Appendix B

Perfectoid rings

The main goal of this appendix is to recall the main structural results about perfectoid rings.

Definition B.1 ([61, Definition 3.6]). A non-archimedean field $(K, |.|_K)$ is a *perfectoid field* if there is a pseudo-uniformizer $\varpi \in K$ such that $\varpi^p \mid p$ in $\mathcal{O}_K = \{x \in K \mid |x| \leq 1\}$ and the p-th power Frobenius map

$$\Phi\colon \mathcal{O}_K/\varpi\mathcal{O}_K \to \mathcal{O}_K/\varpi^p\mathcal{O}_K$$

is an isomorphism.

Definition B.2. A complete valuation ring K^+ is a *perfectoid valuation ring* if its fraction field $K := \mathrm{Frac}(K^+)$ is a perfectoid field with its valuation topology.

A Huber pair (K, K^+) is a *perfectoid field pair* if K is a perfectoid field and K^+ is an open and bounded valuation subring.

Remark B.3. Any perfectoid valuation ring K^+ is automatically microbial (see [64, Lecture 9, Proposition 9.1.3 and Definition 9.1.4]). Note that any rank-1 valuation ring $K^+ \subset K^{++} \subset K$ defines the same topology on K by [17, Ch. VI Section 7.2, Proposition 3]. Therefore, K^{++} must be equal to K°, the set of power-bounded elements. In particular, there is a unique rank-1 valuation ring between K^+ and K that we denote by $\mathcal{O}_K$, and the associated rank-1 valuation on K by $|\,.\,|_K\colon K \to \mathbf{R}_{\geq 0}$.

Lemma B.4 ([61, Proposition 3.8]). *Let K be a non-archimedean field. Then K is a perfectoid field if and only if the following conditions hold:*

(1) *K is not discretely valued,*

(2) $|p|_K < 1$,

(3) *the Frobenius morphism $\Phi\colon \mathcal{O}_K/p\mathcal{O}_K \to \mathcal{O}_K/p\mathcal{O}_K$ is surjective.*

We wish to show that the ideal $\mathfrak{m} = K^{\circ\circ} \subset K^+$ defines an ideal of almost mathematics in K^+. For future reference, it will be convenient to do that in the more general set-up of perfectoid pairs.

Definition B.5 ([61, Definition 3.1]). A complete Tate–Huber pair (R, R^+) is called a *perfectoid pair* if R is a uniform Tate ring that contains a pseudo-uniformizer $\varpi_R \in R^\circ$ such that $\varpi_R^p \mid p$ in R° and moreover, the Frobenius homomorphism $R^\circ/\varpi_R R^\circ \xrightarrow{x \mapsto x^p} R^\circ/\varpi_R^p R^\circ$ is an isomorphism.

A Tate–Huber pair (R, R^+) is a *p-adic perfectoid pair* if it is a perfectoid pair, and $p \neq 0$ in R.

A Tate ring R is a *perfectoid ring* if (R, R°) is a perfectoid pair.

Remark B.6. It is not, a priori, clear that a perfectoid ring R that is a field is a perfectoid field (in the sense of Definition B.1). The problem is to verify that R has a non-archimedean topology on it. This turned out to be always true by [40].

Remark B.7. By [61, Proposition 3.5], a complete Tate ring R of characteristic p is perfectoid if and only if R is perfect as a ring, i.e., the Frobenius morphism is an isomorphism.

Remark B.8. In the definition of a perfectoid pair above, it suffices to require that $R^\circ/\varpi_R R^\circ \xrightarrow{x\mapsto x^p} R^\circ/\varpi_R^p R^\circ$ be surjective. This map actually turns out to be always injective. Moreover, this condition turns out to be equivalent to the surjectivity of the Frobenius map

$$R^\circ/pR^\circ \to R^\circ/pR^\circ.$$

In particular, it is independent of a choice of a pseudo-uniformizer $\varpi_R^p \mid p$, see [61, Remark 3.2] for more detail. Therefore, if R is an algebra over a perfectoid field K with a pseudo-uniformizer $\varpi_K \in \mathcal{O}_K$, one can always take $\varpi_R = \varpi_K$. In particular, every perfectoid ring in the sense of [58, Definition 5.1] is a perfectoid ring in the sense of Definition B.5.

Lemma B.9 ([61, Lemma 3.10]). *Let (R, R^+) be a perfectoid pair. Then there is a pseudo-uniformizer $\varpi \in R^{\circ\circ}$ such that*

(1) *$\varpi^p \mid p$ in R°;*

(2) *ϖ admits a compatible sequence of p^n-th roots $\varpi^{1/p^n} \in R^+$ for $n \geq 0$.*

In this case, $R^{\circ\circ} = \bigcup_{n\geq 0} \varpi^{1/p^n} R^+$.

