Chapter 7

Fourier–Jacobi expansion

Let L be a lattice of signature $(2, n)$ with $n \geq 3$ and Γ be a finite-index subgroup of $\mathrm{O}^+(L)$. We fix a rank 2 primitive isotropic sublattice J of L. In this chapter we study the Fourier–Jacobi expansion of vector-valued modular forms at the J-cusp. From a geometric point of view, the Fourier–Jacobi expansion is the Taylor expansion along the boundary divisor Δ_J of the partial toroidal compactification $\overline{\mathcal{X}(J)}$. The m-th Fourier–Jacobi coefficient is the m-th Taylor coefficient, and is essentially a section of the vector bundle $\mathcal{E}_{\lambda,k} \otimes \Theta_J^{\otimes m}$ over Δ_J, where Θ_J is the conormal bundle of Δ_J. Here we have some special properties beyond general Taylor expansion:

- existence of the projection $\pi_1 \colon \overline{\mathcal{X}(J)} \to \Delta_J$ and the isomorphism

$$\mathcal{E}_{\lambda,k} \simeq \pi_1^*(\mathcal{E}_{\lambda,k}|_{\Delta_J})$$

  (Proposition 5.6), and
- existence of a special generator ω_J of the ideal sheaf of Δ_J which is a linear map on each fiber of π_1.

These properties ensure that the m-th Fourier–Jacobi coefficient as a section of the vector bundle $\mathcal{E}_{\lambda,k} \otimes \Theta_J^{\otimes m}$ over Δ_J is canonically defined (Corollary 7.5) and is invariant under $\overline{\Gamma(J)_{\mathbb{Z}}}$. If we take the (I, ω_J)-trivialization for $I \subset J$, we can pass to a more familiar definition of the Fourier–Jacobi coefficient as a slice in the Fourier expansion at I.

In general, we define vector-valued Jacobi forms as $\overline{\Gamma(J)_{\mathbb{Z}}}$-invariant sections of $\mathcal{E}_{\lambda,k} \otimes \Theta_J^{\otimes m}$ over Δ_J with cusp condition (Definition 7.10). Thus the Fourier–Jacobi coefficients are vector-valued Jacobi forms (Proposition 7.12). Although our approach is geometric, our Jacobi forms in the scalar-valued case are indeed classical Jacobi forms in the sense of Skoruppa [43] if we introduce suitable coordinates and the (I, ω_J)-trivialization (Section 7.4). When $n = 3$, our vector-valued Jacobi forms essentially agree with those considered by Ibukiyama–Kyomura [27] for Siegel modular forms of genus 2.

When J comes from an integral embedding $2U \hookrightarrow L$ and Γ is the so-called stable orthogonal group, the Fourier–Jacobi expansion of scalar-valued modular forms is well understood through the work of Gritsenko [20]. A large part of this chapter can be regarded as a geometric reformulation and a generalization of the calculation in [20, Section 2]. A lot of effort will be paid for keeping introduction of coordinates as minimal as possible (though never zero), or in other words, for describing what is canonical in a canonical way. We believe that this style would be suitable even in the scalar-valued case when working with general (Γ, J), for which simple expression by coordinates is no longer available.

7.1 Fourier–Jacobi and Fourier expansion

We begin with the familiar (but non-canonical) way to define Fourier–Jacobi expansion: slicing the Fourier expansion. The passage to a canonical formulation will be given in Section 7.2.

We choose a rank 1 primitive sublattice I of J, and also a rank 1 sublattice $I' \subset L$ with $(I, I') \neq 0$. Recall from Section 5.1.2 that $U(J)_{\mathbb{R}} = \wedge^2 J_{\mathbb{R}}$ is identified with the isotropic line $(J/I)_{\mathbb{R}} \otimes I_{\mathbb{R}}$ in $U(I)_{\mathbb{R}} = (I^{\perp}/I)_{\mathbb{R}} \otimes I_{\mathbb{R}}$, and that the Siegel domain realization of $\mathcal{D}$ with respect to J can be identified with the restriction of the projection

$$U(I)_{\mathbb{C}} \to U(I)_{\mathbb{C}}/U(J)_{\mathbb{C}} \to U(I)_{\mathbb{C}}/U(J)_{\mathbb{C}}^{\perp}$$

