Appendix C

Strongly sheafy adic spaces

In this appendix, we discuss the notion of strongly sheafy spaces following [33] and [41].

C.1 Preliminary results

In this section, we discuss some results about general Tate–Huber pairs.

Lemma C.1.1. *Let* (A, A^+) *be a complete Tate–Huber pair with a pair of definition* $(A_0 \subset A^+, \varpi)$*, and let* $A \to B$ *be a finite étale morphism. Topologize* B *using its natural* A*-module topology (see* [72, Appendix B.3]*). Then* (B, B^+) *is a complete Tate–Huber pair where* B^+ *is the integral closure of* A^+ *in* B*.*

Proof. Step 1*:* B *is complete in its natural topology*. Since B is finite étale, B is a projective A-module of finite rank. Then there is another finite A-module M such that $B \oplus M \simeq A^{\oplus n}$. Consider the projection $p\colon A^{\oplus n} \to B$, the natural topology on B coincides with the quotient topology (see [72, Lemma B.3.2]). Using the fact that A is a Huber ring, it is not difficult to show that the quotient topology on B should coincide with the subspace topology. Since $A^{\oplus n}$ is complete, we conclude that the natural topology on B is separated. Therefore, the same applies to M since we never used the ring structure on B. Then B is closed in A as a kernel of a continuous homomorphism with a separated target. In particular, B is complete in its subspace (equivalently, quotient) topology, and as discussed above, this topology coincides with the natural topology. So it is complete in its natural topology.

Step 2*:* B *admits a finite set of* A*-module generators* $x_1, \dots, x_n$ *that are integral over* A_0. Pick any finite set $x'_1, \dots, x'_n \in B$ of A-module generators. It suffices to show that $x_i = \varpi^c x'_i \in B$ are integral over A_0 for some integer c. So it is enough to show that, for any $b \in B$, there is an integer c such that $\varpi^c b$ is integral over A_0.

By definition, b is integral over A. So we can find a monic equation

$$b^n + a_{n-1}b^{n-1} + \cdots + a_0 = 0$$

with $a_k \in A$ for $k = 0, \dots, n-1$. Then there is an integer c such that $\varpi^c a_k \in A_0$ for $k = 0, \dots, n-1$. Thus, the equation

$$(\varpi^c b)^n + a_{n-1}\varpi^c(\varpi^c b)^{n-1} + \cdots + a_0\varpi^{cn} = 0$$

shows that $\varpi^c b$ is integral over A_0.

Step 3: *An A_0-subalgebra B_0 of B generated by $x_1, \ldots, x_n$ is finite as an A_0-module.* Clearly, this algebra is finitely generated over A_0 as an algebra and every element is integral. Therefore, it is finite.

Step 4: *B_0 is open in B and the induced topology coincides with the ϖ-adic one.* Choose some A_0-module generators $b_1, \ldots, b_m \in B_0$. Clearly, $B_0[\frac{1}{\varpi}] = B$, so the A-linear morphism

$$q\colon \bigoplus_{i=1}^{m} Ae_i \to B$$

sending e_i to b_i is surjective. By [37, Lemma 2.4 (i)], q is open. In particular, the topology on B is the quotient topology along q. Therefore, B_0 is open in B as $q^{-1}(B_0)$ is a subgroup that contains an open subgroup $\bigoplus_{i=1}^{m} A_0 e_i$. Moreover, the topology on B_0 is ϖ-adic since $B_0 = q(\bigoplus_{i=1}^{m} A_0 e_i)$ with the quotient topology, and the topology on $\bigoplus_{i=1}^{m} A_0 e_i$ is already ϖ-adic.

Step 5: *(B, B^+) is a complete Huber pair.* We have already shown that B is complete in its natural topology and (B_0, ϖ) is a pair of definition for this topology. Therefore, B is a Huber ring. It suffices to show that B^+ is open, integrally closed, and lies in B°. Openness is clear since $B_0 \subset B^+$, and B^+ is integrally closed by definition. One also easily shows that $B^+ \subset B^\circ$ because B^+ is integral over $A^+ \subset A^\circ$. ■

Lemma C.1.2. *Let (A, A^+) and (B, B^+) be as in Lemma C.1.1. Then* Spec $B \to$ Spec A *is surjective if and only if* Spa $(B, B^+) \to$ Spa (A, A^+) *is surjective.*

Proof. First, we assume that Spec $B \to$ Spec A is surjective. In order to show that Spa $(B, B^+) \to$ Spa (A, A^+) is surjective, we need to show that $B \widehat{\otimes}_A \widehat{k(x)} \neq 0$ for any $x \in$ Spa (A, A^+). Now [72, Lemma B.3.5] and Lemma C.1.1 ensure that $B \widehat{\otimes}_A \widehat{k(x)} = B \otimes_A \widehat{k(x)}$. To finish the proof, we note that $B \otimes_A k \neq 0$ for any field k and a homomorphism $A \to k$ (in particular, this holds for $A \to \widehat{k(x)}$).

Now we assume that Spa $(B, B^+) \to$ Spa (A, A^+) is surjective. Then we note that [37, Lemma 1.4] implies that every maximal ideal $\mathfrak{m} \subset A$ admits a valuation $v \in$ Spa (A, A^+) such that $\operatorname{supp}(v) = \mathfrak{m}$. This implies that the image of the morphism Spec $B \to$ Spec A contains all closed points of Spec A. Since étale morphisms are open, we conclude that Spec $B \to$ Spec A must be surjective. ■

Now we discuss the notion of semi-uniform Tate–Huber pairs.

Lemma C.1.3. *Let (A, A^+) be a (possibly noncomplete) Tate–Huber pair. Then A^+ is bounded if and only if A is uniform (i.e., A° is bounded).*

Proof. Clearly, A^+ is bounded if A° is bounded. So we assume that A^+ is bounded and we wish to show that A° is bounded as well. Choose a ring of definition $A_0 \subset A^+$ and a pseudo-uniformizer $\varpi \in A_0$. Since A^+ is bounded, we conclude that there is an integer N such that $A^+ \subset \frac{1}{\varpi^N} A_0$. Now we note that $A^{\circ\circ} \subset A^+$ since A^+ is integrally closed and open. Since ϖ is topologically nilpotent and any element $a \in A^\circ$ is power bounded, we conclude that $\varpi A^\circ \subset A^{\circ\circ} \subset A^+$. Therefore, $A^\circ \subset \frac{1}{\varpi^{N+1}} A_0$, i.e., A° is bounded. ■

The above lemma motivates the following definition:

Definition C.1.4. A (possibly noncomplete) Tate–Huber pair (A, A^+) is *uniform* if $A^+ \subset A$ is bounded.

