Chapter 6

Geometry of Siegel operators

Let L be a lattice of signature $(2, n)$ with $n \geq 3$ and Γ be a finite-index subgroup of $\mathrm{O}^+(L)$. Let $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n)$ be a partition expressing an irreducible representation of $\mathrm{O}(n, \mathbb{C})$. We assume $\lambda \neq 1, \det$. This in particular implies $\lambda_n = 0$, and so, ${}^t\lambda_1 < n$. In Proposition 3.7, we proved that a modular form $f \in M_{\lambda,k}(\Gamma)$ always vanishes at all 0-dimensional cusps. In this chapter we study the restriction of f to a 1-dimensional cusp, an operation usually called the *Siegel operator*.

Let J be a rank 2 primitive isotropic sublattice of L, which we fix throughout this chapter. A traditional way to define the Siegel operator Φ_J at the J-cusp is to choose a 0-dimensional cusp $I \subset J$, take the I-trivialization and the coordinates (τ, z, w) as in Section 5.1.2, and set

$$(\Phi_J f)(\tau) = \lim_{t \to \infty} f(\tau, 0, it), \quad \tau \in \mathbb{H}. \tag{6.1}$$

In this way it is easy to define the Siegel operator, but we have to check the modularity of $\Phi_J f$ and calculate its reduced weight *after* defining it.

In this chapter we take a more geometric approach working directly with the automorphic vector bundle $\mathcal{E}_{\lambda,k}$. This improves the geometric understanding of the Siegel operator, and tells us a priori the modularity of $\Phi_J f$ and its weight. We work with the partial toroidal compactification $\overline{\mathcal{X}(J)}$, rather than with the Baily–Borel compactification, because the boundary structure of $\overline{\mathcal{X}(J)}$ is easier to handle and $\mathcal{E}_{\lambda,k}$ extends to a vector bundle over $\overline{\mathcal{X}(J)}$ as we have seen in Section 5. We also wanted to put the Siegel operator on the same ground as the Fourier–Jacobi expansion (Section 7). Understanding the Siegel operator at the level of toroidal compactification will be useful in some geometric applications.

Let Δ_J be the boundary divisor of $\overline{\mathcal{X}(J)}$ and $\pi_2 \colon \Delta_J \to \mathbb{H}_J$ be the projection to the J-cusp. Let $\mathcal{L}_J$ be the Hodge bundle on $\mathbb{H}_J$. For $V(J) = (J^\perp/J)_{\mathbb{C}}$ we denote by $V(J)_{\lambda'}$ the irreducible representation of $\mathrm{O}(V(J)) \simeq \mathrm{O}(n-2, \mathbb{C})$ with partition $\lambda' = (\lambda_2 \geq \cdots \geq \lambda_{n-1})$. Our result is summarized as follows.

Theorem 6.1. *Let* $\lambda \neq 1, \det$. *There exists a* $\Gamma(J)_{\mathbb{R}}$*-invariant sub vector bundle* $\mathcal{E}_\lambda^J$ *of* $\mathcal{E}_\lambda$ *with the following properties.*

(1) $\mathcal{E}_\lambda^J$ *extends to a sub vector bundle of the canonical extension of* $\mathcal{E}_\lambda$ *over* $\overline{\mathcal{X}(J)}$.

(2) *We have a* $\overline{\Gamma(J)_{\mathbb{R}}}$*-equivariant isomorphism* $\mathcal{E}_\lambda^J|_{\Delta_J} \simeq \pi_2^* \mathcal{L}_J^{\otimes \lambda_1} \otimes V(J)_{\lambda'}$.

(3) *If* f *is a* Γ*-modular form of weight* (λ, k)*, its restriction to* Δ_J *as a section of* $\mathcal{E}_{\lambda,k}$ *takes values in the sub vector bundle* $\mathcal{E}_\lambda^J \otimes \mathcal{L}^{\otimes k}|_{\Delta_J}$ *of* $\mathcal{E}_{\lambda,k}|_{\Delta_J}$.

In particular, we have

$$f|_{\Delta_J} = \pi_2^*(\Phi_J f)$$

for a $V(J)_{\lambda'}$*-valued cusp form* $\Phi_J f$ *of weight* $k+\lambda_1$ *on* $\mathbb{H}_J$ *with respect to the image of* $\Gamma(J)_{\mathbb{Z}} \to \mathrm{SL}(J)$*. If* $f = \sum_l a(l)q^l$ *is the Fourier expansion of* f *at a* 0*-dimensional cusp* $I \subset J$*, the Fourier expansion of* $\Phi_J f$ *at the* I*-cusp of* $\mathbb{H}_J$ *is given by*

$$(\Phi_J f)(\tau) = \sum_{l \in \sigma_J \cap U(I)_{\mathbb{Z}}^{\vee}} a(l)e((l,\tau)), \qquad \tau \in \mathbb{H}_J \subset U(I)_{\mathbb{C}}/U(J)_{\mathbb{C}}^{\perp}, \tag{6.2}$$

where $\sigma_J = (U(J)_{\mathbb{R}})_{\geq 0}$ *is the isotropic ray in* $U(I)_{\mathbb{R}}$ *corresponding to* J*.*

In (6.2), the pairing (l,τ) for $l \in \sigma_J$ and $\tau \in \mathbb{H}_J$ is the natural pairing between $U(J)_{\mathbb{C}}$ and $U(I)_{\mathbb{C}}/U(J)_{\mathbb{C}}^{\perp}$. (This $\tau \in \mathbb{H}_J$ is different from the coordinate $\tau \in \mathbb{H}$ in Section 5.1.2, but rather is identified with the point τl_J there.)

A point here is that the vector bundle $\mathcal{E}_{\lambda,k}$ "reduces" to the sub vector bundle $\mathcal{E}_{\lambda}^{J} \otimes \mathcal{L}^{\otimes k}$ at the boundary divisor Δ_J. This is the difference with the Siegel operator in the scalar-valued case. This reduction corresponds to the reduction $\lambda \rightsquigarrow \lambda_1 \boxtimes \lambda'$ of the weight, and makes it possible to descend $f|_{\Delta_J}$ to $\mathbb{H}_J$. Roughly speaking, this reduction occurs as a result of taking the direct image of $\mathcal{E}_{\lambda,k}$ to the Baily–Borel compactification. In this way, the naive Siegel operator (6.1) can be more geometrically understood as

$$\text{restriction to } \Delta_J + \text{reduction to } \mathcal{E}_{\lambda}^{J} \otimes \mathcal{L}^{\otimes k} + \text{descend to } \mathbb{H}_J.$$

The sub vector bundle $\mathcal{E}_{\lambda}^{J}$ will be taken up again in Section 8.3 from the viewpoint of a filtration on $\mathcal{E}_{\lambda}$.

In Section 6.1 we prepare some calculations related to $\mathcal{E}_{\lambda}^{J}$. In Section 6.2 we define $\mathcal{E}_{\lambda}^{J}$ and prove the properties (1), (2) in Theorem 6.1. The Siegel operator Φ_J is defined in Section 6.3, and the remaining assertions of Theorem 6.1 are proved there.

6.1 Invariant part for a unipotent group

This section is preliminaries for introducing the Siegel operator. We prove that the Fourier coefficients of a modular form in the J-ray are contained in the invariant subspace for a certain unipotent subgroup of $\mathrm{O}(n,\mathbb{C})$, and study this space as a representation of $\mathbb{C}^* \times \mathrm{O}(n-2,\mathbb{C})$.

