Chapter 3

Vector-valued modular forms

In this chapter we define vector-valued orthogonal modular forms (Section 3.2) and explain their Fourier expansions at 0-dimensional cusps (Sections 3.3–3.5). These are the most fundamental parts of this memoir. The rest of this chapter (Sections 3.6–3.8) is devoted to supplementary materials: the passage from O to SO, an example of explicit construction, and an interaction with algebraic cycles.

3.1 Representations of $\mathrm{O}(n, \mathbb{C})$

We begin by recollection of some basic facts from the representation theory for $\mathrm{O}(n, \mathbb{C})$. Our main reference for representation theory is [38, Section 8] (whose main contents are more or less covered by [18, Section 19] and [19, Sections 5.5.5 and 10.2]). In what follows and in Section 3.6, all representations are assumed to be finite dimensional over $\mathbb{C}$.

Irreducible representations of $\mathrm{O}(n, \mathbb{C})$ are labelled by partitions

$$\lambda = (\lambda_1 \geq \cdots \geq \lambda_n \geq 0)$$

such that ${}^t\lambda_1 + {}^t\lambda_2 \leq n$, where ${}^t\lambda$ is the transpose of λ. The irreducible representation corresponding to such a partition λ is constructed as follows. Let $V = \mathbb{C}^n$ be the standard representation of $\mathrm{O}(n, \mathbb{C})$. Let $d = |\lambda| = \sum_i \lambda_i$ be the size of λ. We denote by $V^{[d]}$ the intersection of the kernels of the contraction maps

$$V^{\otimes d} \to V^{\otimes d-2}$$

for all pairs of indices. Vectors in $V^{[d]}$ are called *traceless tensors* or *harmonic tensors* in the literature. The symmetric group $\mathfrak{S}_d$ acts on $V^{\otimes d}$ naturally and preserves $V^{[d]}$. Let $T = T_\lambda^{\downarrow}$ be the column canonical tableau on λ (namely, $1, 2, \ldots, {}^t\lambda_1$ on the first column, ${}^t\lambda_1 + 1, \ldots, {}^t\lambda_1 + {}^t\lambda_2$ on the second column, ...). Let $c_\lambda = b_\lambda a_\lambda \in \mathbb{C}\mathfrak{S}_d$ be the Young symmetrizer associated to T, where

$$a_\lambda = \sum_{\sigma \in H_T} \sigma, \quad b_\lambda = \sum_{\tau \in V_T} \mathrm{sgn}(\tau)\tau$$

as usual. (H_T and V_T are the row and the column Young subgroups of $\mathfrak{S}_d$ for the tableau T, respectively.) We apply the orthogonal Schur functor for λ to V:

$$V_\lambda = c_\lambda \cdot V^{[d]} = V^{[d]} \cap (c_\lambda \cdot V^{\otimes d}).$$

This space V_λ is the irreducible representation of $\mathrm{O}(n, \mathbb{C})$ labelled by the partition λ. Since b_λ maps $V^{\otimes d}$ to $\wedge^{{}^t\lambda_1} V \otimes \cdots \otimes \wedge^{{}^t\lambda_{\lambda_1}} V$, we have

$$V_\lambda \subset \wedge^{{}^t\lambda_1} V \otimes \cdots \otimes \wedge^{{}^t\lambda_{\lambda_1}} V \subset V^{\otimes d}. \tag{3.1}$$

If we take a basis $e_1, \dots, e_n$ of V such that $(e_i, e_j) = 1$ when $i + j = n + 1$ and $(e_i, e_j) = 0$ otherwise, V_λ especially contains the vector

$$(e_1 \wedge \cdots \wedge e_{{}^t\lambda_1}) \otimes (e_1 \wedge \cdots \wedge e_{{}^t\lambda_2}) \otimes \cdots \otimes (e_1 \wedge \cdots \wedge e_{{}^t\lambda_{\lambda_1}}) \tag{3.2}$$

(see [38, Section 8.3.1]).

Example 3.1. (1) The exterior tensor $\wedge^d V$ for $0 \le d \le n$ corresponds to the partition $\lambda = (1^d) = (1, \dots, 1)$. By abuse of notation, we sometimes write $\lambda = 1, \mathrm{St}, \wedge^d, \det$ instead of $\lambda = (0), (1), (1^d), (1^n)$, respectively.

(2) The symmetric tensor $\mathrm{Sym}^d V$ is reducible and decomposes as

$$\begin{aligned}\mathrm{Sym}^d V &= V_{(d)} \oplus \mathrm{Sym}^{d-2} V = \cdots \\ &= V_{(d)} \oplus V_{(d-2)} \oplus \cdots \oplus V_{(1) \text{ or } (0)}.\end{aligned}$$

Geometrically, $V_{(d)}$ is the cohomology $H^0(\mathcal{O}_{Q_{n-2}}(d))$ for the isotropic quadric

$$Q_{n-2} \subset \mathbb{P}V$$

of dimension $n - 2$.

3.2 Automorphic vector bundles

In this section we define automorphic vector bundles and vector-valued modular forms. Let L be a lattice of signature $(2, n)$. For simplicity of exposition we assume $n \ge 3$ so that the Koecher principle holds. (This assumption can be somewhat justified by our calculation of $\mathcal{E}$ in the case $n \le 2$ in Section 2.5.) Let $\lambda = (\lambda_1 \ge \cdots \ge \lambda_n)$ be a partition as in Section 3.1 and let $d = |\lambda|$. Recall that the second Hodge bundle $\mathcal{E}$ is endowed with a canonical quadratic form. Let $\mathcal{E}^{[d]} \subset \mathcal{E}^{\otimes d}$ be the intersection of the kernels of the contractions $\mathcal{E}^{\otimes d} \to \mathcal{E}^{\otimes d-2}$ for all pairs of indices. The fibers of $\mathcal{E}^{[d]}$ consist of traceless tensors in the fibers of $\mathcal{E}^{\otimes d}$. The symmetric group $\mathfrak{S}_d$ acts on $\mathcal{E}^{\otimes d}$ fiberwise and preserves $\mathcal{E}^{[d]}$. We define the vector bundle $\mathcal{E}_\lambda$ by applying the orthogonal Schur functor for λ relatively to $\mathcal{E}$:

$$\mathcal{E}_\lambda = c_\lambda \cdot \mathcal{E}^{[d]} = \mathcal{E}^{[d]} \cap (c_\lambda \cdot \mathcal{E}^{\otimes d}).$$

By construction $\mathcal{E}_\lambda$ is a sub vector bundle of $\mathcal{E}^{\otimes d}$, naturally defined over Q and is $\mathrm{O}(L_{\mathbb{C}})$-invariant.

Let I be a rank 1 primitive isotropic sublattice of L. Recall from Section 2.4 that we have the I-trivialization $\mathcal{E} \simeq V(I) \otimes \mathcal{O}_{\mathcal{D}(I)}$ over $\mathcal{D}(I) = Q - Q \cap \mathbb{P}I_{\mathbb{C}}^{\perp}$. Let $V(I)_\lambda$ be the irreducible representation of $\mathrm{O}(V(I)) \simeq \mathrm{O}(n, \mathbb{C})$ obtained by applying the orthogonal Schur functor for λ to $V(I)$. Since the I-trivialization of $\mathcal{E}$ preserves the quadratic forms, it induces an isomorphism

$$\mathcal{E}_\lambda \simeq V(I)_\lambda \otimes \mathcal{O}_{\mathcal{D}(I)}$$

over $\mathcal{D}(I)$. We call this isomorphism the *I-trivialization* of $\mathcal{E}_\lambda$.

Next for $k \in \mathbb{Z}$ we consider the tensor product

$$\mathcal{E}_{\lambda,k} = \mathcal{E}_\lambda \otimes \mathcal{L}^{\otimes k}.$$

This is an $\mathrm{O}(L_{\mathbb{C}})$-equivariant vector bundle on Q. If we write

$$V(I)_{\lambda,k} = V(I)_\lambda \otimes (I_{\mathbb{C}}^{\vee})^{\otimes k},$$

the I-trivializations of $\mathcal{E}_\lambda$ and $\mathcal{L}^{\otimes k}$ induce an isomorphism

$$\mathcal{E}_{\lambda,k} \simeq V(I)_{\lambda,k} \otimes \mathcal{O}_{\mathcal{D}(I)}$$

over $\mathcal{D}(I)$. This is equivariant with respect to the stabilizer of $I_{\mathbb{C}}$ in $\mathrm{O}(L_{\mathbb{C}})$. We call this isomorphism the *I-trivialization* of $\mathcal{E}_{\lambda,k}$.