Remark C.1.5. [36, Proposition 1] implies that (A, A^+) is uniform if and only if the subspace topology on A^+ coincides with the ϖ-adic topology for a (equivalently, any) choice of a pseudo-uniformizer $\varpi \in A^+$. Lemma C.1.3 implies that it is equivalent to asking that the subspace topology on A° coincides with the ϖ-adic topology.

Lemma C.1.6. *Let A be a (possibly noncomplete) Tate ring. If A is Hausdorff, then A is reduced. In particular, any complete uniform Tate ring is reduced.*

Proof. Let $a \in A$ be a nilpotent element. We choose a pseudo-uniformizer $\varpi \in A^\circ$. Then $\frac{a}{\varpi^n}$ is nilpotent for any $n \geq 0$. In particular, it is bounded, so $\frac{a}{\varpi^n} \in A^\circ$. Thus, $a \in \varpi^n A^\circ$ for any $n \geq 0$. Since A is uniform, we conclude that the topology on A° coincides with the ϖ-adic topology. Since A is Hausdorff, $\bigcap_{n\geq 0} \varpi^n A^\circ = 0$. Thus, $a = 0$ finishing the proof. ■

Definition C.1.7. Let (A, A^+) be a (possibly noncomplete) Tate–Huber pair with a pseudo-uniformizer $\varpi \in A^+$. The *uniformization* of (A, A^+) is the Tate–Huber pair (A_u, A_u^+), where $A_u^+ = A^+$, $A_u = A$, and the topology on A_u is induced from the ϖ-adic topology on A_u^+.

The *uniform completion* of (A, A^+) is the Tate–Huber pair $(\widehat{A}_u, \widehat{A}_u^+)$ obtained as the completion of (A_u, A_u^+) (see [36, Lemma 1.6]).

Remark C.1.8. We leave it to the reader to check that the uniformization is indeed a Tate–Huber pair and that it is independent of the choice of a pseudo-uniformizer $\varpi \in A^+$. In fact, uniformization is a left adjoint functor to the inclusion of uniform Tate–Huber pairs into the category of all Tate–Huber pairs. Likewise, uniform completion is a left adjoint functor to the inclusion of complete uniform Tate–Huber pairs into all Tate–Huber pairs.

Now we discuss the relation between the topology of $\mathrm{Spa}\,(A, A^+)$ and its uniform completion.

Lemma C.1.9. *Let* (A, A^+) *be a (possibly noncomplete) Tate–Huber pair. Then the natural morphisms*

$$\begin{aligned} \mathrm{Spa}\,(A_u, A_u^+) &\to \mathrm{Spa}\,(A, A^+), \\ \mathrm{Spa}\,(\widehat{A}, \widehat{A}^+) &\to \mathrm{Spa}\,(A, A^+), \\ \mathrm{Spa}\,(\widehat{A}_u, \widehat{A}_u^+) &\to \mathrm{Spa}\,(A, A^+) \end{aligned}$$

are homeomorphisms that induce bijections on the sets of rational subdomains.

Proof. First, [36, Proposition 3.9] implies that the natural morphism $\mathrm{Spa}\,(\widehat{A}, \widehat{A}^+) \to \mathrm{Spa}\,(A, A^+)$ is a homeomorphism that induces a bijection on the sets of rational subdomains. Applying the same result to (A_u, A_u^+), we see that it suffices to show the claim for $\mathrm{Spa}\,(A_u, A_u^+) \to \mathrm{Spa}\,(A, A^+)$.

For this, we note that rational subdomains on both sides are indexed by tuples $(f_1, \dots, f_n, g)$ of non-zero elements in A generating the unit ideal, so we conclude that it suffices to show that $\mathrm{Spa}\,(A, A^+) \to \mathrm{Spa}\,(A_u, A_u^+)$ is a bijection. After unraveling the definition, we see that it suffices to show that any continuous (in the usual topology) valuation $v\colon A \to \Gamma_v \cup \{0\}$ is continuous in the topology induced from the ϖ-adic topology on A^+. Since v is continuous, [64, Corollary 9.3.3] implies that $v(\varpi) \in \Gamma_v$ is cofinal and $v(\varpi a) < 1$ for any ring of definition $\varpi \in A_0$. Likewise, loc. cit. implies that it suffices to show that $v(\varpi a) < 1$ for any $a \in A^+$. This follows from [36, Corollary 1.3], which ensures that we can always find a ring of definition $A_0 \subset A^+$ which contains both a and ϖ. ■

Lemma C.1.10. *Let* (A, A^+) *be a complete Tate–Huber pair, and let* $\varpi \in A^+$ *be a pseudo-uniformizer. Then* A^+ *is* ϖ*-adically henselian.*

Proof. First, [36, Corollary 1.3] ensures that A^+ is a filtered colimit of its subrings of definition $A_0 \subset A^+$. Therefore, the result follows from [68, Tag 0ALJ] and [68, Tag 0FWT]. ■

Lemma C.1.11. *Let* (A, A^+) *be a (possibly noncomplete) Tate–Huber pair with a pseudo-uniformizer* $\varpi \in A^+$*. Suppose that* A^+ *is* ϖ*-adically henselian, then the natural functors*

$$\begin{aligned} - \otimes_A \widehat{A}\colon A_{\text{fét}} &\to \widehat{A}_{\text{fét}}, \\ - \otimes_A \widehat{A}_u\colon A_{\text{fét}} &\to \widehat{A}_{u,\text{fét}} \end{aligned}$$

are equivalences. Further, the natural maps $\mathrm{Idem}(A) \to \mathrm{Idem}(\widehat{A})$ *and* $\mathrm{Idem}(A) \to \mathrm{Idem}(\widehat{A}_u)$ *are bijections.*

Proof. The semi-uniform completions of (A, A^+) and $(\widehat{A}, \widehat{A}^+)$ coincide, therefore Lemma C.1.10 implies that it suffices to prove both claims for $\widehat{A}_u$.

The claim about finite étale algebras follows immediately from [26, Proposition 5.4.54] and the observation that $\varpi \in A^+$ is a regular element. Then the claim about idempotents follows from [68, Tag 09XI] and the observation that any idempotent $e \in A$ must lie in A^+ because it is integral over $\mathbf{Z}$. ∎

C.2 Noetherian approximation

The main goal of this section is to show a version of noetherian approximation for complete Tate–Huber pairs. The main result of this section was originally shown in [41, Proposition 2.6.2] in a slightly different language.

