Chapter 2

The two Hodge bundles

In this chapter we study some basic properties of the Hodge bundles $\mathcal{L}$ and $\mathcal{E}$. In Section 2.1 we recall basic facts on the Hermitian symmetric domains of type IV. The Hodge line bundle $\mathcal{L}$ is well known, and we recall it in Section 2.2. In Sections 2.3 and 2.4 we study the second Hodge bundle $\mathcal{E}$. In Section 2.5 we describe $\mathcal{E}$ and $\mathcal{L}$ in the case $n \leq 4$ under the accidental isomorphisms.

2.1 The domain

Let L be a lattice of signature $(2, n)$. Let $Q = Q_L$ be the isotropic quadric in $\mathbb{P}L_{\mathbb{C}}$ defined by the equation $(\omega, \omega) = 0$ for $\omega \in L_{\mathbb{C}}$. We express a point of Q as $[\omega] = \mathbb{C}\omega$. The open set of Q defined by the inequality $(\omega, \bar{\omega}) > 0$ has two connected components. They are interchanged by the complex conjugation $\omega \mapsto \bar{\omega}$. We choose one of them and denote it by $\mathcal{D} = \mathcal{D}_L$. This is the Hermitian symmetric domain attached to L. In Cartan's classification, $\mathcal{D}$ is a Hermitian symmetric domain of type IV. The isotropic quadric Q is the compact dual of $\mathcal{D}$. Points of $\mathcal{D}$ are in one-to-one correspondence with positive-definite planes in $L_{\mathbb{R}}$, by associating

$$\mathcal{D} \ni [\omega] \mapsto H_\omega = \langle \mathrm{Re}(\omega), \mathrm{Im}(\omega) \rangle.$$

The choice of the component $\mathcal{D}$ determines orientation on the positive-definite planes. Note that $(\mathrm{Re}(\omega), \mathrm{Im}(\omega)) = 0$ and $(\mathrm{Re}(\omega), \mathrm{Re}(\omega)) = (\mathrm{Im}(\omega), \mathrm{Im}(\omega))$ by the isotropicity condition $(\omega, \omega) = 0$.

We denote by $\mathrm{O}^+(L_{\mathbb{R}})$ the index 2 subgroup of $\mathrm{O}(L_{\mathbb{R}})$ preserving the component $\mathcal{D}$. Then $\mathrm{O}^+(L_{\mathbb{R}})$ consists of two connected components, the identity component being

$$\mathrm{SO}^+(L_{\mathbb{R}}) = \mathrm{O}^+(L_{\mathbb{R}}) \cap \mathrm{SO}(L_{\mathbb{R}}).$$

The stabilizer K of a point $[\omega] \in \mathcal{D}$ in $\mathrm{O}^+(L_{\mathbb{R}})$ is the same as the stabilizer of the oriented plane H_ω, and is described as

$$K = \mathrm{SO}(H_\omega) \times \mathrm{O}(H_\omega^\perp) \simeq \mathrm{SO}(2, \mathbb{R}) \times \mathrm{O}(n, \mathbb{R}).$$

This is a maximal compact subgroup of $\mathrm{O}^+(L_{\mathbb{R}})$. We have $\mathcal{D} \simeq \mathrm{O}^+(L_{\mathbb{R}})/K$. On the other hand, as explained in (1.2), the stabilizer P of $[\omega]$ in $\mathrm{O}(L_{\mathbb{C}})$ sits in the canonical exact sequence

$$0 \to (\omega^\perp/\mathbb{C}\omega) \otimes \mathbb{C}\omega \to P \to \mathrm{GL}(\mathbb{C}\omega) \times \mathrm{O}(\omega^\perp/\mathbb{C}\omega) \to 1.$$

The reductive part

$$\mathrm{GL}(\mathbb{C}\omega) \times \mathrm{O}(\omega^{\perp}/\mathbb{C}\omega) \simeq \mathbb{C}^* \times \mathrm{O}(n, \mathbb{C})$$

is the complexification of K.

The domain $\mathcal{D}$ has two types of rational boundary components (cusps): 0-dimensional and 1-dimensional cusps. The 0-dimensional cusps correspond to rational isotropic lines in $L_{\mathbb{Q}}$, or equivalently, rank 1 primitive isotropic sublattices I of L. The point $p_I = [I_{\mathbb{C}}]$ of Q is in the closure of $\mathcal{D}$, and this is the 0-dimensional cusp corresponding to I. The 1-dimensional cusps correspond to rational isotropic planes in $L_{\mathbb{Q}}$, or equivalently, rank 2 primitive isotropic sublattices J of L. Each such J determines the line $\mathbb{P}J_{\mathbb{C}}$ on Q. If we remove $\mathbb{P}J_{\mathbb{R}}$ from $\mathbb{P}J_{\mathbb{C}}$, then $\mathbb{P}J_{\mathbb{C}} - \mathbb{P}J_{\mathbb{R}}$ consists of two copies of the upper half plane, one in the closure of $\mathcal{D}$. This component, say $\mathbb{H}_J$, is the 1-dimensional cusp corresponding to J. A 0-dimensional cusp p_I is in the closure of a 1-dimensional cusp $\mathbb{H}_J$ if and only if $I \subset J$.

Let $\mathrm{O}^+(L) = \mathrm{O}(L) \cap \mathrm{O}^+(L_{\mathbb{R}})$ and Γ be a finite-index subgroup of $\mathrm{O}^+(L)$. By Baily–Borel [3], the quotient space

$$\mathcal{F}(\Gamma)^{bb} = \Gamma \backslash \Big(\mathcal{D} \cup \bigcup_J \mathbb{H}_J \cup \bigcup_I p_I \Big)$$

has the structure of a normal projective variety of dimension n. Here the union of $\mathcal{D}$ and the cusps is equipped with the so-called Satake topology. In particular, the quotient

$$\mathcal{F}(\Gamma) = \Gamma \backslash \mathcal{D}$$

is a normal quasi-projective variety. The variety $\mathcal{F}(\Gamma)^{bb}$ is called the *Baily–Borel compactification* of $\mathcal{F}(\Gamma)$.

