Chapter 1

Introduction

In the theory of modular forms of several variables, it is natural and also necessary to study vector-valued modular forms. One way to account for this is that scalar-valued modular forms are concerned only with the 1-dimensional abelian quotient of the maximal compact subgroup K of the Lie group, while the contribution from the whole K emerges if we consider vector-valued modular forms. In more concrete levels, the significance of vector-valued modular forms appears in the study of the cohomology of modular varieties, holomorphic tensors on modular varieties, and constructions of Galois representations, etc. The passage from scalar-valued to vector-valued modular forms is an intrinsic non-abelianization step.

This subject has been well developed for Siegel modular forms since the pioneering work of Freitag, Weissauer and others around the early 1980s (see, e.g., [44] for a survey). In particular, a lot of detailed study have been done in the case of Siegel modular forms of genus 2.

By contrast, despite its potential and expected applications, no systematic theory of vector-valued modular forms for orthogonal groups of signature $(2, n)$ seems to have been developed so far. Only recently its application to holomorphic tensors on modular varieties started to be investigated [34]. The observation that some aspects of the theory of scalar-valued Siegel modular forms of genus 2 have been generalized to orthogonal modular forms, rather than to Siegel modular forms of higher genus, also suggests a promising theory.

Vector-valued orthogonal modular forms will have applications to the geometry and arithmetic of orthogonal modular varieties, and so, especially to the moduli spaces of *K3* surfaces and holomorphic symplectic varieties. Moreover, from the geometric viewpoint of *K3* surfaces, vector-valued modular forms on a period domain of (lattice-polarized) *K3* surfaces are considered as holomorphic invariants related to the family that can be captured by the variation of the Hodge structures on $H^2(K3)$ but typically not by the Hodge line bundle $H^0(K_{K3})$ alone. For example in this direction, the infinitesimal invariants of normal functions for higher Chow cycles in $CH^2(K3, 1)$ give vector-valued modular forms with singularities (Section 3.8). This geometric viewpoint offers another motivation to develop the theory of vector-valued orthogonal modular forms.

The purpose of this memoir is multi-layered:

(1) to lay a foundation of the theory of vector-valued orthogonal modular forms,

(2) to investigate some aspects of the theory in more depth, and

(3) as applications to establish several types of vanishing theorems for vector-valued modular forms of small weight.

Our theory is developed in a full generality in the sense that we work with general arithmetic groups $\Gamma < \mathrm{O}^+(L)$ for general integral quadratic forms L of signature $(2, n)$. The facts that unimodular lattices are rare even up to $\mathbb{Q}$-equivalence (unlike the symplectic case) and that various types of groups Γ appear in the moduli examples urge us to work in this generality.

Our approach is geometric in the sense that we define modular forms as sections of the automorphic vector bundles. Trivializations of the automorphic vector bundles, and thus passage from sections of vector bundles to vector-valued functions, are provided for each 0-dimensional cusp. This approach is suitable for working with general Γ, without losing connection with the more classical style.

In the rest of this introduction, we give a summary of the theory developed in this memoir.

The two Hodge bundles (Section 2)

Let L be an integral quadratic lattice of signature $(2, n)$. We assume $n \geq 3$ for simplicity. The Hermitian symmetric domain $\mathcal{D} = \mathcal{D}_L$ attached to L is defined as an open subset of the isotropic quadric in $\mathbb{P}L_{\mathbb{C}}$. It parametrizes polarized Hodge structures $0 \subset F^2 \subset F^1 \subset L_{\mathbb{C}}$ of weight 2 on L with $\dim F^2 = 1$ and $F^1 = (F^2)^{\perp}$. Over $\mathcal{D}$ we have two fundamental Hodge bundles. The first is the Hodge line bundle

$$\mathcal{L} = \mathcal{O}_{\mathbb{P}L_{\mathbb{C}}}(-1)|_{\mathcal{D}},$$

which geometrically consists of the period lines F^2 in the Hodge filtrations. In terms of representation theory, $\mathcal{L}$ is the homogeneous line bundle associated to the standard character of $\mathbb{C}^* \subset \mathbb{C}^* \times \mathrm{O}(n, \mathbb{C})$, where $\mathbb{C}^* \times \mathrm{O}(n, \mathbb{C})$ is the reductive part of a standard parabolic subgroup of $\mathrm{O}(L_{\mathbb{C}}) \simeq \mathrm{O}(n+2, \mathbb{C})$. Invariant sections of powers of $\mathcal{L}$ are scalar-valued modular forms on $\mathcal{D}$, which have been classically studied.