To motivate the definition below, we want to mention one important subtlety of working with complete Tate–Huber pairs: this category does not admit filtered colimits. However, this issue can be remedied by considering Tate–Huber pairs together with the choice of a ring of definition.

Definition C.2.1. A *Tate–Huber quadruple* (A, A^+, A_0, ϖ) is a quadruple of a Tate–Huber pair (A, A^+), a ring of definition A_0, and a pseudo-uniformizer $\varpi \in A_0$. A *morphism of Tate–Huber quadruples*

$$f\colon (A, A^+, A_0, \varpi) \to (B, B^+, B_0, \pi)$$

is a continuous ring homomorphism $f\colon A \to B$ such that $f(A^+) \subset B^+$, $f(A_0) \subset B_0$, and $f(\varpi) = \pi$.

For the next definition, we fix a filtered system $\{(A_i, A_i^+, A_{i,0}, \varpi), f_{i,j}\}_{i \in I}$ of Tate–Huber quadruples.[1]

Definition C.2.2. The *filtered colimit* of $\{(A_i, A_i^+, A_{i,0}, \varpi), f_{i,j}\}_{i,j \in I}$ is the Tate–Huber quadruple

$$\left(\operatorname{colim}_I A_i, \operatorname{colim}_I A_i^+, \operatorname{colim}_I A_{i,0}, \varpi\right),$$

where we topologize A_i by requiring $\operatorname{colim}_I A_{i,0} \subset \operatorname{colim}_I A_i$ to be a ring of definition with a pseudo-uniformizer ϖ.[2]

The *completed filtered colimit* of $\{(A_i, A_i^+, A_{i,0}, \varpi), f_{i,j}\}_{i \in I}$ is the Tate–Huber quadruple

$$\left(A_\infty, A_\infty^+, A_{\infty,0}, \varpi\right) := \left(\widehat{\operatorname{colim}_I A_i}, \widehat{\operatorname{colim}_I A_i^+}, \widehat{\operatorname{colim}_I A_{i,0}}, \varpi\right),$$

[1] We slightly abuse notation and denote the pseudo-unifomizer in $A_{i,0}$ by the same letter ϖ.

[2] We note that this implies that the subspace topology on $\operatorname{colim}_I A_{i,0}$ is equal to the ϖ-adic topology. We warn the reader that the colimit topology on $A_{i,0}$ is usually *different* from the ϖ-adic one.

where $\widehat{-}$ stands for the topological completion in the sense of [18, Chapitre III, Section 3 n.4] (see also [18, Chapitre III, Section 6 n.5 and Chapitre III,Section 7 n.2]).

Remark C.2.3. [36, Lemma 1.6] gives a very explicit description of completed filtered colimits. Namely, the ring $A_{\infty,0}$ is equal to $(\mathrm{colim}_I\, A_{i,0})^{\wedge}_{\varpi}$, the usual ϖ-adic completion of $\mathrm{colim}_I\, A_{i,0}$, the ring $A^+_\infty = (\mathrm{colim}_I\, A^+_i) \otimes_{\mathrm{colim}_I A_{i,0}} A_{\infty,0}$, and the ring $A_\infty = A^+_\infty[\frac{1}{\varpi}] = A_{\infty,0}[\frac{1}{\varpi}]$.

For the next definition, we fix a filtered system of $\{(A_i, A^+_i), f_{i,j}\}_{i,j\in I}$ of (not necessarily uniform) Tate–Huber pairs with a compatible choice of pseudo-uniformizers $\varpi \in A^+_i$.

Definition C.2.4. The *uniform filtered colimit* of $\{(A_i, A^+_i), f_{i,j}\}_{i,j\in I}$ is the filtered colimit of the Tate–Huber quadruples $\{A_{i,u}, A^+_{i,u}, A^+_{i,u}, \varpi\}_{i\in I}$ (see Definition C.1.7).

The *completed uniform filtered colimit* of $\{(A_i, A^+_i), f_{i,j}\}_{i,j\in I}$ is the completed filtered colimit of the Tate–Huber quadruples $\{A_{i,u}, A^+_{i,u}, A^+_{i,u}, \varpi\}_{i\in I}$.

Remark C.2.5. Remark C.2.3 implies that the completed uniform filtered colimit is explicitly given by the pair $\big(\widehat{(\mathrm{colim}_I\, A^+_i)}[\frac{1}{\varpi}], \widehat{\mathrm{colim}_I\, A^+_i}\big)$, where $\widehat{(-)}$ stands for the ϖ-adic completion.

Now we wish to prove a version of noetherian approximation for complete (uniform) Tate–Huber pairs. Before we do this, we need to invoke the following basic fact:

Lemma C.2.6. *Let (A, A^+) be a complete Tate–Huber pair, and let $I \subset A$ be a closed ideal. Then $(A/I, (A/I)^+)$ is a complete Tate–Huber pair, where $(A/I)^+$ is the integral closure of $A^+/(I \cap A^+)$ in A/I.*

Proof. First, we choose a ring of definition A_0 and a pseudo-uniformizer $\varpi \in A_0$. Then A/I is complete in the quotient topology due to [16, Chapitre IX Section 3 Proposition 1] and [16, Chapitre IX Section 3 Proposition 4]. Then [37, Proposition 2.4 (ii)] ensures that the natural morphism $\pi\colon A \to A/I$ is open. Therefore, $\pi(A_0) \subset A/I$ is an open subset such that the subset topology coincides with the ϖ-adic topology. Furthermore, (the image of) ϖ is clearly a topologically nilpotent unit in A/I. Therefore, we conclude that A/I is a complete Tate ring. Thus, we only need to show that $(A/I)^+$ is open, integrally closed, and lies in $(A/I)^\circ$. It is closed integrally by construction and is open because it contains $\pi(A_0)$. Finally, we note that by construction we have $(A/I)^+ \subset \pi(A^\circ)^+ \subset (A/I)^\circ$, where $\pi(A^\circ)^+$ is the integral closure of $\pi(A^\circ)$ in A/I. ∎

Example C.2.7. Let A be a Tate ring and let $e \in A$ be an idempotent element. Then the ideal eA is closed in A since it is equal to the kernel of the continuous multiplication by $(1-e)$ map.

Finally, we are ready to prove the main results of this section.