2.2 The Hodge line bundle

In this section we recall the first Hodge bundle. Let $\mathcal{O}_Q(-1)$ be the tautological line bundle over $Q \subset \mathbb{P}L_{\mathbb{C}}$. The Hodge line bundle over $\mathcal{D}$ is defined as

$$\mathcal{L} = \mathcal{O}_Q(-1)|_{\mathcal{D}}.$$

This is an $\mathrm{O}^+(L_{\mathbb{R}})$-invariant sub line bundle of $L_{\mathbb{C}} \otimes \mathcal{O}_{\mathcal{D}}$. The fiber of $\mathcal{L}$ over $[\omega] \in \mathcal{D}$ is the line $\mathbb{C}\omega$. By definition $\mathcal{L}$ extends over Q naturally, and we sometimes write $\mathcal{L} = \mathcal{O}_Q(-1)$ when no confusion is likely to occur. A holomorphic section of $\mathcal{L}^{\otimes k}$ over $\mathcal{D}$ invariant under a finite-index subgroup of $\mathrm{O}^+(L)$ and holomorphic at the cusps (in the sense explained later) is called a (scalar-valued) modular form of weight k.

The stabilizer $K \subset \mathrm{O}^+(L_{\mathbb{R}})$ of a point $[\omega] \in \mathcal{D}$ acts on the fiber $\mathcal{L}_{[\omega]}$ of $\mathcal{L}$ as the weight 1 character of $\mathrm{SO}(2, \mathbb{R}) \subset K$. Therefore, if we denote by $W \simeq \mathbb{C}$ the representation space of the weight 1 character of $\mathrm{SO}(2, \mathbb{R})$, we have an $\mathrm{O}^+(L_{\mathbb{R}})$-equivariant isomorphism

$$\mathcal{L} \simeq \mathrm{O}^+(L_{\mathbb{R}}) \times_K \mathcal{L}_{[\omega]} \simeq \mathrm{O}^+(L_{\mathbb{R}}) \times_K W.$$

Similarly, the extension $\mathcal{O}_Q(-1)$ over Q is the homogeneous line bundle corresponding to the weight 1 character of $\mathbb{C}^* \subset \mathbb{C}^* \times \mathrm{O}(n, \mathbb{C})$.

A trivialization of $\mathcal{L}$ can be defined for each 0-dimensional cusp of $\mathcal{D}$ as follows. Let I be a rank 1 primitive isotropic sublattice of L. For later use, it is useful to work over the following enlargement of $\mathcal{D}$:

$$\mathcal{D}(I) = Q - Q \cap \mathbb{P} I_{\mathbb{C}}^{\perp}.$$

This is a Zariski open set of Q containing $\mathcal{D}$. Its complement $Q \cap \mathbb{P} I_{\mathbb{C}}^{\perp}$ is the cone over the isotropic quadric in $\mathbb{P}(I^{\perp}/I)_{\mathbb{C}}$ with vertex $[I_{\mathbb{C}}]$. If $[\omega] \in \mathcal{D}(I)$, the pairing between $I_{\mathbb{C}}$ and $\mathbb{C}\omega$ is nonzero. This defines an isomorphism $\mathbb{C}\omega \to I_{\mathbb{C}}^{\vee}$. Since $\mathbb{C}\omega$ is the fiber of $\mathcal{L} = \mathcal{O}_Q(-1)$ over $[\omega]$, by varying $[\omega]$ we obtain an isomorphism

$$I_{\mathbb{C}}^{\vee} \otimes \mathcal{O}_{\mathcal{D}(I)} \to \mathcal{L} \tag{2.1}$$

of line bundles on $\mathcal{D}(I)$. We call this isomorphism the *I-trivialization* of $\mathcal{L}$. This is equivariant with respect to the stabilizer of $I_{\mathbb{C}}$ in $\mathrm{O}(L_{\mathbb{C}})$. Over Q the I-trivialization has pole of order 1 at the divisor $Q \cap \mathbb{P} I_{\mathbb{C}}^{\perp}$, and hence extends to an isomorphism

$$I_{\mathbb{C}}^{\vee} \otimes \mathcal{O}_Q \to \mathcal{L}(Q \cap \mathbb{P} I_{\mathbb{C}}^{\perp}).$$

In what follows, we work over $\mathcal{D}$. We call the restriction of (2.1) to $\mathcal{D}$ the *I-trivialization* of $\mathcal{L}$ too. If we choose a nonzero vector of $I_{\mathbb{C}}^{\vee}$, it defines a nowhere vanishing section of $\mathcal{L}$ via the I-trivialization. To be more specific, we choose a vector $l \neq 0 \in I$ and let s_l be the section of $\mathcal{L}$ corresponding to the dual vector of l. This section is determined by the condition that the vector

$$s_l([\omega]) \in \mathcal{L}_{[\omega]} = \mathbb{C}\omega$$

has pairing 1 with l. The factor of automorphy of the $\mathrm{O}^+(L_{\mathbb{R}})$-action on $\mathcal{L}$ with respect to the I-trivialization is a function on $\mathrm{O}^+(L_{\mathbb{R}}) \times \mathcal{D}$ which can be written as

$$j(g, [\omega]) = \frac{g \cdot s_l([\omega])}{s_l([g\omega])} = \frac{(g\omega, l)}{(\omega, l)}, \quad g \in \mathrm{O}^+(L_{\mathbb{R}}),\ [\omega] \in \mathcal{D}. \tag{2.2}$$

This gives a more classical style of defining scalar-valued modular forms. Note that if g acts trivially on $I_{\mathbb{R}}$, then $j(g, [\omega]) \equiv 1$.