Let $F = \mathbb{Q}, \mathbb{R}$. Let $W(J)_F \subset \Gamma(J)_F$ be the Heisenberg group and the Jacobi group for J over F defined in Section 5.2. We choose a rank 1 primitive sublattice I of J, and also a rank 1 sublattice I' of L with $(I, I') \neq 0$. Let $\Gamma(I)_F$ be the stabilizer of I as in Section 3.3.2 and let

$$\Gamma(I,J)_F = \Gamma(J)_F \cap \mathrm{Ker}(\Gamma(I)_F \to \mathrm{GL}(I)).$$

By definition $\Gamma(I,J)_F$ consists of isometries of L_F which act trivially on I_F, J_F/I_F and $V(J)_F = (J^{\perp}/J)_F$. As a subgroup of $\Gamma(J)_F$, $\Gamma(I,J)_F$ contains $W(J)_F$, and the quotient $\Gamma(I,J)_F/W(J)_F \simeq F$ is the subgroup of $\Gamma(J)_F/W(J)_F \simeq \mathrm{SL}(J_F)$ which acts trivially on I_F.

As a subgroup of $\Gamma(I)_F$, $\Gamma(I,J)_F$ contains the unipotent radical $U(I)_F$ of $\Gamma(I)_F$ by (5.8). Let $U(J/I)_F$ be the subgroup of $\mathrm{O}(V(I)_F)$ acting trivially on J_F/I_F and $V(J)_F$. Then $U(J/I)_F$ is the image of $\Gamma(I,J)_F$ in $\mathrm{O}(V(I)_F)$. This is also the image of $W(J)_F$ in $\mathrm{O}(V(I)_F)$. From (3.11), we have the exact sequence

$$0 \to U(I)_F \to \Gamma(I,J)_F \to U(J/I)_F \to 0. \tag{6.3}$$

By (1.2), the group $U(J/I)_F$ is the unipotent radical of the stabilizer of J_F/I_F in $\mathrm{O}(V(I)_F)$ and consists of the Eichler transvections of $V(I)_F$ with respect to J_F/I_F. We have a canonical isomorphism

$$U(J/I)_F \simeq V(J)_F \otimes_F (J_F/I_F).$$

We define $U(J/I)_{\mathbb{C}} < \mathrm{O}(V(I))$ similarly.

Now let f be a modular form of weight (λ,k) with respect to Γ, and $f = \sum_l a(l)q^l$ be its Fourier expansion at I. We are interested in the Fourier coefficients $a(l) \in V(I)_{\lambda,k}$ for l in the isotropic ray $\sigma_J = ((J/I)_{\mathbb{R}} \otimes I_{\mathbb{R}})_{\geq 0}$ corresponding to J. We denote by

$$V(I)_\lambda^U = V(I)_\lambda^{U(J/I)_{\mathbb{C}}}$$

the invariant subspace of $V(I)_\lambda$ for the unipotent subgroup $U(J/I)_{\mathbb{C}}$ of $\mathrm{O}(V(I))$, and put

$$V(I)_{\lambda,k}^U = V(I)_\lambda^U \otimes (I_{\mathbb{C}}^{\vee})^{\otimes k} \subset V(I)_{\lambda,k}.$$

Lemma 6.2. *If $l \in U(I)_{\mathbb{Z}}^{\vee} \cap \sigma_J$, then $a(l) \in V(I)_{\lambda,k}^U$.*

Proof. We take the splitting of (6.3) for $F = \mathbb{Q}$ following (3.12), and accordingly express elements of $\Gamma(I,J)_{\mathbb{Q}}$ as (γ_1,α), where $\gamma_1 \in U(J/I)_{\mathbb{Q}} \subset \mathrm{O}(V(I)_{\mathbb{Q}})$ and $\alpha \in U(I)_{\mathbb{Q}}$. (In the notation (3.14), this is $(\gamma_1 \otimes \mathrm{id}_I, 1, \alpha)$.) There exists a finite-index subgroup H of $\Gamma(I,J)_{\mathbb{Q}} \cap \Gamma$ such that $\alpha \in U(I)_{\mathbb{Z}}$ for every element (γ_1,α) of H. The group $\Gamma(I,J)_{\mathbb{Q}}$ acts trivially on the isotropic ray σ_J. Therefore, if $l \in U(I)_{\mathbb{Z}}^{\vee} \cap \sigma_J$, we see from Proposition 3.6 that

$$a(l) = a(\gamma_1 l) = \gamma_1(a(l))$$

for every element (γ_1,α) of H. Here $\gamma_1 \in U(J/I)_{\mathbb{Q}}$ acts on $V(I)_{\lambda,k}$ by its natural action on $V(I)_\lambda$. This equality means that $a(l)$ is contained in the H-invariant subspace $V(I)_{\lambda,k}^H = V(I)_\lambda^H \otimes (I_{\mathbb{C}}^{\vee})^{\otimes k}$ of $V(I)_{\lambda,k}$. The image of H by the projection

$$\Gamma(I,J)_{\mathbb{Q}} \to U(J/I)_{\mathbb{Q}}, \quad (\gamma_1,\alpha) \mapsto \gamma_1,$$

is a full lattice in $U(J/I)_{\mathbb{Q}}$. In particular, it is Zariski dense in $U(J/I)_{\mathbb{C}}$. This shows that $V(I)^H_\lambda = V(I)^U_\lambda$, and so, $a(l) \in V(I)^U_{\lambda,k}$. ■

Let $P(J/I)_{\mathbb{C}}$ be the stabilizer of the isotropic line $J_{\mathbb{C}}/I_{\mathbb{C}} \subset V(I)$ in $\mathrm{O}(V(I))$. Then $U(J/I)_{\mathbb{C}}$ is the unipotent radical of $P(J/I)_{\mathbb{C}}$ and sits in the exact sequence (cf. (1.2))

$$0 \to U(J/I)_{\mathbb{C}} \to P(J/I)_{\mathbb{C}} \to \mathrm{GL}(J_{\mathbb{C}}/I_{\mathbb{C}}) \times \mathrm{O}(V(J)) \to 1. \tag{6.4}$$

Therefore $V(I)^U_\lambda$ is a representation of

$$\mathrm{GL}(J_{\mathbb{C}}/I_{\mathbb{C}}) \times \mathrm{O}(V(J)) \simeq \mathbb{C}^* \times \mathrm{O}(n-2,\mathbb{C}) \simeq \mathrm{SO}(2,\mathbb{C}) \times \mathrm{O}(n-2,\mathbb{C}).$$

Proposition 6.3. *Let* $\lambda \neq \det$. *As a representation of* $\mathbb{C}^* \times \mathrm{O}(V(J))$ *we have*

$$V(I)^U_\lambda \simeq \chi_{\lambda_1} \boxtimes V(J)_{\lambda'},$$

where χ_{λ_1} *is the character of* $\mathbb{C}^*$ *of weight* λ_1 *and* $V(J)_{\lambda'}$ *is the irreducible representation of* $\mathrm{O}(V(J))$ *associated to the partition* $\lambda' = (\lambda_2 \geq \cdots \geq \lambda_{n-1})$.