The Hodge line bundle $\mathcal{L}$ is naturally embedded in $L_{\mathbb{C}} \otimes \mathcal{O}_{\mathcal{D}}$ as an isotropic sub line bundle. The second Hodge bundle is defined as

$$\mathcal{E} = \mathcal{L}^{\perp}/\mathcal{L}.$$

Geometrically this vector bundle consists of the middle graded quotients F^1/F^2 of the Hodge filtrations. In terms of representation theory, $\mathcal{E}$ is the homogeneous vector bundle associated to the standard representation of $\mathrm{O}(n, \mathbb{C}) \subset \mathbb{C}^* \times \mathrm{O}(n, \mathbb{C})$. It is this second Hodge bundle $\mathcal{E}$ that emerges in the theory of vector-valued modular forms on $\mathcal{D}$ and plays a central role in this memoir.

While $\mathcal{L}$ is concerned with scalar-valued modular forms, $\mathcal{E}$ is responsible for the higher rank aspect of the theory of vector-valued modular forms. While $\mathcal{L}$ provides a polarization, $\mathcal{E}$ is an orthogonal vector bundle, and in particular self-dual (but not trivial). Thus $\mathcal{L}$ and $\mathcal{E}$ are rather contrastive.

Vector-valued modular forms (Section 3)

Weights of vector-valued modular forms on $\mathcal{D}$ are expressed by pairs (λ, k), where $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n \geq 0)$ is a partition which corresponds to an irreducible representation V_λ of $\mathrm{O}(n, \mathbb{C})$, and k is an integer which corresponds to a character of $\mathbb{C}^*$. The partition λ satisfies ${}^t\lambda_1 + {}^t\lambda_2 \leq n$, where ${}^t\lambda$ is the transpose of λ. To such a pair (λ, k) we associate the automorphic vector bundle

$$\mathcal{E}_{\lambda,k} = \mathcal{E}_\lambda \otimes \mathcal{L}^{\otimes k},$$

where $\mathcal{E}_\lambda$ is the vector bundle constructed from $\mathcal{E}$ by applying the orthogonal Schur functor associated to λ. Modular forms of weight (λ, k) are defined as holomorphic sections of $\mathcal{E}_{\lambda,k}$ over $\mathcal{D}$ invariant under a finite-index subgroup Γ of $\mathrm{O}^+(L)$ (with cusp conditions when $n \leq 2$). We denote by $M_{\lambda,k}(\Gamma)$ the space of Γ-modular forms of weight (λ, k).

Sometimes it is more appropriate to work with irreducible representations of $\mathrm{SO}(n, \mathbb{C})$ rather than $\mathrm{O}(n, \mathbb{C})$, but in that way we obtain only $\mathrm{SO}^+(L_\mathbb{R})$-equivariant vector bundles. Since in some applications we encounter subgroups Γ of $\mathrm{O}^+(L)$ not contained in $\mathrm{SO}^+(L)$, we decided to work with $\mathrm{O}(n, \mathbb{C})$ at the outset. It is not difficult to switch to $\mathrm{SO}(n, \mathbb{C})$ (see Section 3.6).

Fourier expansion (Section 3)

A first basic point is that $\mathcal{E}_{\lambda,k}$ can be trivialized for each 0-dimensional cusp of $\mathcal{D}$ in a natural way. Let I be a rank 1 primitive isotropic sublattice of L, which corresponds to a 0-dimensional cusp of $\mathcal{D}$. The quotient lattice $I^\perp/I$ is naturally endowed with a hyperbolic quadratic form. Then we have isomorphisms

$$I_\mathbb{C}^\vee \otimes \mathcal{O}_\mathcal{D} \to \mathcal{L}, \quad (I^\perp/I)_\mathbb{C} \otimes \mathcal{O}_\mathcal{D} \to \mathcal{E},$$

canonically associated to I. If we write $V(I)_{\lambda,k} = ((I^\perp/I)_\mathbb{C})_\lambda \otimes (I_\mathbb{C}^\vee)^{\otimes k}$, these induce an isomorphism

$$V(I)_{\lambda,k} \otimes \mathcal{O}_\mathcal{D} \to \mathcal{E}_{\lambda,k},$$

which we call the *I-trivialization* of $\mathcal{E}_{\lambda,k}$. Via this trivialization, modular forms of weight (λ, k) are identified with $V(I)_{\lambda,k}$-valued holomorphic functions f on $\mathcal{D}$ satisfying invariance with the factor of automorphy. Then, after taking the tube domain realization of $\mathcal{D}$ associated to I [40], we obtain the Fourier expansion of f of the form