Lemma C.2.8 (cf., [41, Proposition 2.6.2]). *Let (A, A^+) be a complete Tate–Huber pair, let $A_0 \in A^+$ be a ring of defintion, and let $\varpi \in A_0$ be a pseudo-uniformizer. Then there is a filtered system of Tate–Huber quadruples $\{(A_i, A_i^+, A_{i,0}, \varpi)\}_{i\in I}$ such that*

(1) *each A_i is a strongly noetherian complete Tate algebra;*

(2) *the completed filtered colimit $(A_\infty, A_\infty^+, A_{\infty,0}, \varpi)$ of $\{(A_i, A_i^+, A_{i,0}, \varpi)\}_{i\in I}$ is isomorphic to (A, A^+, A_0, ϖ).*

Proof. The choice of a pseudo-uniformizer $\varpi \in A^+$ defines a map $(\mathbf{Z}(\!(t)\!), \mathbf{Z}[\![t]\!]) \to (A, A^+)$. Then we put I to be the filtered poset of all finite subsets $S \subset A^+$. For each $S \in I$, we consider the unique $(\mathbf{Z}(\!(t)\!), \mathbf{Z}[\![t]\!])$-linear continuous morphism

$$\alpha_S : \big(\mathbf{Z}(\!(t)\!)\langle X_f\rangle_{f\in S}, \mathbf{Z}[\![t]\!]\langle X_f\rangle_{f\in S}\big) \to \big(A, A^+\big)$$

that sends X_f to f. We put $I_S = \operatorname{Ker}\alpha_S$ and $I_S^+ = I_S \cap \mathbf{Z}[\![t]\!]\langle X_f\rangle_{f\in S}$. The ideal I_S is closed because it is the kernel of a continuous morphism. Therefore, Lemma C.2.6 gives us a complete Tate–Huber pair

$$(A_S, A_S^+) := \big(\mathbf{Z}(\!(t)\!)\langle X_f\rangle_{f\in S}/I_S, (\mathbf{Z}(\!(t)\!)\langle X_f\rangle_{f\in S}/I_S)^+\big)$$

that admits an injective continuous morphism $\overline{\alpha}_S : (A_S, A_S^+) \to (A, A^+)$. We finally define

$$A_{S,0} := \big(\mathbf{Z}(\!(t)\!)\langle X_f\rangle_{f\in S}/I_S^+\big) \cap A_0.$$

This subring is clearly open and bounded, so it is a ring of definition due to [36, Proposition 1]. Finally, we put $\varpi \in A_{S,0}$ to be the image of t. Therefore, we note that

$$\big\{(A_S, A_S^+, A_{S,0}, \varpi)\big\}_{S\in I}$$

with natural (injective) transition maps is a filtered system of Tate–Huber quadruples. Using the explicit description of completed filtered colimits from Remark C.2.3, we note that the (uncompleted) filtered colimit of $\big\{(A_S, A_S^+, A_{S,0}, \varpi)\big\}_{S\in I}$ coincides with the completed filtered colimit, and it is isomorphic to (A, A^+, A_0, ϖ). To finish the proof, we only need to show that each A_S is strongly noetherian. This follows from the fact that $\mathbf{Z}(\!(t)\!)$ admits a noetherian ring of definition and A_S is topologically finite type over $\mathbf{Z}(\!(t)\!)$. ■

Lemma C.2.9. *Let (A, A^+) be a complete uniform Tate–Huber pair, and let $\varpi \in A_0$ be a pseudo-uniformizer. Then there is a filtered system of complete uniform Tate–Huber pairs $\{(A_i, A_i^+)\}_{i\in I}$ such that*

(1) *each A_i is strongly noetherian;*

(2) *the completed uniform filtered colimit (A_∞, A_∞^+) of $\big\{(A_i, A_i^+)\big\}_{i\in I}$ is isomorphic to (A, A^+).*

Proof. The proof is similar to that of Lemma C.2.8. We define the index set I as in the proof of Lemma C.2.8. Likewise, for any $S \in I$, we define A_S, A_S^+, and $A_{S,0}$ as in the proof of Lemma C.2.8 as well. Then we wish to show that the Huber-Tate pair (A_S, A_S^+) is uniform. Once we know this fact, the rest of the argument is the same.

Now we show that (A_S, A_S^+) is uniform. We note that $\mathbf{Z}[\![t]\!]$ is excellent due to [68, Tag 07QW]. Therefore, [45, Main Theorem 2] implies that $A_{S,0}$ is also excellent. Then we recall that A_S^+ was defined as the integral closure of $A_{S,0}$ inside A. Since A is reduced due to Lemma C.1.6, [68, Tag 03GH] and [68, Tag 07QV] imply that A_S^+ is a finite $A_{S,0}$-module. Therefore, there is an integer n such that $A_S^+ \subset \frac{1}{\varpi^n} A_{S,0}$, i.e., A_S^+ is bounded. This finishes the proof. ∎

C.3 Étale maps

In this section, we discuss (strongly) étale maps of general complete Tate–Huber pairs. We also show that strongly étale morphisms satisfy approximation along completed (uniform) filtered colimits of complete Tate–Huber pairs.

Definition C.3.1. A morphism $(A, A^+) \to (B, B^+)$ of complete Tate–Huber pairs is a *rational subdomain* if there is a finite set of non-zero elements $f_1, \dots, f_n, g \in A$ which generates the unit ideal in A and $(B, B^+) = \left(A\langle \frac{f_i}{g}\rangle, A\langle \frac{f_i}{g}\rangle^+\right)$ as an (A, A^+)-algebra. We denote by $(A, A^+)_{\mathrm{rsd}}$ the poset[3] of rational subdomains of (A, A^+) (it coincides with the poset of rational subdomains of $\mathrm{Spa}\,(A, A^+)$).

A morphism $(A, A^+) \to (B, B^+)$ of complete Tate–Huber pairs is *strongly finite étale* if $A \to B$ is finite étale and B^+ is the integral closure of A^+ in B. We denote by $(A, A^+)_{\mathrm{s\acute{e}t}}$ the category of finite étale (A, A^+)-pairs and all (A, A^+)-linear morphisms between them.

A morphism $(A, A^+) \to (B, B^+)$ of complete Tate–Huber pairs is *strongly étale* if it can be written as a finite composition of finite étale morphisms and rational subdomains. We denote by $(A, A^+)_{\mathrm{s\acute{e}t}}$ the category of étale (A, A^+)-pairs and all (A, A^+)-linear morphisms between them.