In what follows, we work over $\mathcal{D}$. We use the same notations $\mathcal{E}_\lambda$, $\mathcal{E}_{\lambda,k}$ for the restriction of $\mathcal{E}_\lambda$, $\mathcal{E}_{\lambda,k}$ to $\mathcal{D}$. These are $\mathrm{O}^+(L_{\mathbb{R}})$-equivariant vector bundles on $\mathcal{D}$. Like (2.3), we have an $\mathrm{O}^+(L_{\mathbb{R}})$-equivariant isomorphism

$$\mathcal{E}_\lambda \simeq \mathrm{O}^+(L_{\mathbb{R}}) \times_K (\mathcal{E}_\lambda)_{[\omega]} \simeq \mathrm{O}^+(L_{\mathbb{R}}) \times_K V_\lambda, \tag{3.3}$$

where K is the stabilizer of $[\omega]$ in $\mathrm{O}^+(L_{\mathbb{R}})$. The I-trivialization of $\mathcal{E}_{\lambda,k}$ is defined over $\mathcal{D}$. Let $j(g, [\omega])$ be the factor of automorphy for the $\mathrm{O}^+(L_{\mathbb{R}})$-action on $\mathcal{E}_{\lambda,k}$ with respect to the I-trivialization. This is a $\mathrm{GL}(V(I)_{\lambda,k})$-valued function on $\mathrm{O}^+(L_{\mathbb{R}}) \times \mathcal{D}$. Since the I-trivialization is equivariant with respect to the stabilizer of $I_{\mathbb{R}}$ in $\mathrm{O}^+(L_{\mathbb{R}})$, we especially have the following.

Lemma 3.2. *When $g \in \mathrm{O}^+(L_{\mathbb{R}})$ stabilizes $I_{\mathbb{R}}$, the value of $j(g, [\omega])$ is constant over $\mathcal{D}$, given by the natural action of g on $V(I)_{\lambda,k}$.*

Now let Γ be a finite-index subgroup of $\mathrm{O}^+(L)$. We call a Γ-invariant holomorphic section of $\mathcal{E}_{\lambda,k}$ over $\mathcal{D}$ a *modular form of weight* (λ, k) with respect to Γ. By the I-trivialization, a modular form of weight (λ, k) is identified with a $V(I)_{\lambda,k}$-valued holomorphic function f on $\mathcal{D}$ such that

$$f(\gamma[\omega]) = j(\gamma, [\omega]) f([\omega])$$

for every $\gamma \in \Gamma$ and $[\omega] \in \mathcal{D}$. We denote by $M_{\lambda,k}(\Gamma)$ the space of modular forms of weight (λ, k) with respect to Γ. When $\lambda = (0)$, we especially write $M_{(0),k}(\Gamma) = M_k(\Gamma)$ as usual.

When $-\mathrm{id} \in \Gamma$, the weight (λ, k) satisfies a parity condition. We state it in a slightly generalized form.

Lemma 3.3. *Let $[\omega] \in \mathcal{D}$ and $\Gamma_{[\omega]}$ be the stabilizer of $[\omega]$ in Γ. The value of a Γ-modular form of weight (λ, k) at $[\omega]$ is contained in the $\Gamma_{[\omega]}$-invariant part of $(\mathcal{E}_{\lambda,k})_{[\omega]}$. In particular, when $-\mathrm{id} \in \Gamma$ and $k + |\lambda|$ is odd, we have*

$$M_{\lambda,k}(\Gamma) = 0.$$

Proof. The first assertion follows from the $\Gamma_{[\omega]}$-invariance of the section. As for the second assertion, we note that $-\mathrm{id}$ acts on both $\mathcal{L}$ and $\mathcal{E}$ as the scalar multiplication by -1. Since $\mathcal{E}_\lambda$ is a sub vector bundle of $\mathcal{E}^{\otimes|\lambda|}$, $-\mathrm{id}$ acts on $\mathcal{E}_{\lambda,k}$ as the scalar multiplication by $(-1)^{k+|\lambda|}$. Therefore, when $k + |\lambda|$ is odd, $-\mathrm{id}$ has no nonzero invariant part in every fiber of $\mathcal{E}_{\lambda,k}$. ■

Product of vector-valued modular forms can be given as follows. Suppose that we have a nonzero $\mathrm{O}(n, \mathbb{C})$-homomorphism

$$\varphi : V_{\lambda_1} \otimes V_{\lambda_2} \to V_{\lambda_3} \tag{3.4}$$

for partitions $\lambda_1, \lambda_2, \lambda_3$ for $\mathrm{O}(n, \mathbb{C})$. This uniquely induces an $\mathrm{O}^+(L_{\mathbb{R}})$-equivariant homomorphism

$$\varphi : \mathcal{E}_{\lambda_1,k_1} \otimes \mathcal{E}_{\lambda_2,k_2} \to \mathcal{E}_{\lambda_3,k_1+k_2}.$$

If f_1, f_2 are Γ-modular forms of weight (λ_1, k_1), (λ_2, k_2), respectively, then

$$f_1 \times_\varphi f_2 := \varphi(f_1 \otimes f_2)$$

is a Γ-modular form of weight $(\lambda_3, k_1 + k_2)$. This is the "φ-product" of f_1 and f_2. Note that a homomorphism (3.4) exists exactly when V_{λ_3} appears in the irreducible decomposition of $V_{\lambda_1} \otimes V_{\lambda_2}$, and it is unique up to constant when the multiplicity is 1. This information can be read off from the Littlewood–Richardson numbers [29, 31], see also [38, Section 12].

The map (3.4) also uniquely induces an $\mathrm{O}(V(I))$-homomorphism

$$\varphi_I : V(I)_{\lambda_1,k_1} \otimes V(I)_{\lambda_2,k_2} \to V(I)_{\lambda_3,k_1+k_2}. \tag{3.5}$$

If we denote by ι the relevant I-trivialization maps, then we have

$$\iota(f_1) \times_{\varphi_I} \iota(f_2) = \iota(f_1 \times_\varphi f_2). \tag{3.6}$$

In this sense, φ-product and I-trivialization are compatible.

It will be useful to know how orthogonal weights (λ, k) in $n = 3, 4$ are translated by the accidental isomorphisms. For simplicity we assume ${}^t\lambda_1 < n/2$, namely, ${}^t\lambda_1 = 1$. See Section 3.6 for some justification of this assumption. (There is no essential loss of generality when $n = 3$.) Henceforth we write $\lambda = (d)$ with d a natural number.

Example 3.4. Let $n = 3$. Let $\mathcal{F}$ be the rank 2 Hodge bundle considered in Section 2.5.3. Automorphic vector bundles on Siegel modular 3-folds can be expressed as $\mathrm{Sym}^j \mathcal{F} \otimes \mathcal{L}^{\otimes l}$ with $j \in \mathbb{Z}_{\geq 0}$ and $l \in \mathbb{Z}$. In the literature this is often referred to as weight $(\mathrm{Sym}^j, \det^l)$. This corresponds to the highest weight $(\rho_1, \rho_2) = (j + l, l)$ of $\mathrm{GL}(2, \mathbb{C})$. When $j = 2d$ is even, we have

$$\mathrm{Sym}^{2d} \mathcal{F} \simeq (\mathrm{Sym}^2 \mathcal{F})_{(d)} \simeq \mathcal{E}_{(d)} \otimes \mathcal{L}^{\otimes d}$$

by Lemma 2.8. Therefore

$$\mathrm{Sym}^{2d} \mathcal{F} \otimes \mathcal{L}^{\otimes l} \simeq \mathcal{E}_{(d)} \otimes \mathcal{L}^{\otimes l+d}.$$

Thus we have the following correspondence of weights:

$$\begin{aligned} & \text{orthogonal weight } ((d), k) \\ \leftrightarrow\ & \text{Siegel weight } (\mathrm{Sym}^j, \det^l) \text{ with } (j, l) = (2d, k-d) \\ \leftrightarrow\ & \mathrm{GL}(2, \mathbb{C})\text{-weight } (\rho_1, \rho_2) = (k+d, k-d) \end{aligned}$$

Example 3.5. Let $n = 4$. Let $\mathcal{F}$ and $\mathcal{G}$ be the rank 2 Hodge bundles considered in Section 2.5.4. Automorphic vector bundles on Hermitian modular 4-folds can be expressed as

$$\mathcal{L}^{\otimes k} \otimes \mathrm{Sym}^{j_1} \mathcal{F} \otimes \mathrm{Sym}^{j_2} \mathcal{G}, \quad k \in \mathbb{Z}, \ j_1, j_2 \in \mathbb{Z}_{\geq 0}. \tag{3.7}$$

On the other hand, in [17, Section 2], weights of vector-valued Hermitian modular forms of degree 2 are expressed as $(r, \rho_1 \boxtimes \rho_2)$, where $r \in \mathbb{Z}$ and ρ_1, ρ_2 are symmetric tensors of the standard representation of $\mathrm{GL}(2, \mathbb{C})$. (We are working with $\mathrm{SU}(2, 2)$ rather than $\mathrm{U}(2, 2)$, and we do not consider twist by a character as in [17].) Then $\mathcal{L}$ corresponds to the weight $r = 1$, $\mathcal{F}$ corresponds to the weight $\rho_1 = \mathrm{St}$, and

$$\mathcal{L} \otimes \mathcal{G} \simeq \mathcal{G}^\vee \simeq \iota^* \mathcal{F}$$

corresponds to the weight $\rho_2 = \mathrm{St}$. Thus the vector bundle (3.7) corresponds to the Hermitian weight $(r, \rho_1 \boxtimes \rho_2)$ with $r = k - j_2$, $\rho_1 = \mathrm{Sym}^{j_1}$ and $\rho_2 = \mathrm{Sym}^{j_2}$.

