Appendix D

Achinger's result in the non-noetherian case

Recall that P. Achinger proved a remarkable result [1, Proposition 6.6.1] that says that an affinoid rigid-analytic variety $X = \mathrm{Spa}\,(A, A^+)$ that admits an étale map to a closed unit disk $\mathbf{D}^n_K$, also admits a *finite* étale map to $\mathbf{D}^n_K$ provided that K is the fraction field of a complete DVR R with residue field of characteristic p. This result is an analytic analogue of a more classical result of Kedlaya ([39] and [1, Proposition 5.2.1]), that an affine k-scheme $X = \mathrm{Spec}\, A$ that admits an étale map to an affine space $\mathbf{A}^n_k$ also admits a *finite* étale map to $\mathbf{A}^n_k$ provided that k has characteristic p.

We generalize P. Achinger's result to the non-noetherian setting. The proof essentially follows the ideas of [1], we only need to be slightly more careful at some places due to non-noetherian issues. We also show a version of this result for formal schemes.

Lemma D.1. *Let k be a field of characteristic p, and let A be a finite type k-algebra such that* $\dim A \leq d$ *for some integer d. Suppose that $x_1, \dots, x_d \in A$ are some elements of A, and m is any integer $m \geq 0$. Then there exist elements $y_1, \dots, y_d \in A$ such that the map $f\colon k[T_1, \dots, T_d] \to A$, defined as $f(T_i) = x_i + y_i^{p^m}$ is finite.*

Proof. We extend the set $x_1, \dots, x_d$ to some set of generators $x_1, \dots, x_d, \dots, x_n$ of A as a k-algebra. This defines a presentation $A = k[T_1, \dots, T_d, \dots, T_n]/I$ for some ideal $I \subset k[T_1, \dots, T_r, \dots, T_n]$. We prove the claim by induction on $n - d$.

The case of $n - d = 0$ is trivial as then the map $f\colon k[T_1, \dots, T_d] \to A$, defined by $f(T_i) = x_i$, is surjective. Therefore, it is finite.

Now we do the induction argument, so we suppose that $n - d \geq 1$. We consider the elements

$$x'_i = x_i - x_n^{p^{im'}},\ i = 1, \dots, n-1$$

for some integer $m' \geq m$. Now the assumption $n \geq d + 1$ and Krull's principal ideal theorem imply that we can choose some non-zero element $g \in I$, thus we have an expression

$$g(x'_1 + x_n^{p^{m'}}, x'_2 + x_n^{p^{2m'}}, \dots, x'_{n-1} + x_n^{p^{(n-1)m'}}, x_n) = 0.$$

Now [55, Section 1] implies that there is some large m' such that this expression is a polynomial in x_n with coefficients in $k[x'_1, \dots, x'_{n-1}]$ and a non-zero leading term. We may and do assume that this leading term is 1. So x_n is integral over a subring of R generated by $x'_1, \dots, x'_{n-1}$, we denote this ring by R'. Since $x_i = x'_i + x_n^{p^{im'}}$, we conclude that R is integral over R'. Moreover, R is finite over R' because it is finite type over k. Now we note that [54, Theorem 9.3] implies that $\dim R' \leq \dim R \leq d$, and R' is generated by $x'_1, \dots, x'_{n-1}$ as a k-algebra. So we can use the induction hypothesis to find some elements

$$y'_1, \dots, y'_d \in R'$$

such that the morphism $f' \colon k[T_1, \dots, T_d] \to R'$, defined as $f'(T_i) = x'_i + (y'_i)^{p^m}$, is finite. Therefore, the composite morphism

$$f \colon k[T_1, \dots, T_d] \to R$$

is also finite. We now observe that

$$f(T_i) = x'_i + (y'_i)^{p^m} = x_i + x_n^{p^{im'}} + (y'_i)^{p^m} = x_i + (x_n^{p^{im'-m}} + y'_i)^{p^m}.$$

Therefore, the set $(y_i := x_n^{p^{im'-m}} + y'_i)_{i=1,\dots,d}$ does the job. ■

