Appendix A

Derived complete modules

The main goal of this appendix is to collect some standard results on derived complete modules that seem difficult to find in the literature.

For the rest of the appendix, we fix a ring R with an element $\varpi \in R$.

Definition A.1. A complex $M \in \mathbf{D}(R)$ is *ϖ-adically derived complete* (or just derived complete) if the natural morphism $M \to \mathbf{R}\lim_n [M/\varpi^n]$ is an isomorphism.

Remark A.2. This definition coincides with [68, Tag 091S] due to [68, Tag 091Z].

Lemma A.3. *Let $M \in \mathbf{D}(R)$ be a derived complete complex. Then*

(1) *$M \in \mathbf{D}^{\geq d}(R)$ if $[M/\varpi] \in \mathbf{D}^{\geq d}(R/\varpi)$;*

(2) *$M \in \mathbf{D}^{\leq d}(R)$ if $[M/\varpi] \in \mathbf{D}^{\leq d}(R/\varpi)$.*

Proof. (1): By shifting, we can assume that $d = 0$. Now suppose that $[M/\varpi] \in \mathbf{D}^{\geq 0}(R/\varpi)$. Then we use an exact triangles

$$[M/\varpi] \to [M/\varpi^n] \to [M/\varpi^{n-1}]$$

to ensure that $[M/\varpi^n] \in \mathbf{D}^{\geq 0}(R/\varpi^n)$ for every $n \geq 0$. Now we use that M is derived complete to see that the natural morphism

$$M \to \mathbf{R}\lim_n [M/\varpi^n M]$$

is an isomophism. By passing to cohomology groups (and using that lim has cohomological dimension 1), we see that

$$0 \to \mathrm{R}^1 \lim_n \mathrm{H}^{i-1}\big([M/\varpi^n]\big) \to \mathrm{H}^i\big(M\big) \to \lim_n \mathrm{H}^i\big([M/\varpi^n]\big) \to 0$$

are exact for any integer i. This implies that $\mathrm{H}^i(M) = 0$ for $i \leq 0$, i.e., $M \in \mathbf{D}^{\geq 0}(R)$.

(2): Similarly, we can assume that $d = 0$. Then the same inductive argument shows that $[M/\varpi^n] \in \mathbf{D}^{\leq 0}(R/\varpi^n)$ and we have short exact sequences

$$0 \to \mathrm{R}^1 \lim_n \mathrm{H}^{i-1}\big([M/\varpi^n]\big) \to \mathrm{H}^i\big(M\big) \to \lim_n \mathrm{H}^i\big([M/\varpi^n]\big) \to 0.$$

This implies that $M \in \mathbf{D}^{\leq 1}(R)$ and $\mathrm{H}^1(M) = \mathrm{R}^1 \lim_n \mathrm{H}^0\big([M/\varpi^n]\big)$. Now note that the exact triangle

$$[M/\varpi] \to [M/\varpi^n] \to [M/\varpi^{n-1}]$$

and the fact that $[M/\varpi] \in \mathbf{D}^{\leq 0}(R/\varpi)$ imply that $\mathrm{H}^0\big([M/\varpi^n]\big) \to \mathrm{H}^0\big([M/\varpi^{n-1}]\big)$ is surjective, so $\mathrm{R}^1 \lim_n \mathrm{H}^0\big([M/\varpi^n]\big) = 0$ by the Mittag-Leffler criterion. ∎

Lemma A.4. *Let R be a ring with an ideal of almost mathematics $\mathfrak{m}$ and an element $\varpi \in \mathfrak{m}$. Let $M \in \mathbf{D}(R)$ be a derived ϖ-adically complete complex. Then $\widetilde{\mathfrak{m}} \otimes M$ is also derived ϖ-adically complete complex.*

Proof. Consider the exact triangle

$$\widetilde{\mathfrak{m}} \otimes M \to M \to Q.$$

Since $\widetilde{\mathfrak{m}} \otimes M \to M$ is an almost isomorphism, we see that the cohomology groups of Q are almost zero. In particular, they are ϖ-torsion, so derived complete. Therefore, Q is derived complete (for example, by [68, Tag 091P] and [68, Tag 091S]). Now derived completeness of M and Q implies derived completeness of $\widetilde{\mathfrak{m}} \otimes M$. ■

Lemma A.5. *Let R be a ring with an ideal of almost mathematics $\mathfrak{m}$ and an element $\varpi \in \mathfrak{m}$. Let $M \in \mathbf{D}(R)$ be a ϖ-adically derived complete complex. Then*

(1) $M^a \in \mathbf{D}^{\geq d}(R)^a$ *if* $[M^a/\varpi] \in \mathbf{D}^{\geq a}(R/\varpi R)^a$*;*

(2) $M^a \in \mathbf{D}^{\leq d}(R)^a$ *if* $[M^a/\varpi] \in \mathbf{D}^{\leq a}(R/\varpi R)^a$.

Proof. Lemma A.4 guarantees that $\widetilde{\mathfrak{m}} \otimes M$ is derived ϖ-adically complete. Therefore, the claim follows from Lemma A.3 applied to $\widetilde{\mathfrak{m}} \otimes M$. ■

Now we fix an ringed R-site $(X, \mathcal{O}_X)$.

Definition A.6. A complex $M \in \mathbf{D}(X; \mathcal{O}_X)$ is *ϖ-adically derived complete* (or just derived complete) if the natural morphism $M \to \mathbf{R}\lim_n [M/\varpi^n]$ is an isomorphism.

Remark A.7. This definition coincides with [68, Tag 0999] by [68, Tag 0A0E].

Lemma A.8. *Let $\mathcal{B} \subset \mathrm{Ob}(X)$ be a basis in a ringed site $(X, \mathcal{O}_X)$ and $M \in \mathbf{D}(X; \mathcal{O}_X)$. Then M is ϖ-adically derived complete if and only if $\mathbf{R}\Gamma(U, M)$ is ϖ-adically derived complete for any $U \in \mathcal{B}$.*

Proof. Suppose that M is ϖ-adically derived complete. Then $\mathbf{R}\Gamma(U, M)$ is derived ϖ-adically complete for any $U \in \mathrm{Ob}(X)$ by [68, Tag 0BLX].

Now suppose that $\mathbf{R}\Gamma(U, M)$ is ϖ-adically derived complete for any $U \in \mathcal{B}$, and consider the derived ϖ-adic completion $M \to \widehat{M}$ with the associated distinguished triangle

$$M \to \widehat{M} \to Q.$$

We aim at showing that $Q \simeq 0$. In order to show it, it suffices to establish that $\mathbf{R}\Gamma(U, Q) \simeq 0$ for any $U \in \mathcal{B}$. Now we use [68, Tag 0BLX] to conclude that

$$\mathbf{R}\Gamma(U, \widehat{M}) \simeq \widehat{\mathbf{R}\Gamma(U, M)},$$

so we get the distinguished triangle

$$\mathbf{R}\Gamma(U, M) \to \widehat{\mathbf{R}\Gamma(U, M)} \to \mathbf{R}\Gamma(U, M).$$

Since $\mathbf{R}\Gamma(U, M)$ is derived ϖ-adically complete by the assumption, we see that the morphism

$$\mathbf{R}\Gamma(U, M) \to \widehat{\mathbf{R}\Gamma(U, Q)}$$

is an isomorphism. Therefore, we conclude that $\mathbf{R}\Gamma(U, Q) \simeq 0$. This finishes the proof. ∎