Chapter 4

Witt operators

In this chapter, as a functorial aspect of the theory, we study pullback of vector-valued modular forms to sub orthogonal modular varieties, an operation sometimes called the *Witt operator*. Let L be a lattice of signature $(2, n)$ and L' be a primitive sublattice of L of signature $(2, n')$. We put $K = (L')^{\perp} \cap L$ and $r = \mathrm{rank}(K) = n - n'$. If we write $\mathcal{D}' = \mathcal{D}_{L'}$, then $\mathcal{D}' = \mathbb{P}L'_{\mathbb{C}} \cap \mathcal{D}$. Let f be a vector-valued modular form on $\mathcal{D}$. In Section 4.1 we study the restriction of f to $\mathcal{D}'$. This produces a vector-valued modular form on $\mathcal{D}'$, whose weight (in general reducible) can be known from the branching rule for $\mathrm{O}(n', \mathbb{C}) \subset \mathrm{O}(n, \mathbb{C})$. An immediate consequence is the vanishing of $M_{\lambda,k}(\Gamma)$ in $k \leq 0$ (Proposition 4.4). A more interesting situation is the case when f vanishes identically at $\mathcal{D}'$, which we study in Section 4.2. In that case, we can define the so-called *quasi-pullback* of f, which produces a *cusp* form on $\mathcal{D}'$ (Proposition 4.10). These operations will be useful when studying concrete examples.

Restriction of modular forms to sub modular varieties has been considered classically for scalar-valued Siegel modular forms, going back to Witt [48]. Quasi-pullback has been also considered in this case: see [10, Section 2] for a general treatment.

Quasi-pullback of orthogonal modular forms was first considered for Borcherds products by Borcherds [5, 6], and later for general scalar-valued modular forms by Gritsenko–Hulek–Sankaran [22, Section 8.4] in the case $r = 1$. Our terminology "quasi-pullback" comes from this series of work. The cuspidality of quasi-pullback was first proved in [21, 22] in the scalar-valued case. Our Proposition 4.10 is the vector-valued generalization.

4.1 Ordinary pullback

We embed $\mathrm{O}^+(L'_{\mathbb{R}}) \times \mathrm{O}(K_{\mathbb{R}})$ in $\mathrm{O}^+(L_{\mathbb{R}})$ naturally. This is the stabilizer of $L'_{\mathbb{R}}$ in $\mathrm{O}^+(L_{\mathbb{R}})$. Let Γ be a finite-index subgroup of $\mathrm{O}^+(L)$. Then $\Gamma' = \Gamma \cap \mathrm{O}^+(L')$ is a finite-index subgroup of $\mathrm{O}^+(L')$, and $G = \Gamma \cap \mathrm{O}(K)$ is a finite group. The product $\Gamma' \times G$ is a finite-index subgroup of the stabilizer of L' in Γ.

Let $\mathcal{L}'$, $\mathcal{E}'$ be the Hodge bundles on $\mathcal{D}'$. Since $\mathcal{O}_{\mathbb{P}L_{\mathbb{C}}}(-1)|_{\mathbb{P}L'_{\mathbb{C}}} = \mathcal{O}_{\mathbb{P}L'_{\mathbb{C}}}(-1)$, we have $\mathcal{L}|_{\mathcal{D}'} = \mathcal{L}'$. We also have a natural isomorphism

$$\mathcal{E}|_{\mathcal{D}'} \simeq \mathcal{E}' \oplus (K_{\mathbb{C}} \otimes \mathcal{O}_{\mathcal{D}'}), \tag{4.1}$$

which at each fiber is the decomposition

$$(\omega^{\perp} \cap L_{\mathbb{C}})/\mathbb{C}\omega = ((\omega^{\perp} \cap L'_{\mathbb{C}})/\mathbb{C}\omega) \oplus K_{\mathbb{C}}.$$

This corresponds to the decomposition $\mathrm{St} = \mathrm{St}' \oplus \mathrm{St}''$ of the standard representation of $\mathrm{O}(n, \mathbb{C})$ when restricted to the subgroup $\mathrm{O}(n', \mathbb{C}) \times \mathrm{O}(r, \mathbb{C})$, where St' and St'' are the standard representations of $\mathrm{O}(n', \mathbb{C})$ and $\mathrm{O}(r, \mathbb{C})$, respectively.