$$f(Z) = \sum_{l \in U(I)_\mathbb{Z}^\vee} a(l) \exp(2\pi i (l, Z)), \quad Z \in \mathcal{D}_I,$$

where $a(l) \in V(I)_{\lambda,k}$, $\mathcal{D}_I$ is the tube domain in $(I^{\perp}/I)_{\mathbb{C}} \otimes I_{\mathbb{C}}$, and $U(I)^{\vee}_{\mathbb{Z}}$ is a certain full lattice in $(I^{\perp}/I)_{\mathbb{Q}} \otimes I_{\mathbb{Q}}$. By the Koecher principle, the index vectors l range only over the intersection of $U(I)^{\vee}_{\mathbb{Z}}$ with the closure of the positive cone (a connected component of the locus of vectors of positive norm). We prove that the constant term $a(0)$ always vanishes unless $\lambda = (0), (1^n)$, which correspond to the trivial and the determinant characters, respectively. (In what follows, we write $\lambda = 1, \det$ instead.) Therefore the Siegel operators are interesting only at the 1-dimensional cusps. We can speak of rationality of the Fourier coefficients $a(l)$ because $V(I)_{\lambda,k}$ has a natural $\mathbb{Q}$-structure.

In this way, the choice of a 0-dimensional cusp I determines a passage to a more classical style of defining modular forms. Since there is no distinguished 0-dimensional cusp for a general arithmetic group Γ, we need to treat all 0-dimensional cusps equally. Even after the I-trivialization, it is more suitable to have $V(I)_{\lambda,k}$ as the *canonical* space of values, rather than identifying it with $\mathbb{C}^N$ by choosing a basis. This approach enables us to develop various later constructions in an intrinsic and coherent way (and so, in a full generality) without sacrificing the classical style.

These most basic parts of the theory are developed in Sections 2 and 3. In Section 4, as a functorial aspect of the theory, we study pullback and quasi-pullback of vector-valued modular forms to sub orthogonal modular varieties. This type of operations are sometimes called the *Witt operators*. The consideration of pullbacks leads to an elementary vanishing theorem for $M_{\lambda,k}(\Gamma)$ in $k \leq 0$ (Proposition 4.4). We prove that the quasi-pullback produces *cusp* forms (Proposition 4.10), generalizing a result of Gritsenko–Hulek–Sankaran [22] in the scalar-valued case.

After these foundational parts, this memoir is developed in the following two directions:

(1) Geometric treatment of the Siegel operators and the Fourier–Jacobi expansions at 1-dimensional cusps (Sections 5–9).

(2) Square integrability of modular forms (Sections 10-11).

Both lead, as applications, to vanishing theorems of respective type for modular forms of small weight.

Siegel operator (Section 6)

Let J be a rank 2 primitive isotropic sublattice of L. This corresponds to a 1-dimensional cusp $\mathbb{H}_J$ of $\mathcal{D}$, which is isomorphic to the upper half plane. We take a geometric approach for introducing and studying the Siegel operator and the Fourier–Jacobi expansion at the cusp $\mathbb{H}_J$, by using the partial toroidal compactification over $\mathbb{H}_J$. The Siegel operator is the restriction to the boundary divisor, and the Fourier–Jacobi expansion is the Taylor expansion along it.

The Siegel domain realization of $\mathcal{D}$ with respect to J [40] is a two-step fibration

$$\mathcal{D} \xrightarrow{\pi_1} \mathcal{V}_J \xrightarrow{\pi_2} \mathbb{H}_J,$$

where π_1 is a fibration of upper half planes and π_2 is an affine space bundle. Dividing $\mathcal{D}$ by a rank 1 abelian group $U(J)_{\mathbb{Z}} < \Gamma$, the quotient $\mathcal{X}(J) = \mathcal{D}/U(J)_{\mathbb{Z}}$ is a fibration of punctured discs over $\mathcal{V}_J$. The partial toroidal compactification

$$\mathcal{X}(J) \hookrightarrow \overline{\mathcal{X}(J)}$$

is obtained by filling the origins of the punctured discs [2]. Its boundary divisor Δ_J is naturally identified with $\mathcal{V}_J$. We can extend $\mathcal{E}_{\lambda,k}$ to a vector bundle over $\overline{\mathcal{X}(J)}$ via the I-trivialization for an arbitrary 0-dimensional cusp $I \subset J$, the result being independent of I (Section 5.4). This is an explicit form of Mumford's canonical extension [36] which is suitable for dealing with the Fourier–Jacobi expansion. If f is a Γ-modular form of weight (λ, k), it extends to a holomorphic section of the extended bundle $\mathcal{E}_{\lambda,k}$ over $\overline{\mathcal{X}(J)}$.

