Chapter 5

Canonical extension over 1-dimensional cusps

In this chapter we recall the partial toroidal compactification over a 1-dimensional cusp and the canonical extension of the automorphic vector bundles over it. This provides a geometric basis for the Siegel operator (Section 6) and the Fourier–Jacobi expansion (Section 7). Except for a few calculations in Sections 5.4 and 5.5, most contents of this chapter are essentially expository. We refer the reader to [2] for the general theory of toroidal compactification, to [21, 33, 35] for its specialization to the case of orthogonal modular varieties (especially for more details on the contents of Sections 5.1–5.3), and to [36] for the general theory of canonical extension. Nevertheless, since this chapter is the basis of many later chapters, we tried to keep the presentation as self-contained, explicit, and coherent as possible.

Throughout this chapter, L is a lattice of signature $(2, n)$ with $n \geq 3$. We fix a rank 2 primitive isotropic sublattice J of L, which corresponds to a 1-dimensional cusp of $\mathcal{D} = \mathcal{D}_L$. We write

$$V(J)_F = (J^{\perp}/J) \otimes_{\mathbb{Z}} F$$

for $F = \mathbb{Q}, \mathbb{R}, \mathbb{C}$. This is a quadratic space over F, negative-definite when $F = \mathbb{Q}, \mathbb{R}$. We especially abbreviate $V(J) = V(J)_{\mathbb{C}}$. We also write $U(J)_F = \wedge^2 J_F$. The choice of the component $\mathcal{D}$ determines an orientation of J so that the $\mathbb{R}$-isomorphism $(\omega, \cdot)\colon J_{\mathbb{R}} \to \mathbb{C}$ preserves the orientation for any $[\omega] \in \mathcal{D}$. This determines the positive part of $U(J)_{\mathbb{R}}$.

For $2U = U \oplus U$, where U is the integral hyperbolic plane, we will denote by e_1, f_1 and e_2, f_2 the standard hyperbolic basis of the first and the second components, respectively. We say that an embedding $\iota\colon 2U_F \hookrightarrow L_F$ is compatible with J if it satisfies $\iota(\mathbb{Z}e_1 \oplus \mathbb{Z}e_2) = J$. This defines a lift $V(J)_F \simeq \iota(2U_F)^{\perp} \cap L_F$ of $V(J)_F$ in $J_F^{\perp}$ and hence a splitting

$$L_F \simeq 2U_F \oplus V(J)_F = (J_F \oplus J_F^{\vee}) \oplus V(J)_F, \tag{5.1}$$

where we identify $\iota(\langle f_1, f_2 \rangle)$ with $J_F^{\vee}$. We often choose a rank 1 primitive sublattice I of J. We say that $\iota\colon 2U_F \hookrightarrow L_F$ is compatible with $I \subset J$ if $\iota(\mathbb{Z}e_1 \oplus \mathbb{Z}e_2) = J$ and $\iota(\mathbb{Z}e_1) = I$.

5.1 Siegel domain realization

In this section we recall the Siegel domain realization of $\mathcal{D}$ with respect to the J-cusp and explain its relation with the tube domain realization.

5.1.1 Siegel domain realization

The filtration $J \subset J^{\perp} \subset L$ on L determines the two-step linear projection

$$\mathbb{P}L_{\mathbb{C}} \overset{\pi_1}{\dashrightarrow} \mathbb{P}(L/J)_{\mathbb{C}} \overset{\pi_2}{\dashrightarrow} \mathbb{P}(L/J^{\perp})_{\mathbb{C}}. \tag{5.2}$$

Via the pairing on $L_{\mathbb{C}}$, this is identified with the dual projection

$$\mathbb{P}L_{\mathbb{C}}^{\vee} \dashrightarrow \mathbb{P}(J_{\mathbb{C}}^{\perp})^{\vee} \dashrightarrow \mathbb{P}J_{\mathbb{C}}^{\vee}.$$

The centre of π_1 is $\mathbb{P}J_{\mathbb{C}}$, and the centre of π_2 is $\mathbb{P}V(J)$. The projection π_2 identifies $\mathbb{P}(L/J)_{\mathbb{C}} - \mathbb{P}V(J)$ with an affine space bundle over $\mathbb{P}(L/J^{\perp})_{\mathbb{C}}$. If we choose a lift $V(J) \hookrightarrow J_{\mathbb{C}}^{\perp}$ of $V(J)$, it defines a splitting $(L/J)_{\mathbb{C}} = V(J) \oplus (L/J^{\perp})_{\mathbb{C}}$, and so, defines an isomorphism between the affine space bundle $\mathbb{P}(L/J)_{\mathbb{C}} - \mathbb{P}V(J)$ with the vector bundle $V(J) \otimes \mathcal{O}(1)$ over $\mathbb{P}(L/J^{\perp})_{\mathbb{C}}$.

We restrict (5.2) to the isotropic quadric $Q \subset \mathbb{P}L_{\mathbb{C}}$. The closure of a π_1-fiber is a plane containing $\mathbb{P}J_{\mathbb{C}}$. When this plane is not contained in $\mathbb{P}J_{\mathbb{C}}^{\perp}$, it intersects properly with Q at two distinct lines, one being $\mathbb{P}J_{\mathbb{C}}$. This shows that

$$\pi_1|_Q : Q - Q \cap \mathbb{P}J_{\mathbb{C}}^{\perp} \to \mathbb{P}(L/J)_{\mathbb{C}} - \mathbb{P}V(J)$$

is an affine line bundle.

Next we restrict (5.2) further to an enlargement of the domain $\mathcal{D} \subset Q$. Let $\mathbb{H}_J$ be the connected component of $\mathbb{P}J_{\mathbb{C}}^{\vee} - \mathbb{P}J_{\mathbb{R}}^{\vee}$ consisting of $\mathbb{C}$-linear maps $\phi\colon J_{\mathbb{C}} \to \mathbb{C}$ such that $\phi|_{J_{\mathbb{R}}}\colon J_{\mathbb{R}} \to \mathbb{C}$ is an orientation-preserving $\mathbb{R}$-isomorphism. By the canonical isomorphism $\mathbb{P}J_{\mathbb{C}}^{\vee} \simeq \mathbb{P}J_{\mathbb{C}}$, $\mathbb{H}_J$ corresponds to the J-cusp. We put $\mathcal{V}_J = \pi_2^{-1}(\mathbb{H}_J)$ and $\mathcal{D}(J) = (\pi_1|_Q)^{-1}(\mathcal{V}_J)$. Then $\mathcal{D} \subset \mathcal{D}(J)$. We thus have the extended two-step fibration

$$\mathcal{D} \subset \mathcal{D}(J) \overset{\pi_1}{\longrightarrow} \mathcal{V}_J \overset{\pi_2}{\longrightarrow} \mathbb{H}_J, \tag{5.3}$$

where $\mathcal{V}_J \to \mathbb{H}_J$ is an affine space bundle isomorphic to $V(J) \otimes \mathcal{O}_{\mathbb{H}_J}(1)$, $\mathcal{D}(J) \to \mathcal{V}_J$ is an affine line bundle, and $\mathcal{D} \to \mathcal{V}_J$ is an upper half plane bundle inside $\mathcal{D}(J) \to \mathcal{V}_J$. This is the Siegel domain realization of $\mathcal{D}$ with respect to J. (Up to this point, canonically determined by J.)