Proof. This is purely a representation-theoretic calculation. Let us first rewrite the setting. Let $V = \mathbb{C}^n$ be an n-dimensional quadratic space over $\mathbb{C}$ with a basis $e_1, \dots, e_n$ such that $(e_i, e_j) = 1$ if $i + j = n + 1$ and $(e_i, e_j) = 0$ otherwise. Let P be the stabilizer of the isotropic line $\mathbb{C}e_1$ in $\mathrm{O}(V)$ and let $V' = \langle e_2, \dots, e_{n-1} \rangle$. Then

$$P = (\mathbb{C}^* \times \mathrm{O}(V')) \ltimes U,$$

where $\mathbb{C}^* = \mathrm{SO}(\langle e_1, e_n \rangle) \simeq \mathrm{GL}(\mathbb{C}e_1)$ and U is the group of unipotent matrices

$$\begin{pmatrix} 1 & -v^\vee & -(v,v)/2 \\ 0 & I_{n-2} & v \\ 0 & 0 & 1 \end{pmatrix} \qquad v \in V'.$$

The problem is to calculate the U-invariant part V^U_λ of V_λ as a representation of the reductive part $\mathbb{C}^* \times \mathrm{O}(V')$.

Step 1. There exists a $\mathbb{C}^* \times \mathrm{O}(V')$-equivariant embedding $\chi_{\lambda_1} \boxtimes V'_{\lambda'} \hookrightarrow V^U_\lambda$.

Proof. We write

$$W_0 = \wedge^{{}^t\lambda_1} V \otimes \cdots \otimes \wedge^{{}^t\lambda_{\lambda_1}} V,$$

$$W_0{}' = \wedge^{{}^t\lambda_1 - 1} V' \otimes \cdots \otimes \wedge^{{}^t\lambda_{\lambda_1} - 1} V',$$

$$W_1 = \left(\mathbb{C}e_1 \wedge \wedge^{{}^t\lambda_1 - 1} V'\right) \otimes \cdots \otimes \left(\mathbb{C}e_1 \wedge \wedge^{{}^t\lambda_{\lambda_1} - 1} V'\right).$$

We have a natural $\mathbb{C}^* \times \mathrm{O}(V')$-equivariant isomorphism

$$\iota : \mathbb{C}e_1^{\otimes \lambda_1} \otimes W_0' \xrightarrow{\simeq} W_1 \subset W_0.$$

Recall from (3.1) that $V_\lambda \subset W_0$ and $V'_{\lambda'} \subset W'_0$. (Here we notice that the transpose of λ' is $({}^t\lambda_1 - 1, \dots, {}^t\lambda_{\lambda_1} - 1)$.) We shall show that the image of $\mathbb{C}e_1^{\otimes \lambda_1} \otimes V'_{\lambda'}$ by ι is contained in V_λ^U. Since $\mathbb{C}e_1^{\otimes \lambda_1} \simeq \chi_{\lambda_1}$ as a representation of $\mathbb{C}^*$, this would imply our assertion.

Since U acts on W_1 trivially, it does so on $\iota(\mathbb{C}e_1^{\otimes \lambda_1} \otimes V'_{\lambda'})$. Thus it suffices to see that $\iota(\mathbb{C}e_1^{\otimes \lambda_1} \otimes V'_{\lambda'})$ is contained in V_λ. Recall from (3.2) that V_λ and $V'_{\lambda'}$, respectively, contain the vectors

$$v_\lambda = \bigotimes_{i=1}^{\lambda_1} (e_1 \wedge \cdots \wedge e_{{}^t\lambda_i}), \quad v'_{\lambda'} = \bigotimes_{i=1}^{\lambda_1} (e_2 \wedge \cdots \wedge e_{{}^t\lambda_i}).$$

Since $\iota(e_1^{\otimes \lambda_1} \otimes v'_{\lambda'}) = v_\lambda$, we have $\mathrm{O}(V') \cdot \iota(e_1^{\otimes \lambda_1} \otimes v'_{\lambda'}) \subset V_\lambda$. Taking the linear hull and using the irreducibility of $V'_{\lambda'}$, we see that $\iota(\mathbb{C}e_1^{\otimes \lambda_1} \otimes V'_{\lambda'}) \subset V_\lambda$. ∎

For the proof of Proposition 6.3, it thus suffices to prove $\dim V'_{\lambda'} = \dim V_\lambda^U$. We use the restriction to $\mathrm{SO}(V) \subset \mathrm{O}(V)$. We first consider the case when V_λ remains irreducible as a representation of $\mathrm{SO}(V)$. As recalled in Section 3.6.1, this occurs exactly when n is odd or n is even with ${}^t\lambda_1 \neq n/2$, and V_λ has highest weight

$$\overline{\lambda} = (\overline{\lambda}_1, \dots, \overline{\lambda}_{[n/2]}) = (\lambda_1, \lambda_2 - \lambda_{n-1}, \dots, \lambda_{[n/2]} - \lambda_{n+1-[n/2]})$$

in this case.

Step 2. When V_λ is irreducible as a representation of $\mathrm{SO}(V)$, V_λ^U is irreducible as a representation of $\mathrm{SO}(V')$ with highest weight $\overline{\lambda}' = (\overline{\lambda}_2, \dots, \overline{\lambda}_{[n/2]})$. In particular, we have $\dim V'_{\lambda'} = \dim V_\lambda^U$.

Proof. Let B and B' be the groups of upper triangular matrices in $\mathrm{SO}(V)$ and $\mathrm{SO}(V')$, respectively (the standard Borel subgroups). Let U_0 and U'_0 be the groups of unipotent matrices in B and B', respectively. Then U and U'_0 generate U_0. Therefore we have

$$V_\lambda^{U_0} = (V_\lambda^U)^{U'_0}. \tag{6.5}$$

The space $V_\lambda^{U_0}$ is the highest weight space for the $\mathrm{SO}(V)$-representation V_λ, while $(V_\lambda^U)^{U'_0}$ is the highest weight space for the $\mathrm{SO}(V')$-representation V_λ^U. The irreducibility of V_λ as an $\mathrm{SO}(V)$-representation implies $\dim V_\lambda^{U_0} = 1$, which in turn implies by (6.5) the irreducibility of V_λ^U as a representation of $\mathrm{SO}(V')$.

We shall calculate the highest weight of V_λ^U for $\mathrm{SO}(V')$. Let T and T' be the groups of diagonal matrices in B and B', respectively. Then $T = \mathbb{C}^* \times T'$. The highest weight $\overline{\lambda} = (\overline{\lambda}_1, \dots, \overline{\lambda}_{[n/2]})$ of the $\mathrm{SO}(V)$-representation V_λ is the weight of the T-action on the highest weight space $V_\lambda^{U_0}$. Therefore T' acts by weight $\overline{\lambda}' = (\overline{\lambda}_2, \dots, \overline{\lambda}_{[n/2]})$ on $V_\lambda^{U_0}$. By (6.5), this means that the highest weight of V_λ^U for $\mathrm{SO}(V')$ is $\overline{\lambda}'$. ■

It remains to cover the exceptional case where V_λ gets reducible when restricted to $\mathrm{SO}(V)$, namely, n is even and ${}^t\lambda_1 = n/2$.

Step 3. We have $\dim V'_{\lambda'} = \dim V_\lambda^U$ even when V_λ is reducible as a representation of $\mathrm{SO}(V)$.