Intuitively (and more traditionally), the Siegel operator should be an operation of "restriction to $\mathbb{H}_J$" which produces vector-valued modular forms of some reduced weight on $\mathbb{H}_J$. Geometrically this requires some justification because of the complicated structure around the boundary of the Baily–Borel compactification. We take a somewhat indirect but more geometrically tractable approach, working with the automorphic vector bundle $\mathcal{E}_{\lambda,k}$ over the partial toroidal compactification $\overline{\mathcal{X}(J)}$.

Let $\mathcal{L}_J$ be the Hodge line bundle on $\mathbb{H}_J$. We write $V(J) = (J^{\perp}/J)_{\mathbb{C}}$. For the given partition $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n)$, we denote by $V(J)_{\lambda'}$ the irreducible representation of $\mathrm{O}(V(J)) \simeq \mathrm{O}(n-2, \mathbb{C})$ for the partition $\lambda' = (\lambda_2 \geq \cdots \geq \lambda_{n-1})$.

Theorem 1.1 (Theorem 6.1). *Let* $\lambda \neq 1, \det$. *There exists a sub vector bundle* $\mathcal{E}^J_{\lambda,k}$ *of* $\mathcal{E}_{\lambda,k}$ *such that* $\mathcal{E}^J_{\lambda,k}|_{\Delta_J} \simeq \pi_2^* \mathcal{L}_J^{\otimes k+\lambda_1} \otimes V(J)_{\lambda'}$ *and that the restriction of every modular form* f *of weight* (λ, k) *to* Δ_J *takes values in* $\mathcal{E}^J_{\lambda,k}|_{\Delta_J}$. *In particular, there exists a* $V(J)_{\lambda'}$*-valued cusp form* $\Phi_J f$ *of weight* $k + \lambda_1$ *on* $\mathbb{H}_J$ *such that* $f|_{\Delta_J} = \pi_2^*(\Phi_J f)$.

The map

$$M_{\lambda,k}(\Gamma) \to S_{k+\lambda_1}(\Gamma_J) \otimes V(J)_{\lambda'}, \quad f \mapsto \Phi_J f,$$

is the Siegel operator at the J-cusp, where Γ_J is a suitable subgroup of $\mathrm{SL}(J) \simeq \mathrm{SL}(2, \mathbb{Z})$. If we take the I-trivialization for a 0-dimensional cusp $I \subset J$ and introduce suitable coordinates (τ, z, w) on the tube domain in which the Siegel domain realization is given by

$$(\tau, z, w) \mapsto (\tau, z) \mapsto \tau,$$

the Siegel operator can be expressed as

$$(\Phi_J f)(\tau) = \lim_{t\to\infty} f(\tau, 0, it), \qquad \tau \in \mathbb{H}.$$

In this way, the naive "restriction to $\mathbb{H}_J$" can be geometrically justified at the level of automorphic vector bundles as the combined operation

$$\text{restrict to } \Delta_J + \text{reduce to } \mathcal{E}^J_{\lambda,k} + \text{descend to } \mathbb{H}_J.$$

This a priori tells us the modularity of $\Phi_J f$ with its weight. When $n = 3$, the weight calculation in Theorem 1.1 agrees with the corresponding result for Siegel modular forms of genus 2 [1, 47]. The sub vector bundle $\mathcal{E}^J_{\lambda,k}$ will be taken up in Section 8 again from the viewpoint of a filtration on $\mathcal{E}_{\lambda,k}$.

Fourier–Jacobi expansion (Section 7)

Next we explain the Fourier–Jacobi expansion at the J-cusp. Let Θ_J be the conormal bundle of Δ_J in $\overline{\mathcal{X}(J)}$. After certain choices, we have a special generator ω_J of the ideal sheaf of Δ_J. With this normal coordinate, we can take the Taylor expansion of a modular form $f \in M_{\lambda,k}(\Gamma)$ along Δ_J as a section of the extended bundle $\mathcal{E}_{\lambda,k}$:

$$f = \sum_{m\geq 0} \phi_m \omega_J^m. \tag{1.1}$$

The m-th Taylor coefficient ϕ_m, or rather $\phi_m \otimes \omega_J^{\otimes m}$, is essentially a section of the vector bundle $\mathcal{E}_{\lambda,k} \otimes \Theta_J^{\otimes m}$ over Δ_J. We call (1.1) the *Fourier–Jacobi expansion* of f at the J-cusp, and call the section $\phi_m \otimes \omega_J^{\otimes m}$ of $\mathcal{E}_{\lambda,k} \otimes \Theta_J^{\otimes m}$ for $m > 0$ the m-th *Fourier–Jacobi coefficient* of f. (ϕ_0 is just $f|_{\Delta_J}$ considered above.) Although the choice of ω_J is needed for defining the Fourier–Jacobi expansion, the resulting expansion and the sections of $\mathcal{E}_{\lambda,k} \otimes \Theta_J^{\otimes m}$ are independent of this choice, thus canonically determined by J (Section 7.2). This geometric definition of Fourier–Jacobi expansion, whose advantage is its canonicity, agrees with the more familiar style of defining Fourier–Jacobi expansion by slicing the Fourier expansion (Section 7.1) if we take the (I, ω_J)-trivialization.