2.3 The second Hodge bundle

In this section we define the second Hodge bundle. We have a natural quadratic form on the vector bundle $L_{\mathbb{C}} \otimes \mathcal{O}_{\mathcal{D}}$. By the definition of Q, $\mathcal{L}$ is an isotropic sub line bundle of $L_{\mathbb{C}} \otimes \mathcal{O}_{\mathcal{D}}$, so we have $\mathcal{L} \subset \mathcal{L}^{\perp}$. The second Hodge bundle is defined by

$$\mathcal{E} = \mathcal{L}^{\perp}/\mathcal{L}.$$

This is an $\mathrm{O}^{+}(L_{\mathbb{R}})$-equivariant vector bundle of rank n over $\mathcal{D}$. The fiber of $\mathcal{E}$ over $[\omega] \in \mathcal{D}$ is $\omega^{\perp}/\mathbb{C}\omega$. The quadratic form on $L_{\mathbb{C}} \otimes \mathcal{O}_{\mathcal{D}}$ induces a nondegenerate $\mathrm{O}^{+}(L_{\mathbb{R}})$-invariant quadratic form on $\mathcal{E}$. In other words, $\mathcal{E}$ is an orthogonal vector bundle. In particular, we have $\mathcal{E}^{\vee} \simeq \mathcal{E}$. Since $\mathcal{L}$ is naturally defined on Q, $\mathcal{E}$ is also naturally defined on Q. This is an $\mathrm{O}(L_{\mathbb{C}})$-equivariant vector bundle. By abuse of notation, we often use the same notation $\mathcal{E}$ for this extended vector bundle.

The stabilizer $K \subset \mathrm{O}^{+}(L_{\mathbb{R}})$ of a point $[\omega] \in \mathcal{D}$ acts on the fiber $\mathcal{E}_{[\omega]}$ of $\mathcal{E}$ as the standard $\mathbb{C}$-representation of $\mathrm{O}(n, \mathbb{R}) \subset K$, because we have a natural isomorphism $H_{\omega}^{\perp} \otimes_{\mathbb{R}} \mathbb{C} \simeq \omega^{\perp}/\mathbb{C}\omega$. Therefore, if we denote by $V = \mathbb{C}^{n}$ the standard representation space of $\mathrm{O}(n, \mathbb{C})$, we have an $\mathrm{O}^{+}(L_{\mathbb{R}})$-equivariant isomorphism

$$\mathcal{E} \simeq \mathrm{O}^{+}(L_{\mathbb{R}}) \times_{K} \mathcal{E}_{[\omega]} \simeq \mathrm{O}^{+}(L_{\mathbb{R}}) \times_{K} V. \tag{2.3}$$

Similarly, the extension of $\mathcal{E}$ over Q is the homogeneous vector bundle corresponding to the standard representation of $\mathrm{O}(n, \mathbb{C}) \subset \mathbb{C}^{*} \times \mathrm{O}(n, \mathbb{C})$.

We present some examples where $\mathcal{E}$ and $\mathcal{L}$ appear naturally.

Example 2.1. The "third" Hodge bundle $(L_{\mathbb{C}} \otimes \mathcal{O}_{\mathcal{D}})/\mathcal{L}^{\perp}$ is isomorphic to $\mathcal{L}^{-1}$ by the natural pairing with $\mathcal{L}$.

Example 2.2. The determinant line bundle $\det \mathcal{E} = \wedge^{n} \mathcal{E}$ of $\mathcal{E}$ is isomorphic, as an $\mathrm{O}^{+}(L_{\mathbb{R}})$-equivariant bundle, to the line bundle $\det \otimes \mathcal{O}_{\mathcal{D}}$ associated to the determinant character $\det\colon \mathrm{O}^{+}(L_{\mathbb{R}}) \to \{\pm 1\}$ of $\mathrm{O}^{+}(L_{\mathbb{R}})$. Indeed, by Example 2.1, we have the $\mathrm{O}^{+}(L_{\mathbb{R}})$-equivariant isomorphism

$$\det \mathcal{E} \simeq \det(L_{\mathbb{C}} \otimes \mathcal{O}_{\mathcal{D}}) \otimes \mathcal{L} \otimes \mathcal{L}^{-1} \simeq \det(L_{\mathbb{C}} \otimes \mathcal{O}_{\mathcal{D}}) \simeq \det \otimes \mathcal{O}_{\mathcal{D}}.$$

The line bundle $\det \otimes \mathcal{O}_{\mathcal{D}}$ appears in the study of scalar-valued modular forms with determinant character.

Example 2.3. Let $T_{\mathcal{D}}$ and $\Omega_{\mathcal{D}}^{1}$ be the tangent and cotangent bundles of $\mathcal{D}$, respectively. Then we have the canonical isomorphisms

$$T_{\mathcal{D}} \simeq \mathcal{E} \otimes \mathcal{L}^{-1}, \quad \Omega_{\mathcal{D}}^{1} \simeq \mathcal{E} \otimes \mathcal{L}. \tag{2.4}$$

Indeed, by the Euler sequence for $\mathbb{P}L_{\mathbb{C}}$, we have

$$T_{\mathbb{P}L_{\mathbb{C}}} \simeq \mathcal{O}_{\mathbb{P}L_{\mathbb{C}}}(1) \otimes ((L_{\mathbb{C}} \otimes \mathcal{O}_{\mathbb{P}L_{\mathbb{C}}})/\mathcal{O}_{\mathbb{P}L_{\mathbb{C}}}(-1)).$$

As a sub vector bundle of $T_{\mathbb{P}L_{\mathbb{C}}}|_Q$, we have

$$T_Q \simeq \mathcal{O}_Q(1) \otimes (\mathcal{O}_Q(-1)^{\perp}/\mathcal{O}_Q(-1)) = \mathcal{L}^{-1} \otimes \mathcal{E}.$$

The isomorphism for Ω_Q^1 is obtained by taking the dual.