Lemma D.2. *Let $\mathcal{O}$ be a complete valuation ring of rank-1 with maximal ideal $\mathfrak{m}$ and residue field k. Suppose that $f \colon A \to B$ is a morphism of topologically finitely generated $\mathcal{O}_K$-algebras. Then f is finite if and only if $f \otimes_{\mathcal{O}} k \colon A \otimes_{\mathcal{O}} k \to B \otimes_{\mathcal{O}} k$ is finite.*

Proof. The "only if" part is clear, so we only need to deal with the "if" part. We recall that [53, Lemma (28.P) p. 212] says that $A \to B$ is finite if and only if $A/\pi \to B/\pi$ is finite for some pseudo-uniformizer $\pi \in \mathcal{O}$. So we only need to show that finiteness of $A \otimes_{\mathcal{O}} k \to B \otimes_{\mathcal{O}} k$ implies that there is a pseudo-uniformizer $\pi \in \mathcal{O}$ such that $A/\pi \to B/\pi$ is finite. Then we note that the maximal ideal $\mathfrak{m}$ is a filtered colimit of its finitely generated subideals $\{I_j\}_{j \in J}$. Moreover, the valuation property of the ring $\mathcal{O}$ implies that this colimit is actually direct and that $I_j = (\pi_j)$ is principal for any $j \in J$. We also observe that each π_j is a pseudo-uniformizer since $\mathcal{O}$ is of rank-1. Thus we see that

$$A \otimes_{\mathcal{O}} k \to B \otimes_{\mathcal{O}} k = \operatorname{colim}_{j \in J}(A/\pi_j \to B/\pi_j)$$

and $A/\pi_j \to B/\pi_j$ is a finite type morphism by the assumption that both A and B are topologically finitely generated. Then [68, Tag 07RG] implies that there is $j \in J$ such that $A/\pi_j \to B/\pi_j$ is finite. Therefore, $A \to B$ is finite as well. ■

Before going to the proof of Theorem D.4, we need to show a result on the dimension theory of rigid-analytic spaces that seem to be missing in the literature. It seems that there is no generally accepted definition of a dimension of adic spaces. We define the dimension as $\dim X = \sup_{x \in X} \dim \mathcal{O}_{X,x}$, this is consistent with the definition of dimension in [25, Definition II.10.1.1]. We denote by $X^{\mathrm{cl}} \subset X$ the set of all classical points of X.

Lemma D.3. *Let* $f: X = \mathrm{Spa}(B, B^+) \to Y = \mathrm{Spa}(A, A^+)$ *be an étale morphism of rigid-analytic varieties over a complete rank-*1 *field* K*, then* $\dim B \geq \dim A$*. If* Y *is equidimensional, i.e.,* $\dim \mathcal{O}_{Y,y} = \dim Y$ *for any classical point* $y \in Y^{\mathrm{cl}}$*, then we have an equality* $\dim B = \dim A$*. In particular, if* $f: \mathrm{Spa}(A, A^+) \to \mathbf{D}^d_K$ *is étale, then* $\dim A = d$.

Proof. We note that [25, Proposition II.10.1.9 and Corollary II.10.1.10] imply that

$$\dim X = \dim B = \sup_{x \in X^{\mathrm{cl}}} (\dim \mathcal{O}_{X,x}), \text{ and } \dim Y = \dim A = \sup_{y \in Y^{\mathrm{cl}}} (\dim \mathcal{O}_{Y,y}).$$

Since f is topologically finite type, it sends classical points to classical points. Therefore, [38, Lemma 1.6.4, Corollary 1.7.4, and Proposition 1.7.9] imply that the map $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is finite étale for any $x \in X^{\mathrm{cl}}$. Thus, we see that

$$\dim B = \sup_{x \in X^{\mathrm{cl}}} \left(\dim \mathcal{O}_{X,x}\right) = \sup_{x \in X^{\mathrm{cl}}} \left(\dim \mathcal{O}_{Y,f(x)}\right) \leq \dim Y.$$

It is also clear that this inequality becomes an equality if Y is equidimensional.