to the tube domain $\mathcal{D}_I \subset U(I)_{\mathbb{C}}$ after the tube domain realization $\mathcal{D} \simeq \mathcal{D}_I$. The orientation of J determines the nonnegative part $\sigma_J = (U(J)_{\mathbb{R}})_{\geq 0}$ of $U(J)_{\mathbb{R}}$. Let $v_{J,\Gamma}$ be the positive generator of $U(J)_{\mathbb{Z}} = U(J)_{\mathbb{Q}} \cap \Gamma$. We choose a rational isotropic vector $l_{J,\Gamma} \in U(I)_{\mathbb{Q}}$ such that $(v_{J,\Gamma}, l_{J,\Gamma}) = 1$. Then $v_{J,\Gamma}, l_{J,\Gamma}$ span a rational hyperbolic plane in $U(I)_{\mathbb{Q}}$. We put

$$\omega_J = q^{l_{J,\Gamma}} = e((l_{J,\Gamma}, Z)), \quad Z \in U(I)_{\mathbb{C}}.$$

This is a holomorphic function on $U(I)_{\mathbb{C}}$ invariant under the translation by $U(J)_{\mathbb{Z}}$. Thus we have chosen the auxiliary datum I, I', $l_{J,\Gamma}$. These will be fixed until Lemma 7.4.

Let f be a Γ-modular form of weight (λ, k). We identify f with a $V(I)_{\lambda,k}$-valued holomorphic function on $\mathcal{D}_I$ via the I-trivialization and the tube domain realization, and let $f(Z) = \sum_l a(l) q^l$ be its Fourier expansion. Like the calculation in Section 3.5.2 (see also Remark 3.10), we can rewrite the Fourier expansion as

$$f(Z) = \sum_{m \geq 0} \Big(\sum_{l \in U(J)_{\mathbb{Q}}^{\perp}} a(l + m l_{J,\Gamma}) q^l \Big) \omega_J^m. \tag{7.1}$$

Here l ranges over vectors in $U(J)_{\mathbb{Q}}^{\perp}$ such that $l + m l_{J,\Gamma} \in U(I)_{\mathbb{Z}}^{\vee}$. They form a translation of a full lattice in $U(J)_{\mathbb{Q}}^{\perp}$. Although $l_{J,\Gamma}$ is not necessarily a vector in $U(I)_{\mathbb{Z}}^{\vee}$, this expression still makes sense over the tube domain $\mathcal{D}_I$. We call (7.1) the *Fourier–Jacobi expansion* of f at the J-cusp relative to I, I', $l_{J,\Gamma}$, and usually write it as

$$f = \sum_{m \geq 0} \phi_m \omega_J^m \tag{7.2}$$

with

$$\phi_m = \sum_{l \in U(J)^{\perp}_{\mathbb{Q}}} a(l + m l_{J,\Gamma}) q^l. \tag{7.3}$$

We call ϕ_m the m-th *Fourier–Jacobi coefficient* of f at the J-cusp relative to I, I', $l_{J,\Gamma}$. This is a $V(I)_{\lambda,k}$-valued function on $\mathcal{D}_I$. Since $l \in U(J)^{\perp}_{\mathbb{Q}}$ in (7.3), ϕ_m actually descends to a $V(I)_{\lambda,k}$-valued function on $\Delta_J \subset U(I)_{\mathbb{C}}/U(J)_{\mathbb{C}}$. We often do not specify the precise index lattice in (7.3); it is convenient to allow enlarging it as necessary by putting $a(l + m l_{J,\Gamma}) = 0$ when $l + m l_{J,\Gamma} \notin U(I)^{\vee}_{\mathbb{Z}}$. When $m = 0$, ϕ_0 is the restriction of f to Δ_J and was studied in Section 6. In this chapter we study the case $m > 0$.

7.2 Geometric approach to Fourier–Jacobi expansion

In Sections 7.2 and 7.3 we give a geometric reformulation of the Fourier–Jacobi expansion (7.2). Our starting observation is (compare with Section 3.5.2):

Lemma 7.1. *The Fourier–Jacobi expansion (7.2) gives the Taylor expansion of the $V(I)_{\lambda,k}$-valued holomorphic function f on $\overline{\mathcal{X}(J)}$ along the boundary divisor Δ_J with respect to the normal parameter ω_J, where ϕ_m is the m-th Taylor coefficient as a $V(I)_{\lambda,k}$-valued function on Δ_J.*