5.1.2 Relation with tube domain realization

We choose a rank 1 primitive sublattice I of J. Recall from Section 3.3 that the tube domain realization at the I-cusp (before choosing a base point) is the canonical embedding

$$\mathcal{D} \subset \mathcal{D}(I) \overset{\simeq}{\to} \mathbb{P}(L/I)_{\mathbb{C}} - \mathbb{P}V(I)$$

induced by the projection $\mathbb{P}L_{\mathbb{C}} \dashrightarrow \mathbb{P}(L/I)_{\mathbb{C}}$. Note that $\mathcal{D}(J) \subset \mathcal{D}(I)$. We can factor the projection π_1 in (5.2) as:

$$\mathbb{P}L_{\mathbb{C}} \dashrightarrow \mathbb{P}(L/I)_{\mathbb{C}} \dashrightarrow \mathbb{P}(L/J)_{\mathbb{C}} \dashrightarrow \mathbb{P}(L/J^{\perp})_{\mathbb{C}}.$$

Hence we have the following commutative diagram:

$$\begin{array}{ccccc} \mathbb{P}(L/I)_{\mathbb{C}} - \mathbb{P}V(I) & \longrightarrow & \mathbb{P}(L/J)_{\mathbb{C}} - \mathbb{P}(I^{\perp}/J)_{\mathbb{C}} & \longrightarrow & \mathbb{P}(L/J^{\perp})_{\mathbb{C}} - \mathbb{P}(I^{\perp}/J^{\perp})_{\mathbb{C}} \\ \uparrow & & \uparrow & & \uparrow \\ \mathcal{D} \subset \mathcal{D}(J) & \xrightarrow{\pi_1} & \mathcal{V}_J & \xrightarrow{\pi_2} & \mathbb{H}_J. \end{array}$$

Here the upper row is projections of affine spaces, the left vertical map is the tube domain realization at I, and other vertical maps are natural inclusions. The two squares are cartesian, i.e., $\mathcal{D}(J) \to \mathcal{V}_J \to \mathbb{H}_J$ is the restriction of the upper row over $\mathbb{H}_J$. Thus the Siegel domain realization at J can be given by a decomposition of the tube domain realization at $I \subset J$.

Next we choose a rank 1 isotropic sublattice $I' \subset L$ with $(I, I') \neq 0$ and accordingly a base point of the affine space $\mathbb{P}(L/I)_{\mathbb{C}} - \mathbb{P}V(I)$. This identifies the upper row of the above diagram with the linear maps

$$U(I)_{\mathbb{C}} = (I^{\perp}/I)_{\mathbb{C}} \otimes I_{\mathbb{C}} \to (I^{\perp}/J)_{\mathbb{C}} \otimes I_{\mathbb{C}} \to (I^{\perp}/J^{\perp})_{\mathbb{C}} \otimes I_{\mathbb{C}}.$$

We identify $U(J)_{\mathbb{C}} = \wedge^2 J_{\mathbb{C}}$ with the isotropic line $(J/I)_{\mathbb{C}} \otimes I_{\mathbb{C}}$ in $U(I)_{\mathbb{C}}$. Then this is written as the quotient maps

$$U(I)_{\mathbb{C}} \to U(I)_{\mathbb{C}}/U(J)_{\mathbb{C}} \to U(I)_{\mathbb{C}}/U(J)_{\mathbb{C}}^{\perp}. \tag{5.4}$$

Therefore, after choosing the base point I', the above commutative diagram can be rewritten as

$$\begin{array}{ccccc} U(I)_{\mathbb{C}} & \xrightarrow{\pi_1} & U(I)_{\mathbb{C}}/U(J)_{\mathbb{C}} & \xrightarrow{\pi_2} & U(I)_{\mathbb{C}}/U(J)_{\mathbb{C}}^{\perp} \\ \uparrow & & \uparrow & & \uparrow \\ \mathcal{D} \subset \mathcal{D}(J) & \xrightarrow{\pi_1} & \mathcal{V}_J & \xrightarrow{\pi_2} & \mathbb{H}_J \end{array}$$

where the vertical embeddings are defined by I' and the two squares are cartesian. This gives a simpler (but depending on I, I') expression of the Siegel domain realization.

Finally, we introduce coordinates. Let v_J be the positive generator of $\wedge^2 J \simeq \mathbb{Z}$. We choose an isotropic vector $l_J \in U(I)_{\mathbb{Q}}$ with $(v_J, l_J) = 1$. This defines a splitting $U(I)_{\mathbb{Q}} \simeq U_{\mathbb{Q}} \oplus K_{\mathbb{Q}}$, where $K_{\mathbb{Q}} = V(J)_{\mathbb{Q}} \otimes I_{\mathbb{Q}}$, which determines a splitting of (5.4). Accordingly, we express a point of $U(I)_{\mathbb{C}} \simeq \mathbb{C}l_J \times K_{\mathbb{C}} \times \mathbb{C}v_J$ as

$$Z = (\tau, z, w) = \tau l_J + z + w v_J, \quad z \in K_{\mathbb{C}}, \ \tau, w \in \mathbb{C}. \tag{5.5}$$

In these coordinates, the I-directed Siegel domain realization (5.4) is expressed by

$$(\tau, z, w) \mapsto (\tau, z) \mapsto \tau.$$

The w-component gives coordinates on the π_1-fibers ($\simeq U(J)_{\mathbb{C}}$), and τ gives coordinates on the base $U(I)_{\mathbb{C}}/U(J)_{\mathbb{C}}^{\perp} \simeq U(J)_{\mathbb{C}}^{\vee}$. The images of the embeddings

$$\mathcal{D}(J) \hookrightarrow U(I)_{\mathbb{C}}, \quad \mathcal{V}_J \hookrightarrow U(I)_{\mathbb{C}}/U(J)_{\mathbb{C}}, \quad \mathbb{H}_J \hookrightarrow U(I)_{\mathbb{C}}/U(J)_{\mathbb{C}}^{\perp}$$

are all defined by the inequality $\mathrm{Im}(\tau) > 0$, and the tube domain $\mathcal{D}_I \subset U(I)_{\mathbb{C}}$ is defined by the inequalities

$$-(\mathrm{Im}(z), \mathrm{Im}(z)) < 2\,\mathrm{Im}(\tau)\cdot \mathrm{Im}(w), \quad \mathrm{Im}(\tau) > 0.$$

Thus the choice of I, I', l_J defines a passage from the canonical presentation (5.3) to a more classical presentation of the Siegel domain realization.

Remark 5.1. The choice of I' and l_J is almost equivalent to the choice of an embedding $2U_{\mathbb{Q}} \hookrightarrow L_{\mathbb{Q}}$ compatible with $I_{\mathbb{Q}} \subset J_{\mathbb{Q}}$. More precisely, we choose one of the two generators of $I \simeq \mathbb{Z}$, say v_I. Let $v_I' \in I_{\mathbb{Q}}'$ be the dual vector of v_I in $I_{\mathbb{Q}}'$. We can write $v_J = \tilde{v}_J \otimes v_I$ and $l_J = \tilde{l}_J \otimes v_I$ for some vectors $\tilde{v}_J \in (I_{\mathbb{Q}}')^{\perp} \cap J_{\mathbb{Q}}$ and $\tilde{l}_J \in (I_{\mathbb{Q}}')^{\perp} \cap I_{\mathbb{Q}}^{\perp}$. This defines an embedding $2U_{\mathbb{Q}} \hookrightarrow L_{\mathbb{Q}}$ compatible with $I_{\mathbb{Q}} \subset J_{\mathbb{Q}}$ by sending

$$e_1 \mapsto v_I, \quad f_1 \mapsto v_I', \quad e_2 \mapsto \tilde{v}_J, \quad f_2 \mapsto \tilde{l}_J.$$

5.2 Jacobi group

In this section we describe the rational/real Jacobi group of the J-cusp and its action on the Siegel domain realization.