Finally, we claim that $\mathbf{D}^d_K = \mathrm{Spa}(K\langle T_1, \dots, T_d\rangle, \mathcal{O}_K\langle T_1, \dots, T_d\rangle) = \mathrm{Spa}(A, A^+)$ is equidimensional. Pick any classical point $x \in (\mathbf{D}^d_K)^{\mathrm{cl}}$ and a corresponding maximal ideal $\mathfrak{m}_x \in K\langle T_1, \dots, T_d\rangle$. Then we know that $A_{\mathfrak{m}_x}$ and $\mathcal{O}_{\mathbf{D}^d_K,x}$ are noetherian by [25, Proposition 0.9.3.9, Theorem II.8.3.6], moreover, the isomorphism $\widehat{\mathcal{O}_{\mathbf{D}^d_K,x}} \simeq \widehat{A_{\mathfrak{m}_x}}$ holds by [25, Proposition II.8.3.1]. Therefore, we get

$$\dim \mathcal{O}_{\mathbf{D}^d_K,x} = \dim \widehat{\mathcal{O}_{\mathbf{D}^d_K,x}} = \dim \widehat{A_{\mathfrak{m}_x}} = \dim A_{\mathfrak{m}_x} = d,$$

where the last equality comes from [25, Proposition 0.9.3.9]. ∎

For the rest of the section we fix a complete rank-1 valuation ring $\mathcal{O}$ with fraction field K and characteristic p residue field k. We refer to [38, Section 1.9] for the construction of the adic generic fiber of a topologically finitely generated formal $\mathcal{O}$-scheme. The only thing we mention here is that it sends an affine formal scheme $\mathrm{Spf}\ A$ to the affinoid adic space $\mathrm{Spa}(A \otimes_{\mathcal{O}} K, A^+)$, where A^+ is the integral closure of the image $\mathrm{Im}(A \to A \otimes_{\mathcal{O}} K)$.

Theorem D.4. *In the notation as above, let* $g: \mathrm{Spf}\ A \to \widehat{\mathbf{A}}^d_{\mathcal{O}}$ *be a morphism of flat, topologically finitely generated formal* $\mathcal{O}$*-schemes such that the adic generic fiber* $g_K: \mathrm{Spa}(A \otimes_{\mathcal{O}} K, A^+) \to \mathbf{D}^d_K$ *is étale. Under these assumptions, there is a* finite *morphism* $f: \mathrm{Spf}\ A \to \widehat{\mathbf{A}}^d_{\mathcal{O}}$ *that is étale on adic generic fibers.*

Proof. First of all, we note that Lemma D.3 implies that $\dim A \otimes_{\mathcal{O}} K = d$. Now [25, Theorem 9.2.10] says that there exists a finite injective morphism

$$\varphi: \mathcal{O}\langle T_1, \dots, T_d\rangle \to A$$

with an $\mathcal{O}_K$-flat cokernel. This implies that $K\langle T_1, \dots, T_d\rangle \to A \otimes_{\mathcal{O}} K$ is finite and injective. Flatness of $\operatorname{Coker} \varphi$ implies that the map

$$k[T_1, \dots, T_d] \to A \otimes_{\mathcal{O}} k$$

is also finite and injective, so $\dim A \otimes_{\mathcal{O}} k = d$. Now we finish the proof in two slightly different ways depending on $\operatorname{char} K$.

Case 1: $\operatorname{char} K = p$. We consider the morphism $g^{\#}: \mathcal{O}\langle T_1, \dots, T_d\rangle \to A$ induced by g. We define $x_i := g^{\#}(T_i)$ for $i = 1, \dots, d$. Since $\dim A \otimes_{\mathcal{O}} k = d$, we can apply Lemma D.1 to the residue classes $\overline{x_1}, \dots, \overline{x_d}$ and $m = 1$ to get elements $\overline{y_1}, \dots, \overline{y_d} \in A \otimes_{\mathcal{O}} k$ such that the map

$$\overline{f^{\#}}: k[T_1, \dots, T_d] \to A \otimes_{\mathcal{O}} k, \text{ defined as } \overline{f^{\#}}(T_i) = \overline{x_i} + \overline{y_i}^p \text{ for } i = 1, \dots, d,$$

is finite. We lift $\overline{y_i}$ in an arbitrary way to elements $y_i \in A$, and define

$$f^{\#}: \mathcal{O}\langle T_1, \dots, T_d\rangle \to A$$

as $f^{\#}(T_i) = x_i + y_i^p$ for any $i = 1, \dots, d$. This map is finite by Lemma D.2.