Proof. [61, Lemma 3.10] says that there is a pseudo-uniformizer $\varpi \in R^{\circ\circ} \subset R^+$ such that $\varpi^p \mid p$ in R°, and there is a compatible sequence of the p^n-th roots $\varpi^{1/p^n} \in R^\circ$ for $n \geq 0$. Since R^+ is integrally closed, we conclude that all ϖ^{1/p^n} must lie in R^+. Since $R^{\circ\circ}$ is a radical ideal of R^+ and contains ϖ, it clearly contains $\bigcup_{n\geq 0} \varpi^{1/p^n} R^+$.

Now we pick an element $x \in R^{\circ\circ}$, and wish to show that $x \in \bigcup_{n\geq 0} \varpi^{1/p^n} R^+$. Since x is topologically nilpotent, we can find an integer m such that

$$x^{p^m} \in \varpi R^+.$$

Therefore, $x^{p^m} = \varpi a$ for $a \in R^+$. Thus

$$\left(\frac{x}{\varpi^{1/p^m}}\right)^{p^m} = a \in R^+.$$

Therefore, $\frac{x}{\varpi^{1/p^m}} \in R^+$ because R^+ is integrally closed in R. So $x \in \varpi^{1/p^m} R^+$. ∎

Remark B.10. If (R, R^+) is a p-adic perfectoid pair, then one can choose ϖ such that $\varpi^p R^+ = pR^+$. Indeed, [8, Lemma 3.20] implies that R^+ is perfectoid in the sense of [8, Definition 3.5]. Thus the desired ϖ exists by [8, Lemma 3.9].

Definition B.11. A pseudo-uniformizer $\varpi \in R^+$ of a p-adic perfectoid pair (R, R^+) is *good* if $\varpi R^+ = pR^+$ and ϖ admits a compatible sequence of p-power roots.

For the rest of the appendix, we fix a perfectoid pair (R, R^+) and the ideal $\mathfrak{m} = R^{\circ\circ}$. Our goal is to show that $\mathfrak{m}$ defines a set-up for almost mathematics, i.e., $\widetilde{\mathfrak{m}} = \mathfrak{m} \otimes_{R^+} \mathfrak{m}$ is R^+-flat and $\mathfrak{m}^2 = \mathfrak{m}$.

Lemma B.12. *Let (R, R^+) be a perfectoid pair, and $\mathfrak{m} = R^{\circ\circ}$ the associated ideal of topologically nilpotent elements. Then $\mathfrak{m}$ is flat over R^+ and $\widetilde{\mathfrak{m}} \simeq \mathfrak{m}^2 = \mathfrak{m}$.*

Proof. Lemma B.9 implies that $\mathfrak{m}$ is flat as a colimit of free modules of rank-1.

Now we wish to show that $\mathfrak{m}^2 = \mathfrak{m}$. We take any element $x \in \mathfrak{m}$, by Lemma B.9 we know that $x = \varpi^{1/p^n} a$ for some integer n and $a \in R^+$. Therefore,

$$x = \left(\varpi^{1/p^{n+1}}\right)^{p-1}\left(\varpi^{1/p^{n+1}} a\right) \in \mathfrak{m}^2.$$

Now we consider the short exact sequence

$$0 \to \mathfrak{m} \to R^+ \to R^+/\mathfrak{m} \to 0.$$

By flatness of $\mathfrak{m}$, we know that it remains exact after applying the functor $- \otimes_{R^+} \mathfrak{m}$. Therefore, the sequence

$$0 \to \widetilde{\mathfrak{m}} \to \mathfrak{m} \to \mathfrak{m}/\mathfrak{m}^2 \to 0$$

is exact. Since $\mathfrak{m}^2 = \mathfrak{m}$, we conclude that

$$\widetilde{\mathfrak{m}} \simeq \mathfrak{m}^2 = \mathfrak{m}.$$ ∎

Lemma B.13. *For a perfectoid pair (R, R^+), the natural inclusion $\iota\colon R^+ \to R^\circ$ is an almost isomorphism.*

Proof. Clearly, the map $\iota\colon R^+ \to R^\circ$ is injective, so it suffices to show that its cokernel is almost zero, i.e., annihilated by any $\varepsilon \in \mathfrak{m}$. Pick an element $x \in R^\circ$, then $\varepsilon x \in R^{\circ\circ} \subset R^+$. Therefore, we conclude that $\varepsilon(\operatorname{Coker} \iota) = 0$ finishing the proof. ∎