Proof. In this case, the irreducible summands of V_λ have highest weight $\overline{\lambda} = \lambda$ and $\lambda^\dagger$, respectively. We can argue similarly for each irreducible summand. This shows that V_λ^U as a representation of $\mathrm{SO}(V')$ has two irreducible summands, of highest weight $\lambda' = (\lambda_2, \dots, \lambda_{n/2})$ and $(\lambda^\dagger)' = (\lambda_2, \dots, -\lambda_{n/2})$. On the other hand, $V'_{\lambda'}$ is also reducible as a representation of $\mathrm{SO}(V')$ with highest weight λ' and $(\lambda')^\dagger = (\lambda^\dagger)'$ by Section 3.6.1. This implies that $V_\lambda^U \simeq V'_{\lambda'}$ as $\mathrm{SO}(V')$-representations. ■

The proof of Proposition 6.3 is now complete. ■

6.2 The sub vector bundle $\mathcal{E}_\lambda^J$

Let $\lambda \neq \det$. We define the sub vector bundle $\mathcal{E}_\lambda^J$ of $\mathcal{E}_\lambda$ as the image of $V(I)_\lambda^U \otimes \mathcal{O}_{\mathcal{D}}$ by the I-trivialization $\iota_I \colon V(I)_\lambda \otimes \mathcal{O}_{\mathcal{D}} \to \mathcal{E}_\lambda$.

Lemma 6.4. *The sub vector bundle $\mathcal{E}_\lambda^J$ of $\mathcal{E}_\lambda$ is $\Gamma(J)_{\mathbb{R}}$-invariant. In particular, it does not depend on the choice of I.*

Proof. Let $\gamma \in \Gamma(J)_{\mathbb{R}}$. What has to be shown is that the image of $V(I)_\lambda^U \otimes \mathcal{O}_{\mathcal{D}}$ by the composition homomorphism

$$V(I)_\lambda \otimes \mathcal{O}_{\mathcal{D}} \xrightarrow{\iota_I} \mathcal{E}_\lambda \xrightarrow{\gamma} \mathcal{E}_\lambda \xrightarrow{\iota_I^{-1}} V(I)_\lambda \otimes \mathcal{O}_{\mathcal{D}}$$

is again $V(I)_\lambda^U \otimes \mathcal{O}_{\mathcal{D}}$. This homomorphism coincides with

$$V(I)_\lambda \otimes \mathcal{O}_{\mathcal{D}} \xrightarrow{\gamma} V(\gamma I)_\lambda \otimes \mathcal{O}_{\mathcal{D}} \xrightarrow{\iota_{\gamma I}} \mathcal{E}_\lambda \xrightarrow{\iota_I^{-1}} V(I)_\lambda \otimes \mathcal{O}_{\mathcal{D}},$$

where $\iota_{\gamma I}$ is the γI-trivialization. The image of $V(I)_\lambda^U$ by $\gamma \colon V(I)_\lambda \to V(\gamma I)_\lambda$ is $V(\gamma I)_\lambda^U$, the invariant subspace of $V(\gamma I)_\lambda$ for the unipotent radical $U(\gamma J_{\mathbb{C}}/\gamma I_{\mathbb{C}}) = U(J_{\mathbb{C}}/\gamma I_{\mathbb{C}})$ of the stabilizer of $J_{\mathbb{C}}/\gamma I_{\mathbb{C}}$ in $\mathrm{O}(V(\gamma I))$. Therefore it suffices to show that the homomorphism

$$\iota_{\gamma I}^{-1} \circ \iota_I : V(I)_\lambda \otimes \mathcal{O}_{\mathcal{D}} \to V(\gamma I)_\lambda \otimes \mathcal{O}_{\mathcal{D}}$$

sends $V(I)_\lambda^U \otimes \mathcal{O}_{\mathcal{D}}$ to $V(\gamma I)_\lambda^U \otimes \mathcal{O}_{\mathcal{D}}$.

The problem is pointwise. Let $[\omega] \in \mathcal{D}$. At the fiber of $\mathcal{E}$ over $[\omega]$, the difference of the I-trivialization and the γI-trivialization is the isometry

$$I_{\mathbb{C}}^{\perp}/I_{\mathbb{C}} \to I_{\mathbb{C}}^{\perp} \cap \omega^{\perp} \to \omega^{\perp}/\mathbb{C}\omega \to \gamma I_{\mathbb{C}}^{\perp} \cap \omega^{\perp} \to \gamma I_{\mathbb{C}}^{\perp}/\gamma I_{\mathbb{C}}.$$

This sends the isotropic line $J_{\mathbb{C}}/I_{\mathbb{C}}$ of $I_{\mathbb{C}}^{\perp}/I_{\mathbb{C}}$ as

$$J_{\mathbb{C}}/I_{\mathbb{C}} \to J_{\mathbb{C}} \cap \omega^{\perp} = J_{\mathbb{C}} \cap \omega^{\perp} \to J_{\mathbb{C}}/\gamma I_{\mathbb{C}}.$$

Therefore the induced isomorphism

$$\mathrm{O}(V(I)) \to \mathrm{O}(V(\gamma I))$$

sends the subgroup $U(J/I)_{\mathbb{C}}$ to $U(J_{\mathbb{C}}/\gamma I_{\mathbb{C}})$. It follows that the induced isomorphism

$$(\iota_{\gamma I}^{-1} \circ \iota_I)_{[\omega]} : V(I)_\lambda \to V(\gamma I)_\lambda$$

sends $V(I)_\lambda^U$ to $V(\gamma I)_\lambda^U$. ∎

Recall that the canonical extension of $\mathcal{E}_\lambda$ over the partial toroidal compactification $\overline{\mathcal{X}(J)}$ is defined via the I-trivialization $V(I)_\lambda \otimes \mathcal{O}_{\mathcal{X}(J)} \to \mathcal{E}_\lambda$. Therefore, by construction, $\mathcal{E}_\lambda^J$ extends to a sub vector bundle of the canonical extension of $\mathcal{E}_\lambda$ (again denoted by $\mathcal{E}_\lambda^J$). The I-trivialization $\mathcal{E}_\lambda \to V(I)_\lambda \otimes \mathcal{O}_{\overline{\mathcal{X}(J)}}$ over $\overline{\mathcal{X}(J)}$ sends $\mathcal{E}_\lambda^J$ to $V(I)_\lambda^U \otimes \mathcal{O}_{\overline{\mathcal{X}(J)}}$.

Proposition 6.5. *There exists an* $\mathrm{SL}(J_{\mathbb{R}})$*-equivariant vector bundle* $\Phi_J \mathcal{E}_\lambda$ *on* $\mathbb{H}_J$ *such that we have a* $\overline{\Gamma(J)}_{\mathbb{R}}$*-equivariant isomorphism*

$$\mathcal{E}_\lambda^J|_{\Delta_J} \simeq \pi_2^*(\Phi_J \mathcal{E}_\lambda)$$

of vector bundles on Δ_J.

Proof. Let $j(\gamma, [\omega])$ be the factor of automorphy of the $\Gamma(J)_{\mathbb{R}}$-action on $\mathcal{E}_\lambda^J$ with respect to the I-trivialization $\mathcal{E}_\lambda^J \simeq V(I)_\lambda^U \otimes \mathcal{O}_{\mathcal{D}}$. This is a $\mathrm{GL}(V(I)_\lambda^U)$-valued function on $\Gamma(J)_{\mathbb{R}} \times \mathcal{D}$. We shall prove the following.