In general, we define *vector-valued Jacobi forms* of weight (λ, k) and index $m > 0$ as holomorphic sections of $\mathcal{E}_{\lambda,k} \otimes \Theta_J^{\otimes m}$ over $\Delta_J = \mathcal{V}_J$ which is invariant under the integral Jacobi group and satisfies a certain cusp condition (Definition 7.10). The m-th Fourier–Jacobi coefficient of a modular form of weight (λ, k) is such a vector-valued Jacobi form (Proposition 7.12). In the scalar-valued case, our geometric definition agrees with the classical definition of Jacobi forms [20, 43] after introducing suitable coordinates and trivialization (Section 7.4). When $n = 3$, our vector-valued Jacobi forms essentially agree with those considered by Ibukiyama–Kyomura [27] for Siegel modular forms of genus 2.

Filtrations associated to 1-dimension cusps (Section 8)

While a 0-dimensional cusp of $\mathcal{D}$ provides a trivialization of $\mathcal{E}_{\lambda,k}$ which enables the Fourier expansion, we will show that a 1-dimensional cusp introduces a filtration on $\mathcal{E}_{\lambda,k}$ which is useful when studying the Fourier–Jacobi expansion. To start with, we observe that for each 1-dimensional cusp J, the second Hodge bundle $\mathcal{E}$ has an isotropic sub line bundle $\mathcal{E}_J$ canonically determined by J. This defines the filtration

$$0 \subset \mathcal{E}_J \subset \mathcal{E}_J^{\perp} \subset \mathcal{E}$$

associated to the J-cusp, which we call the J-*filtration*. Its graded quotients are respectively isomorphic to

$$\mathcal{E}_J \simeq \pi^*\mathcal{L}_J, \quad \mathcal{E}_J^{\perp}/\mathcal{E}_J \simeq (J^{\perp}/J)_{\mathbb{C}} \otimes \mathcal{O}_{\mathcal{D}}, \quad \mathcal{E}/\mathcal{E}_J^{\perp} \simeq \pi^*\mathcal{L}_J^{-1},$$

where $\pi = \pi_2 \circ \pi_1$ is the projection from $\mathcal{D}$ to $\mathbb{H}_J$. The J-filtration is translated to a constant filtration on $V(I) \otimes \mathcal{O}_{\mathcal{D}}$ by the I-trivialization for every adjacent 0-dimensional cusp $I \subset J$ (Proposition 8.3).

The J-filtration on $\mathcal{E}$ induces a (decreasing) filtration on a general automorphic vector bundle $\mathcal{E}_{\lambda,k}$, also called the J-filtration, whose graded quotient in level r is isomorphic to a direct sum of copies of $\pi^*\mathcal{L}_J^{\otimes k+r}$. Representation-theoretic calculations show that the J-filtration on $\mathcal{E}_{\lambda,k}$ has length $\leq 2\lambda_1 + 1$ (from level $-\lambda_1$ to λ_1), and that the sub vector bundle $\mathcal{E}_{\lambda,k}^J$ of $\mathcal{E}_{\lambda,k}$ in Theorem 1.1 is exactly the last (= level λ_1) sub vector bundle in the J-filtration (Proposition 8.13). Moreover, we have a duality between the graded quotients in level r and $-r$.

We give two applications of the J-filtration. The first is decomposition of vector-valued Jacobi forms. We prove that a vector-valued Jacobi form of weight (λ, k) decomposes, in a certain sense, into a tuple of scalar-valued Jacobi forms of various weights in the range $[k - \lambda_1, k + \lambda_1]$ (Proposition 8.15). More precisely, what is proved is that certain graded pieces are scalar-valued Jacobi forms, so this result does not mean that the theory of vector-valued Jacobi forms reduces to the scalar-valued theory. Nevertheless, this decomposition theorem enables us to derive some basic results for vector-valued Jacobi forms from those for scalar-valued ones. For example, we deduce that vector-valued Jacobi forms of weight (λ, k) with

$$k + \lambda_1 < n/2 - 1$$

vanish (Corollary 8.18). In the case of Siegel modular forms of genus 2 (namely, $n = 3$), the fact that vector-valued Jacobi forms decompose into scalar-valued Jacobi forms was first found by Ibukiyama and Kyomura [27]. Their method is different, using differential operators, but it might be plausible that their decomposition agrees with that of us.