Proof. Since the function f is invariant under the translation by $U(J)_{\mathbb{Z}} \subset U(I)_{\mathbb{Z}}$, it descends to a function on $\mathcal{X}(J) \simeq \mathcal{D}_I/U(J)_{\mathbb{Z}}$. Since $(l_{J,\Gamma}, v_{J,\Gamma}) = 1$ for the positive generator $v_{J,\Gamma}$ of $U(J)_{\mathbb{Z}}$, the function $\omega_J = e((l_{J,\Gamma}, Z))$ descends to a function on $\mathcal{X}(J)$ and extends holomorphically over $\overline{\mathcal{X}(J)}$, with the boundary divisor Δ_J defined by $\omega_J = 0$. In particular, ω_J generates the ideal sheaf of Δ_J. On the other hand, as explained above, the Fourier–Jacobi coefficient ϕ_m is the pullback of a $V(I)_{\lambda,k}$-valued function on $\Delta_J \subset U(I)_{\mathbb{C}}/U(J)_{\mathbb{C}}$ (again denoted by ϕ_m). Thus $f = \sum_m (\pi_1^* \phi_m)\omega_J^m$ gives the Taylor expansion of f along Δ_J with respect to the normal parameter ω_J, in which the $V(I)_{\lambda,k}$-valued function ϕ_m on Δ_J is the m-th Taylor coefficient. ∎

Recall from Section 5.3 that $\overline{\mathcal{X}(J)}$ is an open set of the relative torus embedding $\overline{\mathcal{T}(J)} = \mathcal{T}(J) \times_{T(J)} \overline{T(J)}$ which has the structure of a line bundle on Δ_J. Since $\mathcal{D}(J) \subset \mathcal{D}(I) \simeq U(I)_{\mathbb{C}}$, the function ω_J on $\overline{\mathcal{X}(J)}$ extends over $\overline{\mathcal{T}(J)}$ naturally. It is a linear map on each fiber of $\overline{\mathcal{T}(J)} \to \Delta_J$. Indeed, the fact that ω_J preserves the scalar multiplication follows from the equality

$$e((l_{J,\Gamma}, \alpha v_{J,\Gamma} + Z)) = e(\alpha) \cdot e((l_{J,\Gamma}, Z)), \quad \alpha \in \mathbb{C},$$

and similarly for the sum. The following property will be used in Section 7.3.

Lemma 7.2. *For each $\gamma \in \overline{\Gamma(J)}_{\mathbb{Z}}$ we have $\gamma^*\omega_J = (\pi_1^* j_\gamma) \cdot \omega_J$ for a nowhere vanishing function j_γ on Δ_J.*

Proof. Since γ acts on $\overline{\mathcal{T}(J)} \to \Delta_J$ as an equivariant action on the line bundle (see Section 5.3), $\gamma^*\omega_J$ is also linear on each fiber. Therefore $\gamma^*\omega_J/\omega_J$ is the pullback of a function on Δ_J. See also Corollary 7.15 for a computational proof. ■

Let us reformulate Lemma 7.1 by passing from vector-valued functions to sections of vector bundles. Let $\mathcal{I} = \mathcal{I}_{\Delta_J}$ be the ideal sheaf of Δ_J and $\Theta_J = \mathcal{I}/\mathcal{I}^2$ be the conormal bundle of Δ_J. As explained above, ω_J generates $\mathcal{I}$ over $\overline{\mathcal{X}(J)}$. In particular, it generates Θ_J over Δ_J. We have

$$\mathcal{I}^m/\mathcal{I}^{m+1} \simeq \Theta_J^{\otimes m} = \mathcal{O}_{\Delta_J}\omega_J^{\otimes m}$$

for every $m \geq 0$. In what follows, we write $\mathcal{E}_{\lambda,k}|_{\Delta_J} \otimes \Theta_J^{\otimes m} = \mathcal{E}_{\lambda,k} \otimes \Theta_J^{\otimes m}$ for simplicity. The I-trivialization $\mathcal{E}_{\lambda,k}|_{\Delta_J} \simeq V(I)_{\lambda,k} \otimes \mathcal{O}_{\Delta_J}$ of $\mathcal{E}_{\lambda,k}|_{\Delta_J}$ and the trivialization of $\Theta_J^{\otimes m}$ by $\omega_J^{\otimes m}$ define an isomorphism

$$\mathcal{E}_{\lambda,k} \otimes \Theta_J^{\otimes m} \simeq V(I)_{\lambda,k} \otimes \mathcal{O}_{\Delta_J}.$$

We call this isomorphism the *(I, ω_J)-trivialization* of $\mathcal{E}_{\lambda,k} \otimes \Theta_J^{\otimes m}$. Via this isomorphism, we regard the $V(I)_{\lambda,k}$-valued function ϕ_m over Δ_J as a section of $\mathcal{E}_{\lambda,k} \otimes \Theta_J^{\otimes m}$ over Δ_J. Specifically, the process is to multiply the function ϕ_m by $\omega_J^{\otimes m}$, and then regard $\phi_m \otimes \omega_J^{\otimes m}$ as a section of $\mathcal{E}_{\lambda,k} \otimes \Theta_J^{\otimes m}$ by the I-trivialization.