Now, by Lemma 2.9, we have

$$\mathcal{E}_{(d)} \simeq \mathrm{Sym}^d \mathcal{F} \otimes \mathrm{Sym}^d \mathcal{G}.$$

Therefore the weights correspond as follows:

$$\begin{aligned} & \text{orthogonal weight } ((d), k) \\ \leftrightarrow\quad & \text{Hermitian weight } (k-d, \mathrm{Sym}^d \boxtimes \mathrm{Sym}^d) \end{aligned}$$

In [17, Sections 3 and 4], some examples in the case $d = 1$ are studied in detail.

3.3 Tube domain realization

In this section we recall the tube domain realization of $\mathcal{D}$ associated to a 0-dimensional cusp. We refer the reader to [21, 33, 35] for some more details. This section is preliminaries for the Fourier expansion (Section 3.4).

We choose a rank 1 primitive isotropic sublattice I of L, which is fixed throughout Sections 3.3–3.5. Recall that this corresponds to the 0-dimensional cusp $[I_{\mathbb{C}}]$ of $\mathcal{D}$. The $\mathbb{Z}$-module $(I^{\perp}/I) \otimes_{\mathbb{Z}} I$ is canonically endowed with the structure of a hyperbolic lattice, from the quadratic form on $I^{\perp}/I$ and the standard quadratic form $I \times I \to I^{\otimes 2} \simeq \mathbb{Z}$ on I in which the generators of I have norm 1. For $F = \mathbb{Q}, \mathbb{R}, \mathbb{C}$ we write

$$U(I)_F = (I^{\perp}/I)_F \otimes_F I_F = V(I)_F \otimes_F I_F.$$

This is a quadratic space over F, hyperbolic when $F = \mathbb{Q}, \mathbb{R}$.

3.3.1 Tube domain realization

The linear projection $\mathbb{P}L_{\mathbb{C}} \dashrightarrow \mathbb{P}(L/I)_{\mathbb{C}}$ from the point $[I_{\mathbb{C}}] \in Q$ defines an isomorphism

$$\mathcal{D}(I) \to \mathbb{P}(L/I)_{\mathbb{C}} - \mathbb{P}V(I). \tag{3.8}$$

We choose, as an auxiliary data, a rank 1 sublattice $I' \subset L$ such that $(I, I') \neq 0$. This determines a base point of the affine space $\mathbb{P}(L/I)_{\mathbb{C}} - \mathbb{P}V(I)$ and hence an isomorphism

$$\mathbb{P}(L/I)_{\mathbb{C}} - \mathbb{P}V(I) \to V(I) \otimes_{\mathbb{C}} I_{\mathbb{C}} = U(I)_{\mathbb{C}}. \tag{3.9}$$

Since the quadratic form on $U(I)_{\mathbb{R}}$ is hyperbolic, the set of vectors $v \in U(I)_{\mathbb{R}}$ with $(v, v) > 0$ consists of two connected components. The choice of the component $\mathcal{D}$ determines one of them, which we denote by $\mathcal{C}_I$ (the positive cone). Let

$$\mathcal{D}_I = \{Z \in U(I)_{\mathbb{C}} \mid \mathrm{Im}(Z) \in \mathcal{C}_I\}$$

be the tube domain associated to $\mathcal{C}_I$. Then the composition of (3.8) and (3.9) gives an isomorphism

$$\mathcal{D} \xrightarrow{\simeq} \mathcal{D}_I \subset U(I)_{\mathbb{C}}. \tag{3.10}$$

This is the tube domain realization of $\mathcal{D}$ associated to I. If we change I', this isomorphism is shifted by the translation by a vector of $U(I)_{\mathbb{Q}}$.

3.3.2 Stabilizer

Next we recall the structure of the stabilizer of the I-cusp. Let $F = \mathbb{Q}, \mathbb{R}$. We denote by $\Gamma(I)_F$ the stabilizer of I in $\mathrm{O}^+(L_F)$ (not the stabilizer of I_F). Elements of $\Gamma(I)_F$ act on $U(I)_F$ as isometries. Let $\mathrm{O}^+(U(I)_F)$ be the subgroup of $\mathrm{O}(U(I)_F)$ preserving the positive cone $\mathcal{C}_I$. By (1.2), $\Gamma(I)_F$ sits in the canonical exact sequence

$$0 \to U(I)_F \to \Gamma(I)_F \to \mathrm{O}^+(U(I)_F) \times \mathrm{GL}(I) \to 1. \tag{3.11}$$

Here the subgroup $U(I)_F$ consists of the Eichler transvections of L_F with respect to the isotropic line I_F. The adjoint action of $\Gamma(I)_F$ on $U(I)_F$ via (3.11) coincides with the natural action of $\Gamma(I)_F$ on $(I^\perp/I)_F \otimes I_F$.

The choice of I' determines the lift $V(I)_F \simeq (I_F \oplus I'_F)^\perp$ of $V(I)_F$ in $I_F^\perp$, and thus a splitting $L_F \simeq U_F \oplus V(I)_F$. This determines a section of (3.11)

$$\mathrm{O}^+(U(I)_F) \times \mathrm{GL}(I) \hookrightarrow \Gamma(I)_F,$$

by letting $\mathrm{O}^+(U(I)_F) \simeq \mathrm{O}^+(V(I)_F)$ act on the lifted component $V(I)_F \subset L_F$ and mapping $\mathrm{GL}(I) = \{\pm 1\}$ to $\{\pm \mathrm{id}\}$. In this way, from the choice of I', we obtain a splitting of (3.11):

$$\Gamma(I)_F \simeq (\mathrm{O}^+(U(I)_F) \times \mathrm{GL}(I)) \ltimes U(I)_F, \tag{3.12}$$

where $\mathrm{O}^+(U(I)_F)$ acts on $U(I)_F$ in the natural way and $\mathrm{GL}(I)$ acts on $U(I)_F$ trivially. This splitting is compatible with the tube domain realization in the following sense. We translate the $\Gamma(I)_F$-action on $\mathcal{D}$ to action of $\Gamma(I)_F$ on $\mathcal{D}_I$ via the tube domain realization (3.10) defined by (the same) I'. Then,

- the unipotent radical $U(I)_F \subset \Gamma(I)_F$ acts on $\mathcal{D}_I$ as the translation by $U(I)_F$ on $U(I)_{\mathbb{C}}$,
- the lifted group $\mathrm{O}^+(U(I)_F)$ in (3.12) acts on $\mathcal{D}_I$ by its linear action on $U(I)_{\mathbb{C}}$,
- the lifted group $\mathrm{GL}(I) = \{\pm \mathrm{id}\}$ acts trivially.

Now let Γ be a finite-index subgroup of $\mathrm{O}^+(L)$. We write

$$\Gamma(I)_{\mathbb{Z}} = \Gamma(I)_{\mathbb{Q}} \cap \Gamma, \quad U(I)_{\mathbb{Z}} = U(I)_{\mathbb{Q}} \cap \Gamma, \quad \overline{\Gamma(I)}_{\mathbb{Z}} = \Gamma(I)_{\mathbb{Z}}/U(I)_{\mathbb{Z}}.$$

The group $\Gamma(I)_{\mathbb{Z}}$ is the stabilizer of I in Γ. The exact sequence

$$0 \to U(I)_{\mathbb{Z}} \to \Gamma(I)_{\mathbb{Z}} \to \overline{\Gamma(I)}_{\mathbb{Z}} \to 1 \tag{3.13}$$

is naturally embedded in (3.11). The group $U(I)_{\mathbb{Z}}$ is a full lattice in $U(I)_{\mathbb{Q}}$. It defines the algebraic torus

$$T(I) = U(I)_{\mathbb{C}}/U(I)_{\mathbb{Z}}.$$

Then the tube domain realization (3.10) induces an isomorphism

$$\mathcal{D}/U(I)_{\mathbb{Z}} \xrightarrow{\simeq} \mathcal{D}_I/U(I)_{\mathbb{Z}} \subset T(I).$$