Vanishing theorem I (Section 9)

It is a classical fact that there is no nonzero scalar-valued modular form of weight $0 < k < n/2 - 1$ on $\mathcal{D}$. Two proofs of this fact are well known. The first uses vanishing of Jacobi forms (cf. [20, 43]), and the second uses classification of unitary representations. We give two generalizations of this classical vanishing theorem to the vector-valued case, corresponding to these two approaches.

Our first vanishing theorem belongs to the Jacobi form approach, and is obtained as the second application of the J-filtration. We assume that L has Witt index 2, i.e., $\mathcal{D}$ has a 1-dimensional cusp. This is always satisfied when $n \geq 5$.

Theorem 1.2 (Theorem 9.1). *Let* $\lambda \neq 1, \det$. *If* $k < \lambda_1 + n/2 - 1$, *then* $M_{\lambda,k}(\Gamma) = 0$. *In particular,* $M_{\lambda,k}(\Gamma) = 0$ *whenever* $k < n/2$.

As a consequence, we obtain the following vanishing theorem for holomorphic tensors on the modular variety $\Gamma \backslash \mathcal{D}$.

Corollary 1.3 (Theorem 9.5). *Let* X *be the regular locus of* $\Gamma \backslash \mathcal{D}$. *Then we have*

$$H^0(X, (\Omega^1_X)^{\otimes k}) = 0$$

for all $0 < k < n/2 - 1$.

Moreover, we obtain a classification of possible types of holomorphic tensors of the next few degrees up to $n/2$ (Proposition 9.6). The vanishing bound $k < n/2 - 1$ is optimal as a general bound.

The proof of Theorem 1.2 is built on the results of Sections 7 and 8, and proceeds as follows. We apply the classical vanishing theorem of scalar-valued Jacobi forms of weight $< n/2 - 1$ [20,43] to the first graded quotient of the J-filtration on $\mathcal{E}_{\lambda,k}$. This implies that the Fourier–Jacobi coefficients of $f \in M_{\lambda,k}(\Gamma)$ take values in a certain sub vector bundle of $\mathcal{E}_{\lambda,k} \otimes \Theta_J^{\otimes m}$. Passing to the Fourier expansion at $I \subset J$, we see that the Fourier coefficients of f are contained in a proper subspace of $V(I)_{\lambda,k}$. Finally, running J over all 1-dimensional cusps containing I, we conclude that the Fourier coefficients are zero.

In the case of Siegel modular forms of genus 2, the idea to use Jacobi forms to deduce a vanishing theorem for vector-valued modular forms seems to go back to Ibukiyama [25, Section 6]. Our proof of Theorem 1.2 can be regarded as a generalization of his argument.

In this way, we have the unified viewpoint that the Siegel operator is concerned with the last sub vector bundle in the J-filtration, while the proof of Theorem 1.2 makes use of the first graded quotient. We expect that a closer look at the intermediate pieces of the J-filtration would tell us more.

Square integrability (Section 10)

We now turn to our second line of investigation. We can explicitly define and calculate an invariant Hermitian metric on $\mathcal{E}$ (and on $\mathcal{L}$, which is well known). They are essentially the Hodge metrics. They induce an invariant Hermitian metric $(\ ,\ )_{\lambda,k}$ on a general automorphic vector bundle $\mathcal{E}_{\lambda,k}$. Apart from the matter of convergence, this defines the Petersson inner product on $M_{\lambda,k}(\Gamma)$:

$$(f, g) = \int_{\Gamma\backslash\mathcal{D}} (f, g)_{\lambda,k} \mathrm{vol}_{\mathcal{D}}, \quad f, g \in M_{\lambda,k}(\Gamma),$$

where $\mathrm{vol}_{\mathcal{D}}$ is the invariant volume form on $\mathcal{D}$. When f or g is a cusp form, this integral converges as usual. Conversely, we prove the following. Let

$$\overline{\lambda} = (\overline{\lambda}_1, \dots, \overline{\lambda}_{[n/2]}) = (\lambda_1 - \lambda_n, \lambda_2 - \lambda_{n-1}, \dots, \lambda_{[n/2]} - \lambda_{n+1-[n/2]})$$

be the highest weight for $\mathrm{SO}(n, \mathbb{C})$ associated to λ. We write $|\overline{\lambda}| = \sum_i \overline{\lambda}_i$.