Remark C.3.2. In what follows, we will freely use the fact that the category of complete Tate–Huber pairs admits pushouts. Explicitly, the pushout $(B, B^+) \otimes_{A,A^+} (C, C^+)$ is given by

$$\left(B \widehat{\otimes}_A C, (B \widehat{\otimes}_A C)^+\right),$$

where $B \widehat{\otimes}_A C$ is the completed tensor product, and $(B \widehat{\otimes}_A C)^+$ is the integral closure of (the image of) $B^+ \widehat{\otimes}_{A^+} C^+$ in $B \widehat{\otimes}_A C$.

[3] [37, Proposition 1.3] implies that there is at most one unique continuous (A, A^+)-linear morphism between two rational subdomains over (A, A^+).

Remark C.3.3. By definition, strongly étale maps are closed under composition. Lemma C.1.1 implies that strongly finite étale maps are closed under pushouts of complete Tate–Huber pairs (since completion is not needed). Therefore, all strongly étale maps are also closed under pushouts in the category of complete Tate–Huber pairs.

Remark C.3.4. Lemma C.1.1 ensures that there is an equivalence $(A, A^+)_{\text{sfét}} \simeq A_{\text{fét}}$ for any complete Tate–Huber pair (A, A^+).

Now we wish to show that the category of strongly étale (A, A^+)-pairs satisfies approximation with respect to completed filtered colimits. It will be convenient to first prove a version of this result for completed uniform filtered colimits. For this, we need a number of preliminary lemmas.

Definition C.3.5. A morphism $f\colon (A, A^+) \to (B, B^+)$ of complete Tate–Huber pairs is a *clopen immersion* if $A \to B$ is a topological quotient morphism, B^+ is equal to the integral closure of A^+, and $\operatorname{Ker} f$ is generated by an idempotent element $e_f \in A$.

Remark C.3.6. If $f\colon (A, A^+) \to (B, B^+)$ is a clopen immersion, then Lemma C.2.6 implies that

$$(B, B^+) = \big(A/e_f A, \big(A^+/(e_f A \cap A^+)\big)^+\big)$$

as complete Tate–Huber pairs.

Remark C.3.7. We note that the idempotent $e_f \in A$ in Definition C.3.5 is unique if exists. In particular, two clopen immersions $f\colon (A, A^+) \to (B, B^+)$ and $g\colon (A, A^+) \to (B, B^+)$ coincide if and only if $e_f = e_g$.

For the purpose of the next definition, we fix a complete Tate–Huber pair (A, A^+) and an (A, A^+)-linear morphism $f\colon (B, B^+) \to (C, C^+)$ of complete Tate–Huber pairs.

Definition C.3.8. The *graph* of f is the unique continuous (A, A^+)-linear morphism

$$\Gamma_f\colon \big(B \widehat{\otimes}_A C, (B \widehat{\otimes}_A C)^+\big) \to (C, C^+)$$

which sends $b \otimes c$ to $f(b)c$.

The diagonal of f is the morphism $\Delta_f\colon \big(C \widehat{\otimes}_B C, (C \widehat{\otimes}_B C)^+\big) \to (C, C^+)$ that sends $c \otimes c'$ to cc'.

Lemma C.3.9. *Let $f\colon (A, A^+) \to (B, B^+)$ and $g\colon (B, B^+) \to (C, C^+)$ be morphisms of complete Tate–Huber pairs such that g and $h := g \circ f$ are strongly étale. Then the morphisms Δ_g and Γ_g are clopen immersions.*

Proof. First, we note that it suffices to prove the claim for Δ_g (for all g). This follows from the following pushout square (for simplicity, we suppress the $+$-rings in the diagram below):

$$\begin{array}{ccc} B\widehat{\otimes}_A B & \xrightarrow{\Delta_f} & B \\ {\scriptstyle \mathrm{id}\otimes g}\downarrow & & \downarrow{\scriptstyle g} \\ B\widehat{\otimes}_A C & \xrightarrow[\Gamma_g]{} & C \end{array}$$

and the observation that clopen immersions are preserved by pushouts. Now we show that Δ_g is a clopen immersion if Δ_f and Δ_h are so. For this, we consider the following diagram (for simplicity, we suppress the $+$-rings in the diagram below):

$$\begin{array}{ccccc} B\widehat{\otimes}_A B & \xrightarrow{\Delta_f} & B & & \\ {\scriptstyle g\otimes g}\downarrow & & \downarrow & & \\ C\widehat{\otimes}_A C & \xrightarrow{\alpha} & C\widehat{\otimes}_B C & \xrightarrow{\Delta_g} & C, \end{array} \qquad C\widehat{\otimes}_A C \xrightarrow{\Delta_h} C \tag{C.3.1}$$

where the left square is a pushout square. Now, if Δ_f is a clopen immersion, then α is a clopen immersion as well. Now since α and Δ_h are clopen immersions, then Δ_g is a clopen immersion as well (with $e_{\Delta_g} = \alpha(e_{\Delta_h})$). Therefore, we reduce the question to showing that f and h are clopen immersions. In other words, we can assume that g is a strongly étale morphism.

Now we use Diagram (C.3.1) and the observation that clopen immersions are preserved by compositions to conclude that it suffices to prove the result separately for strongly finite étale morphisms and rational subdomains. If g is a rational subdomain, then Δ_g is clearly an isomorphism. Therefore, it suffices to assume that g is a strongly finite étale map. In this case, Lemma C.1.1 and [72, Lemma B.3.5] imply that $B\widehat{\otimes}_A B = B \otimes_A B$. Therefore, the result follows from the algebro-geometric claim that $\operatorname{Spec} B \to \operatorname{Spec}(B \otimes_A B)$ is a clopen immersion for a finite étale $A \to B$. ∎

Theorem C.3.10. *Let $\{(A_i, A_i^+)\}_{i\in I}$ be a filtered system of complete uniform Tate–Huber pairs, and let $(A_\infty, A_\infty^+) = \left(\widehat{\operatorname{colim}_I A_i^+}\left[\frac{1}{\varpi}\right], \widehat{\operatorname{colim}_I A_i^+}\right)$ be its completed uniform filtered colimit. Then*

1. *the natural map $|\operatorname{Spa}(A_\infty, A_\infty^+)| \to \lim_I |\operatorname{Spa}(A_i, A_i^+)|$ is a homeomorphism of spectral spaces;*
2. *the natural map $\operatorname{colim}_I (A_i, A_i^+)_{\mathrm{rsd}} \to (A_\infty, A_\infty^+)_{\mathrm{rsd}}$ is a bijection;*
3. *the natural functor $2\text{-}\operatorname{colim}_I (A_i, A_i^+)_{\mathrm{sfét}} \to (A_\infty, A_\infty^+)_{\mathrm{sfét}}$ is an equivalence;*
4. *the natural functor $2\text{-}\operatorname{colim}_I (A_i, A_i^+)_{\mathrm{sét}} \to (A_\infty, A_\infty^+)_{\mathrm{sét}}$ is an equivalence.*