Proposition 7.3. *The Taylor expansion of sections of $\mathcal{E}_{\lambda,k}$ over $\overline{\mathcal{X}(J)}$ along the boundary divisor Δ_J with respect to the normal parameter ω_J and with the pullback*

$$\pi_1^*\colon H^0(\Delta_J, \mathcal{E}_{\lambda,k}|_{\Delta_J}) \hookrightarrow H^0(\overline{\mathcal{X}(J)}, \mathcal{E}_{\lambda,k}) \tag{7.4}$$

defines an embedding

$$H^0(\overline{\mathcal{X}(J)}, \mathcal{E}_{\lambda,k}) \hookrightarrow \prod_{m\geq 0} H^0(\Delta_J, \mathcal{E}_{\lambda,k} \otimes \Theta_J^{\otimes m}), \quad f \mapsto (\phi_m \otimes \omega_J^{\otimes m})_m, \tag{7.5}$$

where ϕ_m are the sections of $\mathcal{E}_{\lambda,k}|_{\Delta_J}$ with $f = \sum_m (\pi_1^*\phi_m)\omega_J^m$. If we send a modular form $f \in M_{\lambda,k}(\Gamma)$ as a section of $\mathcal{E}_{\lambda,k}$ by this map, its image is the Fourier–Jacobi coefficients of f regarded as sections of $\mathcal{E}_{\lambda,k} \otimes \Theta_J^{\otimes m}$ via the (I, ω_J)-trivialization.

Here the pullback map (7.4) is defined by the isomorphism

$$\mathcal{E}_{\lambda,k} \simeq \pi_1^*(\mathcal{E}_{\lambda,k}|_{\Delta_J})$$

in Proposition 5.6. Via the I-trivialization, this is just the pullback of $V(I)_{\lambda,k}$-valued functions by $\pi_1\colon \overline{\mathcal{X}(J)} \to \Delta_J$ (see Remark 5.7). The existence of this pullback map is one of key properties in the Fourier–Jacobi expansion.

Proof. The exact sequence of sheaves

$$0 \to \mathcal{I}^{m+1}\mathcal{E}_{\lambda,k} \to \mathcal{I}^{m}\mathcal{E}_{\lambda,k} \to \mathcal{E}_{\lambda,k} \otimes \Theta_J^{\otimes m} \to 0$$

on $\overline{\mathcal{X}(J)}$ defines the canonical exact sequence

$$0 \to H^0(\overline{\mathcal{X}(J)}, \mathcal{I}^{m+1}\mathcal{E}_{\lambda,k}) \to H^0(\overline{\mathcal{X}(J)}, \mathcal{I}^{m}\mathcal{E}_{\lambda,k}) \to H^0(\Delta_J, \mathcal{E}_{\lambda,k} \otimes \Theta_J^{\otimes m}). \tag{7.6}$$

The generator ω_J^m of $\mathcal{I}^m$ and the pullback map

$$\pi_1^*\colon H^0(\Delta_J, \mathcal{E}_{\lambda,k}|_{\Delta_J}) \to H^0(\overline{\mathcal{X}(J)}, \mathcal{E}_{\lambda,k})$$

define the splitting map

$$H^0(\Delta_J, \mathcal{E}_{\lambda,k} \otimes \Theta_J^{\otimes m}) \hookrightarrow H^0(\overline{\mathcal{X}(J)}, \mathcal{I}^m\mathcal{E}_{\lambda,k}), \quad \phi \otimes \omega_J^{\otimes m} \mapsto \omega_J^m \cdot \pi_1^*\phi \tag{7.7}$$

of (7.6). Here $\omega_J^{\otimes m}$ in the source is a section of $\Theta_J^{\otimes m}$ over Δ_J, while ω_J^m in the target is a section of the sheaf $\mathcal{I}^m$ over $\overline{\mathcal{X}(J)}$. This defines a splitting of the filtration $(H^0(\overline{\mathcal{X}(J)}, \mathcal{I}^m\mathcal{E}_{\lambda,k}))_m$ on $H^0(\overline{\mathcal{X}(J)}, \mathcal{E}_{\lambda,k})$ and thus an embedding

$$H^0(\overline{\mathcal{X}(J)}, \mathcal{E}_{\lambda,k}) \hookrightarrow \prod_{m\geq 0} H^0(\Delta_J, \mathcal{E}_{\lambda,k} \otimes \Theta_J^{\otimes m}).$$