The group $\overline{\Gamma(I)_{\mathbb{Z}}}$ acts on $\mathcal{D}/U(I)_{\mathbb{Z}} \simeq \mathcal{D}_I/U(I)_{\mathbb{Z}}$. Let $\overline{\gamma} \in \overline{\Gamma(I)_{\mathbb{Z}}}$ and let $\gamma \in \Gamma(I)_{\mathbb{Z}}$ be its lift. According to the splitting (3.12), we express γ as

$$\gamma = (\gamma_1, \varepsilon, \alpha), \quad \gamma_1 \in \mathrm{O}^+(U(I)_{\mathbb{Z}}),\ \varepsilon = \pm\,\mathrm{id},\ \alpha \in U(I)_{\mathbb{Q}}. \tag{3.14}$$

Here γ_1, a priori an element of $\mathrm{O}^+(U(I)_{\mathbb{Q}})$, is contained in $\mathrm{O}^+(U(I)_{\mathbb{Z}})$ because the adjoint action of $\Gamma(I)_{\mathbb{Z}}$ on $U(I)_{\mathbb{Q}}$ preserves the lattice $U(I)_{\mathbb{Z}}$. Then the action of $\overline{\gamma}$ on $\mathcal{D}_I/U(I)_{\mathbb{Z}}$ is given by the linear action by γ_1 plus the translation by $[\alpha] \in U(I)_{\mathbb{Q}}/U(I)_{\mathbb{Z}}$. Note that $\overline{\gamma}$ is determined by (γ_1, ε) because the projection

$$\overline{\Gamma(I)_{\mathbb{Z}}} \to \mathrm{O}^+(U(I)_{\mathbb{Q}}) \times \mathrm{GL}(I)$$

is injective. Nevertheless, the translation component $[\alpha]$ could be nontrivial because (3.13) may not necessarily split.

3.4 Fourier expansion

Let I and I' be as in Section 3.3. Let f be a modular form of weight (λ, k) on $\mathcal{D}$ with respect to a finite-index subgroup Γ of $\mathrm{O}^+(L)$. By the I-trivialization $\mathcal{E}_{\lambda,k} \simeq V(I)_{\lambda,k} \otimes \mathcal{O}_{\mathcal{D}}$ and the tube domain realization $\mathcal{D} \simeq \mathcal{D}_I$, we regard f as a $V(I)_{\lambda,k}$-valued holomorphic function on the tube domain $\mathcal{D}_I$ (again denoted by f). The subgroup $U(I)_{\mathbb{Z}}$ of $\Gamma(I)_{\mathbb{Z}}$ acts on $\mathcal{D}_I$ by translation, and acts on $V(I)_{\lambda,k}$ trivially. By Lemma 3.2, this shows that the function f is invariant under the translation by the lattice $U(I)_{\mathbb{Z}}$. Therefore it admits a Fourier expansion of the form

$$f(Z) = \sum_{l \in U(I)_{\mathbb{Z}}^{\vee}} a(l) q^l, \quad q^l = e((l, Z)),$$

for $Z \in \mathcal{D}_I$, where $a(l) \in V(I)_{\lambda,k}$ and $U(I)_{\mathbb{Z}}^{\vee} \subset U(I)_{\mathbb{Q}}$ is the dual lattice of $U(I)_{\mathbb{Z}}$. This series is convergent when $\mathrm{Im}(Z)$ is sufficiently large. The Fourier coefficients $a(l)$ can be expressed as

$$a(l) = \int_{U(I)_{\mathbb{R}}/U(I)_{\mathbb{Z}}} f(Z_0 + v) e(-(Z_0 + v, l)) dv, \tag{3.15}$$

where dv is the flat volume form on $U(I)_{\mathbb{R}}$ normalized so that $U(I)_{\mathbb{R}}/U(I)_{\mathbb{Z}}$ has volume 1.

The Koecher principle says that we have $a(l) \neq 0$ only when l is in the closure of the positive cone $\mathcal{C}_I$, which is the dual cone of $\mathcal{C}_I$. See, e.g., [44, p. 191] for a proof of the Koecher principle in the vector-valued Siegel modular case. The present case can be proved similarly by using (3.15) and Proposition 3.6 below. See also [8, Proposition 4.15] for the scalar-valued case. In general, when $n \leq 2$, the condition $a(l) \neq 0 \Rightarrow l \in \overline{\mathcal{C}_I}$ is the holomorphicity condition required around the I-cusp.

The modular form f is called a *cusp form* if $a(l) \neq 0$ only when $l \in \mathcal{C}_I$ at every 0-dimensional cusp I. We denote by

$$S_{\lambda,k}(\Gamma) \subset M_{\lambda,k}(\Gamma)$$

the subspace of cusp forms.

It should be noted that the Fourier expansion depends on the choice of I'. If we change I', the tube domain realization is shifted by the translation by a vector of $U(I)_{\mathbb{Q}}$, say v_0. Then we need to replace $f(Z)$ by $f(Z+v_0)$, and the Fourier coefficient $a(l)$ is replaced by $e((l,v_0)) \cdot a(l)$. In what follows, when we speak of Fourier expansion at the I-cusp, the choice of I' (and hence of the tube domain realization $\mathcal{D} \to \mathcal{D}_I$) is subsumed.

The Fourier coefficients satisfy the following symmetry under $\overline{\Gamma(I)_{\mathbb{Z}}}$.

Proposition 3.6. *Let $\overline{\gamma} \in \overline{\Gamma(I)_{\mathbb{Z}}}$. Let $\gamma = (\gamma_1, \varepsilon, \alpha)$ be its lift in $\Gamma(I)_{\mathbb{Z}}$ expressed as in* (3.14)*. Then we have*

$$a(\gamma_1 l) = e(-(\gamma_1 l, \alpha)) \cdot \gamma(a(l)) \tag{3.16}$$

for every $l \in U(I)_{\mathbb{Z}}^{\vee}$.

Proof. By Lemma 3.2, the factor of automorphy for γ is given by its natural action on $V(I)_{\lambda,k}$. Therefore we have

$$f(\gamma(Z)) = \gamma(f(Z)), \quad Z \in \mathcal{D}_I,$$

where γ acts on $\mathcal{D}_I$ via the tube domain realization $\mathcal{D} \simeq \mathcal{D}_I$. We compute the Fourier expansion of both sides. Since $\gamma(Z) = \gamma_1 Z + \alpha$, we have

$$\begin{aligned} f(\gamma(Z)) &= \sum_l a(l) e((l, \gamma_1 Z + \alpha)) \\ &= \sum_l a(l) e((l,\alpha)) e((\gamma_1^{-1} l, Z)) \\ &= \sum_l a(\gamma_1 l) e((\gamma_1 l, \alpha)) e((l, Z)). \end{aligned}$$

In the last equality we rewrote l as $\gamma_1 l$. Comparing this with

$$\gamma(f(Z)) = \sum_l \gamma(a(l)) e((l, Z)),$$

we obtain $\gamma(a(l)) = e((\gamma_1 l, \alpha)) a(\gamma_1 l)$. ■

In the right-hand side of (3.16), the action of γ on $a(l) \in V(I)_{\lambda,k}$ is determined by (γ_1, ε). More precisely, γ acts on $I_{\mathbb{C}}$ by $\varepsilon \in \{\pm 1\}$, and on $V(I) = U(I)_{\mathbb{C}} \otimes I_{\mathbb{C}}^{\vee}$ by $\gamma_1 \otimes \varepsilon$.

Proposition 3.6 implies the vanishing of the constant term $a(0)$ in most cases.

Proposition 3.7. *Assume that* $\lambda \neq 1, \det$. *Then* $a(0) = 0$.

Proof. We apply Proposition 3.6 to $l = 0$ and elements $\overline{\gamma}$ in the subgroup

$$\{\overline{\gamma} \in \overline{\Gamma(I)_{\mathbb{Z}}} \mid \varepsilon = 1, \det \gamma_1 = 1\} \tag{3.17}$$

of $\overline{\Gamma(I)_{\mathbb{Z}}}$. By trivializing $I \simeq \mathbb{Z}$, we identify $V(I)_{\lambda,k}$ with $V(I)_{\lambda}$. We also identify $\mathrm{SO}(U(I)_{\mathbb{Q}})$ with $\mathrm{SO}(V(I)_{\mathbb{Q}})$ naturally. Then elements $\overline{\gamma}$ of the group (3.17) act on $V(I)_{\lambda,k}$ by the action of $\gamma_1 \in \mathrm{SO}(V(I)_{\mathbb{Q}})$ on $V(I)_{\lambda}$. Therefore, by Proposition 3.6, we find that $a(0) = \gamma_1(a(0)) \in V(I)_{\lambda}$ for every such $\overline{\gamma}$. The mapping $\overline{\gamma} \mapsto \gamma_1$ embeds the group (3.17) into $\mathrm{SO}(V(I)_{\mathbb{Q}})$, and the image is an arithmetic subgroup of $\mathrm{SO}(V(I)_{\mathbb{Q}})$. By the density theorem of Borel [7] (see also [41, Corollary 5.15]), it is Zariski dense in $\mathrm{SO}(V(I))$. Therefore the vector $a(0) \in V(I)_{\lambda}$ is invariant under the action of $\mathrm{SO}(V(I))$ on $V(I)_{\lambda}$. However, by our assumption $\lambda \neq 1, \det$, the $\mathrm{SO}(n, \mathbb{C})$-representation V_{λ} contains no nonzero invariant vector (cf. Section 3.6). Therefore $a(0) = 0$. ∎