(1) For fixed γ, the function $j(\gamma, [\omega])$ of $[\omega]$ is constant on each fiber of $\mathcal{D} \to \mathbb{H}_J$.

(2) $j(\gamma, [\omega]) = \mathrm{id}$ if $\gamma \in W(J)_{\mathbb{R}}$.

Since $\Gamma(J)_{\mathbb{R}}/W(J)_{\mathbb{R}} \simeq \mathrm{SL}(J_{\mathbb{R}})$, these properties ensure that $j(\gamma, [\omega])$ descends to a $\mathrm{GL}(V(I)_\lambda^U)$-valued function on $\mathrm{SL}(J_{\mathbb{R}}) \times \mathbb{H}_J$. This function defines the factor of automorphy of an $\mathrm{SL}(J_{\mathbb{R}})$-equivariant vector bundle $\Phi_J \mathcal{E}_\lambda$ on $\mathbb{H}_J$ such that $\mathcal{E}_\lambda^J \simeq \pi^*(\Phi_J \mathcal{E}_\lambda)$ as $\Gamma(J)_{\mathbb{R}}$-equivariant vector bundles on $\mathcal{D}$. This gives an isomorphism $\mathcal{E}_\lambda^J|_{\Delta_J} \simeq \pi_2^*(\Phi_J \mathcal{E}_\lambda)$ over Δ_J.

We first check the property (2). Since $W(J)_{\mathbb{R}}$ acts on $I_{\mathbb{R}}$ trivially, we see from Lemma 3.2 that the factor of automorphy of the $W(J)_{\mathbb{R}}$-action on $\mathcal{E}_\lambda$ with respect to the I-trivialization is given by the natural action of $W(J)_{\mathbb{R}}$ on $V(I)_\lambda$. Since the image of $W(J)_{\mathbb{R}}$ in $\mathrm{O}(V(I)_{\mathbb{R}})$ is equal to $U(J/I)_{\mathbb{R}}$, $W(J)_{\mathbb{R}}$ acts on $V(I)_\lambda^U$ trivially by definition. This implies (2).

Next we verify the property (1). The fibers of $\mathcal{D} \to \mathcal{V}_J$ are contained in $U(J)_{\mathbb{C}}$-orbits in $\mathcal{D}(J) \supset \mathcal{D}$, and the fibers of $\Delta_J \to \mathbb{H}_J$ are $W(J)_{\mathbb{R}}/U(J)_{\mathbb{R}}$-orbits. In particular, the constancy on the fibers of $\mathcal{D} \to \mathcal{V}_J$ would follow from the constancy on $U(J)_{\mathbb{R}}$-orbits and the identity theorem in complex analysis for $U(J)_{\mathbb{R}} \subset U(J)_{\mathbb{C}}$. Thus we are reduced to checking the constancy on $W(J)_{\mathbb{R}}$-orbits. Let $\gamma \in \Gamma(J)_{\mathbb{R}}$ and $g \in W(J)_{\mathbb{R}}$. Then we can calculate

$$\begin{aligned} j(\gamma, g([\omega])) &= j(\gamma g, [\omega]) \circ j(g, [\omega])^{-1} = j(\gamma g, [\omega]) \\ &= j(\gamma g \gamma^{-1}, \gamma([\omega])) \circ j(\gamma, [\omega]) = j(\gamma, [\omega]). \end{aligned}$$

In the second and the last equalities we used the property (2) proved above, with the normality of $W(J)_{\mathbb{R}}$ in $\Gamma(J)_{\mathbb{R}}$ in the last equality. The property (1) is thus proved. ■

Remark 6.6. By construction, $\Phi_J \mathcal{E}_\lambda$ is endowed with a trivialization

$$V(I)_\lambda^U \otimes \mathcal{O}_{\mathbb{H}_J} \simeq \Phi_J \mathcal{E}_\lambda,$$

whose pullback agrees with the I-trivialization $V(I)_\lambda^U \otimes \mathcal{O}_{\Delta_J} \simeq \mathcal{E}_\lambda^J|_{\Delta_J}$ of $\mathcal{E}_\lambda^J$ over Δ_J.

We can calculate the weights of $\Phi_J \mathcal{E}_\lambda$ by using Proposition 6.3. Let $\mathcal{L}_J$ be the Hodge bundle on $\mathbb{H}_J$.

Proposition 6.7. *There exists an* $\mathrm{SL}(J_{\mathbb{R}})$*-equivariant isomorphism*

$$\Phi_J \mathcal{E}_\lambda \simeq \mathcal{L}_J^{\otimes \lambda_1} \otimes V(J)_{\lambda'},$$

of vector bundles on $\mathbb{H}_J$.

The proof of Proposition 6.7 is divided into several steps. Let us formulate the first half as preparatory lemmas as follows. Let $P(J)$ be the stabilizer of $J_{\mathbb{C}}$ in $\mathrm{O}(L_{\mathbb{C}})$. We write $Q(J) = Q - Q \cap \mathbb{P} J_{\mathbb{C}}^{\perp}$. Recall that $\mathcal{E}_\lambda$ is naturally defined over Q as an $\mathrm{O}(L_{\mathbb{C}})$-equivariant vector bundle.

Lemma 6.8. *The vector bundle* $\mathcal{E}_\lambda^J$ *extends to a* $P(J)$*-invariant sub vector bundle of* $\mathcal{E}_\lambda$ *over* $Q(J)$ *(again denoted by* $\mathcal{E}_\lambda^J$*).*

Proof. For each $\mathbb{C}$-line $I' \subset J_{\mathbb{C}}$, the I'-trivialization $\iota_{I'}\colon V(I')_\lambda \otimes \mathcal{O} \to \mathcal{E}_\lambda$ is defined over $Q(I') = Q - Q \cap \mathbb{P}(I')^{\perp}$. The same argument as the second half of the proof of Lemma 6.4 shows that for two $\mathbb{C}$-lines $I_1, I_2 \subset J_{\mathbb{C}}$, we have

$$\iota_{I_1}(V(I_1)_\lambda^U \otimes \mathcal{O}) = \iota_{I_2}(V(I_2)_\lambda^U \otimes \mathcal{O})$$

over $Q(I_1) \cap Q(I_2)$. Therefore, by gluing the image of $\iota_{I'}$ for all $\mathbb{C}$-lines $I' \subset J_{\mathbb{C}}$, we obtain a sub vector bundle of $\mathcal{E}_\lambda$ over $Q(J) = \bigcup_{I'} Q(I')$ which extends $\mathcal{E}_\lambda^J$. Since $\gamma \in P(J)$ sends $\iota_{I'}(V(I')_\lambda^U \otimes \mathcal{O})$ to $\iota_{\gamma I'}(V(\gamma I')_\lambda^U \otimes \mathcal{O})$ (cf. the proof of Lemma 6.4), this sub vector bundle is $P(J)$-invariant. ■

Lemma 6.9. *Let $D_J = Q(J) \cap \mathbb{P}I_{\mathbb{C}}^{\perp}$. The I-trivialization $V(I)_\lambda^U \otimes \mathcal{O}_{\mathcal{D}(I)} \to \mathcal{E}_\lambda^J$ over $\mathcal{D}(I)$ extends to an isomorphism*

$$V(I)_\lambda^U \otimes \mathcal{O}_{Q(J)} \to \mathcal{E}_\lambda^J \otimes \mathcal{O}_{Q(J)}(\lambda_1 D_J) \tag{6.6}$$

over $Q(J)$, which is equivariant with respect to the stabilizer of $I_{\mathbb{C}}$ in $P(J)$.