Explicitly, this is given by writing a section f of $\mathcal{E}_{\lambda,k}$ over $\overline{\mathcal{X}(J)}$ as

$$f = \sum_m (\pi_1^*\phi_m)\omega_J^m$$

with ϕ_m a section of $\mathcal{E}_{\lambda,k}|_{\Delta_J}$, and sending f to the collection $(\phi_m \otimes \omega_J^{\otimes m})_m$ of sections.

Since π_1^* is just the ordinary pullback after the I-trivialization, the equation $f = \sum_m (\pi_1^*\phi_m)\omega_J^m$ when f is a modular form coincides with the Fourier–Jacobi expansion (7.2) of f after the I-trivialization. Thus the I-trivialization of ϕ_m is the m-th Fourier–Jacobi coefficient (7.3). It follows that the section $\phi_m \otimes \omega_J^{\otimes m}$ is identified with the Fourier–Jacobi coefficient by the (I, ω_J)-trivialization. ∎

At first glance, the Taylor expansion (7.5) may seem non-canonical because the lifting map (7.7) uses the special normal parameter ω_J, which as a function on $\overline{\mathcal{X}(J)}$ depends on the choice of $l_{J,\Gamma}$, I', I. In fact, it *is* canonical.

Lemma 7.4. *The map* (7.7)*, and hence the Taylor expansion* (7.5)*, does not depend on the choice of* $l_{J,\Gamma}$, I', I.

Proof. Let $\tilde{\omega}_J$ be the special normal parameter constructed from another such data $(\tilde{I}, \tilde{I}', \tilde{l}_{J,\Gamma})$. Both ω_J and $\tilde{\omega}_J$ extend over $\overline{\mathcal{T}(J)}$ and are linear at each fiber of the projection $\pi_1 \colon \overline{\mathcal{T}(J)} \to \Delta_J$. Therefore we have $\widetilde{\omega}_J/\omega_J = \pi_1^*\xi$ for a nowhere vanishing holomorphic function ξ on Δ_J. Then the map (7.7) defined by using $\widetilde{\omega}_J$ in place of ω_J sends $\phi \otimes \omega_J^{\otimes m}$ as

$$\phi \otimes \omega_J^{\otimes m} = (\xi^{-m}\phi) \otimes \widetilde{\omega}_J^{\otimes m} \mapsto \widetilde{\omega}_J^m \cdot \pi_1^*(\xi^{-m}\phi) = \omega_J^m \cdot \pi_1^*\phi.$$

This coincides with the map using ω_J. ∎

This in particular implies the following.

Corollary 7.5. *The m-th Fourier–Jacobi coefficient of a modular form, viewed as a section of $\mathcal{E}_{\lambda,k} \otimes \Theta_J^{\otimes m}$ over Δ_J via the (I, ω_J)-trivialization, does not depend on the choice of $l_{J,\Gamma}$, I', I.*

This means that we obtain the same section of $\mathcal{E}_{\lambda,k} \otimes \Theta_J^{\otimes m}$ even if we start from the Fourier expansion at another 0-dimensional cusp $\widetilde{I} \subset J$.

To summarize, the Fourier–Jacobi expansion of a modular form f as a section of $\mathcal{E}_{\lambda,k}$ is a canonical Taylor expansion along Δ_J which uses but does not depend on the choice of a special normal parameter ω_J. The m-th Fourier–Jacobi coefficient is canonically determined as a section of $\mathcal{E}_{\lambda,k} \otimes \Theta_J^{\otimes m}$. If we take the (I, ω_J)-trivialization, this section is identified with the $V(I)_{\lambda,k}$-valued function (7.3) defined as a slice in the Fourier expansion of f at the I-cusp.

7.3 Vector-valued Jacobi forms

We want to refine Proposition 7.3 by taking the invariant part for the integral Jacobi group $\Gamma(J)_{\mathbb{Z}}$ and imposing cusp condition. This leads us to define vector-valued Jacobi forms in a geometric style. In what follows, we let $m > 0$ and consider the vector bundle $\mathcal{E}_{\lambda,k} \otimes \Theta_J^{\otimes m}$ over Δ_J, leaving modular forms on $\mathcal{D}$ for a while.