Now we note that $X := \operatorname{Spa}(A \otimes_{\mathcal{O}} K, A^+)$ is smooth over K, so [15, Proposition 2.6] says that étaleness of $f_K: X \to \mathbf{D}_K^d$ is equivalent to the bijectivity of the map

$$f_K^* \Omega^1_{\mathbf{D}_K^d/K} \to \Omega^1_{X/K}.$$

This easily follows from étaleness of g_K and the fact that $d(x_i + y_i^p) = d(x_i)$ in characteristic p.

Case 2: $\operatorname{char} K = 0$. We denote $\operatorname{Spf} A$ by $\mathfrak{X}$ and we denote its adic generic fiber $\operatorname{Spa}(A \otimes_{\mathcal{O}} K, A^+)$ by X. Then we use [15, Proposition 2.6] once again to see that the map

$$g_K^* \Omega^1_{\mathbf{D}_K^d/K} \to \Omega^1_{X/K}$$

is an isomorphism. Since $(\widehat{\Omega}^1_{\mathfrak{X}/\mathcal{O}})_K \simeq \Omega^1_{X/K}$ and the same for $\widehat{\mathbf{A}}_{\mathcal{O}}^d$ and $\mathbf{D}_K^d$, we conclude that the fundamental short exact sequence ([25, Proposition I.3.6.3, Proposition I.5.2.5 and Theorem I.5.2.6])

$$g^* \widehat{\Omega}^1_{\widehat{\mathbf{A}}_{\mathcal{O}}^d/\mathcal{O}} \to \widehat{\Omega}^1_{\mathfrak{X}/\mathcal{O}} \to \widehat{\Omega}^1_{\mathfrak{X}/\widehat{\mathbf{A}}_{\mathcal{O}}^d} \to 0$$

implies that $\big(\widehat{\Omega}^1_{\mathfrak{X}/\widehat{\mathbf{A}}_{\mathcal{O}}^d}\big)_K = 0$. Furthermore, we know that

$$\widehat{\Omega}^1_{\mathfrak{X}/\widehat{\mathbf{A}}_{\mathcal{O}}^d} \cong \big(\widehat{\Omega}^1_{A/\mathcal{O}\langle T_1, \dots, T_d\rangle}\big)^{\Delta}$$

for a finite A-module $\widehat{\Omega}^1_{A/\mathcal{O}\langle T_1, \dots, T_d\rangle}$ (see [25, Corollary I.5.1.11]). We denote this module by $\widehat{\Omega}^1_g$ for the rest of the proof, and recall that the condition $\big(\widehat{\Omega}^1_{\mathfrak{X}/\widehat{\mathbf{A}}_{\mathcal{O}}^d}\big)_K = 0$

is equivalent to $\widehat{\Omega}^1_g \otimes_{\mathcal{O}} K = 0$. Using finiteness of $\widehat{\Omega}^1_g$ and adhesiveness of A, we conclude that there is an integer k such that

$$p^k \widehat{\Omega}^1_g = 0$$

as p is a pseudo-uniformizer in $\mathcal{O}$. Now, similarly to the case of $\mathrm{char} K = p$, we consider the morphism

$$g^{\#}\colon \mathcal{O}\langle T_1, \dots, T_d\rangle \to A$$

and define $x_i := g^{\#}(T_i)$ for $i = 1, \dots, d$. Again, using that $\dim A \otimes_{\mathcal{O}} k = d$, we can apply Lemma D.1 to the residue classes $\overline{x_1}, \dots, \overline{x_d}$ and $m = k + 1$ to get elements $\overline{y_1}, \dots, \overline{y_d} \in A \otimes_{\mathcal{O}} k$ such that the map

$$\overline{f^{\#}}\colon k[T_1, \dots, T_d] \to A \otimes_{\mathcal{O}} k, \text{ defined as } \overline{f^{\#}}(T_i) = \overline{x_i} + \overline{y_i}^{p^{k+1}} \text{ for } i = 1, \dots, d,$$

is finite. We lift $\overline{y_i}$ to some elements $y_i \in A$ and define

$$f^{\#}\colon \mathcal{O}\langle T_1, \dots, T_d\rangle \to A$$

by $f^{\#}(T_i) = x_i + y_i^{p^{k+1}}$. The map $f^{\#}$ is finite by Lemma D.2.

