$$

for each $i \in I$*.*

Proof. First, (5) follows directly from Lemma 7.8.7. Therefore, we only need to prove (1)–(4). These claims are local on $\mathfrak{X}$, so we can assume that $\mathfrak{X} = \mathrm{Spf}\ A$ is affine. Then it suffices to show that, for every étale morphism $\mathrm{Spf}\ B_0 \to \mathrm{Spf}\ A_0$ with adic generic fiber $Y \to X$,

$$\mathrm{H}^i\left(Y_v^\diamondsuit, \mathcal{E}|_{Y^\diamond}\right)$$

is almost coherent for $i \geq 0$,

$$\mathrm{H}^i\left(Y_v^\diamondsuit, \mathcal{E}|_{Y^\diamond}\right) \simeq^a 0$$

for $i > 2d$ (resp. for $i > d$ if $\mathcal{E}$ is small), and the natural morphism

$$\mathrm{H}^i\left(X_v^\diamondsuit, \mathcal{E}\right) \otimes_{A_0} B_0 \to \mathrm{H}^i\left(Y_v^\diamondsuit, \mathcal{E}|_{Y^\diamond}\right)$$

is an isomorphism (see Lemma 5.1.8 and its proof). The first two claims follow from Lemma 7.13.2, while the last one follows from Lemma 7.13.3 (and A_0-flatness of B_0). ■

Let us also mention a version of Theorem 7.1.11 for the pro-étale site of X as defined in [59] and [60]. It will be convenient to have this reference in our future work. In what follows, $\widehat{\mathcal{O}}_X^+$ is the completed integral structure sheaf on $X_{\text{proét}}$ (see [59, Definition 4.1]), and

$$\nu' \colon \left(X_{\text{proét}}, \widehat{\mathcal{O}}_X^+\right) \to \left(\mathfrak{X}_{\text{Zar}}, \mathcal{O}_{\mathfrak{X}}\right)$$

is the evident morphism of ringed sites.

Theorem 7.13.5. *Let $\mathfrak{X}$ be an admissible formal $\mathcal{O}_K$-scheme with adic generic fiber X of dimension d and mod-p fiber $\mathfrak{X}_0$. Then*

(1) $\mathbf{R}\nu'_*\left(\mathcal{O}_X^+/p\right) \in \mathbf{D}^+_{\text{qc,acoh}}(\mathfrak{X}_0)$ *and* $\mathbf{R}\nu'_*\left(\mathcal{O}_X^+/p\right)^a \in \mathbf{D}^{[0,d]}_{\text{acoh}}(\mathfrak{X}_0)^a$*;*

(2) *if $\mathfrak{X} = \mathrm{Spf}\ A$ is affine, then the natural map*

$$\mathrm{H}^i\left(\widetilde{X_{\text{proét}}, \mathcal{O}_X^+}/p\right) \to \mathrm{R}^i\nu'_*\left(\mathcal{O}_X^+/p\right)$$

is an isomorphism for every $i \geq 0$;

(3) *the formation of $\mathrm{R}^i\nu'_*\left(\mathcal{O}_X^+/p\right)$ commutes with étale base change, i.e., for any étale morphism $\mathfrak{f} \colon \mathfrak{Y} \to \mathfrak{X}$ with adic generic fiber $f \colon Y \to X$, the natural morphism*

$$\mathfrak{f}_0^*\left(\mathrm{R}^i\nu'_{\mathfrak{X},*}\left(\mathcal{O}_X^+/p\right)\right) \to \mathrm{R}^i\nu'_{\mathfrak{Y},*}\left(\mathcal{O}_Y^+/p\right)$$

is an isomorphism for any $i \geq 0$.

Proof. By [59, Corollary 3.17], $\mathbf{R}\nu'_*\left(\mathcal{O}_X^+/p\right) \simeq \mathbf{R}t_*\left(\mathcal{O}_{X_{\text{ét}}}^+/p\right)$. So the results follow formally from Theorem 7.12.4. ■

Theorem 7.13.6. *Let $\mathfrak{X}$ be an admissible formal $\mathcal{O}_K$-scheme with adic generic fiber X of dimension d. Then*

(1) $\mathbf{R}\nu'_*\widehat{\mathcal{O}}^+_X \in \mathbf{D}^+_{\mathrm{qc,acoh}}(\mathfrak{X})$ *and* $\big(\mathbf{R}\nu'_*\widehat{\mathcal{O}}^+_X\big)^a \in \mathbf{D}^{[0,d]}_{\mathrm{acoh}}(\mathfrak{X})^a$*;*

(2) *if* $\mathfrak{X} = \operatorname{Spf} A$ *is affine, then the natural map*

$$\mathrm{H}^i\big(X_{\mathrm{pro\acute{e}t}}, \widehat{\mathcal{O}}^+_X\big)^\Delta \to \mathrm{R}^i\nu'_*\widehat{\mathcal{O}}^+_X$$

is an isomorphism for every $i \geq 0$*;*

(3) *the formation of* $\mathrm{R}^i\nu'_*(\mathcal{E})$ *commutes with étale base change, i.e., for any étale morphism* $\mathfrak{f}\colon \mathfrak{Y} \to \mathfrak{X}$ *with adic generic fiber* $f\colon Y \to X$*, the natural morphism*

$$\mathfrak{f}^*\big(\mathrm{R}^i\nu'_{\mathfrak{X},*}(\widehat{\mathcal{O}}^+_X)\big) \to \mathrm{R}^i\nu'_{\mathfrak{Y},*}\big(\widehat{\mathcal{O}}^+_Y\big)$$

is an isomorphism for any $i \geq 0$.

Proof. The proof is identical to the proof of Theorem 7.13.4 once one establishes that the sheaf $\widehat{\mathcal{O}}^+_X$ is p-adically derived complete. For this, see [8, Remark 5.5]. ■