Let λ be a partition expressing an irreducible representation V_{λ} of $\mathrm{O}(n, \mathbb{C})$. We denote by

$$V_{\lambda} \simeq \bigoplus_{\alpha} V'_{\lambda'(\alpha)} \boxtimes V''_{\lambda''(\alpha)} \tag{4.2}$$

the irreducible decomposition as a representation of $\mathrm{O}(n', \mathbb{C}) \times \mathrm{O}(r, \mathbb{C})$, where $V'_{\lambda'(\alpha)}$ (resp., $V''_{\lambda''(\alpha)}$) is the irreducible representation of $\mathrm{O}(n', \mathbb{C})$ (resp., $\mathrm{O}(r, \mathbb{C})$) with partition $\lambda'(\alpha)$ (resp., $\lambda''(\alpha)$). See [30, 32] for an explicit description of this restriction rule in terms of the Littlewood–Richardson numbers. Let k be an integer.

Proposition 4.1. *Restriction of modular forms to $\mathcal{D}' \subset \mathcal{D}$ defines a linear map*

$$M_{\lambda,k}(\Gamma) \to \bigoplus_{\alpha} M_{\lambda'(\alpha),k}(\Gamma') \otimes (K_{\mathbb{C}})^{G}_{\lambda''(\alpha)}, \quad f \mapsto f|_{\mathcal{D}'}.$$

This maps cusp forms to cusp forms.

For the proof of Proposition 4.1, we need to calculate the Fourier expansion of $f|_{\mathcal{D}'}$. We take a rank 1 primitive isotropic sublattice I of L'. Let $U(I)_{\mathbb{Z}} \subset U(I)_{\mathbb{Q}}$ be as in Section 3.3 and we define $U(I)'_{\mathbb{Z}} \subset U(I)'_{\mathbb{Q}}$ similarly for (L', Γ'). Then $U(I)'_{\mathbb{Q}} \subset U(I)_{\mathbb{Q}}$ and $U(I)'_{\mathbb{Z}} \subset U(I)_{\mathbb{Z}}$. If we write $K'_{\mathbb{Q}} = K_{\mathbb{Q}} \otimes I_{\mathbb{Q}}$, we have

$$U(I)_{\mathbb{Q}} = U(I)'_{\mathbb{Q}} \oplus K'_{\mathbb{Q}}.$$

The tube domain realization with respect to I (with I' also taken from L') identifies $\mathcal{D}' \subset \mathcal{D}$ with

$$\mathcal{D}'_I = \mathcal{D}_I \cap U(I)'_{\mathbb{C}} \subset \mathcal{D}_I.$$

Lemma 4.2. *Let $f(Z) = \sum_{l \in U(I)^{\vee}_{\mathbb{Z}}} a(l) q^l$ be the Fourier expansion of $f \in M_{\lambda,k}(\Gamma)$ at the I-cusp of $\mathcal{D}$. Then we have*

$$f|_{\mathcal{D}'_I}(Z') = \sum_{l' \in (U(I)'_{\mathbb{Z}})^{\vee}} b(l')(q')^{l'}, \quad (q')^{l'} = e((l', Z')), \tag{4.3}$$

for $Z' \in \mathcal{D}'_I$, where

$$b(l') = \sum_{\substack{l'' \in K'_{\mathbb{Q}} \\ l' + l'' \in U(I)^{\vee}_{\mathbb{Z}}}} a(l' + l'').$$

Proof. Let $\pi\colon U(I)_{\mathbb{Q}} \to U(I)'_{\mathbb{Q}}$ be the orthogonal projection. This maps $U(I)^{\vee}_{\mathbb{Z}}$ to a sublattice of $(U(I)'_{\mathbb{Z}})^{\vee}$. For $l \in U(I)^{\vee}_{\mathbb{Z}}$, the restriction of the function $q^l = e((l, Z))$ to $\mathcal{D}'_I \subset \mathcal{D}_I$ is $(q')^{\pi(l)} = e((\pi(l), Z'))$. Then our assertion follows by substituting $q^l = (q')^{\pi(l)}$ in $f = \sum_l a(l)q^l$. Note that the sum defining $b(l')$ is actually a finite sum by the condition $l' + l'' \in \overline{\mathcal{C}_I}$ (the cusp condition for f) and the fact that $K'_{\mathbb{Q}}$ is negative-definite. ■

Now we prove Proposition 4.1.