Let $F = \mathbb{Q}, \mathbb{R}$. Let $\Gamma(J)_F$ be the subgroup of the stabilizer of J_F in $\mathrm{O}(L_F)$ acting trivially on $\wedge^2 J_F$ and $V(J)_F$. We call $\Gamma(J)_F$ the *Jacobi group* for J over F. (It is certainly useful to take into account the action on $V(J)_F$, but here we refrain from doing so for simplicity of exposition.) The Jacobi group has the canonical filtration

$$U(J)_F \subset W(J)_F \subset \Gamma(J)_F$$

defined by

$$W(J)_F = \mathrm{Ker}(\Gamma(J)_F \to \mathrm{SL}(J_F)),$$
$$U(J)_F = \mathrm{Ker}(\Gamma(J)_F \to \mathrm{GL}(J_F^{\perp})).$$

The group $U(J)_F$ consists of the Eichler transvections $E_{l\otimes l'}$ for $l, l' \in J_F$. Since $E_{l'\otimes l} = E_{-l\otimes l'}$, $U(J)_F$ is canonically isomorphic to $\wedge^2 J_F$. This justifies our use of the notation $U(J)_F$. We also have the canonical isomorphism

$$V(J)_F \otimes J_F \to W(J)_F/U(J)_F, \quad m \otimes l \mapsto E_{\tilde{m}\otimes l} \mod U(J)_F,$$

where $\widetilde{m} \in J_F^{\perp}$ is a lift of $m \in V(J)_F$. The linear space $V(J)_F \otimes J_F$ has a canonical $U(J)_F$-valued symplectic form as the tensor product of the quadratic form on $V(J)_F$ and the canonical $\wedge^2 J_F$-valued symplectic form on J_F. We thus have the canonical exact sequences

$$\begin{aligned} &0 \to W(J)_F \to \Gamma(J)_F \to \mathrm{SL}(J_F) \to 1, \\ &0 \to U(J)_F \to W(J)_F \to V(J)_F \otimes J_F \to 0. \end{aligned} \tag{5.6}$$

The group $U(J)_F$ is the centre of $\Gamma(J)_F$, and $W(J)_F$ is the unipotent radical of $\Gamma(J)_F$. The first sequence (5.6) splits if we choose an embedding $2U_F \hookrightarrow L_F$ compatible with J_F and hence a splitting $L_F \simeq (J_F \oplus J_F^{\vee}) \oplus V(J)_F$ as in (5.1):

$$\Gamma(J)_F \simeq \mathrm{SL}(J_F) \ltimes W(J)_F. \tag{5.7}$$

Here the lifted group $\mathrm{SL}(J_F) \subset \Gamma(J)_F$ acts on the component $J_F \oplus J_F^{\vee}$ in the natural way. The adjoint action of $\mathrm{SL}(J_F)$ on $W(J)_F/U(J)_F \simeq V(J)_F \otimes J_F$ is the tensor product of the natural action of $\mathrm{SL}(J_F)$ on J_F and the trivial action on $V(J)_F$. The group $W(J)_F$ is isomorphic to the Heisenberg group attached to the symplectic space $V(J)_F \otimes J_F$ with centre $U(J)_F$. We call $W(J)_F$ the *Heisenberg group* for J over F.

If I is a rank 1 primitive sublattice of J, we have

$$U(J)_F \subset U(I)_F \subset \Gamma(J)_F, \tag{5.8}$$

as can be seen from the definitions. In $U(I)_F = (I^{\perp}/I)_F \otimes I_F$, $U(J)_F$ corresponds to the isotropic line $(J/I)_F \otimes I_F$. We also have $W(J)_F \subset \Gamma(I)_F$ and

$$U(I)_F \cap W(J)_F = U(J)_F^{\perp} = (J^{\perp}/I)_F \otimes I_F.$$

The image of $W(J)_F$ in $\mathrm{O}(V(I)_F)$ is the group of Eichler transvections of $V(I)_F$ with respect to the isotropic line $(J/I)_F$.

The Jacobi group $\Gamma(J)_F$ preserves the Siegel domain realization (5.3) by definition. The actions of the factors $U(J)_F$, $W(J)_F/U(J)_F$, $\mathrm{SL}(J_F)$ of $\Gamma(J)_F$ on the spaces in (5.3) are described as follows.

(1) The group $U(J)_F$ acts on $\mathcal{V}_J$ trivially. The projection $\mathcal{D}(J) \to \mathcal{V}_J$ is a principal $U(J)_{\mathbb{C}}$-bundle, where $U(J)_{\mathbb{C}} = \wedge^2 J_{\mathbb{C}}$ is the group of Eichler transvections $E_{l\otimes l'}$ with $l, l' \in J_{\mathbb{C}}$.

(2) The Heisenberg group $W(J)_F$ acts on $\mathbb{H}_J$ trivially. The quotient $W(J)_F/U(J)_F$ acts on the fibers of $\mathcal{V}_J \to \mathbb{H}_J$ by translation. More precisely, if τ is a point of $\mathbb{H}_J \subset \mathbb{P}J_{\mathbb{C}}^{\vee}$ and $J_{\mathbb{C}} = J^{1,0} \oplus J^{0,1}$ is the corresponding Hodge decomposition of $J_{\mathbb{C}}$ (where $J^{1,0}$ is the kernel), the fiber of $\mathcal{O}_{\mathbb{H}_J}(1)$ over τ is $J_{\mathbb{C}}/J^{1,0}$. So the fiber $(\mathcal{V}_J)_\tau$ of $\mathcal{V}_J$ over τ is an affine space for $V(J) \otimes_{\mathbb{C}} (J_{\mathbb{C}}/J^{1,0})$. On the other hand, we have a natural projection $V(J)_{\mathbb{R}} \otimes_{\mathbb{R}} J_{\mathbb{R}} \to V(J) \otimes_{\mathbb{C}} (J_{\mathbb{C}}/J^{1,0})$ which is

an $\mathbb{R}$-isomorphism. Then the action of an element of $W(J)_{\mathbb{R}}/U(J)_{\mathbb{R}} \simeq V(J)_{\mathbb{R}} \otimes_{\mathbb{R}} J_{\mathbb{R}}$ on the affine space $(\mathcal{V}_J)_\tau$ is the translation by its projection image in $V(J) \otimes_{\mathbb{C}} (J_{\mathbb{C}}/J^{1,0})$.

(3) To describe the action of $\mathrm{SL}(J_F)$, we take an embedding $2U_F \hookrightarrow L_F$ compatible with J_F. As explained before, this induces an isomorphism $\mathcal{V}_J \simeq V(J) \otimes \mathcal{O}_{\mathbb{H}_J}(1)$ and a lift $\mathrm{SL}(J_F) \hookrightarrow \Gamma(J)_F$. Then the lifted group $\mathrm{SL}(J_F)$ acts on $\mathcal{V}_J$ by its equivariant action on $\mathcal{O}_{\mathbb{H}_J}(1)$.