Tautologically, the identity of $\mathcal{D}$ can be regarded as the period map

$$[\omega] \mapsto \mathcal{L}_{[\omega]}$$

for the universal variation $0 \subset \mathcal{L} \subset \mathcal{L}^{\perp} \subset L_{\mathbb{C}} \otimes \mathcal{O}_{\mathcal{D}}$ of Hodge structures on $\mathcal{D}$. Then the isomorphism $T_{\mathcal{D}} \simeq \mathcal{L}^{-1} \otimes \mathcal{E}$ is nothing but the differential of this tautological period map (cf. [46, Section 10.1]). By taking the adjunctions of

$$T_{\mathcal{D}} \simeq \mathcal{L}^{-1} \otimes \mathcal{E},$$

we obtain the homomorphisms

$$\mathcal{L} \otimes T_{\mathcal{D}} \xrightarrow{\simeq} \mathcal{E}, \quad \mathcal{E} \otimes T_{\mathcal{D}} \to \mathcal{L}^{-1}. \tag{2.5}$$

These are familiar forms in the context of variation of Hodge structures. Here the second homomorphism is given by the pairing on $\mathcal{E}$:

$$\mathcal{E} \otimes T_{\mathcal{D}} \simeq \mathcal{E} \otimes \mathcal{E} \otimes \mathcal{L}^{-1} \to \mathcal{L}^{-1}.$$

Example 2.4. Adjunctions of (2.5) induce the following complex of vector bundles on $\mathcal{D}$ (the *Koszul complex*):

$$\mathcal{L} \to \mathcal{E} \otimes \Omega_{\mathcal{D}}^1 \to \mathcal{L}^{-1} \otimes \Omega_{\mathcal{D}}^2. \tag{2.6}$$

Here the second homomorphism is the composition

$$\mathcal{E} \otimes \Omega_{\mathcal{D}}^1 \simeq \mathcal{L}^{-1} \otimes \Omega_{\mathcal{D}}^1 \otimes \Omega_{\mathcal{D}}^1 \xrightarrow{\wedge} \mathcal{L}^{-1} \otimes \Omega_{\mathcal{D}}^2.$$

By (2.4), the Koszul complex is identified with the complex

$$\mathcal{L} \otimes \left(\mathcal{O}_{\mathcal{D}} \to \mathcal{E}^{\otimes 2} \xrightarrow{\wedge} \wedge^2 \mathcal{E}\right),$$

where $\mathcal{O}_{\mathcal{D}} \to \mathcal{E}^{\otimes 2}$ is the embedding defined by the quadratic form on $\mathcal{E}$. This shows that (2.6) is indeed a complex, and its middle cohomology sheaf is isomorphic to

$$(\mathrm{Sym}^2 \mathcal{E}/\mathcal{O}_{\mathcal{D}}) \otimes \mathcal{L} \simeq \mathcal{E}_{(2)} \otimes \mathcal{L},$$

where $\mathcal{E}_{(2)}$ is the automorphic vector bundle associated to the representation $\mathrm{Sym}^2 \mathbb{C}^n/\mathbb{C}$ of $\mathrm{O}(n, \mathbb{C})$ (see Section 3.2). The Koszul complex will be taken up in Section 3.8.

2.4 I-trivialization of the second Hodge bundle

In this section we define a trivialization of $\mathcal{E}$ associated to each 0-dimensional cusp. This is the starting point of various later constructions.

Let I be a rank 1 primitive isotropic sublattice of L. The quadratic form on L induces a hyperbolic quadratic form on the $\mathbb{Z}$-module $I^{\perp}/I$. We write

$$V(I)_F = (I^{\perp}/I) \otimes_{\mathbb{Z}} F$$

for $F = \mathbb{Q}, \mathbb{R}, \mathbb{C}$. This is a quadratic space over F. We especially abbreviate $V(I) = V(I)_{\mathbb{C}}$. We consider the following sub vector bundle of $L_{\mathbb{C}} \otimes \mathcal{O}_{\mathcal{D}(I)}$:

$$I^{\perp} \cap \mathcal{L}^{\perp} = (I_{\mathbb{C}}^{\perp} \otimes \mathcal{O}_{\mathcal{D}(I)}) \cap \mathcal{L}^{\perp}.$$

The fiber of $I^{\perp} \cap \mathcal{L}^{\perp}$ over $[\omega] \in \mathcal{D}(I)$ is the subspace $I_{\mathbb{C}}^{\perp} \cap \omega^{\perp}$ of $L_{\mathbb{C}}$. The projection $\mathcal{L}^{\perp} \to \mathcal{E}$ induces a homomorphism $I^{\perp} \cap \mathcal{L}^{\perp} \to \mathcal{E}$, and the projection $I_{\mathbb{C}}^{\perp} \to V(I)$ induces a homomorphism $I^{\perp} \cap \mathcal{L}^{\perp} \to V(I) \otimes \mathcal{O}_{\mathcal{D}(I)}$.

Lemma 2.5. *The homomorphisms $I^{\perp} \cap \mathcal{L}^{\perp} \to \mathcal{E}$ and $I^{\perp} \cap \mathcal{L}^{\perp} \to V(I) \otimes \mathcal{O}_{\mathcal{D}(I)}$ are isomorphisms. Therefore we obtain an isomorphism*

$$V(I) \otimes \mathcal{O}_{\mathcal{D}(I)} \to \mathcal{E} \tag{2.7}$$

of vector bundles on $\mathcal{D}(I)$. This is equivariant with respect to the stabilizer of $I_{\mathbb{C}}$ in $\mathrm{O}(L_{\mathbb{C}})$, and preserves the quadratic forms on both sides.