As in Sections 7.1 and 7.2, we choose a rank 1 primitive sublattice I of J, a rank 1 sublattice $I' \subset L$ with $(I, I') \neq 0$, and an isotropic vector $l_{J,\Gamma} \in U(I)_{\mathbb{Q}}$ with $(l_{J,\Gamma}, v_{J,\Gamma}) = 1$. ($I$ will be fixed until Definition 7.10, and I', $l_{J,\Gamma}$ will be fixed until Lemma 7.9.) We keep the same notation as in Section 7.1. Since $U(I)_{\mathbb{Z}} \subset \Gamma(J)_{\mathbb{Z}}$ by (5.8), the group $\overline{\Gamma(J)}_{\mathbb{R}}$ contains $U(I)_{\mathbb{Z}}/U(J)_{\mathbb{Z}}$ as a subgroup. As recalled in Section 7.1, I' determines an embedding $\Delta_J \hookrightarrow U(I)_{\mathbb{C}}/U(J)_{\mathbb{C}}$. The action of $U(I)_{\mathbb{Z}}/U(J)_{\mathbb{Z}}$ on Δ_J is given by the translation on $U(I)_{\mathbb{C}}/U(J)_{\mathbb{C}}$.

We consider the action of $U(I)_{\mathbb{Z}}/U(J)_{\mathbb{Z}}$ on the vector bundle $\mathcal{E}_{\lambda,k} \otimes \Theta_J^{\otimes m}$. The I-trivialization $\mathcal{E}_{\lambda,k}|_{\Delta_J} \simeq V(I)_{\lambda,k} \otimes \mathcal{O}_{\Delta_J}$ over Δ_J is equivariant with respect to the subgroup $(\Gamma(I)_{\mathbb{R}} \cap \Gamma(J)_{\mathbb{R}})/U(J)_{\mathbb{Z}}$ of $\overline{\Gamma(J)}_{\mathbb{R}}$. In particular, it is equivariant with respect to $U(I)_{\mathbb{Z}}/U(J)_{\mathbb{Z}}$. Since $U(I)_{\mathbb{Z}}/U(J)_{\mathbb{Z}}$ acts trivially on $V(I)_{\lambda,k}$, the factor of automorphy for the I-trivialization of $\mathcal{E}_{\lambda,k}|_{\Delta_J}$ is trivial on this group. On the other hand, as for the ω_J-trivialization of Θ_J, we note the following.

Lemma 7.6. *There exists a finite-index sublattice Λ_0 of $U(I)_{\mathbb{Z}}/U(J)_{\mathbb{Z}}$ such that $\gamma^*\omega_J = \omega_J$ for every $\gamma \in \Lambda_0$. In particular, the factor of automorphy for the (I, ω_J)-trivialization $\mathcal{E}_{\lambda,k} \otimes \Theta_J^{\otimes m} \simeq V(I)_{\lambda,k} \otimes \mathcal{O}_{\Delta_J}$ of $\mathcal{E}_{\lambda,k} \otimes \Theta_J^{\otimes m}$ is trivial on the group Λ_0.*

Proof. Recall that $v \in U(I)_{\mathbb{R}}$ acts on the tube domain $\mathcal{D}_I \simeq \mathcal{D}$ as the translation by v, say t_v. Then

$$t_v^*\omega_J = e((l_{J,\Gamma}, Z + v)) = e((l_{J,\Gamma}, v)) \cdot \omega_J.$$

Therefore, if we put

$$\Lambda_0 = \{v \in U(I)_{\mathbb{Z}}/U(J)_{\mathbb{Z}} \mid (l_{J,\Gamma}, v + U(J)_{\mathbb{Z}}) \subset \mathbb{Z}\}, \tag{7.8}$$

we have $t_v^*\omega_J = \omega_J$ for every $v \in \Lambda_0$. Since $(U(I)_{\mathbb{Z}}, l_{J,\Gamma}) \subset \mathbb{Q}$ and $U(I)_{\mathbb{Z}}$ is finitely generated, we have $(U(I)_{\mathbb{Z}}, l_{J,\Gamma}) \subset N^{-1}\mathbb{Z}$ for some natural number N. This shows that Λ_0 is of finite index in $U(I)_{\mathbb{Z}}/U(J)_{\mathbb{Z}}$. ∎