5.3 Partial toroidal compactification

Let Γ be a finite-index subgroup of $\mathrm{O}^+(L)$. We take the intersection of $\Gamma(J)_{\mathbb{Q}}$, $W(J)_{\mathbb{Q}}$, $U(J)_{\mathbb{Q}}$ with Γ and denote them by

$$\Gamma(J)_{\mathbb{Z}} = \Gamma(J)_{\mathbb{Q}} \cap \Gamma, \quad W(J)_{\mathbb{Z}} = W(J)_{\mathbb{Q}} \cap \Gamma, \quad U(J)_{\mathbb{Z}} = U(J)_{\mathbb{Q}} \cap \Gamma.$$

By the orientation on J, we have a distinguished isomorphism $U(J)_{\mathbb{Z}} \simeq \mathbb{Z}$. We also denote by $\Gamma(J)^*_{\mathbb{Z}}$ the stabilizer of J in Γ. The integral Jacobi group $\Gamma(J)_{\mathbb{Z}}$ is of finite index in $\Gamma(J)^*_{\mathbb{Z}}$ because

$$\Gamma(J)^*_{\mathbb{Z}}/\Gamma(J)_{\mathbb{Z}} \hookrightarrow \mathrm{O}(J^\perp/J)$$

and $\mathrm{O}(J^\perp/J)$ is a finite group. If Γ is neat, we have $\Gamma(J)^*_{\mathbb{Z}} = \Gamma(J)_{\mathbb{Z}}$.

We put

$$\overline{\Gamma(J)}_{\mathbb{Z}} = \Gamma(J)_{\mathbb{Z}}/U(J)_{\mathbb{Z}}, \quad \overline{\Gamma(J)}_F = \Gamma(J)_F/U(J)_{\mathbb{Z}}$$

for $F = \mathbb{Q}, \mathbb{R}$. These quotients make sense because $U(J)_F$ is the centre of $\Gamma(J)_F$. By definition we have the canonical exact sequence

$$0 \to W(J)_{\mathbb{Z}}/U(J)_{\mathbb{Z}} \to \overline{\Gamma(J)}_{\mathbb{Z}} \to \Gamma(J)_{\mathbb{Z}}/W(J)_{\mathbb{Z}} \to 1,$$

which is canonically embedded in the quotient of (5.6) by $U(J)_F$: more specifically, $\Gamma(J)_{\mathbb{Z}}/W(J)_{\mathbb{Z}}$ is embedded in $\mathrm{SL}(J)$ as a finite-index subgroup, and $W(J)_{\mathbb{Z}}/U(J)_{\mathbb{Z}}$ is embedded in $V(J)_{\mathbb{Q}} \otimes J_{\mathbb{Q}}$ as a full lattice.

Let $T(J) = U(J)_{\mathbb{C}}/U(J)_{\mathbb{Z}} \simeq \mathbb{C}^*$ be the 1-dimensional torus defined by $U(J)_{\mathbb{Z}}$. We denote by $\overline{T(J)} \simeq \mathbb{C}$ the natural partial compactification of $T(J)$. We take the quotient of $\mathcal{D} \subset \mathcal{D}(J)$ by $U(J)_{\mathbb{Z}}$:

$$\mathcal{X}(J) = \mathcal{D}/U(J)_{\mathbb{Z}}, \quad \mathcal{T}(J) = \mathcal{D}(J)/U(J)_{\mathbb{Z}}.$$

Then $\mathcal{T}(J)$ is a principal $T(J)$-bundle over $\mathcal{V}_J$, which contains $\mathcal{X}(J)$ as a fibration of punctured discs. Let $\overline{\mathcal{T}(J)} = \mathcal{T}(J) \times_{T(J)} \overline{T(J)}$ be the relative torus embedding. This has the structure of a line bundle on $\mathcal{V}_J$: the scalar multiplication on each fiber is given by the action of $T(J) \simeq \mathbb{C}^*$, and the sum is determined by the scalar multiplication because the fiber is 1-dimensional. The group $\overline{\Gamma(J)_{\mathbb{R}}}$ acts on $\mathcal{T}(J)$ naturally, and this extends to an action on $\overline{\mathcal{T}(J)}$. The fact that $\Gamma(J)_{\mathbb{R}}$ commutes with $U(J)_{\mathbb{C}}$ implies that the action of $\overline{\Gamma(J)_{\mathbb{R}}}$ on $\overline{\mathcal{T}(J)}$ is an equivariant action on the line bundle.

Let $\overline{\mathcal{X}(J)}$ be the interior of the closure of $\mathcal{X}(J)$ in $\overline{\mathcal{T}(J)}$. We call $\overline{\mathcal{X}(J)}$ the *partial toroidal compactification* of $\mathcal{X}(J)$. This is a disc bundle over $\mathcal{V}_J$ obtained by filling the origins in the punctured disc bundle $\mathcal{X}(J) \to \mathcal{V}_J$. Let Δ_J be the boundary divisor of $\overline{\mathcal{X}(J)}$. This is naturally isomorphic to $\mathcal{V}_J$. We denote by Θ_J the conormal bundle of Δ_J in $\overline{\mathcal{X}(J)}$. This is a $\overline{\Gamma(J)_{\mathbb{R}}}$-equivariant line bundle on Δ_J. (Although the subgroup $U(J)_{\mathbb{R}}/U(J)_{\mathbb{Z}}$ of $\overline{\Gamma(J)_{\mathbb{R}}}$ acts on Δ_J trivially, it acts on the fibers of Θ_J by rotations.)

Lemma 5.2. *We have a natural $\overline{\Gamma(J)_{\mathbb{R}}}$-equivariant isomorphism $\Theta^{\vee} \simeq \overline{\mathcal{T}(J)}$ of line bundles on Δ_J.*

Proof. Since Δ_J is the zero section of the line bundle $\overline{\mathcal{T}(J)}$, its normal bundle in $\overline{\mathcal{X}(J)}$ is the same as the normal bundle in $\overline{\mathcal{T}(J)}$, which is isomorphic to $\overline{\mathcal{T}(J)}$ itself. ■

The partial compactification $\overline{\mathcal{X}(J)}$ already appears in essence in the partial compactifications $\mathcal{X}(I)^{\Sigma}$ for $I \subset J$ considered in Section 3.5.1. Recall that the isotropic ray $\sigma_J = (U(J)_{\mathbb{R}})_{\geq 0}$ appears in every $\Gamma(I)_{\mathbb{Z}}$-admissible fan Σ as in Section 3.5.1. Since $U(J)_{\mathbb{Z}} \subset U(I)_{\mathbb{Z}}$, we have a natural étale map $\mathcal{X}(J) \to \mathcal{X}(I)$ which is a free quotient map by $U(I)_{\mathbb{Z}}/U(J)_{\mathbb{Z}}$.

Lemma 5.3. *The map $\mathcal{X}(J) \to \mathcal{X}(I)$ extends to an étale map $\overline{\mathcal{X}(J)} \to \mathcal{X}(I)^{\Sigma}$. The image of Δ_J is a Zariski open set of the boundary divisor of $\mathcal{X}(I)^{\Sigma}$ associated to the isotropic ray σ_J.*

Proof. Since $\mathcal{D}(J) \subset \mathcal{D}(I)$, we have the following commutative diagram (cf. Section 5.1.2):

$$\begin{array}{ccccc} \mathcal{T}(J) & \hookrightarrow & \mathcal{D}(I)/U(J)_{\mathbb{Z}} & \longrightarrow & T(I) \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{V}_J & \hookrightarrow & \mathcal{D}(I)/U(J)_{\mathbb{C}} & \longrightarrow & T(I)/T(J). \end{array}$$

Here the vertical maps are principal $T(J)$-bundles, and the two right horizontal maps are free quotients by $U(I)_{\mathbb{Z}}/U(J)_{\mathbb{Z}}$. The two squares are cartesian: the right is the pullback of a principal $T(J)$-bundle to a $U(I)_{\mathbb{Z}}/U(J)_{\mathbb{Z}}$-cover, and the left is the restriction to an open set. Since the upper row is $T(J)$-equivariant, it extends to

$$\overline{\mathcal{T}(J)} \hookrightarrow (\mathcal{D}(I)/U(J)_{\mathbb{Z}}) \times_{T(J)} \overline{T(J)} \to T(I) \times_{T(J)} \overline{T(J)}.$$

The second map is still a free quotient by $U(I)_{\mathbb{Z}}/U(J)_{\mathbb{Z}}$. The image of $\Delta_J \subset \overline{\mathcal{T}(J)}$ by this map is an open set of the (unique) boundary divisor of $T(I) \times_{T(J)} \overline{T(J)}$.