Remark 3.8. Since $V(I)_{\mathbb{C}}$ and $I_{\mathbb{C}}$ have the natural $\mathbb{Q}$-structures $V(I)_{\mathbb{Q}}$ and $I_{\mathbb{Q}}$, respectively, the $\mathbb{C}$-linear space $V(I)_{\lambda,k}$ has the natural $\mathbb{Q}$-structure

$$V(I)_{\mathbb{Q},\lambda} \otimes (I_{\mathbb{Q}}^{\vee})^{\otimes k},$$

where $V(I)_{\mathbb{Q},\lambda} = c_{\lambda} \cdot V(I)_{\mathbb{Q}}^{[d]}$ is the $\mathbb{Q}$-representation of $\mathrm{O}(V(I)_{\mathbb{Q}})$ obtained by applying the orthogonal Schur functor to $V(I)_{\mathbb{Q}}$. Thus we can speak of rationality and algebraicity of the Fourier coefficients $a(l)$. (Rationality depends on the choice of I', but algebraicity does not because the transition constant $e((l, v_0))$ is a root of unity.) When the homomorphism φ_I in (3.5) is defined over $\mathbb{Q}$, the φ-product of two modular forms with rational Fourier coefficients at the I-cusp again has rational Fourier coefficients by (3.6).

3.5 Geometry of Fourier expansion

Let I and I' be as in Sections 3.3 and 3.4. In this section we recall the partial toroidal compactifications of $\mathcal{D}/U(I)_{\mathbb{Z}}$ following [2] and explain the Fourier expansion from that point of view.

3.5.1 Partial toroidal compactification

We write $\mathcal{X}(I) = \mathcal{D}/U(I)_{\mathbb{Z}}$. The tube domain realization identifies $\mathcal{X}(I)$ with the open set $\mathcal{D}_I/U(I)_{\mathbb{Z}}$ of the torus $T(I)$. Let

$$\mathcal{C}_I^+ = \mathcal{C}_I \cup \bigcup_v \mathbb{R}_{\geq 0} v$$

be the union of the positive cone $\mathcal{C}_I$ and the rays $\mathbb{R}_{\geq 0} v$ generated by rational isotropic vectors v in $\overline{\mathcal{C}_I}$. Let $\Sigma_I = (\sigma_\alpha)$ be a rational polyhedral cone decomposition of $\mathcal{C}_I^+$, namely, a fan in $U(I)_{\mathbb{R}}$ whose support is $\mathcal{C}_I^+$. Note that every rational isotropic ray in $\mathcal{C}_I^+$ must be included in Σ_I. We will often abbreviate $\Sigma_I = \Sigma$ when I is clear from the context. The fan Σ is said to be $\Gamma(I)_{\mathbb{Z}}$*-admissible* if it is preserved by the $\Gamma(I)_{\mathbb{Z}}$-action on $U(I)_{\mathbb{R}}$ and there are only finitely many cones up to the $\Gamma(I)_{\mathbb{Z}}$-action. The fan Σ is called *regular* if each cone σ_α is generated by a part of a $\mathbb{Z}$-basis of $U(I)_{\mathbb{Z}}$. It is possible to choose Σ to be $\Gamma(I)_{\mathbb{Z}}$-admissible and regular [2, 14].

Let Σ be a $\Gamma(I)_{\mathbb{Z}}$-admissible fan. It determines a $\overline{\Gamma(I)_{\mathbb{Z}}}$-equivariant torus embedding $T(I) \hookrightarrow T(I)^\Sigma$. The toric variety $T(I)^\Sigma$ is normal; it is nonsingular if Σ is regular. The cones σ in Σ correspond to the boundary strata of $T(I)^\Sigma$, say Δ_σ. A stratum Δ_σ is in the closure of another stratum Δ_τ if and only if τ is a face of σ. The stratum Δ_σ is isomorphic to the quotient torus of $T(I)$ defined by the quotient lattice $U(I)_{\mathbb{Z}}/(U(I)_{\mathbb{Z}} \cap \langle \sigma \rangle)$, where $\langle \sigma \rangle$ is the $\mathbb{R}$-span of σ. In particular, the rays $\mathbb{R}_{\geq 0} v$ in Σ correspond to the boundary strata of codimension 1, say Δ_v. If we take v to be a primitive vector of $U(I)_{\mathbb{Z}}$, the stratum Δ_v is isomorphic to the quotient torus of $T(I)$ defined by $U(I)_{\mathbb{Z}}/\mathbb{Z}v$. The variety $T(I)^\Sigma$ is nonsingular along Δ_v. If we take a vector $l \in U(I)_{\mathbb{Z}}^\vee$ with $(v, l) = 1$, then $q^l = e((l, Z))$ is a character of $T(I)$ and extends holomorphically over Δ_v. The divisor Δ_v is defined by $q^l = 0$. More generally, a character q^l of $T(I)$, where $l \in U(I)_{\mathbb{Z}}^\vee$ extends holomorphically around a boundary stratum Δ_σ (i.e., extends over Δ_σ and the strata Δ_τ which contains Δ_σ in its closure) if and only if $(l, \sigma) \geq 0$, or in other words, l is in the dual cone of σ. If moreover l has positive pairing with the relative interior of σ, the extended function vanishes identically at Δ_σ.

Now let $\mathcal{X}(I)^\Sigma$ be the interior of the closure of $\mathcal{X}(I)$ in $T(I)^\Sigma$. We call $\mathcal{X}(I)^\Sigma$ the *partial toroidal compactification* of $\mathcal{X}(I)$ defined by the fan Σ. As a partial compactification of $\mathcal{X}(I) = \mathcal{D}/U(I)_{\mathbb{Z}}$, this does not depend on the choice of I'. When a cone $\sigma \in \Sigma$ is not an isotropic ray, its relative interior is contained in $\mathcal{C}_I$, and the corresponding boundary stratum Δ_σ of $T(I)^\Sigma$ is totally contained in $\mathcal{X}(I)^\Sigma$. On the other hand, when $\sigma = \mathbb{R}_{\geq 0} v$ is an isotropic ray, only an open subset of Δ_v is contained in $\mathcal{X}(I)^\Sigma$. (This will be glued with the boundary of the partial toroidal compactification over the corresponding 1-dimensional cusp: see Section 5.3.) By abuse of notation, we still write Δ_v for the boundary stratum in $\mathcal{X}(I)^\Sigma$ in this case.

3.5.2 Fourier expansion and Taylor expansion

Let $f(Z) = \sum_l a(l)q^l$ be the Fourier expansion of a Γ-modular form of weight (λ, k) at the I-cusp. This can be viewed as the expansion of the $V(I)_{\lambda,k}$-valued function f on $\mathcal{X}(I)$ by the characters of $T(I)$.

Lemma 3.9. *The function f on $\mathcal{X}(I)$ extends holomorphically over $\mathcal{X}(I)^{\Sigma}$. When $\lambda \neq 1, \det$ and σ is not an isotropic ray, f vanishes at the corresponding boundary stratum Δ_σ. When f is a cusp form, it vanishes at every boundary stratum Δ_σ.*

Proof. Since the dual cone of $\overline{\mathcal{C}_I}$ is $\overline{\mathcal{C}_I}$ itself, $\overline{\mathcal{C}_I}$ is contained in the dual cone of every cone σ in Σ. Therefore, if $l \in U(I)_{\mathbb{Z}}^{\vee} \cap \overline{\mathcal{C}_I}$, then l is contained in the dual cone of every σ, which implies that the function q^l extends holomorphically over $\mathcal{X}(I)^{\Sigma}$. By the cusp condition in the Fourier expansion, this shows that the function f extends holomorphically over $\mathcal{X}(I)^{\Sigma}$.

When σ is not an isotropic ray, its relative interior is contained in $\mathcal{C}_I$. Hence any nonzero vector $l \in U(I)_{\mathbb{Z}}^{\vee} \cap \overline{\mathcal{C}_I}$ has positive pairing with the relative interior of σ. This shows that the corresponding character q^l vanishes at the boundary stratum Δ_σ. It follows that $f|_{\Delta_\sigma}$ is the constant $a(0)$. By Proposition 3.7, this vanishes when $\lambda \neq 1, \det$.