We are only left to show that the induced map

$$f\colon X \to \widehat{\mathbf{A}}^d_{\mathcal{O}}$$

is étale on adic generic fibers. Next we claim that $p^k(\widehat{\Omega}^1_f) = 0$. Indeed, we use [25, Proposition I.5.1.10] to trivialize $\widehat{\Omega}^1_{\mathcal{O}\langle T_1,\dots,T_d\rangle/\mathcal{O}} \simeq \bigoplus_{i=1}^d dT_i \mathcal{O}\langle T_1, \dots, T_d\rangle$, so we have the fundamental exact sequence

$$\bigoplus_{i=1}^d A dT_i \xrightarrow{dT_i \mapsto d(x_i + y_i^{p^{k+1}})} \widehat{\Omega}^1_{A/\mathcal{O}} \to \widehat{\Omega}^1_f \to 0$$

as $d(y_i^{p^{k+1}})$ is divisible by p^{k+1}. Therefore, we see that modulo p^{k+1}, this sequence is equal to

$$\bigoplus_{i=1}^d A/p^{k+1} dT_i \xrightarrow{dT_i \to d(x_i)} \widehat{\Omega}^1_{A/\mathcal{O}}/p^{k+1} \to \widehat{\Omega}^1_f/p^{k+1} \to 0.$$

Thus, we see that $\widehat{\Omega}^1_f/p^{k+1} \simeq \widehat{\Omega}^1_g/p^{k+1}$. In particular,

$$\left(p^k \widehat{\Omega}^1_f\right)/p\left(p^k \widehat{\Omega}^1_f\right) = \left(p^k \widehat{\Omega}^1_g\right)/p\left(p^k \widehat{\Omega}^1_g\right) = 0$$

by the choice of k. Therefore, $p^k \widehat{\Omega}^1_f = 0$ by [53, Lemma 28.P p. 212]. By passing to the adic generic fiber, we get the map $f_K\colon X \to \mathbf{D}^d_K$ such that

$$d(f_K)\colon f_K^* \Omega^1_{\mathbf{D}^d_K/K} \to \Omega^1_{X/K}$$

is surjective. However, we recall that X and $\mathbf{D}_K^d$ are both smooth rigid-analytic varieties of (pure) dimension d. Thus $d_{f_K^*}$ is a surjective map of vector bundles of the same dimension d, so it must be an isomorphism. Finally, [15, Proposition 2.6] implies that f_K is étale. ∎

Corollary D.5. *Let K be a complete rank-1 valuation field with valuation ring $\mathcal{O}_K$, and residue field k of characteristic p. Suppose that $g\colon X = \mathrm{Spa}\,(A, A^+) \to \mathbf{D}_K^d$ is an étale morphism of affinoid rigid-analytic K-varieties. Then there exists a* finite *étale morphism $f\colon X \to \mathbf{D}_K^d$.*

Proof. We note that [37, Lemma 4.4] implies that $A^+ = A^\circ$, so the map g corresponds to the map

$$g^{\#}\colon (K\langle T_1, \dots, T_d\rangle, \mathcal{O}_K\langle T_1, \dots, T_d\rangle) \to (A, A^\circ)$$

of Tate–Huber pairs. Theorem D.4 implies that it suffices to show that the image of $\mathcal{O}_K\langle T_1, \dots, T_d\rangle$ lies inside some ring of definition $A_0 \subset A$.

Since A is topologically finitely generated, we can extend $g^{\#}$ to a surjection

$$\varphi\colon K\langle T_1, \dots, T_d, X_1, \dots, X_n\rangle \twoheadrightarrow A.$$

Then

$$A_0 \coloneqq \varphi(\mathcal{O}_K\langle T_1, \dots, T_d, X_1, \dots, X_n\rangle)$$

is bounded and it is open as a consequence of the Banach open mapping theorem ([37, Lemma 2.4 (i)]). Thus, it is a ring of definition containing $g^{\sharp}\big(\mathcal{O}_K\langle T_1, \dots, T_d\rangle\big)$. ∎