Since $T(I) \times_{T(J)} \overline{T(J)}$ is the torus embedding of $T(I)$ associated to the ray σ_J, it is a Zariski open set of $T(I)^{\Sigma}$. Thus we obtain an étale map $\overline{\mathcal{T}(J)} \to T(I)^{\Sigma}$ which maps Δ_J to an open set of the boundary divisor of $T(I)^{\Sigma}$ corresponding to σ_J. ■

5.4 Canonical extension

In this section, which is the central part of Section 5, we extend the automorphic vector bundles $\mathcal{E}_{\lambda,k}$ over $\overline{\mathcal{X}(J)}$. This is an explicit form of Mumford's canonical extension [36] which is suitable for dealing with the Fourier–Jacobi expansion. We use the same notations $\mathcal{L}$, $\mathcal{E}$, $\mathcal{E}_{\lambda}$, $\mathcal{E}_{\lambda,k}$ for the descends of these vector bundles to $\mathcal{X}(J)$. They are $\overline{\Gamma(J)_{\mathbb{R}}}$-equivariant vector bundles on $\mathcal{X}(J)$.

We choose an adjacent 0-dimensional cusp $I \subset J$. Since $U(J)_{\mathbb{Z}} \subset \Gamma(I)_{\mathbb{R}}$, the I-trivialization of $\mathcal{E}_{\lambda,k}$ over $\mathcal{D}$ descends to an isomorphism $\mathcal{E}_{\lambda,k} \simeq V(I)_{\lambda,k} \otimes \mathcal{O}_{\mathcal{X}(J)}$ over $\mathcal{X}(J) = \mathcal{D}/U(J)_{\mathbb{Z}}$. Thus we still have the I-trivialization over $\mathcal{X}(J)$. This is equivariant with respect to $(\Gamma(I)_{\mathbb{R}} \cap \Gamma(J)_{\mathbb{R}})/U(J)_{\mathbb{Z}}$. We extend $\mathcal{E}_{\lambda,k}$ to a vector bundle over $\overline{\mathcal{X}(J)}$ (still use the same notation) by requiring that this isomorphism extends to

$$\mathcal{E}_{\lambda,k} \simeq V(I)_{\lambda,k} \otimes \mathcal{O}_{\overline{\mathcal{X}(J)}}.$$

We call it the *canonical extension* of $\mathcal{E}_{\lambda,k}$ over $\overline{\mathcal{X}(J)}$. This is the pullback of the canonical extension over $\mathcal{X}(I)^{\Sigma}$ defined in Section 3.5.3 by the gluing map $\overline{\mathcal{X}(J)} \to \mathcal{X}(I)^{\Sigma}$ in Lemma 5.3. By construction, the frame of $\mathcal{E}_{\lambda,k}$ over $\mathcal{X}(J)$ corresponding to a basis of $V(I)_{\lambda,k}$ via the I-trivialization extends to a frame of the extended bundle over $\overline{\mathcal{X}(J)}$.

Proposition 5.4. *The canonical extension of $\mathcal{E}_{\lambda,k}$ over $\overline{\mathcal{X}(J)}$ defined above does not depend on the choice of I. The action of $\overline{\Gamma(J)_{\mathbb{R}}}$ on $\mathcal{E}_{\lambda,k}$ over $\mathcal{X}(J)$ extends to action on the canonical extension of $\mathcal{E}_{\lambda,k}$ over $\overline{\mathcal{X}(J)}$.*

The proof of this proposition amounts to the following assertion.

Lemma 5.5. *The factor of automorphy of the $\Gamma(J)_{\mathbb{R}}$-action on $\mathcal{E}_{\lambda,k}$ with respect to the I-trivialization is constant on each fiber of $\pi_1\colon \mathcal{D} \to \mathcal{V}_J$. In particular, if I' is another $\mathbb{R}$-line in $J_{\mathbb{R}}$, the difference of the I-trivialization and the I'-trivialization at $[\omega] \in \mathcal{D}$ as the composition map*

$$V(I)_{\lambda,k} \to (\mathcal{E}_{\lambda,k})_{[\omega]} \to V(I')_{\lambda,k} \tag{5.9}$$

is constant on each π_1-fiber.

Proof. Let $j(\gamma, [\omega])$ be the factor of automorphy in question. This is a $\mathrm{GL}(V(I)_{\lambda,k})$-valued function on $\Gamma(J)_{\mathbb{R}} \times \mathcal{D}$. What has to be shown is that $j(\gamma, [\omega]) = j(\gamma, [\omega'])$ if $\pi_1([\omega]) = \pi_1([\omega'])$. We consider the natural extension of $\mathcal{E}_{\lambda,k}$ over $\mathcal{D}(J)$, on which the group $U(J)_{\mathbb{C}} \cdot \Gamma(J)_{\mathbb{R}}$ acts equivariantly. Note that $U(J)_{\mathbb{C}}$ commutes with $\Gamma(J)_{\mathbb{R}}$. We can write $[\omega'] = g[\omega]$ for some $g \in U(J)_{\mathbb{C}}$. Since $U(J)_{\mathbb{C}}$ acts trivially on $I_{\mathbb{C}}$ and $V(I)$, we have $j(g, \cdot) \equiv \mathrm{id}$. Therefore

$$j(\gamma, g[\omega]) = j(\gamma g, [\omega]) = j(g\gamma, [\omega]) = j(\gamma, [\omega]).$$

As for the second assertion, we choose $\gamma \in \Gamma(J)_{\mathbb{R}}$ with $\gamma(I_{\mathbb{R}}) = I'$. Then (5.9) coincides with the isomorphism

$$\gamma \circ j(\gamma^{-1}, [\omega]) : V(I)_{\lambda,k} \to V(I)_{\lambda,k} \to V(I')_{\lambda,k}.$$

Hence the constancy of $j(\gamma^{-1}, [\omega])$ over π_1-fibers implies that of (5.9). ■

Now we can prove Proposition 5.4.

Proof of Proposition 5.4. Let I, I' be two rank 1 primitive sublattices of J. By the second assertion of Lemma 5.5, the difference of the I-trivialization and the I'-trivialization

$$V(I)_{\lambda,k} \otimes \mathcal{O}_{\mathcal{X}(J)} \to \mathcal{E}_{\lambda,k} \to V(I')_{\lambda,k} \otimes \mathcal{O}_{\mathcal{X}(J)}, \tag{5.10}$$

viewed as a $\mathrm{GL}(n, \mathbb{C})$-valued holomorphic function on $\mathcal{X}(J)$ via basis of $V(I)_{\lambda,k}$ and $V(I')_{\lambda,k}$, is constant on each fiber of $\mathcal{X}(J) \to \mathcal{V}_J$. Therefore it extends to a $\mathrm{GL}(n, \mathbb{C})$-valued holomorphic function over $\overline{\mathcal{X}(J)}$. This implies that (5.10) extends to an isomorphism

$$V(I)_{\lambda,k} \otimes \mathcal{O}_{\overline{\mathcal{X}(J)}} \to V(I')_{\lambda,k} \otimes \mathcal{O}_{\overline{\mathcal{X}(J)}}$$

over $\overline{\mathcal{X}(J)}$. Thus the two extensions agree.