Proof of Proposition 4.1. From the expression (3.3) and the decomposition (4.2), we see that

$$\mathcal{E}_\lambda|_{\mathcal{D}'} \simeq \bigoplus_{\alpha} \mathcal{E}'_{\lambda'(\alpha)} \otimes (K_{\mathbb{C}})_{\lambda''(\alpha)} \tag{4.4}$$

as an $\mathrm{O}^+(L'_{\mathbb{R}}) \times \mathrm{O}(K_{\mathbb{R}})$-equivariant vector bundle on $\mathcal{D}'$. With the isomorphism $\mathcal{L}|_{\mathcal{D}'} = \mathcal{L}'$, we obtain

$$\mathcal{E}_{\lambda,k}|_{\mathcal{D}'} \simeq \bigoplus_{\alpha} \mathcal{E}'_{\lambda'(\alpha),k} \otimes (K_{\mathbb{C}})_{\lambda''(\alpha)}.$$

If f is a Γ-invariant section of $\mathcal{E}_{\lambda,k}$ over $\mathcal{D}$, this shows that $f|_{\mathcal{D}'}$ is a $\Gamma' \times G$-invariant section of $\bigoplus_{\alpha} \mathcal{E}'_{\lambda'(\alpha),k} \otimes (K_{\mathbb{C}})_{\lambda''(\alpha)}$ over $\mathcal{D}'$. Hence it is a Γ'-invariant section of $\bigoplus_{\alpha} \mathcal{E}'_{\lambda'(\alpha),k} \otimes (K_{\mathbb{C}})^G_{\lambda''(\alpha)}$ over $\mathcal{D}'$.

Holomorphicity of $f|_{\mathcal{D}'}$ at the cusps of $\mathcal{D}'$ holds automatically when $n' \geq 3$ by the Koecher principle. In general, this can be seen from Lemma 4.2 as follows. Let I and $K'_{\mathbb{Q}}$ be as in Lemma 4.2. Since $K'_{\mathbb{Q}}$ is negative-definite, the orthogonal projection $U(I)_{\mathbb{R}} \to U(I)'_{\mathbb{R}}$ maps the positive cone $\mathcal{C}_I$ of $U(I)_{\mathbb{R}}$ to the positive cone $\mathcal{C}'_I$ of $U(I)'_{\mathbb{R}}$, and maps $\overline{\mathcal{C}_I}$ to $\overline{\mathcal{C}'_I}$. Hence the vectors l' in (4.3) actually range over $(U(I)'_{\mathbb{Z}})^{\vee} \cap \overline{\mathcal{C}'_I}$. This proves the holomorphicity of $f|_{\mathcal{D}'}$ around the I-cusp of $\mathcal{D}'$. Since I is arbitrary, f is holomorphic at all cusps of $\mathcal{D}'$. When f is a cusp form, the vectors l' range over $(U(I)'_{\mathbb{Z}})^{\vee} \cap \mathcal{C}'_I$ for the same reason. This means that $f|_{\mathcal{D}'}$ is a cusp form. This proves Proposition 4.1. ■

Example 4.3. Let us look at a typical example. Let $\lambda = \mathrm{St}$. As noticed before, this decomposes as $\mathrm{St} = \mathrm{St}' \oplus \mathrm{St}''$ when restricted to $\mathrm{O}(n', \mathbb{C}) \times \mathrm{O}(r, \mathbb{C})$, which corresponds to the decomposition (4.1). Therefore restriction to $\mathcal{D}'$ gives a linear map

$$M_{\mathrm{St},k}(\Gamma) \to M_{\mathrm{St}',k}(\Gamma') \oplus (M_k(\Gamma') \otimes K_{\mathbb{C}}^G).$$