Proof. We choose an arbitrary embedding $2U_{\mathbb{C}} \hookrightarrow L_{\mathbb{C}}$ compatible with $I_{\mathbb{C}} \subset J_{\mathbb{C}}$ in the sense of Section 5 and accordingly take a lift $\mathrm{GL}(J_{\mathbb{C}}) \hookrightarrow P(J)$ of $\mathrm{GL}(J_{\mathbb{C}})$. Let $T \simeq \mathbb{C}^*$ be the subgroup of $\mathrm{GL}(J_{\mathbb{C}})$ consisting of matrices $\left(\begin{smallmatrix}1 & 0\\ 0 & \alpha\end{smallmatrix}\right)$, $\alpha \in \mathbb{C}^*$, with respect to the basis e_1, e_2 of $J_{\mathbb{C}}$. (e_1 spans $I_{\mathbb{C}}$ and e_2 spans $J_{\mathbb{C}}/I_{\mathbb{C}}$.) The image of T in $P(J/I)_{\mathbb{C}}$ is a lift of $\mathrm{GL}(J_{\mathbb{C}}/I_{\mathbb{C}})$ in (6.4). Then

$$V(I) = \mathbb{C}e_2 \oplus V(J) \oplus \mathbb{C}f_2$$

is the weight decomposition for T, where $\mathbb{C}e_2$, $V(J)$, $\mathbb{C}f_2$ have weight 1, 0, -1, respectively. A general T-orbit $C^\circ = T[\omega]$ in $\mathcal{D}(I)$ gives a flow converging to the point $p = [f_2]$ of D_J as $\alpha \to 0$ from a normal direction. Let $C = C^\circ \cup p \simeq \mathbb{C}$ be the closure of such a T-orbit in $Q(J)$. The proof of Lemma 6.9 is based on the following assertion.

Claim 6.10. *The I-trivialization $\mathcal{E}|_{C^\circ} \simeq V(I) \otimes \mathcal{O}_{C^\circ}$ over C° extends to an isomorphism*

$$\mathcal{E}|_C \simeq \mathbb{C}e_2 \otimes \mathcal{O}_C(-p) \oplus V(J) \otimes \mathcal{O}_C \oplus \mathbb{C}f_2 \otimes \mathcal{O}_C(p)$$

over C.

We postpone the proof of this claim for a while and continue the proof of Lemma 6.9. From Claim 6.10, we see that if $V(I)_\lambda = \bigoplus_r V(r)$ is the weight decomposition for T with $V(r)$ the weight r subspace, the I-trivialization of $\mathcal{E}_\lambda$ over C° extends to an isomorphism

$$\mathcal{E}_\lambda|_C \simeq \bigoplus_r V(r) \otimes \mathcal{O}_C(-rp)$$

over C. Since $V(I)_\lambda^U \subset V(\lambda_1)$ by Proposition 6.3, we obtain

$$\mathcal{E}_\lambda^J|_C \simeq V(I)_\lambda^U \otimes \mathcal{O}_C(-\lambda_1 p).$$

Finally, if we vary the embedding $2U_{\mathbb{C}} \hookrightarrow L_{\mathbb{C}}$, then the point $p = [f_2]$ runs over D_J. This implies the assertion of Lemma 6.9. ■

We give the postponed proof of Claim 6.10.

Proof of Claim 6.10. Let $v \in V(I)$ be a weight vector for T with weight $r \in \{-1, 0, 1\}$ and let s_v be the corresponding section of $\mathcal{E}$. We calculate the limit behavior of s_v on the T-orbit $C^\circ = T[\omega]$ as $\alpha \to 0$. We write $\gamma_\alpha = \left(\begin{smallmatrix} 1 & 0 \\ 0 & \alpha \end{smallmatrix}\right) \in T$. We lift $V(I) \hookrightarrow I^\perp_{\mathbb{C}}$ by the given embedding $U_{\mathbb{C}} \hookrightarrow L_{\mathbb{C}}$. By Lemma 2.6 for $l = e_1 \in I$, we have

$$\begin{aligned} s_v(\gamma_\alpha[\omega]) &= v - (v, s_{e_1}(\gamma_\alpha[\omega]))e_1 \mod \mathbb{C}\gamma_\alpha(\omega) \\ &= v - (v, \gamma_\alpha(s_{e_1}([\omega])))e_1 \mod \mathbb{C}\gamma_\alpha(\omega) \\ &= v - (\gamma_\alpha^{-1}(v), s_{e_1}([\omega]))e_1 \mod \mathbb{C}\gamma_\alpha(\omega) \\ &= v - \alpha^{-r}(v, s_{e_1}([\omega]))e_1 \mod \mathbb{C}\gamma_\alpha(\omega). \end{aligned} \tag{6.7}$$

We take the $\mathbb{C}e_2$-trivialization of $\mathcal{E}$ and express $s_v(\gamma_\alpha[\omega])$ as a $V(\mathbb{C}e_2)$-valued function. We identify $V(\mathbb{C}e_2) = \mathbb{C}e_1 \oplus V(J) \oplus \mathbb{C}f_1$ naturally. Then, according to the weight r of v, we have

$$s_v(\gamma_\alpha[\omega]) = \begin{cases} v + C_1(v)e_1 & v \in V(J), \\ \alpha^{-1}C_2e_1 & v = e_2, \\ \alpha C_2^{-1} f_1 + \alpha C_3 e_1 + \alpha v_0 & v = f_2 \end{cases}$$

as a $V(\mathbb{C}e_2)$-valued function. Here $C_1(v)$ is a linear function on $V(J)$, $C_2 \neq 0$ and C_3 are constants, and $v_0 \in V(J)$ is some constant vector. These expressions in the cases $v \in V(J)$ and $v = e_2$ are apparent from (6.7), because the vector in (6.7) is already perpendicular to e_2 in these cases. The case $v = f_2$ follows from the conditions

$$(s_{f_2}, s_{e_2}) = 1, \quad (s_{f_2}, s_{f_2}) = 0, \quad (s_{f_2}, s_w) = 0 \quad \text{for } w \in V(J).$$

(This can also be calculated by using the coordinates (τ, z, w) in Section 5.1.2.) The assertion of Claim 6.10 now follows from these expressions. ∎

Now we can complete the proof of Proposition 6.7.

Proof of Proposition 6.7. We pass from $Q(J)$ to $\mathbb{P}J^\vee_{\mathbb{C}}$. By the same argument as the proof of Proposition 6.5 with $\Gamma(J)_{\mathbb{R}}$ replaced by $P(J)$ and $W(J)_{\mathbb{R}}$ replaced by the kernel of $P(J) \to \mathrm{GL}(J_{\mathbb{C}}) \times \mathrm{O}(V(J))$, we find that the $P(J)$-equivariant vector bundle $\mathcal{E}^J_\lambda$ on $Q(J)$ descends to a $\mathrm{GL}(J_{\mathbb{C}}) \times \mathrm{O}(V(J))$-equivariant vector bundle on $\mathbb{P}J^\vee_{\mathbb{C}}$. This is an extension of $\Phi_J\mathcal{E}_\lambda$, and we denote it again by $\Phi_J\mathcal{E}_\lambda$. Let $p_I = I^\perp \cap \mathbb{P}J^\vee_{\mathbb{C}}$ be the I-cusp of $\mathbb{H}_J$. Since D_J is the fiber of $Q(J) \to \mathbb{P}J^\vee_{\mathbb{C}}$ over p_I, we find that the isomorphism (6.6) descends to an isomorphism

$$V(I)^U_\lambda \otimes \mathcal{O}_{\mathbb{P}J^\vee_{\mathbb{C}}} \to \Phi_J\mathcal{E}_\lambda \otimes \mathcal{O}_{\mathbb{P}J^\vee_{\mathbb{C}}}(\lambda_1 p_I). \tag{6.8}$$

This is equivariant with respect to the stabilizer of $I_{\mathbb{C}}$ in $\mathrm{GL}(J_{\mathbb{C}})$ and $\mathrm{O}(V(J))$. Note that these groups act on $V(I)^U_\lambda$ by the representation in Proposition 6.3.