Theorem 1.4 (Theorem 10.1). *Let $\lambda \neq 1, \det$ and assume that $k \geq n + |\overline{\lambda}| - 1$. Then a modular form f of weight (λ, k) is a cusp form if and only if $(f, f) < \infty$.*

This holds also for $\lambda = 1, \det$ at least when L has Witt index 2 (Remark 10.13). In fact, Theorem 10.1 contains one more result that any modular form of weight (λ, k) with $k < n - |\overline{\lambda}| - 1$ and $\lambda \neq 1, \det$ is square integrable, but this is rather an intermediate step in the proof of our second vanishing theorem.

Vanishing theorem II (Section 11)

Our study of square integrability is partly motivated by the following vanishing theorem. Let $\mathrm{corank}(\lambda)$ be the maximal index $1 \leq i \leq [n/2]$ such that $\overline{\lambda}_1 = \overline{\lambda}_2 = \cdots = \overline{\lambda}_i$. Let $S_{\lambda,k}(\Gamma) \subset M_{\lambda,k}(\Gamma)$ be the subspace of cusp forms.

Theorem 1.5 (Theorem 11.1). *Let $\lambda \neq 1, \det$. If $k < n + \lambda_1 - \mathrm{corank}(\lambda) - 1$, there is no nonzero square integrable modular form of weight (λ, k). In particular,*

(1) $S_{\lambda,k}(\Gamma) = 0$ *if* $k < n + \lambda_1 - \mathrm{corank}(\lambda) - 1$.

(2) $M_{\lambda,k}(\Gamma) = 0$ *if* $k < n - |\overline{\lambda}| - 1$.

Although $\lambda_1 + n/2 - 1 < n + \lambda_1 - \mathrm{corank}(\lambda) - 1$, Theorem 1.5 does not supersede Theorem 1.2 because it is about square integrable modular forms. It depends on (λ, k) which bound in Theorem 1.2 or Theorem 1.5 (2) is larger. The two vanishing theorems are rather complementary.

The proof of Theorem 1.5 is parallel to Weissauer's vanishing theorem [47] for Siegel modular forms. If we have a square integrable modular form, we can construct a unitary highest weight module for $\mathrm{SO}^+(L_{\mathbb{R}})$ by a standard procedure. Then the bound $k < n + \lambda_1 - \mathrm{corank}(\lambda) - 1$ is derived from the classification of unitary highest weight modules [12, 13, 28]. The more specific conclusions (1), (2) are consequences of the square integrability theorem (Theorem 10.1).

Terminology and notation

Let us summarize some frequently used terminology and notation.

(1) By a *lattice* we mean a free $\mathbb{Z}$-module L of finite rank equipped with a nondegenerate symmetric bilinear form $(\cdot,\cdot) : L \times L \to \mathbb{Z}$. (Sometimes we still use the word "lattice" when the bilinear form is only $\mathbb{Q}$-valued.) The dual lattice $\mathrm{Hom}(L, \mathbb{Z})$ of L is written as $L^\vee$. A sublattice M of L is called *primitive* if L/M is free. We denote by $M^\perp$ the orthogonal complement of M in L. A sublattice I of L is called *isotropic* if $(I, I) \equiv 0$. The lattice L is called an *even lattice* if $(l, l) \in 2\mathbb{Z}$ for every $l \in L$. The orthogonal group of a lattice L is denoted by $\mathrm{O}(L)$. For $F = \mathbb{Q}, \mathbb{R}, \mathbb{C}$ we write

$$L_F = L \otimes_{\mathbb{Z}} F.$$

This is a quadratic space over F. Its orthogonal group is denoted by $\mathrm{O}(L_F)$. The special orthogonal group, namely, the subgroup of $\mathrm{O}(L_F)$ of determinant 1, is denoted by $\mathrm{SO}(L_F)$. A lattice L in a $\mathbb{Q}$-quadratic space V is called a *full lattice* in V if $V = L_{\mathbb{Q}}$. For a rational number $\alpha \neq 0$ we write $L(\alpha)$ for the α-scaling of L, namely, the same underlying $\mathbb{Z}$-module with the bilinear form multiplied by α. In the context of lattices, the symbol U will stand for the integral hyperbolic plane, namely, the even unimodular lattice of signature $(1, 1)$.

(2) Let G be a group acting on a set X and let Y be a subset of X. By the *stabilizer* of Y in G, we mean the subgroup of G consisting of elements g such that $g(Y) = Y$.