Extendability of the $\overline{\Gamma(J)}_{\mathbb{R}}$-action on $\mathcal{E}_{\lambda,k}$ can be verified as follows. Let $\gamma \in \Gamma(J)_{\mathbb{R}}$. The γ-action on $\mathcal{E}_{\lambda,k}$ sends a frame corresponding to a basis of $V(I)_{\lambda,k}$ via the I-trivialization to a frame corresponding to a basis of $V(\gamma I)_{\lambda,k}$ via the γI-trivialization. By Lemma 5.5 again, the latter extends to a frame over $\overline{\mathcal{X}(J)}$ also in the I-trivialization. Thus γ sends an extendable frame to an extendable frame. This means that the γ-action extends over $\overline{\mathcal{X}(J)}$. ■

The fact that the canonical extension comes with an I-trivialization (but independent of it) enables us to develop the theory of Fourier–Jacobi expansion (Section 7) in an intrinsic but still explicit way. The following property will play a fundamental role in Section 7.

Proposition 5.6. *Let $\pi_1 \colon \overline{\mathcal{X}(J)} \to \mathcal{V}_J \simeq \Delta_J$ be the projection. Then we have a $\overline{\Gamma(J)}_{\mathbb{R}}$-equivariant isomorphism $\mathcal{E}_{\lambda,k} \simeq \pi_1^*(\mathcal{E}_{\lambda,k}|_{\Delta_J})$ over $\overline{\mathcal{X}(J)}$.*

Proof. We fix a rank 1 primitive sublattice $I \subset J$ and let $j(\gamma, [\omega])$ be the factor of automorphy of the $\Gamma(J)_{\mathbb{R}}$-action on $\mathcal{E}_{\lambda,k}$ with respect to the I-trivialization. By Lemma 5.5, the $\mathrm{GL}(V(I)_{\lambda,k})$-valued function $j(\gamma, [\omega])$ on $\Gamma(J)_{\mathbb{R}} \times \mathcal{X}(J)$ descends to a $\mathrm{GL}(V(I)_{\lambda,k})$-valued function on $\Gamma(J)_{\mathbb{R}} \times \Delta_J$. This gives the factor of automorphy of the $\Gamma(J)_{\mathbb{R}}$-action on $\mathcal{E}_{\lambda,k}|_{\Delta_J}$ with respect to the I-trivialization

$$\mathcal{E}_{\lambda,k}|_{\Delta_J} \simeq V(I)_{\lambda,k} \otimes \mathcal{O}_{\Delta_J}.$$

The fact that its pullback agrees with the factor of automorphy of $\mathcal{E}_{\lambda,k}$ implies that the composition

$$\pi_1^*(\mathcal{E}_{\lambda,k}|_{\Delta_J}) \to \pi_1^*(V(I)_{\lambda,k} \otimes \mathcal{O}_{\Delta_J}) \simeq V(I)_{\lambda,k} \otimes \mathcal{O}_{\overline{\mathcal{X}(J)}} \to \mathcal{E}_{\lambda,k}$$

gives a $\Gamma(J)_{\mathbb{R}}$-equivariant isomorphism $\pi_1^*(\mathcal{E}_{\lambda,k}|_{\Delta_J}) \to \mathcal{E}_{\lambda,k}$ over $\overline{\mathcal{X}(J)}$, where the first isomorphism is the pullback of the I-trivialization over Δ_J, and the last isomorphism is the I-trivialization over $\overline{\mathcal{X}(J)}$. ■

Remark 5.7. By the proof, we have the following commutative diagram:

$$\begin{array}{ccc} \pi_1^*(\mathcal{E}_{\lambda,k}|_{\Delta_J}) & \longrightarrow & \mathcal{E}_{\lambda,k} \\ \downarrow & & \downarrow \\ \pi_1^*(V(I)_{\lambda,k} \otimes \mathcal{O}_{\Delta_J}) & \longrightarrow & V(I)_{\lambda,k} \otimes \mathcal{O}_{\overline{\mathcal{X}(J)}}. \end{array}$$

Here the upper arrow is the isomorphism in Proposition 5.6, the vertical arrows are the I-trivializations, and the lower arrow is the natural isomorphism.

Remark 5.8. Although the canonical extension at the level of $\overline{\mathcal{X}(J)}$ still has a trivialization (by construction), this no longer holds when passing to the full toroidal compactifications (Section 5.6). Around Δ_J we need to further take the quotient by $\overline{\Gamma(J)_{\mathbb{Z}}}$, which does not preserve the trivialization.

5.5 The Hodge line bundle at the boundary

In this section we study the Hodge line bundle $\mathcal{L}$ relative to the J-cusp and show that its canonical extension can be understood more directly. Let

$$\mathcal{L}_J = \mathcal{O}_{\mathbb{H}_J}(-1) = \mathcal{O}_{\mathbb{P}(L/J^{\perp})_{\mathbb{C}}}(-1)|_{\mathbb{H}_J}$$

be the Hodge bundle over the upper half plane $\mathbb{H}_J$. The group $\Gamma(J)_{\mathbb{R}}$ acts on $\mathcal{L}_J$ equivariantly via the natural map $\Gamma(J)_{\mathbb{R}} \to \mathrm{SL}(J_{\mathbb{R}})$. Let $\pi = \pi_2 \circ \pi_1 \colon \mathcal{D} \to \mathbb{H}_J$ be the projection from $\mathcal{D}$ to $\mathbb{H}_J$.

Lemma 5.9. *We have a* $\Gamma(J)_{\mathbb{R}}$*-equivariant isomorphism* $\mathcal{L} \simeq \pi^*\mathcal{L}_J$ *over* $\mathcal{D}$.

Proof. Recall that π is restriction of the projection $\mathbb{P}L_{\mathbb{C}} \dashrightarrow \mathbb{P}(L/J^{\perp})_{\mathbb{C}}$. Since this is induced by the linear map $L_{\mathbb{C}} \to (L/J^{\perp})_{\mathbb{C}}$, we have a natural isomorphism

$$\pi^*\mathcal{O}_{\mathbb{P}(L/J^{\perp})_{\mathbb{C}}}(-1) \simeq \mathcal{O}_{\mathbb{P}L_{\mathbb{C}}}(-1)$$

over $\mathbb{P}L_{\mathbb{C}} - \mathbb{P}J_{\mathbb{C}}^{\perp}$. Restricting this isomorphism to $\mathcal{D}$, we obtain $\mathcal{L} \simeq \pi^*\mathcal{L}_J$. Since the projection $L_{\mathbb{C}} \to (L/J^{\perp})_{\mathbb{C}}$ is $\Gamma(J)_{\mathbb{R}}$-equivariant, so is the isomorphism $\mathcal{L} \simeq \pi^*\mathcal{L}_J$. ∎

The fiber of $\pi^*\mathcal{L}_J$ over $[\omega] \in \mathcal{D}$ is the image of the projection $\mathbb{C}\omega \to (L/J^{\perp})_{\mathbb{C}}$, and the isomorphism $\mathcal{L} \to \pi^*\mathcal{L}_J$ over $[\omega]$ is identified with the natural map $\mathbb{C}\omega \to \mathrm{Im}(\mathbb{C}\omega \to (L/J^{\perp})_{\mathbb{C}})$.