Let ϕ be a $\overline{\Gamma(J)_{\mathbb{Z}}}$-invariant section of the vector bundle $\mathcal{E}_{\lambda,k} \otimes \Theta_J^{\otimes m}$ over Δ_J. By the (I, ω_J)-trivialization of $\mathcal{E}_{\lambda,k} \otimes \Theta_J^{\otimes m}$, we regard ϕ as a $V(I)_{\lambda,k}$-valued holomorphic function on Δ_J. By Lemma 7.6, the function ϕ is invariant under the translation by the lattice Λ_0. Therefore it admits a Fourier expansion of the form

$$\phi(Z) = \sum_{l \in \Lambda} a(l) q^l, \quad Z \in \Delta_J \subset U(I)_{\mathbb{C}}/U(J)_{\mathbb{C}}, \tag{7.9}$$

where $a(l) \in V(I)_{\lambda,k}$, $q^l = e((l, Z))$, and Λ is a full lattice in $U(J)^{\perp}_{\mathbb{Q}}$ (which is the dual space of $U(I)_{\mathbb{Q}}/U(J)_{\mathbb{Q}}$).

At this point, Λ can be taken to be the dual lattice of Λ_0 defined by (7.8), but we can replace Λ by its arbitrary overlattice (or even the whole $U(J)^{\perp}_{\mathbb{Q}}$) by setting $a(l) = 0$ if $l \notin \Lambda_0^{\vee}$. It is sometimes convenient to enlarge Λ in this way. For this reason, we do not specify the lattice Λ in (7.9).

Remark 7.7. The dual lattice of Λ_0 in $U(J)^{\perp}_{\mathbb{Q}}$ can be explicitly written as

$$\Lambda_0^{\vee} = \langle U(I)_{\mathbb{Z}}^{\vee}, \mathbb{Z}l_{J,\Gamma} \rangle \cap U(J)^{\perp}_{\mathbb{Q}}.$$

We do not use this information.

Replacing Λ by its overlattice, we assume that Λ is of the split form

$$\Lambda = \mathbb{Z}(\beta_1 v_{J,\Gamma}) \oplus K,$$

where $\beta_1 > 0$ is a rational number and K is a full lattice in $l^{\perp}_{J,\Gamma} \cap U(J)^{\perp}_{\mathbb{Q}}$. Note that K is negative-definite. Accordingly, we can rewrite the Fourier expansion of ϕ as

$$\phi(Z) = \sum_{n \in \beta_1\mathbb{Z}} \sum_{l \in K} a(n, l) q^l q^n_{J,\Gamma}, \quad q_{J,\Gamma} = e((v_{J,\Gamma}, Z)), \tag{7.10}$$

for $Z \in \Delta_J \subset U(I)_{\mathbb{C}}/U(J)_{\mathbb{C}}$.

Definition 7.8. We say that ϕ is *holomorphic at the I-cusp* of $\mathbb{H}_J$ if $a(n,l) \neq 0$ only when $2nm \geq |(l,l)|$. We say that ϕ *vanishes at the I-cusp* if $a(n,l) \neq 0$ only when $2nm > |(l,l)|$.

The expression (7.10) of the Fourier expansion of ϕ depends on the choice of I', $l_{J,\Gamma}$, Λ. Specifically,

- I' determines the embedding $\Delta_J \hookrightarrow U(I)_{\mathbb{C}}/U(J)_{\mathbb{C}}$.
- $l_{J,\Gamma}$ determines the normal parameter ω_J which determines the trivialization of $\Theta_J^{\otimes m}$. The vector $l_{J,\Gamma}$ also determines the splitting $U(J)_{\mathbb{Q}}^{\perp} = U(J)_{\mathbb{Q}} \oplus K_{\mathbb{Q}}$ of the index space $U(J)_{\mathbb{Q}}^{\perp}$.
- Λ is the index lattice in the Fourier expansion which is taken to be a split form.

However, we can prove the following.

Lemma 7.9. *Definition 7.8 does not depend on the choice of I', $l_{J,\Gamma}$, Λ.*

Proof. We verify this for the holomorphicity condition. The case of vanishing condition is similar.

(1) If we change I', its effect is the translation on $\Delta_J \subset U(I)_{\mathbb{C}}/U(J)_{\mathbb{C}}$ by a vector of $U(I)_{\mathbb{Q}}/U(J)_{\mathbb{Q}}$. This multiplies each Fourier coefficient $a(n,l)$ by a nonzero constant, so its vanishing/nonvanishing does not change.

(2) The condition $2nm \geq |(l,l)|$ is the same as the condition

$$(ml_{J,\Gamma} + v, ml_{J,\Gamma} + v) \geq 0 \tag{7.11}$$

for the vector $v = nv_{J,\Gamma} + l$ of $U(J)_{\mathbb{Q}}^{\perp}$ which corresponds to the index (n,l). With $l_{J,\Gamma}$ fixed, this condition does not depend on the lattice Λ.