The first component $M_{\mathrm{St},k}(\Gamma) \to M_{\mathrm{St}',k}(\Gamma')$ can be considered as the main component of the restriction, but we also obtain some scalar-valued modular forms in $M_k(\Gamma') \otimes K_{\mathbb{C}}^G$ as "extra" components. When G fixes no nonzero vector of K, these extra components vanish. For example, this happens when Γ contains a reflection and L' is the fixed lattice of this reflection.

As an application of Proposition 4.1, we obtain the following elementary vanishing theorem. Although this will be superseded later (Section 9), we present it here because it can be proved easily and is already informative.

Proposition 4.4. *When $k < 0$, we have $M_{\lambda,k}(\Gamma) = 0$. Moreover, we have $M_{\lambda,0}(\Gamma) = 0$ when $\lambda \neq 1, \det$.*

Proof. Let $f \in M_{\lambda,k}(\Gamma)$ with $k < 0$. We consider restriction of f to 1-dimensional domains $\mathcal{D}_{L'} \subset \mathcal{D}$ for sublattices $L' \subset L$ of signature $(2,1)$. As a representation of $\mathrm{O}(1,\mathbb{C}) = \{\pm \mathrm{id}\}$, V_λ is a direct sum of copies of the trivial character and the determinant character. By Proposition 4.1 and the calculation in Section 2.5.1, we see that $f|_{\mathcal{D}_{L'}}$ is a tuple of scalar-valued modular forms of weight $2k < 0$ on the upper half plane $\mathcal{D}_{L'}$. Since there is no nonzero elliptic modular form of negative weight, we find that f vanishes identically at $\mathcal{D}_{L'}$. Now, if we vary L', then $\mathcal{D}_{L'}$ run over a dense subset of $\mathcal{D}$. Therefore $f \equiv 0$.

When $f \in M_{\lambda,0}(\Gamma)$ with $\lambda \neq 1, \det$, by combining Proposition 3.7 and Lemma 4.2, we see that $f|_{\mathcal{D}_{L'}}$ is a tuple of scalar-valued cusp forms of weight 0 on $\mathcal{D}_{L'}$, which vanish identically. Therefore $f \equiv 0$ similarly. ■

The idea to deduce a vanishing theorem by considering restriction to sub modular varieties is classical. In the case of Siegel modular forms, this goes back to Freitag [15].

Proposition 4.4 in particular implies the following.

Proposition 4.5. *Let $n \geq 3$. Assume that $\langle \Gamma, -\mathrm{id} \rangle$ does not contain a reflection. Let X be the regular locus of $\mathcal{F}(\Gamma) = \Gamma \backslash \mathcal{D}$. Then $H^0(X, T_X^{\otimes k}) = 0$ for every $k > 0$.*

Proof. Let $\pi\colon \mathcal{D} \to \mathcal{F}(\Gamma)$ be the projection and $X' \subset X$ be the locus where π is unramified. By [21], the absence of reflection in $\langle \Gamma, -\mathrm{id} \rangle$ implies that π is unramified in codimension 1, so the complement of $\pi^{-1}(X')$ in $\mathcal{D}$ has codimension ≥ 2. Since we can pull back sections of $T_{X'}^{\otimes k}$ by the étale map

$$\pi^{-1}(X') \to X',$$

we see that

$$H^0(X, T_X^{\otimes k}) = H^0(X', T_{X'}^{\otimes k}) = H^0\big(\pi^{-1}(X'), T_{\pi^{-1}(X')}^{\otimes k}\big)^\Gamma = H^0(\mathcal{D}, T_{\mathcal{D}}^{\otimes k})^\Gamma.$$

