

Abstract

This memoir is devoted to the theory of vector-valued modular forms for orthogonal groups of signature $(2, n)$. Our purpose is multi-layered: (1) to lay a foundation of the theory of vector-valued orthogonal modular forms; (2) to develop some aspects of the theory in more depth such as geometry of the Siegel operators, filtrations associated to 1-dimensional cusps, decomposition of vector-valued Jacobi forms, square integrability, etc.; and (3) as applications derive several types of vanishing theorems for vector-valued modular forms of small weight. Our vanishing theorems imply in particular vanishing of holomorphic tensors of degree less than $n/2 - 1$ on orthogonal modular varieties, which is optimal as a general bound. The fundamental ingredients of the theory are the two Hodge bundles. The first is the Hodge line bundle which already appears in the theory of scalar-valued modular forms. The second Hodge bundle emerges in the vector-valued theory and plays a central role. It corresponds to the non-abelian part $O(n, \mathbb{R})$ of the maximal compact subgroup of $O(2, n)$. The main focus of this memoir is centred around the properties and the role of the second Hodge bundle in the theory of vector-valued orthogonal modular forms.

Mathematics Subject Classification (2020). Primary 11F55; Secondary 11F50

Keywords. orthogonal modular forms, vector-valued modular forms, Siegel operator, Fourier–Jacobi expansion, vector-valued Jacobi forms, Petersson metric, vanishing theorem

Acknowledgements. It is our pleasure to thank Eberhard Freitag and Tomoyoshi Ibukiyama for sharing with us their perspectives on vector-valued modular forms.

Funding. This work is supported by KAKENHI 21H00971 and 20H00112.