(3) Let V be a nondegenerate quadratic space over $F = \mathbb{Q}, \mathbb{R}, \mathbb{C}$. Let I be an isotropic line in V, and $P(I)$ be the stabilizer of I in $\mathrm{O}(V)$. Then we have the canonical exact sequence

$$0 \to (I^\perp/I) \otimes_F I \to P(I) \to \mathrm{GL}(I) \times \mathrm{O}(I^\perp/I) \to 1. \tag{1.2}$$

Here $P(I) \to \mathrm{GL}(I)$ and $P(I) \to \mathrm{O}(I^\perp/I)$ are the natural maps, and the map $(I^\perp/I) \otimes_F I \to P(I)$ sends a vector $m \otimes l$ of $(I^\perp/I) \otimes_F I$ to the isometry $E_{m\otimes l}$ of V defined by

$$E_{m\otimes l}(v) = v - (\tilde{m}, v)l + (l, v)\tilde{m} - \frac{1}{2}(m, m)(l, v)l, \quad v \in V. \tag{1.3}$$

Here $\tilde{m} \in I^\perp$ is a lift of $m \in I^\perp/I$. In particular, when $v \in I^\perp$, (1.3) is simplified to

$$E_{m\otimes l}(v) = v - (m, v)l.$$

The isometries $E_{m\otimes l}$ are sometimes called the *Eichler transvections*. If we take a basis $e_1, \dots, e_n$ of V such that $I = \langle e_1 \rangle$, $I^\perp = \langle e_1, \dots, e_{n-1} \rangle$ and $(e_1, e_n) = 1$, $(e_i, e_n) = 0$ for $i > 1$, then $E_{m\otimes e_1}$ is expressed by the matrix

$$\begin{pmatrix} 1 & -m^\vee & -(m,m)/2 \\ 0 & I_{n-2} & m \\ 0 & 0 & 1 \end{pmatrix},$$

where we regard $m \in \langle e_2, \dots, e_{n-1} \rangle \simeq I^\perp / I$. The group $(I^\perp/I) \otimes_F I$ of Eichler transvections is the unipotent radical of $P(I)$.

(4) We will not distinguish between vector bundles and locally free sheaves on a complex manifold X. The fiber of a vector bundle $\mathcal{F}$ over a point $x \in X$ is denoted by $\mathcal{F}_x$ (not the germ of the sheaf). A collection of sections of a vector bundle $\mathcal{F}$ is called a *frame* of $\mathcal{F}$ when it defines an isomorphism $\mathcal{O}_X^{\oplus r} \simeq \mathcal{F}$, i.e., it forms a basis in every fiber. The dual vector bundle of $\mathcal{F}$ is denoted by $\mathcal{F}^\vee$.

(5) Let X be a complex manifold and G be a group acting on X. Let $\mathcal{F}$ be a G-equivariant vector bundle on X. Suppose that $\mathcal{F}$ is endowed with an isomorphism

$$\iota \colon V \otimes \mathcal{O}_X \to \mathcal{F}$$

for a $\mathbb{C}$-linear space V. Then the *factor of automorphy* of the G-action on $\mathcal{F}$ with respect to the trivialization ι is the $\mathrm{GL}(V)$-valued function on $G \times X$ defined by

$$j(g, x) = \iota_{gx}^{-1} \circ g \circ \iota_x : V \to \mathcal{F}_x \to \mathcal{F}_{gx} \to V$$

for $g \in G$, $x \in X$. Here the middle map is the equivariant action by g. If Γ is a subgroup of G, a Γ-invariant section of $\mathcal{F}$ over X is identified via ι with a V-valued holomorphic function f on X satisfying $f(\gamma x) = j(\gamma, x) f(x)$ for every $\gamma \in \Gamma$ and $x \in X$.

(6) We write $e(z) = \exp(2\pi i z)$ for $z \in \mathbb{C}/\mathbb{Z}$. We use the symbol $\mathbb{H}$ for the upper half plane $\{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\}$.

Organization

The logical dependence between the chapters is as follows. A dotted arrow means that the dependence is weak.

$$\begin{tikzcd}
 & & \text{Section } 4 & \text{Section } 6 \arrow[d, dotted] \arrow[dr, dotted] & & \\
\text{Section } 2 \arrow[r] & \text{Section } 3 \arrow[r] \arrow[ur] \arrow[dr] & \text{Section } 5 \arrow[r] \arrow[ur] \arrow[d, dotted] & \text{Section } 7 \arrow[r] & \text{Section } 8 \arrow[r] & \text{Section } 9 \\
 & & \text{Section } 10 \arrow[r] & \text{Section } 11 & &
\end{tikzcd}$$