Since $T_{\mathcal{D}} \simeq \mathcal{E} \otimes \mathcal{L}^{-1}$ by (2.4), we find that

$$H^0(X, T_X^{\otimes k}) = H^0(\mathcal{D}, \mathcal{E}^{\otimes k} \otimes \mathcal{L}^{\otimes -k})^\Gamma = \bigoplus_i M_{\lambda(i),-k}(\Gamma),$$

where $\lambda(i)$ run over the irreducible summands of $\mathrm{St}^{\otimes k}$. By Proposition 4.4, the last space vanishes when $-k < 0$. ■

4.2 Quasi-pullback

In this section we show that when $f|_{\mathcal{D}'} \equiv 0$, we can still obtain a nonzero *cusp* form on $\mathcal{D}'$ by considering the Taylor expansion of f along $\mathcal{D}'$. We assume $n' \geq 3$ for simplicity of exposition, but the results below hold also when $n' \leq 2$ (see the proof of Proposition 4.10).

We first describe the normal bundle $\mathcal{N} = \mathcal{N}_{\mathcal{D}'/\mathcal{D}}$ of $\mathcal{D}'$ in $\mathcal{D}$.

Lemma 4.6. *We have $\mathcal{N} \simeq (\mathcal{L}')^{-1} \otimes K_{\mathbb{C}}$ as an $\mathrm{O}^+(L'_{\mathbb{R}}) \times \mathrm{O}(K_{\mathbb{R}})$-equivariant vector bundle on $\mathcal{D}'$.*

Proof. By (2.4) and (4.1), we have natural isomorphisms

$$\begin{aligned} T_{\mathcal{D}}|_{\mathcal{D}'} &\simeq (\mathcal{E} \otimes \mathcal{L}^{-1})|_{\mathcal{D}'} \simeq (\mathcal{E}' \oplus (K_{\mathbb{C}} \otimes \mathcal{O}_{\mathcal{D}'})) \otimes (\mathcal{L}')^{-1} \\ &\simeq T_{\mathcal{D}'} \oplus ((\mathcal{L}')^{-1} \otimes K_{\mathbb{C}}). \end{aligned}$$

This implies

$$\mathcal{N} \simeq (\mathcal{L}')^{-1} \otimes K_{\mathbb{C}}. \qquad \blacksquare$$

Let $\mathcal{I}$ be the ideal sheaf of $\mathcal{D}' \subset \mathcal{D}$ and $\nu \geq 0$. By Lemma 4.6 we have

$$\mathcal{I}^{\nu}/\mathcal{I}^{\nu+1}|_{\mathcal{D}'} \simeq \mathrm{Sym}^{\nu} \mathcal{N}^{\vee} \simeq (\mathcal{L}')^{\otimes \nu} \otimes \mathrm{Sym}^{\nu} K_{\mathbb{C}}^{\vee} \tag{4.5}$$

as an $\mathrm{O}^+(L'_{\mathbb{R}}) \times \mathrm{O}(K_{\mathbb{R}})$-equivariant vector bundle on $\mathcal{D}'$. Therefore we have the exact sequence

$$0 \to \mathcal{I}^{\nu+1}\mathcal{E}_{\lambda,k} \to \mathcal{I}^{\nu}\mathcal{E}_{\lambda,k} \to \mathcal{E}_{\lambda}|_{\mathcal{D}'} \otimes (\mathcal{L}')^{\otimes k+\nu} \otimes \mathrm{Sym}^{\nu} K_{\mathbb{C}}^{\vee} \to 0 \tag{4.6}$$

of sheaves on $\mathcal{D}$. By (4.4) we have an $\mathrm{O}^+(L'_{\mathbb{R}}) \times \mathrm{O}(K_{\mathbb{R}})$-equivariant isomorphism

$$\mathcal{E}_{\lambda}|_{\mathcal{D}'} \otimes (\mathcal{L}')^{\otimes k+\nu} \otimes \mathrm{Sym}^{\nu} K_{\mathbb{C}}^{\vee} \simeq \bigoplus_{\alpha} \mathcal{E}'_{\lambda'(\alpha),k+\nu} \otimes (K_{\mathbb{C}})_{\lambda''(\alpha)} \otimes \mathrm{Sym}^{\nu} K_{\mathbb{C}}^{\vee}.$$