The projection $\mathcal{D} \to \mathbb{H}_J$ descends to $\mathcal{X}(J) \to \mathbb{H}_J$ and extends to $\overline{\mathcal{X}(J)} \to \mathbb{H}_J$ naturally. We denote it again by $\pi\colon \overline{\mathcal{X}(J)} \to \mathbb{H}_J$. The isomorphism in Lemma 5.9 descends to a $\overline{\Gamma(J)}_{\mathbb{R}}$-equivariant isomorphism $\mathcal{L} \simeq \pi^*\mathcal{L}_J|_{\mathcal{X}(J)}$ over $\mathcal{X}(J)$. We have respective extension of both sides over $\overline{\mathcal{X}(J)}$: for $\mathcal{L}$ the canonical extension constructed in Section 5.4, and for $\pi^*\mathcal{L}_J|_{\mathcal{X}(J)}$ the natural extension $\pi^*\mathcal{L}_J$. It turns out that these two extensions agree, as the following proposition shows.

Proposition 5.10. *The isomorphism* $\mathcal{L} \simeq \pi^*\mathcal{L}_J|_{\mathcal{X}(J)}$ *over* $\mathcal{X}(J)$ *extends to a* $\overline{\Gamma(J)}_{\mathbb{R}}$*-equivariant isomorphism between the canonical extension of* $\mathcal{L}$ *and* $\pi^*\mathcal{L}_J$ *over* $\overline{\mathcal{X}(J)}$*. In particular, we have* $\mathcal{L}|_{\Delta_J} \simeq \pi_2^*\mathcal{L}_J$ *over* Δ_J.

Proof. We choose a rank 1 primitive sublattice $I \subset J$. The canonical extension of $\mathcal{L}$ is defined via the I-trivialization of $\mathcal{L}$, which we denote by $\iota_I\colon \mathcal{L} \simeq I_{\mathbb{C}}^{\vee} \otimes \mathcal{O}_{\mathcal{X}(J)}$. On the other hand, we also have a trivialization $\iota_I'\colon \mathcal{L}_J \simeq I_{\mathbb{C}}^{\vee} \otimes \mathcal{O}_{\mathbb{H}_J}$ of $\mathcal{L}_J = \mathcal{O}_{\mathbb{H}_J}(-1)$ over $\mathbb{H}_J \subset \mathbb{P}(L/J^{\perp})_{\mathbb{C}}$ induced by the pairing between $(L/J^{\perp})_{\mathbb{C}}$ and $I_{\mathbb{C}}$. The natural extension $\pi^*\mathcal{L}_J$ of $\pi^*\mathcal{L}_J|_{\mathcal{X}(J)}$ over $\overline{\mathcal{X}(J)}$ coincides with the extension via the trivialization

$$\pi^*\mathcal{L}_J|_{\mathcal{X}(J)} \xrightarrow{\pi^*\iota_I'} \pi^*(I_{\mathbb{C}}^{\vee} \otimes \mathcal{O}_{\mathbb{H}_J})|_{\mathcal{X}(J)} = I_{\mathbb{C}}^{\vee} \otimes \mathcal{O}_{\mathcal{X}(J)}, \tag{5.11}$$

because $\pi^*\iota_I'$ is defined over $\overline{\mathcal{X}(J)}$.

We observe that the composition of (5.11) with the isomorphism $\mathcal{L} \simeq \pi^*\mathcal{L}_J|_{\mathcal{X}(J)}$ in Lemma 5.9 coincides with the I-trivialization ι_I of $\mathcal{L}$: this is just the remark that taking the pairing of a vector $\omega \in L_{\mathbb{C}}$ with $I_{\mathbb{C}}$ (this is ι_I) is the same as projecting ω to $(L/J^{\perp})_{\mathbb{C}}$ (this is $\mathcal{L} \to \pi^*\mathcal{L}_J$) and then taking pairing with $I_{\mathbb{C}}$ (this is $\pi^*\iota_I'$). From this coincidence, we see that the isomorphism in Lemma 5.9 extends to an isomorphism over $\overline{\mathcal{X}(J)}$ from the extension of $\mathcal{L}$ via ι_I (this is the canonical extension of $\mathcal{L}$) to the extension of $\pi^*\mathcal{L}_J|_{\mathcal{X}(J)}$ via $\pi^*\iota_I'$ (this is $\pi^*\mathcal{L}_J$). The $\overline{\Gamma(J)}_{\mathbb{R}}$-equivariance holds by continuity. ∎

Thus the canonical extension of $\mathcal{L}$ defined in Section 5.4 via the I-trivialization can be understood more directly as the *canonical* (verbatim) extension $\pi^*\mathcal{L}_J$ of $\pi^*\mathcal{L}_J|_{\mathcal{X}(J)}$.

Remark 5.11. By the proof of Proposition 5.10, $\mathcal{L}_J$ is endowed with the I-trivialization $I_{\mathbb{C}}^{\vee} \otimes \mathcal{O}_{\mathbb{H}_J} \to \mathcal{L}_J$ induced by the pairing between $(L/J^{\perp})_{\mathbb{C}}$ and $I_{\mathbb{C}}$, and its pullback by π agrees with the I-trivialization of $\mathcal{L}$ via the isomorphism $\mathcal{L} \simeq \pi^*\mathcal{L}_J$.

5.6 Toroidal compactification

In this section we recall the (full) toroidal compactifications of the modular variety $\mathcal{F}(\Gamma) = \Gamma\backslash\mathcal{D}$ following [2]. While this provides a background for our geometric approach, logically it will be used only in Section 10 in a rather auxiliary way, so the reader may skip it for the moment.

The data for constructing a toroidal compactification of $\mathcal{F}(\Gamma)$ is a collection $\Sigma = (\Sigma_I)$ of $\Gamma(I)_{\mathbb{Z}}$-admissible rational polyhedral cone decomposition of $\mathcal{C}_I^+ \subset U(I)_{\mathbb{R}}$ in the sense of Section 3.5.1, one for each Γ-equivalence class of rank 1 primitive isotropic sublattices I of L. Two fans Σ_I, $\Sigma_{I'}$ for different Γ-equivalence classes I, I' are independent, and no choice is required for rank 2 isotropic sublattices J (it is canonical). Then the toroidal compactification is defined by

$$\mathcal{F}(\Gamma)^{\Sigma} = \Big(\mathcal{D} \sqcup \bigsqcup_{I} \mathcal{X}(I)^{\Sigma_I} \sqcup \bigsqcup_{J} \overline{\mathcal{X}(J)}\Big)\Big/ \sim,$$

where I (resp., J) run over all primitive isotropic sublattices of L of rank 1 (resp., rank 2), and $\sim$ is the equivalence relation generated by the following étale maps.

(1) The γ-action $\mathcal{D} \to \mathcal{D}$, $\mathcal{X}(I)^{\Sigma_I} \to \mathcal{X}(\gamma I)^{\Sigma_{\gamma I}}$, $\overline{\mathcal{X}(J)} \to \overline{\mathcal{X}(\gamma J)}$ for $\gamma \in \Gamma$.

(2) The gluing maps $\mathcal{D} \to \mathcal{X}(I)^{\Sigma_I}$, $\mathcal{D} \to \overline{\mathcal{X}(J)}$ and $\overline{\mathcal{X}(J)} \to \mathcal{X}(I)^{\Sigma_I}$ for $I \subset J$ as in Lemma 5.3.