Proof. Let us denote by $\left(\overline{A}, \overline{A}^+\right) = \left(\operatorname{colim}_I A_i^+\left[\frac{1}{\varpi}\right], \operatorname{colim}_I A_i^+\right)$ the uncompleted uniform filtered colimit of (A_i, A_i^+). Then we easily see that the natural morphism $|\operatorname{Spa}(\overline{A}, \overline{A}^+)| \to \lim_I |\operatorname{Spa}(A_i, A_i^+)|$ is a homeomorphism. Now [36, Proposition 3.9] implies that the natural map $|\operatorname{Spa}(A_\infty, A_\infty^+)| \to |\operatorname{Spa}(\overline{A}, \overline{A}^+)|$ is a homeomorphism that induces a bijection on rational subdomains. This already proves (1). To see (2), we use that the homeomorphism $|\operatorname{Spa}(\overline{A}, \overline{A}^+)| \simeq |\operatorname{Spa}(A_\infty, A_\infty^+)|$ induces a bijection on rational subdomains. Since every rational subdomain of $|\operatorname{Spa}(\overline{A}, \overline{A}^+)|$ is defined at a finite level, we conclude that the natural morphism

$$\operatorname{colim}_I (A_i, A_i^+)_{\mathrm{rsd}} \to (A_\infty, A_\infty^+)_{\mathrm{rsd}}$$

is surjective. The map is also injective due to and [68, Tag 0A30]. This finishes the proof of (2).

(3) follows from Lemma C.1.11, Remark C.3.4, and a standard (algebraic) approximation for finite étale algebras.

Now we show (4). First, we set up some notation. For any complete Tate–Huber (A_i, A_i^+)-pair (B_i, B_i^+) and $i' > i \in I \sqcup \{\infty\}$, we put

$$(B_i', B_i'^+) := \big(B_i \widehat{\otimes}_{A_i} A_{i'}, (B_i \widehat{\otimes}_{A_i} A_{i'})^+\big).$$

Observation. For any compatible sequence of complete Tate–Huber (A_i, A_i^+)-pairs (B_i, B_i^+), the uniform completion of (B_∞, B_∞^+) and the compeleted uniform filtered colimit of $\{(B_i, B_i')\}_{i \in I}$ are isomorphic as (A, A^+)-pairs.

In what follows, we will freely use this observation. Finally, we are ready to start the proof.

Step 0: *Essential surjectivity.* Using *Observation*, Lemma C.1.9, and Lemma C.1.11, we can inductively reduce the question to showing that any rational subdomain (resp. finite étale pair) over (A_∞, A_∞^+) comes from a finite level. This follows directly from (1) (resp. (2)).

Step 1: *Faithfulness.* We start with fixing two systems of compatible morphisms $f_i, g_i \colon (B_i, B_i^+) \to (C_i, C_i^+)$ in $(A_i, A_i^+)_{\text{sét}}$ and then wish to show that, if their pushouts to (A_∞, A_∞^+) coincide, then $f_i = g_i$ for some $i \gg 0$. For this, we set $f_\infty, g_\infty \colon (B_\infty, B_\infty^+) \to (C_\infty, C_\infty^+)$ to be the morphisms induced by f_i and g_i respectively.

The graphs $\Gamma_{f_i}, \Gamma_{g_i}, \Gamma_{f_\infty}$, and Γ_{g_∞} are clopen immersions due to Lemma C.3.9. We notice moreover that $f_i = g_i$ (resp. $f_\infty = g_\infty$) if and only if $\Gamma_{f_i} = \Gamma_{g_i}$ (resp. $\Gamma_{f_\infty} = \Gamma_{g_\infty}$). Furthermore, Remark C.3.7 implies that $\Gamma_{f_i} = \Gamma_{g_i}$ (resp. $\Gamma_{f_\infty} = \Gamma_{g_\infty}$) if and only if $e_{\Gamma_{f_i}} = e_{\Gamma_{g_i}}$ (resp. $e_{\Gamma_{f_\infty}} = e_{\Gamma_{g_\infty}}$). Thus, we reduce the question to showing that if two idempotents $e, e' \in A_i$ become equal in A_∞, they are already equal in A_j for some $j > i$. This follows from Lemma C.1.11 and usual properties of filtered colimits. This finishes the proof of faithfulness.

Step 2: *Fullness.* We start with two compatible sequences (B_i, B_i^+), (C_i, C_i^+) of elements in $(A_i, A_i^+)_{\text{sét}}$ and a continuous (A_∞, A_∞^+)-linear morphism $f_\infty \colon (B_\infty, B_\infty^+) \to (C_\infty, C_\infty^+)$, we wish to show that it is defined over (A_i, A_i^+) for some $i \in I$. For this, we can freely replace I with $I_{\geq i_0}$ for some i_0 to assume that I has a minimal element $i_0 \in I$.

We write the morphism $g_{i_0}\colon (A_{i_0}, A_{i_0}^+) \to (B_{i_0}, B_{i_0}^+)$ as a composition of n morphisms, each of which is finite étale or a rational subdomain. We argue by induction on n. If $n = 0$, then $(A_{i_0}, A_{i_0}^+) = (B_{i_0}, B_{i_0}^+)$ and then the result is obvious (the morphism f_∞ must be the structure morphism $(A_\infty, A_\infty^+) \to (C_\infty, C_\infty^+)$, so it descends to any $i \in I$).

Now we do the induction step. We write g_{i_0} as a composition

$$\left(A_{i_0}, A_{i_0}^+\right) \xrightarrow{h_{i_0}} \left(B'_{i_0}, B'^+_{i_0}\right) \xrightarrow{g'_{i_0}} \left(B_{i_0}, B_{i_0}\right),$$

where g'_{i_0} is either strongly finite étale or a rational subdomain, and h_{i_0} is a composition of at most $n-1$ finite étale and rational subdomain morphisms. By induction, we know that there is $i \in I$ such that the morphism $f'_\infty := f_\infty \circ g'_\infty\colon (B'_\infty, B'^+_\infty) \to (C_\infty, C_\infty^+)$ is defined over $i \in I$. So we can replace i_0 with i to assume that there is a morphism

$$f'_{i_0}\colon \left(B'_{i_0}, B'^+_{i_0}\right) \to \left(C_{i_0}, C_{i_0}^+\right)$$

such that its pushout to (A_∞, A_∞^+) is equal to f'_∞.