Proof. At the fibers over a point $[\omega] \in \mathcal{D}(I)$, the two homomorphisms are given by the linear maps $I_{\mathbb{C}}^{\perp} \cap \omega^{\perp} \to \omega^{\perp}/\mathbb{C}\omega$ and $I_{\mathbb{C}}^{\perp} \cap \omega^{\perp} \to (I^{\perp}/I)_{\mathbb{C}}$, respectively. The source and the target have the same dimension ($= n$) for both maps, so it suffices to check the injectivity of these two maps. This is equivalent to $I_{\mathbb{C}}^{\perp} \cap \mathbb{C}\omega = 0$ and $\omega^{\perp} \cap I_{\mathbb{C}} = 0$, respectively, and both follow from the nondegeneracy $(I_{\mathbb{C}}, \mathbb{C}\omega) \neq 0$ for $[\omega] \in \mathcal{D}(I)$.

Since both $I_{\mathbb{C}}^{\perp} \cap \omega^{\perp} \to \omega^{\perp}/\mathbb{C}\omega$ and $I_{\mathbb{C}}^{\perp} \cap \omega^{\perp} \to (I^{\perp}/I)_{\mathbb{C}}$ preserve the quadratic forms, so does the composition

$$\omega^{\perp}/\mathbb{C}\omega \to (I^{\perp}/I)_{\mathbb{C}}.$$

Hence (2.7) preserves the quadratic forms. The equivariance of (2.7) can be verified similarly. ∎

We call the isomorphism (2.7) and its restriction to $\mathcal{D}$ the *I-trivialization* of $\mathcal{E}$. This is a trivialization as an orthogonal vector bundle. See Claim 6.10 for the boundary behavior of this isomorphism at a Zariski open set of the divisor $Q \cap \mathbb{P}I_{\mathbb{C}}^{\perp}$.

For later use, we calculate the sections of $\mathcal{E}$ corresponding to vectors of $V(I)$. We choose a vector $l \neq 0$ of I and let s_l be the corresponding section of $\mathcal{L}$ as defined in Section 2.2.

Lemma 2.6. *Let v be a vector of $V(I)$. We define a section of $I^{\perp} \cap \mathcal{L}^{\perp}$ by*

$$s_v([\omega]) = \tilde{v} - (\tilde{v}, s_l([\omega]))l, \quad [\omega] \in \mathcal{D}(I),$$

where $\tilde{v} \in I_{\mathbb{C}}^{\perp}$ is a lift of $v \in V(I)$ and we regard $s_l([\omega]) \in \mathbb{C}\omega \subset L_{\mathbb{C}}$. Then the image of s_v in $\mathcal{E}$ is the section of $\mathcal{E}$ which corresponds by the I-trivialization to the constant section of $V(I) \otimes \mathcal{O}_{\mathcal{D}(I)}$ with value v.

Proof. It is straightforward to check that $s_v([\omega])$ does not depend on the choice of the lift $\tilde{v}$ and that $(s_v([\omega]), \omega) = (s_v([\omega]), l) = 0$. Thus s_v is indeed a section of $I^{\perp} \cap \mathcal{L}^{\perp}$. Since $s_v([\omega]) \equiv \tilde{v} \mod I_{\mathbb{C}}$ as a vector of $I_{\mathbb{C}}^{\perp}$, the image of $s_v([\omega])$ in $V(I)$ is v. This proves our assertion. ∎

2.5 Accidental isomorphisms

When $n \leq 4$, orthogonal modular varieties are isomorphic to other types of classical modular varieties by the so-called accidental isomorphisms. In this section we explain how the second Hodge bundle $\mathcal{E}$ in $n \leq 4$ is translated under the accidental isomorphism. (This is well known for $\mathcal{L}$; we also include it for completeness.) This correspondence is the basis of comparing vector-valued orthogonal modular forms in $n = 3, 4$ with vector-valued Siegel and Hermitian modular forms, respectively. We explain the translation from both algebro-geometric and representation-theoretic viewpoints. Since the contents of this section will be used only sporadically in the rest of this memoir, the reader may skip it for the moment.

2.5.1 Modular curves

When $n = 1$, the accidental isomorphism between the real Lie groups is $\mathrm{PSL}(2, \mathbb{R}) \simeq \mathrm{SO}^{+}(1, 2)$. Its complexification is the isomorphism $\mathrm{PSL}(2, \mathbb{C}) \simeq \mathrm{SO}(3, \mathbb{C})$. This lifts to $\mathrm{SL}(2, \mathbb{C}) \simeq \mathrm{Spin}(3, \mathbb{C})$. The isomorphism between the compact duals is provided by the anti-canonical embedding $\mathbb{P}^1 \hookrightarrow \mathbb{P}^2$ of $\mathbb{P}^1$, which maps $\mathbb{P}^1$ to a conic $Q \subset \mathbb{P}^2$. This gives an isomorphism between the upper half plane and the type IV domain in $n = 1$. The line bundle $\mathcal{L} = \mathcal{O}_Q(-1)$ on Q is identified with $\mathcal{O}_{\mathbb{P}^1}(-2)$ on $\mathbb{P}^1$. This means that orthogonal modular forms of weight k correspond to elliptic modular forms of weight $2k$.

The reductive part of a standard parabolic subgroup of $\mathrm{SL}(2, \mathbb{C})$ is the 1-dimensional torus T consisting of diagonal matrices $\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$ of determinant 1. The corresponding group in $\mathrm{PSL}(2, \mathbb{C})$ is $T/-1$. The weight 2 character $\alpha \mapsto \alpha^2$ of T defines an isomorphism $T/-1 \simeq \mathbb{C}^*$. This explains $\mathcal{O}_Q(-1) \simeq \mathcal{O}_{\mathbb{P}^1}(-2)$ from representation theory.