Claim 6.11. *The element* $g(\alpha) = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$ *of* $\mathrm{GL}(J_{\mathbb{C}})$, $\alpha \in \mathbb{C}^*$, *acts on the fiber of* $\Phi_J \mathcal{E}_\lambda$ *over* p_I *as the scalar multiplication by* α^{λ_1}.

We prove Claim 6.11. By Proposition 6.3, the matrices $\begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}$ in $\mathrm{GL}(J_{\mathbb{C}})$ act on $V(I)^U_\lambda$ as the scalar multiplication by α^{λ_1}. Moreover, the matrices $\begin{pmatrix} \beta & 0 \\ 0 & 1 \end{pmatrix}$ act on $V(I)$ trivially. It follows that $g(\alpha)$ acts on $V(I)^U_\lambda$ as the scalar multiplication by $\alpha^{-\lambda_1}$. On the other hand, since the tangent space of $p_I \in \mathbb{P}J^\vee_{\mathbb{C}}$ is

$$\mathrm{Hom}(I^\perp \cap J^\vee_{\mathbb{C}}, J^\vee_{\mathbb{C}}/(I^\perp \cap J^\vee_{\mathbb{C}})) \simeq \mathrm{Hom}((J/I)^\vee_{\mathbb{C}}, I^\vee_{\mathbb{C}}),$$

the element $g(\alpha)$ acts on it by the multiplication by α^{-2}. Hence $g(\alpha)$ acts on the fiber of $\mathcal{O}_{\mathbb{P}J^\vee_{\mathbb{C}}}(-\lambda_1 p_I)$ over p_I as the multiplication by $\alpha^{2\lambda_1}$. By the isomorphism (6.8), we find that $g(\alpha)$ acts on the fiber of $\Phi_J \mathcal{E}_\lambda$ over p_I as the scalar multiplication by $\alpha^{-\lambda_1} \cdot \alpha^{2\lambda_1} = \alpha^{\lambda_1}$. This proves Claim 6.11.

We go back to the proof of Proposition 6.7. The torus consisting of the matrices $g(\alpha)$ is the reductive part of the stabilizer of p_I in $\mathrm{SL}(J_{\mathbb{C}})$. Therefore Claim 6.11 implies that $\Phi_J \mathcal{E}_\lambda$ is isomorphic to a direct sum of copies of $\mathcal{L}_J^{\otimes \lambda_1}$ as an $\mathrm{SL}(J_{\mathbb{C}})$-equivariant vector bundle on $\mathbb{P}J^\vee_{\mathbb{C}}$. Moreover, by the isomorphism (6.8) and Proposition 6.3, the action of $\mathrm{O}(V(J))$ on the fibers of $\Phi_J \mathcal{E}_\lambda$ is isomorphic to the representation $V(J)_{\lambda'}$. Therefore $\Phi_J \mathcal{E}_\lambda \simeq \mathcal{L}_J^{\otimes \lambda_1} \otimes V(J)_{\lambda'}$ as $\mathrm{SL}(J_{\mathbb{C}}) \times \mathrm{O}(V(J))$-equivariant vector bundles on $\mathbb{P}J^\vee_{\mathbb{C}}$. This finishes the proof of Proposition 6.7. ■

Remark 6.12. By the proof, the vector bundle $\Phi_J \mathcal{E}_\lambda$ is in fact $\mathrm{SL}(J_{\mathbb{R}}) \times \mathrm{O}(V(J)_{\mathbb{R}})$-linearized, and the isomorphism $\Phi_J \mathcal{E}_\lambda \simeq \mathcal{L}_J^{\otimes \lambda_1} \otimes V(J)_{\lambda'}$ is $\mathrm{SL}(J_{\mathbb{R}}) \times \mathrm{O}(V(J)_{\mathbb{R}})$-equivariant.

6.3 The Siegel operator

Combining the arguments so far, we can now define the Siegel operator at the J-cusp.

Proposition 6.13. *Let* $f \in M_{\lambda,k}(\Gamma)$ *with* $\lambda \neq 1, \det$. *There exists a cusp form* $\Phi_J f$ *on* $\mathbb{H}_J$ *with values in* $\Phi_J \mathcal{E}_\lambda \otimes \mathcal{L}_J^{\otimes k} \simeq \mathcal{L}_J^{\otimes \lambda_1 + k} \otimes V(J)_{\lambda'}$ *and invariant under the image of* $\Gamma(J)_{\mathbb{Z}} \to \mathrm{SL}(J)$ *such that* $f|_{\Delta_J} = \pi_2^*(\Phi_J f)$. *If* $f = \sum_l a(l) q^l$ *is the Fourier expansion of* f *at a* 0*-dimensional cusp* $I \subset J$, *the Fourier expansion of* $\Phi_J f$ *at the* I*-cusp of* $\mathbb{H}_J$ *is given by*

$$(\Phi_J f)(\tau) = \sum_{l \in \sigma_J \cap U(I)^\vee_{\mathbb{Z}}} a(l) e((l, \tau)), \quad \tau \in \mathbb{H}_J \subset U(I)_{\mathbb{C}}/U(J)^\perp_{\mathbb{C}}.$$

Here we recall that $\mathcal{L}_J$ and $\Phi_J \mathcal{E}_\lambda$ on $\mathbb{H}_J$ are endowed with I-trivializations whose pullback agree with the I-trivializations of $\mathcal{L}$ and $\mathcal{E}^J_\lambda$, respectively (Remarks 5.11 and 6.6). These define the I-trivialization

$$\Phi_J \mathcal{E}_\lambda \otimes \mathcal{L}_J^{\otimes k} \simeq V(I)^U_{\lambda,k} \otimes \mathcal{O}_{\mathbb{H}_J}$$

of $\Phi_J\mathcal{E}_\lambda \otimes \mathcal{L}_J^{\otimes k}$ whose pullback agrees with the I-trivialization of $\mathcal{E}_\lambda^J \otimes \mathcal{L}^{\otimes k}$. The Fourier expansion of $\Phi_J f$ is done with respect to this trivialization.