Note that $K_{\mathbb{C}}^{\vee} \simeq K_{\mathbb{C}}$ canonically by the pairing on K. Taking global sections in (4.6), and then the $\Gamma' \times G$-invariant part, we obtain the exact sequence

$$\begin{aligned} 0 &\to H^0(\mathcal{D}, \mathcal{I}^{\nu+1}\mathcal{E}_{\lambda,k})^{\Gamma' \times G} \to H^0(\mathcal{D}, \mathcal{I}^{\nu}\mathcal{E}_{\lambda,k})^{\Gamma' \times G} \\ &\to \bigoplus_{\alpha} M_{\lambda'(\alpha),k+\nu}(\Gamma') \otimes ((K_{\mathbb{C}})_{\lambda''(\alpha)} \otimes \mathrm{Sym}^{\nu} K_{\mathbb{C}})^{G}. \end{aligned} \tag{4.7}$$

By definition, a modular form $f \in M_{\lambda,k}(\Gamma)$ vanishes to order $\geq \nu$ along $\mathcal{D}'$ if it is a section of the subsheaf $\mathcal{I}^{\nu}\mathcal{E}_{\lambda,k}$ of $\mathcal{E}_{\lambda,k}$. The *vanishing order* of f along $\mathcal{D}'$ is the largest ν for which f is a section of $\mathcal{I}^{\nu}\mathcal{E}_{\lambda,k}$.

Definition 4.7. Let $f \in M_{\lambda,k}(\Gamma)$ and ν be the vanishing order of f at $\mathcal{D}'$. We define the *quasi-pullback* of f

$$f\|_{\mathcal{D}'} \in \bigoplus_{\alpha} M_{\lambda'(\alpha),k+\nu}(\Gamma') \otimes ((K_{\mathbb{C}})_{\lambda''(\alpha)} \otimes \mathrm{Sym}^{\nu} K_{\mathbb{C}})^{G}$$

as the image of f by the last map in (4.7).

By the exactness of (4.7) and the definition of the vanishing order, we have $f\|_{\mathcal{D}'} \not\equiv 0$. Note that the vanishing order ν contributes to the increase $k \rightsquigarrow k+\nu$ of the scalar weight. When $\nu = 0$, the quasi-pullback is just the ordinary pullback considered in Section 4.1.

Example 4.8. When $r = 1$, ignoring the symmetry by $G \subset \{\pm \mathrm{id}\}$, the quasi-pullback $f\|_{\mathcal{D}'}$ belongs to $\bigoplus_{\alpha} M_{\lambda'(\alpha),k+\nu}(\Gamma')$. Explicitly, $f\|_{\mathcal{D}'}$ is given by the restriction of $f/(\cdot,\delta)^{\nu}$ to $\mathcal{D}'$, where δ is a nonzero vector of K and $(\cdot,\delta)$ is the section of $\mathcal{O}(1)$ defined by the pairing with δ.

Example 4.9. The quasi-pullback of a Borcherds product f considered by Borcherds [5,6] is defined as $f/\prod_{\delta}(\delta,\cdot)|_{\mathcal{D}'}$, where δ run over primitive vectors in K (with multiplicity) such that f vanishes at $\delta^{\perp} \cap \mathcal{D}$. This is a single scalar-valued modular form (again a Borcherds product), while our quasi-pullback produces a tuple of scalar-valued modular forms, or more canonically, a $\mathrm{Sym}^{\nu} K_{\mathbb{C}}$-valued modular form. The relationship is as follows.