The full orthogonal group $\mathrm{O}(3,\mathbb{C})$ is $\mathrm{SO}(3,\mathbb{C}) \times \{\pm \mathrm{id}\}$. By Example 2.2, the second Hodge bundle $\mathcal{E}$ is the line bundle associated to the determinant character $\det\colon \mathrm{O}(3,\mathbb{C}) \to \{\pm 1\}$. This is nontrivial as an $\mathrm{O}(3,\mathbb{C})$-line bundle, but trivial as an $\mathrm{SO}(3,\mathbb{C})$-line bundle. Therefore $\mathcal{E}$ cannot be detected at the side of $\mathrm{SL}(2,\mathbb{C})$.

2.5.2 Hilbert modular surfaces

When $n = 2$, the accidental isomorphism between the real Lie groups is

$$\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})/(-1,-1) \simeq \mathrm{SO}^+(2,2).$$

Its complexification is

$$\mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C})/(-1,-1) \simeq \mathrm{SO}(4,\mathbb{C}).$$

This lifts to $\mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C}) \simeq \mathrm{Spin}(4,\mathbb{C})$. The isomorphism between the compact duals is provided by the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$ of $\mathbb{P}^1 \times \mathbb{P}^1$, which maps $\mathbb{P}^1 \times \mathbb{P}^1$ to a quadric surface $Q \subset \mathbb{P}^3$. This gives an isomorphism between the product of two upper half planes and the type IV domain in $n = 2$. Since the Segre embedding is defined by $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,1)$, the Hodge line bundle $\mathcal{L} = \mathcal{O}_Q(-1)$ on Q is identified with $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-1,-1)$ on $\mathbb{P}^1 \times \mathbb{P}^1$. This means that orthogonal modular forms of weight k correspond to Hilbert modular forms of weight (k,k).

We explain the representation-theoretic aspect. The reductive part of a standard parabolic subgroup of $\mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C})$ is the 2-dimensional torus $T_1 \times T_2$ consisting of pairs (α,β) of diagonal matrices in each $\mathrm{SL}(2,\mathbb{C})$. The corresponding group in $\mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C})/(-1,-1)$ is $T_1 \times T_2/(-1,-1)$. We have natural isomorphisms

$$T_1 \times T_2/(-1,-1) \simeq \mathbb{C}^* \times \mathbb{C}^* \simeq \mathbb{C}^* \times \mathrm{SO}(2,\mathbb{C}), \tag{2.8}$$

where the first isomorphism is induced by

$$T_1 \times T_2 \to \mathbb{C}^* \times \mathbb{C}^*, \quad (\alpha,\beta) \mapsto (\alpha\beta, \alpha^{-1}\beta).$$

This is the isomorphism between the reductive parts of standard parabolic subgroups of $\mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C})/(-1,-1)$ and $\mathrm{SO}(4,\mathbb{C})$. The pullback of the weight 1 character of $\mathbb{C}^* \subset \mathbb{C}^* \times \mathrm{SO}(2,\mathbb{C})$ to $T_1 \times T_2$ by (2.8) is the tensor product $\chi_1 \boxtimes \chi_2$ of the weight 1 characters χ_1, χ_2 of T_1, T_2. This explains $\mathcal{O}_Q(-1) \simeq \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-1,-1)$ from representation theory.

The second Hodge bundle $\mathcal{E}$ is described as follows.

Lemma 2.7. *We have an* $\mathrm{O}(4,\mathbb{C})$*-equivariant isomorphism*

$$\mathcal{E} \simeq \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-1,1) \oplus \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,-1). \tag{2.9}$$

Proof. Let $\pi_i \colon \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ be the i-th projection. Then

$$\Omega^1_{\mathbb{P}^1\times\mathbb{P}^1} \simeq \pi_1^*\Omega^1_{\mathbb{P}^1} \oplus \pi_2^*\Omega^1_{\mathbb{P}^1} \simeq \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-2,0) \oplus \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(0,-2).$$

By (2.4) and $\mathcal{L}^{-1} \simeq \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,1)$, we have

$$\mathcal{E} \simeq \Omega^1_{\mathbb{P}^1\times\mathbb{P}^1} \otimes \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,1) \simeq \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-1,1) \oplus \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,-1).$$

This proves (2.9). ■

Note that $\mathrm{O}(4,\mathbb{C})$ is the semi-product $\mathfrak{S}_2 \ltimes \mathrm{SO}(4,\mathbb{C})$, where $\mathfrak{S}_2$ switches the two $\mathrm{SL}(2,\mathbb{C})$. This involution switches the two rulings of $Q \simeq \mathbb{P}^1 \times \mathbb{P}^1$, and acts on the right-hand side of (2.9) by switching the two components.

At the level of representations, the isomorphism (2.9) comes from the following correspondence. Let χ be the weight 1 character of $\mathrm{SO}(2,\mathbb{C}) \simeq \mathbb{C}^*$. The 2-dimensional standard representation of $\mathrm{SO}(2,\mathbb{C})$ is $\chi \oplus \chi^{-1}$. The pullback of χ to $T_1 \times T_2$ by (2.8) is the character $\chi_1^{-1} \boxtimes \chi_2$. Hence the pullback of the standard representation of $\mathrm{SO}(2,\mathbb{C})$ to $T_1 \times T_2$ is $(\chi_1^{-1} \boxtimes \chi_2) \oplus (\chi_1 \boxtimes \chi_2^{-1})$. This explains (2.9) from representation theory.

By Lemma 2.7, a general automorphic vector bundle $\mathcal{E}_{\lambda,k}$ on Q decomposes into a direct sum of various line bundles $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(a,b)$. This means that vector-valued orthogonal modular forms in $n = 2$ decompose into tuples of scalar-valued Hilbert modular forms of various weights, so we have nothing new here.