Finally, if f is a cusp form, we have $a(l) \neq 0$ only when $l \in \mathcal{C}_I$. Such a vector l has positive pairing with the relative interior of every cone $\sigma \in \Sigma$, and so, q^l vanishes at Δ_σ. Therefore f vanishes at the boundary of $\mathcal{X}(I)^{\Sigma}$. ■

Let us explain that the Fourier expansion gives Taylor expansion along each boundary divisor. Let $\sigma = \mathbb{R}_{\geq 0} v$ be a ray in Σ with $v \in U(I)_{\mathbb{Z}}$ primitive. We can rewrite the Fourier expansion of f as

$$f(Z) = \sum_{m \geq 0} \sum_{\substack{l \in U(I)_{\mathbb{Z}}^{\vee} \\ (l,v)=m}} a(l)q^l. \tag{3.18}$$

We choose a vector $l_0 \in U(I)_{\mathbb{Z}}^{\vee}$ with $(l_0, v) = 1$ and put $q_0 = q^{l_0}$. The boundary divisor Δ_v is defined by $q_0 = 0$. We put

$$\phi_m = \sum_{\substack{l \in U(I)_{\mathbb{Z}}^{\vee} \\ (l,v)=m}} a(l)q^{l-ml_0} = \sum_{l \in v^{\perp} \cap U(I)_{\mathbb{Z}}^{\vee}} a(l + ml_0)q^l.$$

Note that $v^{\perp} \cap U(I)_{\mathbb{Z}}^{\vee}$ is the dual lattice of $U(I)_{\mathbb{Z}}/\mathbb{Z}v$ and hence is the character lattice of the quotient torus Δ_v. Therefore ϕ_m is (the pullback of) a $V(I)_{\lambda,k}$-valued function on Δ_v. Then (3.18) can be rewritten as

$$f(Z) = \sum_{m \geq 0} \phi_m q_0^m.$$

This is the Taylor expansion of f along the divisor Δ_v with normal parameter q_0, and ϕ_m (as a function on Δ_v) is the m-th Taylor coefficient. In particular, the restriction of f to Δ_v is given by ϕ_0:

$$f|_{\Delta_v} = \phi_0 = \sum_{l \in v^{\perp} \cap U(I)_{\mathbb{Z}}^{\vee}} a(l) q^l.$$

When $(v, v) \neq 0$, this reduces to $a(0)$ because $v^{\perp} \cap \overline{\mathcal{C}_I} = \{0\}$ holds (cf. the proof of Lemma 3.9). On the other hand, when $(v, v) = 0$, this reduces to

$$f|_{\Delta_v} = \sum_{l \in \mathbb{Q}v \cap U(I)_{\mathbb{Z}}^{\vee}} a(l) q^l \tag{3.19}$$

because $v^{\perp} \cap \overline{\mathcal{C}_I} = \mathbb{R}_{\geq 0} v$.

Remark 3.10. Sometimes it is useful to allow l_0 from an overlattice of $U(I)_{\mathbb{Z}}^{\vee}$, e.g., when considering the Fourier–Jacobi expansion (Section 7). Then q_0 and ϕ_m are still defined, as functions on a finite cover of $T(I)$.

3.5.3 Canonical extension

In Sections 3.4 and 3.5, we regarded modular forms as $V(I)_{\lambda,k}$-valued functions via the I-trivialization. Let us go back to the viewpoint of sections of $\mathcal{E}_{\lambda,k}$. The vector bundle $\mathcal{E}_{\lambda,k}$ on $\mathcal{D}$ descends to a vector bundle on $\mathcal{X}(I) = \mathcal{D}/U(I)_{\mathbb{Z}}$, which we again denote by $\mathcal{E}_{\lambda,k}$. We extend it over $\mathcal{X}(I)^{\Sigma}$ as follows.

Since the I-trivialization $\mathcal{E}_{\lambda,k} \simeq V(I)_{\lambda,k} \otimes \mathcal{O}_{\mathcal{D}}$ is equivariant with respect to $U(I)_{\mathbb{Z}}$ which acts on $V(I)_{\lambda,k}$ trivially, it descends to an isomorphism

$$\mathcal{E}_{\lambda,k} \simeq V(I)_{\lambda,k} \otimes \mathcal{O}_{\mathcal{X}(I)}$$

over $\mathcal{X}(I)$. Then we can extend $\mathcal{E}_{\lambda,k}$ to a vector bundle over $\mathcal{X}(I)^{\Sigma}$ (still denoted by $\mathcal{E}_{\lambda,k}$) so that this isomorphism extends to $\mathcal{E}_{\lambda,k} \simeq V(I)_{\lambda,k} \otimes \mathcal{O}_{\mathcal{X}(I)^{\Sigma}}$ over $\mathcal{X}(I)^{\Sigma}$. In other words, the extension is defined so that the frame of $\mathcal{E}_{\lambda,k}$ over $\mathcal{X}(I)$ corresponding to a basis of $V(I)_{\lambda,k}$ by the I-trivialization extends to a frame of the extended bundle $\mathcal{E}_{\lambda,k}$. This is an explicit form of Mumford's canonical extension [36]. By construction, a section f of $\mathcal{E}_{\lambda,k}$ over $\mathcal{X}(I)$ extends to a holomorphic section of the extended bundle $\mathcal{E}_{\lambda,k}$ over $\mathcal{X}(I)^{\Sigma}$ if and only if f viewed as a $V(I)_{\lambda,k}$-valued function via the I-trivialization extends holomorphically over $\mathcal{X}(I)^{\Sigma}$. Then Lemma 3.9 can be restated as follows.

Lemma 3.11. *A modular form $f \in M_{\lambda,k}(\Gamma)$ as a section of $\mathcal{E}_{\lambda,k}$ over $\mathcal{X}(I)$ extends to a holomorphic section of the extended bundle $\mathcal{E}_{\lambda,k}$ over $\mathcal{X}(I)^{\Sigma}$. When $\lambda \neq 1, \det$ and σ is not an isotropic ray, this extended section vanishes at Δ_{σ}. When f is a cusp form, this section vanishes at every Δ_{σ}.*

3.6 Special orthogonal groups

In the theory of orthogonal modular forms, there is an option at the outset: which Lie group to mainly work with. The full orthogonal group O, or the special orthogonal group SO, or the spin group Spin, or even the pin group Pin. We decided to start with O for two reasons: (1) in some applications we need to consider subgroups Γ of $\mathrm{O}^+(L)$ not contained in $\mathrm{SO}^+(L)$, and (2) the explicit construction by the orthogonal Schur functor for $\mathcal{E}$ will be useful at some points.

On the other hand, it is sometimes more convenient to work with SO. In this section we explain the switch from O to SO. The contents of this section will be used only in Sections 6.1, 10 and 11, so the reader may skip it for the moment.

3.6.1 Representations of $\mathrm{SO}(n, \mathbb{C})$

We first recall some basic facts from the representation theory of $\mathrm{SO}(n, \mathbb{C})$ following [38, Sections 4 and 8] and [18, Section 19]. Irreducible representations of $\mathrm{SO}(n, \mathbb{C})$ are labelled by their highest weights. When $n = 2m$ is even, the highest weights are expressed by m-tuples $\rho = (\rho_1, \ldots, \rho_m)$ of integers, nonnegative for $i < m$, such that $\rho_1 \geq \cdots \geq \rho_{m-1} \geq |\rho_m|$. We write $\rho^\dagger = (\rho_1, \ldots, \rho_{m-1}, -\rho_m)$ for such ρ. When $n = 2m + 1$ is odd, the highest weights are expressed by m-tuples $\rho = (\rho_1, \ldots, \rho_m)$ of nonnegative integers such that $\rho_1 \geq \cdots \geq \rho_m \geq 0$. We denote by W_ρ the irreducible representation of $\mathrm{SO}(n, \mathbb{C})$ with highest weight ρ. The dual representation $W_\rho^\vee$ is isomorphic to W_ρ itself when n is odd or $4|n$, while it is isomorphic to $W_{\rho^\dagger}$ in the case $n \equiv 2 \bmod 4$.

By the Weyl unitary trick, W_ρ remains irreducible as a representation of

$$\mathrm{SO}(n, \mathbb{R}) \subset \mathrm{SO}(n, \mathbb{C}),$$

and the above classification is the same as the classification of irreducible $\mathbb{C}$-representations of $\mathrm{SO}(n, \mathbb{R})$.