The denominator $\prod_{\delta}(\delta,\cdot)$ is a section of $\mathcal{I}^{\nu} \cdot \mathcal{O}(\nu)$ over $\mathcal{D}$. This corresponds to a sheaf homomorphism $\iota\colon \mathcal{L}^{\otimes \nu} \to \mathcal{I}^{\nu}$. By a property of Borcherds products, f is a section of the subsheaf $\iota(\mathcal{L}^{\otimes \nu}) \cdot \mathcal{L}^{\otimes k}$ of $\mathcal{I}^{\nu} \cdot \mathcal{L}^{\otimes k}$. Let $\bar{\iota}\colon (\mathcal{L}')^{\otimes \nu} \to \mathrm{Sym}^{\nu} \mathcal{N}^{\vee}$ be the embedding induced by $\iota|_{\mathcal{D}'}$ and (4.5). Under the isomorphism

$$\mathrm{Sym}^{\nu} \mathcal{N}^{\vee} \simeq (\mathcal{L}')^{\otimes \nu} \otimes \mathrm{Sym}^{\nu} K_{\mathbb{C}}^{\vee},$$

this corresponds to the vector $\prod_{\delta}(\cdot,\delta)$ of $\mathrm{Sym}^{\nu} K_{\mathbb{C}}^{\vee}$, which in turn corresponds to the vector $\prod_{\delta}\delta$ of $\mathrm{Sym}^{\nu} K_{\mathbb{C}}$. Then $f\|_{\mathcal{D}'}$ as a section of $\mathrm{Sym}^{\nu} \mathcal{N}^{\vee} \otimes (\mathcal{L}')^{\otimes k}$ takes values in the sub line bundle $\bar{\iota}((\mathcal{L}')^{\otimes \nu}) \otimes (\mathcal{L}')^{\otimes k} \simeq (\mathcal{L}')^{\otimes k+\nu}$. This section of $(\mathcal{L}')^{\otimes k+\nu}$ is the quasi-pullback in [5,6].

Next we prove the cuspidality of quasi-pullback. In the case $\lambda = 0$ and $r = 1$, this is due to Gritsenko–Hulek–Sankaran [22, Theorem 8.18].

Proposition 4.10. *Let $f \in M_{\lambda,k}(\Gamma)$ and ν be the vanishing order of f at $\mathcal{D}'$. Suppose that $\nu > 0$. Then $f\|_{\mathcal{D}'}$ is a cusp form. Thus*

$$f\|_{\mathcal{D}'} \in \bigoplus_{\alpha} S_{\lambda'(\alpha),k+\nu}(\Gamma') \otimes ((K_{\mathbb{C}})_{\lambda''(\alpha)} \otimes \mathrm{Sym}^{\nu} K_{\mathbb{C}})^{G}.$$

For the proof of Proposition 4.10, we calculate the Fourier expansion of $f\|_{\mathcal{D}'}$. We work under the same setting and notation as in the proof of Lemma 4.2. We choose a basis of $K'_{\mathbb{Q}}$. According to the decomposition $U(I)_{\mathbb{Q}} = U(I)'_{\mathbb{Q}} \oplus K'_{\mathbb{Q}}$, we express a point of $U(I)_{\mathbb{C}}$ as $Z = (Z', z_1, \dots, z_r)$ with $Z' \in U(I)'_{\mathbb{C}}$ and $z_i \in \mathbb{C}$. Then $\mathcal{D}'_I \subset \mathcal{D}_I$ is defined by $z_1 = \cdots = z_r = 0$. The coordinates $z_1, \dots, z_r$ give a trivialization of the conormal bundle $\mathcal{N}^\vee$ of $\mathcal{D}'_I$. The quasi-pullback $f\|_{\mathcal{D}'}$ as a $V(I)_{\lambda,k} \otimes \mathrm{Sym}^\nu \mathbb{C}^r$-valued function on $\mathcal{D}'_I$ is given, up to constants, by the Taylor coefficients of f along $\mathcal{D}'_I$ in degree ν:

$$f\|_{\mathcal{D}'}(Z') = \left(\frac{\partial^\nu f}{\partial z_1^{\nu_1} \cdots \partial z_r^{\nu_r}}(Z', 0) \right)_{\nu_1 + \cdots + \nu_r = \nu}$$

We calculate the Fourier expansion of the Taylor coefficients. In what follows, we identify $(K'_{\mathbb{Q}})^\vee \simeq \mathbb{Q}^r$ by the dual basis of the chosen basis of $K'_{\mathbb{Q}}$ and express vectors of $(K'_{\mathbb{Q}})^\vee$ as $(n_1, \dots, n_r)$, $n_i \in \mathbb{Q}$.