2.5.3 Siegel modular 3-folds

When $n = 3$, the accidental isomorphism between the real Lie groups is

$$\mathrm{PSp}(4,\mathbb{R}) \simeq \mathrm{SO}^+(2,3).$$

Its complexification is $\mathrm{PSp}(4,\mathbb{C}) \simeq \mathrm{SO}(5,\mathbb{C})$, which lifts to $\mathrm{Sp}(4,\mathbb{C}) \simeq \mathrm{Spin}(5,\mathbb{C})$. The isomorphism between the compact duals is provided by the Plücker embedding $\mathrm{LG}(2,4) \hookrightarrow \mathbb{P}V = \mathbb{P}^4$ of the Lagrangian Grassmannian $\mathrm{LG}(2,4)$. Here V is the 5-dimensional irreducible representation of $\mathrm{Sp}(4,\mathbb{C})$ appearing in $\wedge^2\mathbb{C}^4$. The Plücker embedding maps $\mathrm{LG}(2,4)$ to a 3-dimensional quadric $Q \subset \mathbb{P}^4$, and hence gives an isomorphism between the Siegel upper half space of genus 2 and the type IV domain in $n = 3$.

Let $\mathcal{F}$ be the rank 2 universal sub vector bundle over $\mathrm{LG}(2,4)$. (This is the weight 1 Hodge bundle for Siegel modular 3-folds.) Since the Plücker embedding is defined by $\mathcal{O}_{\mathrm{LG}}(1) = \det \mathcal{F}^\vee$, the Hodge line bundle $\mathcal{L} = \mathcal{O}_Q(-1)$ on Q is identified with $\det \mathcal{F}$ on $\mathrm{LG}(2,4)$. This means that orthogonal modular forms of weight k correspond to Siegel modular forms of weight k.

We explain the representation-theoretic aspect. The reductive part of a standard parabolic subgroup of $\mathrm{Sp}(4,\mathbb{C})$ is isomorphic to $\mathrm{GL}(2,\mathbb{C})$. The corresponding group in $\mathrm{PSp}(4,\mathbb{C})$ is $\mathrm{GL}(2,\mathbb{C})/-1$. We have a natural isomorphism

$$\mathrm{GL}(2,\mathbb{C})/-1 \simeq \mathbb{C}^* \times \mathrm{PGL}(2,\mathbb{C}) \simeq \mathbb{C}^* \times \mathrm{SO}(3,\mathbb{C}), \tag{2.10}$$

where $\mathrm{GL}(2,\mathbb{C}) \to \mathbb{C}^*$ in the first isomorphism is the determinant character, and $\mathrm{PGL}(2,\mathbb{C}) \simeq \mathrm{SO}(3,\mathbb{C})$ in the second isomorphism is the accidental isomorphism in $n = 1$. This gives the isomorphism between the reductive parts of standard parabolic subgroups of $\mathrm{PSp}(4,\mathbb{C})$ and $\mathrm{SO}(5,\mathbb{C})$. By construction, the pullback of the weight 1 character of $\mathbb{C}^*$ to $\mathrm{GL}(2,\mathbb{C})$ by (2.10) is the determinant character of $\mathrm{GL}(2,\mathbb{C})$. This explains $\mathcal{L} \simeq \det \mathcal{F}$ from representation theory.

The second Hodge bundle $\mathcal{E}$ is described as follows.

Lemma 2.8. *We have an* $\mathrm{SO}(5,\mathbb{C})$*-equivariant isomorphism*

$$\mathcal{E} \simeq \mathrm{Sym}^2 \mathcal{F} \otimes \mathcal{L}^{-1}. \tag{2.11}$$

Proof. It is known (see, e.g., [44, Section 14]) that we have an $\mathrm{Sp}(4,\mathbb{C})$-equivariant isomorphism

$$\Omega^1_{\mathrm{LG}} \simeq \mathrm{Sym}^2 \mathcal{F}.$$

Then (2.11) follows from the isomorphism $\mathcal{E} \simeq \Omega^1_{\mathrm{LG}} \otimes \mathcal{L}^{-1}$ in (2.4). ■

Note that $\mathcal{F}$ is not $\mathrm{SO}(5,\mathbb{C})$-linearized but $\mathrm{Sym}^2 \mathcal{F}$ is. At the level of representations, the isomorphism (2.11) comes from the following fact: the symmetric square of the standard representation of $\mathrm{GL}(2,\mathbb{C})$, when viewed as a representation of $\mathbb{C}^* \times \mathrm{SO}(3,\mathbb{C})$ via (2.10), is isomorphic to the tensor product of the weight 1 character of $\mathbb{C}^*$ and the standard representation of $\mathrm{SO}(3,\mathbb{C})$.

The full orthogonal group $\mathrm{O}(5,\mathbb{C})$ is $\mathrm{SO}(5,\mathbb{C}) \times \{\pm \mathrm{id}\}$. As an $\mathrm{O}(5,\mathbb{C})$-vector bundle, we have

$$\mathcal{E} \simeq \mathrm{Sym}^2 \mathcal{F} \otimes \mathcal{L}^{-1} \otimes \det.$$

The twist by det cannot be detected at the side of $\mathrm{Sp}(4,\mathbb{C})$.

2.5.4 Hermitian modular 4-folds

When $n = 4$, the accidental isomorphism between the real Lie groups is

$$\mathrm{SU}(2,2)/-1 \simeq \mathrm{SO}^+(2,4).$$

The complexification is $\mathrm{SL}(4,\mathbb{C})/-1 \simeq \mathrm{SO}(6,\mathbb{C})$. This lifts to

$$\mathrm{SL}(4,\mathbb{C}) \simeq \mathrm{Spin}(6,\mathbb{C}).$$

The isomorphism between the compact duals is provided by the Plücker embedding $\mathrm{G}(2,4) \hookrightarrow \mathbb{P}(\wedge^2\mathbb{C}^4) = \mathbb{P}^5$ of the Grassmannian $\mathrm{G}(2,4)$. This maps $\mathrm{G}(2,4)$ to a 4-dimensional quadric $Q \subset \mathbb{P}^5$, and gives an isomorphism between the Hermitian upper half space of degree 2 and the type IV domain in $n = 4$.