The restriction rule from $\mathrm{O}(n, \mathbb{C})$ to $\mathrm{SO}(n, \mathbb{C})$ is as follows [38, Proposition 8.24]. Let $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n \geq 0)$ be a partition expressing an irreducible representation V_λ of $\mathrm{O}(n, \mathbb{C})$. We define a highest weight $\overline{\lambda}$ for $\mathrm{SO}(n, \mathbb{C})$ by

$$\overline{\lambda} = (\lambda_1 - \lambda_n, \lambda_2 - \lambda_{n-1}, \ldots, \lambda_{[n/2]} - \lambda_{n+1-[n/2]}).$$

Note that $\overline{\lambda}$ itself can be viewed as a partition for $\mathrm{O}(n, \mathbb{C})$. When n is odd or $n = 2m$ is even with ${}^t\lambda_1 \neq m$, the $\mathrm{O}(n, \mathbb{C})$-representation V_λ remains irreducible as a representation of $\mathrm{SO}(n, \mathbb{C})$, with highest weight $\overline{\lambda}$. The vector defined in (3.2) is a highest weight vector. Thus $V_\lambda \simeq W_{\overline{\lambda}}$ as a representation of $\mathrm{SO}(n, \mathbb{C})$ in this case. In particular, since the highest weight for the partition $\overline{\lambda}$ is $\overline{\lambda}$ itself, we have $V_\lambda \simeq V_{\overline{\lambda}}$ as $\mathrm{SO}(n, \mathbb{C})$-representations. More specifically, when ${}^t\lambda_1 < n/2$ we have $\overline{\lambda} = \lambda$, while when ${}^t\lambda_1 > n/2$ we have $V_\lambda \simeq V_{\overline{\lambda}} \otimes \det$ as $\mathrm{O}(n,\mathbb{C})$-representations. (In the latter case, the partitions λ and $\overline{\lambda}$ are called *associated* in [18, 38].)

In the remaining case, namely, when $n = 2m$ is even and ${}^t\lambda_1 = m$, V_λ gets reducible when restricted to $\mathrm{SO}(n,\mathbb{C})$. More precisely,

$$V_\lambda \simeq W_{\overline{\lambda}} \oplus W_{\overline{\lambda}^\dagger} \tag{3.20}$$

as a representation of $\mathrm{SO}(n,\mathbb{C})$. Note that $\overline{\lambda} = \lambda$ and $\lambda_m \neq 0$ in this case. Since $\overline{\lambda} \neq \overline{\lambda}^\dagger$, this decomposition is unique. In this case, V_λ is the induced representation from the representation $W_{\overline{\lambda}}$ of $\mathrm{SO}(n,\mathbb{C}) \subset \mathrm{O}(n,\mathbb{C})$.

3.6.2 Automorphic vector bundles

We go back to the automorphic vector bundles on $\mathcal{D}$. We choose a base point $[\omega_0] \in \mathcal{D}$. Let $K \simeq \mathrm{SO}(2,\mathbb{R}) \times \mathrm{O}(n,\mathbb{R})$ and $SK \simeq \mathrm{SO}(2,\mathbb{R}) \times \mathrm{SO}(n,\mathbb{R})$ be the stabilizers of $[\omega_0]$ in $\mathrm{O}^+(L_\mathbb{R})$ and in $\mathrm{SO}^+(L_\mathbb{R})$, respectively (cf. Section 2.1).

Proposition 3.12. *The following holds.*

(1) *If either n is odd or $n = 2m$ is even with ${}^t\lambda_1 \neq m$, then $\mathcal{E}_\lambda$ remains irreducible as an $\mathrm{SO}^+(L_\mathbb{R})$-equivariant vector bundle, and we have*

$$\mathcal{E}_\lambda \simeq \mathrm{SO}^+(L_\mathbb{R}) \times_{SK} W_{\overline{\lambda}}.$$

In particular, we have $\mathcal{E}_\lambda \simeq \mathcal{E}_{\overline{\lambda}}$ as $\mathrm{SO}^+(L_\mathbb{R})$-equivariant vector bundles.

(2) *If n is even and ${}^t\lambda_1 = n/2$, then $\mathcal{E}_\lambda$ as an $\mathrm{SO}^+(L_\mathbb{R})$-vector bundle decomposes into the direct sum of two non-isomorphic vector bundles:*

$$\mathcal{E}_\lambda \simeq \mathcal{E}^+_\lambda \oplus \mathcal{E}^-_\lambda \tag{3.21}$$

with each component isomorphic to $\mathrm{SO}^+(L_\mathbb{R}) \times_{SK} W_{\overline{\lambda}}$ and $\mathrm{SO}^+(L_\mathbb{R}) \times_{SK} W_{\overline{\lambda}^\dagger}$, respectively.

Proof. By (3.3), we have $\mathcal{E}_\lambda \simeq \mathrm{O}^+(L_\mathbb{R}) \times_K V_\lambda$ as an $\mathrm{O}^+(L_\mathbb{R})$-equivariant vector bundle. Therefore

$$\mathcal{E}_\lambda \simeq \mathrm{SO}^+(L_\mathbb{R}) \times_{SK} V_\lambda$$

as an $\mathrm{SO}^+(L_\mathbb{R})$-equivariant vector bundle. Note that the representation of $\mathrm{O}(n,\mathbb{R}) \simeq \mathrm{O}(H^\perp_{\omega_0}) \subset K$ on $V_\lambda = (\omega_0^\perp/\mathbb{C}\omega_0)_\lambda \simeq (H^\perp_{\omega_0} \otimes_\mathbb{R} \mathbb{C})_\lambda$ extends to a representation of $\mathrm{O}(n,\mathbb{C}) \simeq \mathrm{O}(H^\perp_{\omega_0} \otimes_\mathbb{R} \mathbb{C})$ naturally. Then our assertions follow from the restriction rule for $\mathrm{SO}(n,\mathbb{C}) \subset \mathrm{O}(n,\mathbb{C})$. ∎

At each fiber of the vector bundle, the decomposition (3.21) is the irreducible decomposition of $(\omega^\perp/\mathbb{C}\omega)_\lambda$ as a representation of $\mathrm{SO}(\omega^\perp/\mathbb{C}\omega)$. The I-trivialization respects the decomposition (3.21) in the following sense. As a representation of

$\mathrm{SO}(V(I))$, $V(I)_\lambda$ decomposes according to (3.20), which we denote by $V(I)_\lambda = W(I)_{\bar{\lambda}} \oplus W(I)_{\bar{\lambda}^\dagger}$. By the uniqueness of the decomposition (3.20), the I-trivialization $\mathcal{E}_\lambda \simeq V(I)_\lambda \otimes \mathcal{O}_{\mathcal{D}}$ sends the decomposition (3.21) of $\mathcal{E}_\lambda$ to the decomposition

$$V(I)_\lambda \otimes \mathcal{O}_{\mathcal{D}} = (W(I)_{\bar{\lambda}} \otimes \mathcal{O}_{\mathcal{D}}) \oplus (W(I)_{\bar{\lambda}^\dagger} \otimes \mathcal{O}_{\mathcal{D}})$$

of $V(I)_\lambda \otimes \mathcal{O}_{\mathcal{D}}$. Thus we have the I-trivializations

$$\mathcal{E}_\lambda^+ \simeq W(I)_{\bar{\lambda}} \otimes \mathcal{O}_{\mathcal{D}}, \quad \mathcal{E}_\lambda^- \simeq W(I)_{\bar{\lambda}^\dagger} \otimes \mathcal{O}_{\mathcal{D}} \tag{3.22}$$

of each component $\mathcal{E}_\lambda^+$, $\mathcal{E}_\lambda^-$.

3.7 Rankin–Cohen brackets

In this section, as an example of explicit construction of vector-valued modular forms, we define the Rankin–Cohen bracket of two scalar-valued modular forms. This is a general method: see, e.g., [9, 16, 17, 26, 42] for the case of some other types of modular forms, where Rankin–Cohen bracket is a successful technique for explicitly describing some modules of vector-valued modular forms.

Let f, g be nonzero scalar-valued modular forms of weight k, l, respectively, for $\Gamma < \mathrm{O}^+(L)$. We define the Rankin–Cohen bracket of f and g by

$$\{f, g\} = (g^{k+1}/f^{l-1}) \otimes d(f^l/g^k).$$

Here g^{k+1}/f^{l-1} is a meromorphic section of

$$\mathcal{L}^{\otimes l(k+1)-k(l-1)} = \mathcal{L}^{\otimes k+l},$$

and $d(f^l/g^k)$ is the exterior differential of the meromorphic function f^l/g^k on $\mathcal{D}$. Thus $d(f^l/g^k)$ is a meromorphic 1-form on $\mathcal{D}$. It is immediate to see that $\{g, f\} = -\{f, g\}$. When $k = l$, the Rankin–Cohen bracket reduces to the more simple expression

$$\begin{aligned}\{f, g\} &= (g^{k+1}/f^{k-1}) \otimes k(f/g)^{k-1} \cdot d(f/g) \\ &= kg^2 \otimes d(f/g).\end{aligned}$$

Proposition 3.13. *The Rankin–Cohen bracket $\{f, g\}$ is a modular form of weight $(\mathrm{St}, k + l + 1)$ for Γ. We have $\{f, g\} \neq 0$ unless when f^l is a constant multiple of g^k.*