Lemma 4.11. *Let $f(Z) = \sum_l a(l) q^l$ be the Fourier expansion of f. Let $(\nu_1, \dots, \nu_r)$ be an index with $\nu_1 + \cdots + \nu_r = \nu$. Then we have*

$$\frac{\partial^\nu f}{\partial z_1^{\nu_1} \cdots \partial z_r^{\nu_r}}(Z', 0) = (2\pi\sqrt{-1})^\nu \sum_{l' \in (U(I)'_{\mathbb{Z}})^\vee} b(l')(q')^{l'},$$

where $(q')^{l'} = e((l', Z'))$ and

$$b(l') = \sum_{\substack{(n_1,\dots,n_r) \in \mathbb{Q}^r \\ l' + (n_1,\dots,n_r) \in U(I)^\vee_{\mathbb{Z}}}} n_1^{\nu_1} \cdots n_r^{\nu_r} \cdot a(l' + (n_1, \dots, n_r)).$$

Here, by convention, $0^0 = 1$ but $0^m = 0$ when $m > 0$.

Note that the sum defining $b(l')$ is actually a finite sum for the same reason as in Lemma 4.2.

Proof. We can rewrite the Fourier expansion of f as

$$\begin{aligned} & f(Z', z_1, \dots, z_r) \\ & \quad = \sum_{l'} \sum_{(n_1,\dots,n_r)} a(l' + (n_1, \dots, n_r)) \cdot e((l' + (n_1, \dots, n_r), (Z', z_1, \dots, z_r))) \\ & \quad = \sum_{l'} \sum_{(n_1,\dots,n_r)} a(l' + (n_1, \dots, n_r)) \cdot e((l', Z')) \cdot \prod_{i=1}^{r} e(n_i z_i). \end{aligned}$$

Here l' ranges over $(U(I)'_{\mathbb{Z}})^\vee$ and $(n_1, \dots, n_r)$ ranges over vectors in $\mathbb{Q}^r = (K'_{\mathbb{Q}})^\vee$ such that $l' + (n_1, \dots, n_r) \in U(I)^\vee_{\mathbb{Z}}$. Since we have

$$\frac{\partial^\nu \prod_i e(n_i z_i)}{\partial z_1^{\nu_1} \cdots \partial z_r^{\nu_r}} = (2\pi\sqrt{-1})^\nu \prod_i n_i^{\nu_i} \cdot e(n_i z_i),$$

we see that

$$\frac{\partial^\nu f}{\partial z_1^{\nu_1} \cdots \partial z_r^{\nu_r}}(Z', z_1, \dots, z_r)$$
$$= (2\pi\sqrt{-1})^\nu \sum_{l'} \sum_{(n_1,\dots,n_r)} a(l' + (n_1, \dots, n_r)) \cdot (q')^{l'} \cdot \prod_i n_i^{\nu_i} \cdot e(n_i z_i).$$

Substituting $z_1 = \cdots = z_r = 0$, this proves Lemma 4.11. ■

Now we complete the proof of Proposition 4.10.

Proof of Proposition 4.10. Let l' be a vector in $\overline{\mathcal{C}'_I} \cap (U(I)'_{\mathbb{Z}})^\vee$ with $(l', l') = 0$. For $(n_1, \dots, n_r) \in \mathbb{Q}^r$, we have $l' + (n_1, \dots, n_r) \in \overline{\mathcal{C}_I}$ only when $(n_1, \dots, n_r) = (0, \dots, 0)$ because $K'_{\mathbb{Q}}$ is negative-definite and perpendicular to $U(I)'_{\mathbb{Q}}$. By Lemma 4.11, this shows that

$$b(l') = 0^{\nu_1} \cdots 0^{\nu_r} \cdot a(l') = 0$$

because $(\nu_1, \dots, \nu_r) \neq (0, \dots, 0)$ by the assumption $\nu > 0$. This proves Proposition 4.10. ■