By [2, Section III.5], $\mathcal{F}(\Gamma)^{\Sigma}$ is a compact Moishezon space which contains $\mathcal{F}(\Gamma)$ as a Zariski open set and has a morphism $\mathcal{F}(\Gamma)^{\Sigma} \to \mathcal{F}(\Gamma)^{bb}$ to the Baily–Borel compactification. We have natural maps

$$\mathcal{X}(I)^{\Sigma_I}/\overline{\Gamma(I)}_{\mathbb{Z}} \to \mathcal{F}(\Gamma)^{\Sigma}, \quad \overline{\mathcal{X}(J)}/(\Gamma(J)_{\mathbb{Z}}^{*}/U(J)_{\mathbb{Z}}) \to \mathcal{F}(\Gamma)^{\Sigma}. \tag{5.12}$$

These maps are isomorphism in a neighbourhood of the locus of boundary points lying over the I-cusp and the J-cusp, respectively (see [2, p. 175]). We may choose Σ so that $\mathcal{F}(\Gamma)^{\Sigma}$ is projective. When Γ is neat and each fan Σ_I is regular, i.e., every cone is generated by a part of a $\mathbb{Z}$-basis of $U(I)_{\mathbb{Z}}$, then $\mathcal{F}(\Gamma)^{\Sigma}$ is nonsingular [2, Section III.7].

Next we explain the canonical extension of $\mathcal{E}_{\lambda,k}$ over $\mathcal{F}(\Gamma)^{\Sigma}$ (cf. [36]). We assume that Γ is neat and Σ is regular. Then not only Γ itself but also the subquotients $\overline{\Gamma(I)_{\mathbb{Z}}}$ and $\Gamma(J)_{\mathbb{Z}}^{*}/U(J)_{\mathbb{Z}} = \overline{\Gamma(J)_{\mathbb{Z}}}$ are torsion-free, so the quotient map

$$\mathcal{D} \sqcup \bigsqcup_{I} \mathcal{X}(I)^{\Sigma_I} \sqcup \bigsqcup_{J} \overline{\mathcal{X}(J)} \to \mathcal{F}(\Gamma)^{\Sigma}$$

is étale. The vector bundle $\mathcal{E}_{\lambda,k}$ is initially defined on $\mathcal{D}$ and hence on

$$\mathcal{D} \sqcup \bigsqcup_{I} \mathcal{X}(I) \sqcup \bigsqcup_{J} \mathcal{X}(J).$$

In Sections 3.5.3 and 5.4, we constructed the canonical extension of $\mathcal{E}_{\lambda,k}$ over $\mathcal{X}(I)^{\Sigma_I}$ and $\overline{\mathcal{X}(J)}$, respectively. By construction we have a natural isomorphism

$$p^{*}\mathcal{E}_{\lambda,k} \simeq \mathcal{E}_{\lambda,k}$$

for a gluing map p in (2) above. Moreover, we have a natural isomorphism $\gamma^{*}\mathcal{E}_{\lambda,k} \simeq \mathcal{E}_{\lambda,k}$ for the action of $\gamma \in \Gamma$: this is evident for $\mathcal{D}$ and $\mathcal{X}(I)^{\Sigma_I}$, while it is assured by Proposition 5.4 for $\overline{\mathcal{X}(J)}$. Since these isomorphisms are compatible with each other, the extended vector bundle $\mathcal{E}_{\lambda,k}$ on

$$\mathcal{D} \sqcup \bigsqcup_{I} \mathcal{X}(I)^{\Sigma_I} \sqcup \bigsqcup_{J} \overline{\mathcal{X}(J)}$$

descends to a vector bundle on $\mathcal{F}(\Gamma)^{\Sigma}$. We denote it again by $\mathcal{E}_{\lambda,k}$. This is the same as extending $\mathcal{E}_{\lambda,k}$ on $\mathcal{F}(\Gamma)$ over the boundary of $\mathcal{F}(\Gamma)^{\Sigma}$ by using the local charts (5.12).

Proposition 5.12. *For Γ neat, we have $M_{\lambda,k}(\Gamma) = H^0(\mathcal{F}(\Gamma)^{\Sigma}, \mathcal{E}_{\lambda,k})$.*

Proof. We have the natural inclusion

$$H^0(\mathcal{F}(\Gamma)^{\Sigma}, \mathcal{E}_{\lambda,k}) \hookrightarrow H^0(\mathcal{F}(\Gamma), \mathcal{E}_{\lambda,k}) = M_{\lambda,k}(\Gamma).$$

It is sufficient to see that this is surjective. Let $f \in M_{\lambda,k}(\Gamma)$. As a section of $\mathcal{E}_{\lambda,k}$ over $\mathcal{X}(I)$, f extends holomorphically over $\mathcal{X}(I)^{\Sigma_I}$ by Lemma 3.11. By the gluing, f extends holomorphically over $\overline{\mathcal{X}(J)}$. Therefore, as a section of $\mathcal{E}_{\lambda,k}$ over $\mathcal{F}(\Gamma)$, f extends holomorphically over $\mathcal{F}(\Gamma)^{\Sigma}$. ∎

Let us remark an immediate consequence of this interpretation. We go back to a general finite-index subgroup Γ of $\mathrm{O}^{+}(L)$. For a fixed λ, the direct sum $\bigoplus_{k\geq 0} M_{\lambda,k}(\Gamma)$ is a module over the ring $\bigoplus_{k\geq 0} M_{k}(\Gamma)$ of scalar-valued modular forms.

Proposition 5.13. *For each λ, the module $\bigoplus_{k} M_{\lambda,k}(\Gamma)$ is finitely generated over the ring $\bigoplus_{k} M_{k}(\Gamma)$.*

Proof. We may assume that Γ is neat by replacing the given Γ by its neat subgroup of finite index. We take a smooth toroidal compactification $\mathcal{F}(\Gamma)^{\Sigma}$ as above and let $\pi\colon \mathcal{F}(\Gamma)^{\Sigma} \to \mathcal{F}(\Gamma)^{bb}$ be the projection to the Baily–Borel compactification. Then $\mathcal{L}^{\otimes n} = \pi^*\mathcal{O}(1)$ for an ample line bundle $\mathcal{O}(1)$ on $\mathcal{F}(\Gamma)^{bb}$ by [36, Proposition 3.4 (b)]. (In fact, $\mathcal{L}$ itself descends, but we do not need that.) It suffices to show that for each $0 \leq k_0 < n$, the module $\bigoplus_k M_{\lambda,k_0+nk}(\Gamma)$ is finitely generated over $\bigoplus_k M_{nk}(\Gamma)$. By Proposition 5.12, we have

$$\begin{aligned}\bigoplus_{k\geq 0} M_{\lambda,k_0+nk}(\Gamma) &= \bigoplus_{k\geq 0} H^0(\mathcal{F}(\Gamma)^{\Sigma}, \mathcal{E}_{\lambda,k_0} \otimes \pi^*\mathcal{O}(k)) \\ &\simeq \bigoplus_{k\geq 0} H^0(\mathcal{F}(\Gamma)^{bb}, \pi_*\mathcal{E}_{\lambda,k_0} \otimes \mathcal{O}(k)),\end{aligned}$$

where the second isomorphism follows from the projection formula for π. Since $\mathcal{F}(\Gamma)^{bb}$ is projective, the last module is finitely generated over the ring

$$\bigoplus_{k} H^0(\mathcal{F}(\Gamma)^{bb}, \mathcal{O}(k)) = \bigoplus_{k} M_{nk}(\Gamma)$$

by a general theorem of Serre (see, e.g., [37, p. 128]). ∎