Consider the following diagram:

$$\begin{array}{ccccc} & & \left(C_i, C_i^+\right) & & \\ & \nearrow & \uparrow{\scriptstyle f'_i} & \overset{f_i}{\nwarrow} & \\ \left(A_i, A_i\right) & \longrightarrow & \left(B'_i, B'^+_i\right) & \underset{g'_i}{\longrightarrow} & \left(B_i, B_i^+\right) \end{array} \tag{C.3.2}$$

for $i \geq i_0 \in I$. The proof of faithfulness boils down to constructing a morphism f_i such that Diagram (C.3.2) commutes and the pushout of f_i to (A_∞, A_∞^+) is equal to f_∞. For this, we consider two cases.

Case 1: *g'_{i_0} is a rational subdomain.* In this case, [37, Proposition 1.3] implies that, for each $i \in I \sqcup \infty$, there is at most one f_i which makes Diagram (C.3.2) commute. Furthermore, it exists if and only if

$$\left|\operatorname{Spa}(f'_i)\right|\colon \left|\operatorname{Spa}(C_i, C_i^+)\right| \to \left|\operatorname{Spa}(B'_i, B'^+_i)\right|$$

factors through $|\operatorname{Spa}(B_i, B_i^+)| \subset |\operatorname{Spa}(B'_i, B'^+_i)|$. Loc. cit. implies that $|\operatorname{Spa}(f'_\infty)|$ factors through $|\operatorname{Spa}(B_\infty, B_\infty^+)| \subset |\operatorname{Spa}(B'_\infty, B'^+_\infty)|$. So, *Observation*, Lemma C.1.9, Part (1), [68, Tag 0A2S], and [68, Tag 0A2X] imply that there is an index $i \in I$ such that

$$\left|\operatorname{Spa}(f'_i)\right|\colon \left|\operatorname{Spa}(C_i, C_i^+)\right| \to \left|\operatorname{Spa}(B'_i, B'^+_i)\right|$$

factors through the inclusion $|\operatorname{Spa}(B_i, B_i^+)| \subset |\operatorname{Spa}(B'_i, B'^+_i)|$. This defines a morphism $f_i\colon (B_i, B_i^+) \to (C_i, C_i^+)$ which makes Diagram (C.3.2) commute. Furthermore, its pushout to (A_∞, A_∞^+) equals f_∞ due to its uniqueness.

Case 2: *g'_{i_0} is a finite étale morphism.* Then we consider the pushout diagram

$$\begin{array}{ccc} (C_i, C_i^+) & \xrightarrow{\alpha_i} & (D_i, D_i^+) := (B_i \widehat{\otimes}_{B'_i} C_i, (B_i \widehat{\otimes}_{B'_i} C_i)^+) \\ {\scriptstyle f'_i}\uparrow & & \uparrow{\scriptstyle f''_i} \\ (B'_i, B'^+_i) & \xrightarrow[g'_i]{} & (B_i, B_i^+), \end{array}$$

and notice that morphisms f_i that make Diagram (C.3.2) commute are in bijection with morphisms $\beta_i \colon (D_i, D_i^+) \to (C_i, C_i^+)$ such that $\beta_i \circ \alpha_i = \mathrm{id}$, i.e., they are in bijection with sections of α_i. Now we note that α_i are finite étale as pushouts of finite étale morphisms. Therefore, we note that the question boils down to showing that any section of a finite étale morphism $\alpha_\infty \colon (C_\infty, C_\infty^+) \to (D_\infty, D_\infty^+)$ comes from a finite level. This follows from *Observation*, Lemma C.1.11, and Part (2) (applied to the filtered system $\{(C_i, C_i^+)\}_{i \in I}$). ■

Corollary C.3.11. *Let $\left(A, A^+\right)$ be a complete Tate–Huber pair with the uniform completion $\left(\widehat{A}_u, \widehat{A}_u^+\right)$. Then the natural functor*

$$\left(A, A^+\right)_{\text{sét}} \to \left(\widehat{A}_u, \widehat{A}_u^+\right)_{\text{sét}}$$

is an equivalence.

Proof. This follows directly from Theorem C.3.10 applied to the constant filtered system $\{(A, A^+)\}$. ■

Corollary C.3.12. *Let $\left\{(A_i, A_i^+, A_{i,0}, \varpi)\right\}_{i \in I}$ be a filtered system of complete Tate–Huber quadruples and set*

$$\left(A_\infty, A_\infty^+, A_{\infty,0}, \varpi\right) := \left(\widehat{\mathrm{colim}_I\, A_i}, \widehat{\mathrm{colim}_I\, A_i^+}, \widehat{\mathrm{colim}_I\, A_{i,0}}, \varpi\right)$$

its completed filtered colimit. Then

(1) *the natural map* $|\mathrm{Spa}\,(A_\infty, A_\infty^+)| \to \lim_I |\mathrm{Spa}\,(A_i, A_i^+)|$ *is a homeomorphism of spectral spaces;*

(2) *the natural map* $\mathrm{colim}_I (A_i, A_i^+)_{\mathrm{rsd}} \to (A_\infty, A_\infty^+)_{\mathrm{rsd}}$ *is a bijection;*

(3) *the natural functor* $2\text{-}\mathrm{colim}_I (A_i, A_i^+)_{\text{sfét}} \to (A_\infty, A_\infty^+)_{\text{sfét}}$ *is an equivalence;*

(4) *the natural functor* $2\text{-}\mathrm{colim}_I (A_i, A_i^+)_{\text{sét}} \to (A_\infty, A_\infty^+)_{\text{sét}}$ *is an equivalence.*

Proof. This follows directly from Theorem C.3.10 and Corollary C.3.11 by replacing A_i, A_i^+, A_∞, and A_∞^+ by their uniform completions. ■

C.4 Strongly sheafy adic spaces

In this section, we define the notion of a strongly sheafy adic space. We also define the étale structure sheaves on such spaces.

Definition C.4.1 ([33, Definition 4.1]). A complete Tate ring A is *strongly sheafy* if $A\langle T_1, \ldots, T_d\rangle$ is sheafy for any integer $d \geq 0$.

A Tate-affinoid (pre-)adic space $X = \mathrm{Spa}\,(A, A^+)$ is *strongly sheafy* if A is strongly sheafy.

An adic space X is *strongly sheafy* if there is an open covering of X by strongly sheafy Tate-affinoids.

Example C.4.2. A strongly noetherian Tate ring A is strongly sheafy (see [37, Theorem 2.2]). Likewise, a sousperfectoid Tate ring A is strongly sheafy as well (see [33, Definition 7.1 and Corollary 7.4]).