Proof. We choose a rank 1 primitive sublattice $I \subset J$ and let $f = \sum_l a(l)q^l$ be the Fourier expansion of f at I. By (3.19) and the gluing map $\overline{\mathcal{X}(J)} \to \mathcal{X}(I)^\Sigma$ in Lemma 5.3, we see that

$$f|_{\Delta_J} = \sum_{l\in\sigma_J\cap U(I)_{\mathbb{Z}}^\vee} a(l)q^l \tag{6.9}$$

as a $V(I)_{\lambda,k}$-valued function on $\Delta_J \subset U(I)_{\mathbb{C}}/U(J)_{\mathbb{C}}$. By Lemma 6.2, the function $f|_{\Delta_J}$ takes values in $V(I)_{\lambda,k}^U$. This in turn implies that $f|_{\Delta_J}$ as a section of $\mathcal{E}_{\lambda,k}|_{\Delta_J}$ takes values in the sub vector bundle

$$\mathcal{E}_\lambda^J \otimes \mathcal{L}^{\otimes k}|_{\Delta_J} \simeq \pi_2^*(\Phi_J\mathcal{E}_\lambda \otimes \mathcal{L}_J^{\otimes k}) \simeq \pi_2^*\mathcal{L}_J^{\otimes\lambda_1+k} \otimes V(J)_{\lambda'}.$$

Since the section $f|_{\Delta_J}$ is $\overline{\Gamma(J)}_{\mathbb{Z}}$-invariant, it is in particular $W(J)_{\mathbb{Z}}/U(J)_{\mathbb{Z}}$-invariant, and so, it descends to a section of the vector bundle $\bar{\pi}_2^*\mathcal{L}_J^{\otimes\lambda_1+k} \otimes V(J)_{\lambda'}$ over $\Delta_J/(W(J)_{\mathbb{Z}}/U(J)_{\mathbb{Z}})$, where

$$\bar{\pi}_2 : \Delta_J/(W(J)_{\mathbb{Z}}/U(J)_{\mathbb{Z}}) \to \mathbb{H}_J$$

is the projection. Since $\bar{\pi}_2$ is a proper map (family of abelian varieties), we find that $f|_{\Delta_J}$ is constant on each π_2-fiber. Therefore $f|_{\Delta_J} = \pi_2^*(\Phi_J f)$ for a section $\Phi_J f$ of $\mathcal{L}_J^{\otimes\lambda_1+k} \otimes V(J)_{\lambda'}$ over $\mathbb{H}_J$. Since $f|_{\Delta_J}$ is $\overline{\Gamma(J)}_{\mathbb{Z}}$-invariant, $\Phi_J f$ is invariant under the image of $\Gamma(J)_{\mathbb{Z}} \to \mathrm{SL}(J)$.

The fact that the pullback of the I-trivialization of $\Phi_J\mathcal{E}_\lambda \otimes \mathcal{L}_J^{\otimes k}$ agrees with the I-trivialization of $\mathcal{E}_\lambda^J \otimes \mathcal{L}^{\otimes k}$ implies that the pullback of $\Phi_J f$ as a $V(I)_{\lambda,k}^U$-valued function by $\Delta_J \to \mathbb{H}_J$ is equal to $f|_{\Delta_J}$ as a $V(I)_{\lambda,k}^U$-valued function. Therefore $\Phi_J f$ as a $V(I)_{\lambda,k}^U$-valued function on $\mathbb{H}_J$ is given by the right-hand side of (6.9):

$$\Phi_J f = \sum_{l\in\sigma_J\cap U(I)_{\mathbb{Z}}^\vee} a(l)q^l.$$

Here q^l for $l \in \sigma_J \cap U(I)_{\mathbb{Z}}^\vee$ is naturally viewed as a function on $\mathbb{H}_J \subset U(I)_{\mathbb{C}}/U(J)_{\mathbb{C}}^\perp$ by the pairing between $U(J)_{\mathbb{C}}$ and $U(I)_{\mathbb{C}}/U(J)_{\mathbb{C}}^\perp$. This gives the Fourier expansion of $\Phi_J f$ at the I-cusp of $\mathbb{H}_J$. By Proposition 3.7, $\Phi_J f$ vanishes at the I-cusp. Since this holds at every cusp of $\mathbb{H}_J$, we see that $\Phi_J f$ is a cusp form. ∎

Let Γ_J be the image of $\Gamma(J)_{\mathbb{Z}}$ in $\mathrm{SL}(J) \simeq \mathrm{SL}(2,\mathbb{Z})$. We call the map

$$M_{\lambda,k}(\Gamma) \to S_{\lambda_1+k}(\Gamma_J) \otimes V(J)_{\lambda'}, \quad f \mapsto \Phi_J f, \tag{6.10}$$

the *Siegel operator* at the J-cusp.

We look at some examples. We use the same notation as in the proof of Proposition 6.3.

Example 6.14. Let $\lambda = (1^d)$ for $0 < d < n$, namely, $V_\lambda = \wedge^d V$. Then

$$\left(\wedge^d V\right)^U = \mathbb{C}e_1 \wedge \left(\wedge^{d-1}\langle e_1, \dots, e_{n-1}\rangle\right) \simeq \mathbb{C}e_1 \otimes \wedge^{d-1}\langle e_2, \dots, e_{n-1}\rangle.$$

In this case, $\Phi_J f$ is a $\binom{n-2}{d-1}$-tuple of scalar-valued cusp forms of weight $k+1$.

Example 6.15. Let $\lambda = (d)$, namely, $V_{(d)}$ is the main irreducible component of $\mathrm{Sym}^d V$ (see Example 3.1 (2)). We have

$$V_{(d)}^U = \mathbb{C}e_1^{\otimes d} \subset \mathrm{Sym}^d V.$$

In this case, $\Phi_J f$ is a single scalar-valued cusp form of weight $k+d$.

The Siegel operator for vector-valued Siegel modular forms is studied in [47, Section 2]. The case of genus 2 is also studied in [1, Section 1]. Let us observe that the weight calculation in Example 6.15 in the case $n = 3$ agrees with the results of [1,47] for Siegel modular forms of genus 2.

Example 6.16. Let $n = 3$. In [1,47], it is proved that the Siegel operator for a Siegel modular form of genus 2 and weight $(\mathrm{Sym}^j, \det^l)$ produces a scalar-valued cusp form of weight $j + l$ on the 1-dimensional cusp.

On the other hand, when $j = 2d$ is even, we saw in Example 3.4 that the Siegel weight $(\mathrm{Sym}^{2d}, \det^l)$ corresponds to the orthogonal weight $(\lambda, k) = ((d), d+l)$. According to Example 6.15, $\Phi_J f$ is a cusp form of weight $d + (d+l) = j + l$. This agrees with the above results of [1,47].

In general, the Siegel operator in the form of (6.10) is still not surjective for the following obvious reason. Let $\Gamma(J)_{\mathbb{Z}}^*$ be the stabilizer of J in Γ, and let Γ_J^* be the image of $\Gamma(J)_{\mathbb{Z}}^*$ in $\mathrm{SL}(J) \times \mathrm{O}(J^\perp/J)$. Then $\Gamma_J = \Gamma_J^* \cap \mathrm{SL}(J)$ is of finite index and normal in Γ_J^*. Let

$$G = \Gamma_J^*/\Gamma_J \simeq \Gamma(J)_{\mathbb{Z}}^*/\Gamma(J)_{\mathbb{Z}}.$$

The modular forms are not only $\Gamma(J)_{\mathbb{Z}}$-invariant but also $\Gamma(J)_{\mathbb{Z}}^*$-invariant. Therefore, in view of Remark 6.12, we see that the image of the map (6.10) is contained in the G-invariant part of $S_{\lambda_1+k}(\Gamma_J) \otimes V(J)_{\lambda'}$.