Proof. Since g^{k+1}/f^{l-1} and $d(f^l/g^k)$ are meromorphic sections of $\mathcal{L}^{\otimes k+l}$ and $\Omega^1_{\mathcal{D}} \simeq \mathcal{E} \otimes \mathcal{L}$, respectively, $\{f, g\}$ is a meromorphic section of $\mathcal{E} \otimes \mathcal{L}^{\otimes k+l+1}$, i.e., has weight $(\mathrm{St}, k + l + 1)$. The Γ-invariance is obvious from the definition. It remains to check the holomorphicity over $\mathcal{D}$. We take a frame s of $\mathcal{L}$ and write $f = \tilde{f} s^{\otimes k}$, $g = \tilde{g} s^{\otimes l}$ with $\tilde{f}$, $\tilde{g}$ holomorphic functions on $\mathcal{D}$. Then

$$\begin{aligned}\{f, g\} &= (\tilde{g}^{k+1}/\tilde{f}^{l-1}) s^{\otimes k+l} \otimes d(\tilde{f}^l/\tilde{g}^k) \\ &= s^{\otimes k+l} \otimes (l(d\tilde{f})\tilde{g} - k(d\tilde{g})\tilde{f}).\end{aligned}$$

From this expression, we find that $\{f, g\}$ is holomorphic. The nonvanishing assertion is apparent. ∎

When $f = 0$ or $g = 0$, we simply set $\{f, g\} = 0$. Then the Rankin–Cohen bracket defines a bilinear map

$$M_k(\Gamma) \times M_l(\Gamma) \to M_{\mathrm{St},k+l+1}(\Gamma).$$

When $k = l$, this induces $\wedge^2 M_k(\Gamma) \to M_{\mathrm{St},2k+1}(\Gamma)$ by the anti-commutativity.

3.8 Higher Chow cycles on *K3* surfaces

One of the geometric significance of vector-valued modular forms on $\mathcal{D}$ is the appearance of the middle graded piece of the Hodge filtration, while scalar-valued modular forms are concerned only with the last piece. Thus the connection between modular forms and geometry related to the variation of Hodge structures on $\mathcal{D}$ shows up fully. In this section we present such an example of geometric construction of vector-valued modular forms with singularities. This section is independent of the rest of the memoir.

Let $\pi\colon X \to B$ be a smooth family of *K3* surfaces. We say that $\pi\colon X \to B$ is *lattice-polarized* with period lattice L if we have a sub local system Λ_{NS} of $R^2\pi_*\mathbb{Z}$ whose fibers are primitive hyperbolic sublattices of the Néron–Severi lattices of the π-fibers X_b and the fibers of $\Lambda_T = \Lambda_{NS}^{\perp}$ are isometric to L. Let $\widetilde{B}$ be an unramified cover of B, where the local system Λ_T can be trivialized (e.g., the universal cover) and let $\widetilde{X} = X \times_B \widetilde{B}$. After choosing a base point $o \in \widetilde{B}$ and an isometry

$$(\Lambda_T)_o \simeq L,$$

we have the period map

$$\widetilde{\mathcal{P}} : \widetilde{B} \to \mathcal{D}, \quad b \mapsto [H^{2,0}(\widetilde{X}_b) \subset L_{\mathbb{C}}].$$

If Γ is a finite-index subgroup of $\mathrm{O}^+(L)$ which contains the monodromy group of Λ_T, $\widetilde{\mathcal{P}}$ descends to a holomorphic map

$$\mathcal{P} : B \to \mathcal{F}(\Gamma).$$

When B is algebraic, $\mathscr{P}$ is a morphism of algebraic varieties by Borel's extension theorem.

Let $Z = (Z_b)$ be a family of higher Chow cycles in $CH^2(X_b, 1)$. By this, we mean that

- Z is a higher Chow cycle of type $(2, 1)$ on the total space X, i.e., a codimension 2 cycle on $X \times \mathbb{A}^1$ which meets $X \times \{0\}$ and $X \times \{1\}$ properly and satisfies $Z|_{X\times\{0\}} = Z|_{X\times\{1\}}$, and
- the restriction $Z_b = Z|_{X_b}$ to each fiber X_b is well defined, i.e., without using the moving lemma, Z already intersects with $X_b \times \mathbb{A}^1$ properly and gives a higher Chow cycle on X_b.

The normal function ν_Z of Z is defined as a holomorphic section of the fibration of the generalized intermediate Jacobians $\mathscr{H}/(F^2\mathscr{H} + R^2\pi_*\mathbb{Z})$. Here $\mathscr{H} = R^2\pi_*\mathbb{C} \otimes \mathscr{O}_B$ and $(F^p\mathscr{H})_p$ is the Hodge filtration on $\mathscr{H}$. The infinitesimal invariant $\delta\nu_Z$ of ν_Z is defined as a section of the middle cohomology sheaf of the Koszul complex

$$F^2\mathscr{H} \to (F^1\mathscr{H}/F^2\mathscr{H}) \otimes \Omega_B^1 \to (\mathscr{H}/F^1\mathscr{H}) \otimes \Omega_B^2 \tag{3.23}$$

over B. See [11,45] for more details and examples.

We explain the connection with vector-valued modular forms. We first consider the case where $\widetilde{B} = B$ is an analytic open set of $\mathscr{D}$ and the period map $B \to \mathscr{D}$ coincides with the inclusion map. Then we can identify

$$F^2\mathscr{H} = \mathscr{L}|_B, \quad F^1\mathscr{H}/F^2\mathscr{H} = \mathscr{E}|_B \oplus (\Lambda_{NS} \otimes_{\mathbb{Z}} \mathscr{O}_B), \quad \mathscr{H}/F^1\mathscr{H} = \mathscr{L}^{-1}|_B.$$

The Koszul complex (3.23) is the direct sum of the complex

$$0 \to \Lambda_{NS} \otimes \Omega_B^1 \to 0$$

and the modular Koszul complex (2.6) restricted to B:

$$\mathscr{L} \to \mathscr{E} \otimes \Omega_B^1 \to \mathscr{L}^{-1} \otimes \Omega_B^2.$$

According to this decomposition, we can write $\delta\nu_Z$ as $((\delta\nu_Z)_{\mathrm{pol}}, (\delta\nu_Z)_{\mathrm{prim}})$, where $(\delta\nu_Z)_{\mathrm{pol}}$ is a section of $\Lambda_{NS} \otimes \Omega_B^1$ and $(\delta\nu_Z)_{\mathrm{prim}}$ is a section of the middle cohomology sheaf of the modular Koszul complex over B. By the calculation in Example 2.4, we see that

$$(\delta\nu_Z)_{\mathrm{prim}} \in H^0(B, \mathscr{E}_{(2)} \otimes \mathscr{L}),$$

namely, $(\delta\nu_Z)_{\mathrm{prim}}$ is a local modular form of weight $(\lambda, k) = ((2), 1)$ over B.

Now we consider the case where the family $\pi\colon X \to B$ is algebraic, $-\mathrm{id} \notin \Gamma$, and the algebraic period map $\mathscr{P}\colon B \to \mathscr{F}(\Gamma)$ is birational. By removing some divisors from B if necessary, we may assume that $\mathscr{P}$ is an open immersion and $\mathscr{D} \to \mathscr{F}(\Gamma)$ is unramified over $B \subset \mathscr{F}(\Gamma)$. Then we may take $\widetilde{B}$ to be a Γ-invariant Zariski open set of $\mathcal{D}$. In this case, the Koszul complex (3.23) over B is the direct sum of $0 \to \Lambda_{NS} \otimes \Omega^1_B \to 0$ and the descent of the modular Koszul complex (2.6) from $\widetilde{B} \subset \mathcal{D}$ to $B \subset \mathcal{F}(\Gamma)$. Let Z be a family of higher Chow cycles on $X \to B$ as above. According to the decomposition of the Koszul complex over B, we can write

$$\delta\nu_Z = ((\delta\nu_Z)_{\mathrm{pol}}, (\delta\nu_Z)_{\mathrm{prim}})$$

as in the local case. Then the pullback of the primitive part $(\delta\nu_Z)_{\mathrm{prim}}$ to $\widetilde{B}$ is a Γ-invariant holomorphic section of $\mathcal{E}_{(2)} \otimes \mathcal{L}$ over $\widetilde{B}$. By a vanishing theorem proved later (Theorem 9.1), there is no nonzero holomorphic modular form of weight $((2), 1)$ on $\mathcal{D}$. Hence, if $(\delta\nu_Z)_{\mathrm{prim}}$ does not vanish identically, it must have a singularity at some component of the complement of $\widetilde{B}$ in $\mathcal{D}$. In other words, the primitive part $(\delta\nu_Z)_{\mathrm{prim}}$ of the infinitesimal invariant $\delta\nu_Z$ of Z is a modular form of weight $((2), 1)$ with singularities.