The reductive part of a standard parabolic subgroup of $\mathrm{SL}(4,\mathbb{C})$ is the group

$$G = \left\{ \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix} \,\middle|\, g_1, g_2 \in \mathrm{GL}(2,\mathbb{C}),\ \det g_2 = \det g_1^{-1} \right\}.$$

The corresponding group in $\mathrm{SL}(4,\mathbb{C})/-1$ is $G/-1$. We have a natural isomorphism

$$G/-1 \simeq \mathbb{C}^* \times (\mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C})/(-1,-1)) \simeq \mathbb{C}^* \times \mathrm{SO}(4,\mathbb{C}). \tag{2.12}$$

Here the first isomorphism sends $(g_1, g_2) \in G$ to $(\det g_1, \pm\alpha^{-1}g_1, \pm\alpha g_2)$, where α is one of the square roots of $\det g_1$, and the second isomorphism is given by the accidental isomorphism in $n = 2$. This is the isomorphism between the reductive parts of standard parabolic subgroups of $\mathrm{SL}(4,\mathbb{C})/-1$ and $\mathrm{SO}(6,\mathbb{C})$.

Let $\mathcal{F}$, $\mathcal{G}$ be the universal sub and quotient vector bundles on $\mathrm{G}(2,4)$, respectively. Since the Plücker embedding is defined by $\mathcal{O}_{\mathrm{G}(2,4)}(1) = \det\mathcal{G} = (\det\mathcal{F})^{-1}$, the Hodge line bundle $\mathcal{L} = \mathcal{O}_Q(-1)$ is isomorphic to $\det\mathcal{F}$. Thus orthogonal modular forms of weight k correspond to Hermitian modular forms of weight k. At the level of representations, this comes from the fact that the pullback of the weight 1 character of $\mathbb{C}^*$ to G by (2.12) is the character of G given by $(g_1, g_2) \mapsto \det g_1$.

The second Hodge bundle $\mathcal{E}$ is described as follows.

Lemma 2.9. *We have an* $\mathrm{SO}(6,\mathbb{C})$*-equivariant isomorphism*

$$\mathcal{E} \simeq \mathcal{F} \otimes \mathcal{G}. \tag{2.13}$$

Proof. We have a canonical isomorphism $T_{\mathrm{G}(2,4)} \simeq \mathcal{F}^\vee \otimes \mathcal{G}$. The natural symplectic form $\mathcal{F} \otimes \mathcal{F} \to \det\mathcal{F}$ induces an isomorphism $\mathcal{F}^\vee \simeq \mathcal{F} \otimes \mathcal{L}^{-1}$. Therefore, by (2.4), we have

$$\mathcal{E} \simeq T_{\mathrm{G}(2,4)} \otimes \mathcal{L} \simeq \mathcal{F}^\vee \otimes \mathcal{G} \otimes \mathcal{L} \simeq \mathcal{F} \otimes \mathcal{G}.$$

This proves (2.13). ∎

Note that each $\mathcal{F}$, $\mathcal{G}$ is not $\mathrm{SO}(6,\mathbb{C})$-linearized, but $\mathcal{F} \otimes \mathcal{G}$ is. At the level of representations, the isomorphism (2.13) comes from the following correspondence. Let V_i, $i = 1, 2$, be the representation of G obtained as the pullback of the standard representation of $\mathrm{GL}(2,\mathbb{C})$ by the i-th projection $G \to \mathrm{GL}(2,\mathbb{C})$, $(g_1, g_2) \mapsto g_i$. Then V_1, V_2 correspond to the homogeneous vector bundles $\mathcal{F}$, $\mathcal{G}$, respectively. Each V_1, V_2 is not a representation of $G/-1$, but $V_1 \otimes V_2$ is. Then, as a representation of $\mathbb{C}^* \times (\mathrm{SL}(2,\mathbb{C})^2/(-1,-1))$ via the first isomorphism in (2.12), $V_1 \otimes V_2$ is isomorphic to the external tensor product of the standard representations of the two $\mathrm{SL}(2,\mathbb{C})$ (with weight 0 on $\mathbb{C}^*$). This in turn is the standard representation of $\mathrm{SO}(4,\mathbb{C})$ via the second isomorphism in (2.12). This explains the isomorphism (2.13) from representation theory.

Finally, $\mathrm{O}(6,\mathbb{C})$ is the semi-product $\mathfrak{S}_2 \ltimes \mathrm{SO}(6,\mathbb{C})$, where $\mathfrak{S}_2 = \langle \iota \rangle$ acts on $\mathrm{G}(2,4)$ by the following involution: choose a symplectic form on $\mathbb{C}^4$ (say the standard one), and sends 2-dimensional subspaces $W \subset \mathbb{C}^4$ to $W^\perp \subset \mathbb{C}^4$. This involution exchanges the two $\mathbb{P}^3$-families of planes on $\mathrm{G}(2,4) = Q$. (This is essentially the involution $Z \mapsto Z'$ in [17, Section 1] on the Hermitian upper half space.) The involution ι acts on the vector bundle $\mathcal{F} \otimes \mathcal{G} \simeq \mathcal{F}^\vee \otimes \mathcal{G}^\vee$ by $\iota^*\mathcal{F} \simeq \mathcal{G}^\vee$ and $\iota^*\mathcal{G} \simeq \mathcal{F}^\vee$. Then (2.13) is an $\mathrm{O}(6,\mathbb{C})$-equivariant isomorphism.