Remark C.4.3. [33, Proposition 5.5] and (the proof of) [33, Theorem 5.6] imply that, if (A, A^+) is a sheafy complete Tate–Huber pair and $\mathrm{Spa}\,(A, A^+)$ is a strongly sheafy adic space, then A is a strongly sheafy Tate ring.

Remark C.4.4. [33, Theorem 5.6 and Definition 5.4] imply that, if (A, A^+) is a strongly sheafy Tate–Huber ring and $(A, A^+) \to (B, B^+)$ is a strongly étale morphism, then (B, B^+) is strongly sheafy as well.

The above remark allows us to make the following definition:

Definition C.4.5. A morphism of strongly sheafy Tate-affinoids $\mathrm{Spa}\,(B, B^+) \to \mathrm{Spa}\,(A, A^+)$ is *strongly finite étale* if $(A, A^+) \to (B, B^+)$ is strongly finite étale (in the sense of Definition C.3.1).

A morphism of strongly sheafy Tate-affinoids $\mathrm{Spa}\,(B, B^+) \to \mathrm{Spa}\,(A, A^+)$ is *(affinoid) strongly étale* if $(A, A^+) \to (B, B^+)$ is strongly étale (in the sense of Definition C.3.1).

Remark C.4.6. We note that finite disjoint unions of rational subdomains $\bigsqcup_{i\in I} X_i \to X = \mathrm{Spa}\,(A, A^+)$ are strongly étale as they can be decomposed as a composition $\bigsqcup_{i\in I} X_i \to \bigsqcup_{i\in I} X \to X$, where the first morphism is a rational subdomain and the second morphism is strongly finite étale.

Remark C.4.7. More generally, if $\{X_i \to X\}_{i\in I}$ is a finite family of strongly étale morphisms, then $\bigsqcup_{i\in I} X_i \to X$ is a strongly étale morphism as well.

Lemma C.4.8. *Let Y be a strongly sheafy adic space, let X be a pre-adic space (in the sense of* [41, Definition 8.2.3]*), and let $f: X \to Y$ be an étale morphism (in the sense of* [41, Definition 8.2.16]*). Then X is a strongly sheafy adic space.*

Proof. The claim is local in the analytic topology on both X and Y. Therefore, we can assume that $X = \operatorname{Spa}\widetilde{(B, B^+)}$, $Y = \operatorname{Spa}\widetilde{(A, A^+)}$ for a strongly sheafy Tate ring A, and the morphism $(A, A^+) \to (B, B^+)$ is strongly étale. Then (B, B^+) is strongly sheafy due to Remark C.4.4. ∎

Finally, we can define the étale integral and structure (pre-)sheaves on strongly sheafy spaces:

Definition C.4.9. Let X be a strongly sheafy adic space. The *étale structure pre-sheaf* $\mathcal{O}_{X_{\text{ét}}}$ is a pre-sheaf of rings on $X_{\text{ét}}$ defined via the assignment

$$\left(Y \xrightarrow{\text{étale}} X\right) \mapsto \mathcal{O}_Y(Y)$$

with evident transition maps. The *integral étale structure pre-sheaf* $\mathcal{O}^+_{X_{\text{ét}}}$ is a pre-sheaf of rings on $X_{\text{ét}}$ defined via the assignment

$$\left(Y \xrightarrow{\text{étale}} X\right) \mapsto \mathcal{O}^+_Y(Y)$$

with evident transition maps.

Before we show that $\mathcal{O}_{X_{\text{ét}}}$ and $\mathcal{O}^+_{X_{\text{ét}}}$ are sheaves, we need to prove the following basic lemma:

Lemma C.4.10. *Let* $\left\{\varphi_i : (A, A^+) \to (B_i, B_i^+)\right\}_{i \in I}$ *be a family of morphisms of complete Tate–Huber pairs such that* $\bigcup_{i \in I} |\operatorname{Spa}(\varphi_i)|\left(|\operatorname{Spa}(B_i, B_i^+)|\right) = |\operatorname{Spa}(A, A^+)|$, *and let* $a \in A$. *Then* $a \in A^+$ *if and only if* $\varphi_i(a) \in B_i^+$.

Proof. If $a \in A^+$, then clearly $\varphi_i(a) \in B_i^+$ for every $i \in I$. Now we assume that $\varphi_i(a) \in B_i^+$ for all $i \in I$ and wish to show that $a \in A^+$. First, [36, Lemma 3.3 (i)] (or [37, Proposition 1.6 (iv)]) implies that

$$A^+ = \left\{f \in A \mid v(f) \le 1 \ \forall v \in \operatorname{Spa}(A, A^+)\right\}.$$

Therefore, we wish to show that $v(a) \le 1$ for any $v \in \operatorname{Spa}(A, A^+)$. For this, we choose $i \in I$ and a $w_i \in \operatorname{Spa}(B_i, B_i^+)$ such that $\operatorname{Spa}(\varphi_i)(w) = v$. Then we know that

$$v(a) = w(\varphi_i(a)) \le 1.$$

This finishes the proof. ∎

Lemma C.4.11. *For a strongly sheafy adic space* X, *the étale pre-sheaves* $\mathcal{O}_{X_{\text{ét}}}$ *and* $\mathcal{O}^+_{X_{\text{ét}}}$ *are sheaves.*

Proof. Let $\left\{Y_i \to Y\right\}_{i \in I}$ be a covering in $X_{\text{ét}}$. We wish to verify the sheaf axiom for $\mathcal{O}_{X_{\text{ét}}}$ and $\mathcal{O}^+_{X_{\text{ét}}}$ with respect to this covering. Lemma C.4.8 implies that the usual (analytic) pre-sheaves $\mathcal{O}^+_Y$, $\mathcal{O}^+_{Y_i}$, $\mathcal{O}_Y$, and $\mathcal{O}_{Y_i}$ are sheaves (in the analytic topology).

Therefore, we can verify the sheaf condition analytically locally on Y and Y_i. Therefore, we can assume that all spaces involved are strongly sheafy Tate-affinoids and all morphisms are strongly étale. In this case, sheafiness of $\mathcal{O}_{X_{\text{ét}}}$ follows from the last sentence of [33, Proposition 5.5] and (the proof of) [33, Theorem 5.6]. Then sheafiness of $\mathcal{O}^+_{X_{\text{ét}}}$ follows from sheafiness of $\mathcal{O}_{X_{\text{ét}}}$ and Lemma C.4.10. ∎