(3) Finally, if we change $l_{J,\Gamma}$, the new vector can be written as

$$l'_{J,\Gamma} = l_{J,\Gamma} + l_0 - 2^{-1}(l_0, l_0)v_{J,\Gamma}$$

for some vector $l_0 \in K_{\mathbb{Q}}$. Since the normal parameter $\omega_J = e((l_{J,\Gamma}, Z))$ is replaced by

$$\omega'_J = e((l'_{J,\Gamma}, Z)) = q^{l_0} \cdot q_{J,\Gamma}^{-(l_0,l_0)/2} \cdot \omega_J,$$

we have to multiply the function ϕ by $q^{-ml_0} \cdot q_{J,\Gamma}^{m(l_0,l_0)/2}$ when passing from the ω_J-trivialization to the ω'_J-trivialization of $\Theta_J^{\otimes m}$. Also

$$K_{\mathbb{Q}} = l_{J,\Gamma}^{\perp} \cap U(J)_{\mathbb{Q}}^{\perp}$$

is replaced by $K'_{\mathbb{Q}} = (l'_{J,\Gamma})^{\perp} \cap U(J)_{\mathbb{Q}}^{\perp}$, for which we have the natural isometry

$$K_{\mathbb{Q}} \to K'_{\mathbb{Q}}, \quad l \mapsto l' := l - (l, l_0)v_{J,\Gamma}.$$

Therefore the new Fourier expansion is

$$\begin{aligned}\phi' &:= \phi \cdot q^{-ml_0} \cdot q_{J,\Gamma}^{m(l_0,l_0)/2} \\ &= \sum_{n\in\mathbb{Q}} \sum_{l\in K_{\mathbb{Q}}} a(n,l) q^{l-ml_0} q_{J,\Gamma}^{n+m(l_0,l_0)/2} \\ &= \sum_{n\in\mathbb{Q}} \sum_{l\in K_{\mathbb{Q}}} a(n,l) q^{l'-ml'_0} q_{J,\Gamma}^{n+(l,l_0)-m(l_0,l_0)/2}.\end{aligned}$$

In the last equality we used

$$l - ml_0 = (l - ml_0)' + (l - ml_0, l_0) v_{J,\Gamma}.$$

This means that $a(n,l)$ is equal to the Fourier coefficient of ϕ' of index

$$(n + (l, l_0) - m(l_0, l_0)/2, l' - ml'_0) \in \mathbb{Q} \oplus K'_{\mathbb{Q}}.$$

The holomorphicity condition $2nm \geq -(l,l)$ for ϕ can be rewritten as

$$2m(n + (l, l_0) - m(l_0, l_0)/2) \geq -(l' - ml'_0, l' - ml'_0).$$

This is the holomorphicity condition for ϕ'. ■

Lemma 7.9 ensures that Definition 7.8 is well defined for a $\overline{\Gamma(J)_{\mathbb{Z}}}$-invariant section of $\mathcal{E}_{\lambda,k} \otimes \Theta_J^{\otimes m}$.

Definition 7.10. We denote by

$$J_{\lambda,k,m}(\Gamma(J)_{\mathbb{Z}}) \subset H^0(\Delta_J, \mathcal{E}_{\lambda,k} \otimes \Theta_J^{\otimes m})$$

the space of $\overline{\Gamma(J)_{\mathbb{Z}}}$-invariant sections ϕ of $\mathcal{E}_{\lambda,k} \otimes \Theta_J^{\otimes m}$ over Δ_J which are holomorphic at every cusp $I \subset J$ of $\mathbb{H}_J$ in the sense of Definition 7.8. We call such a section ϕ a *Jacobi form* of weight (λ, k) and index m for the integral Jacobi group $\Gamma(J)_{\mathbb{Z}}$. We call ϕ a *Jacobi cusp form* if it vanishes at every cusp $I \subset J$. When $\lambda = (0)$, we especially write $J_{(0),k,m}(\Gamma(J)_{\mathbb{Z}}) = J_{k,m}(\Gamma(J)_{\mathbb{Z}})$.

For later use (Section 7.4), we note the following.

Lemma 7.11. *Let γ be an element of $\Gamma(J)_{\mathbb{Q}}$ which stabilizes J. A $\overline{\Gamma(J)_{\mathbb{Z}}}$-invariant section ϕ of $\mathcal{E}_{\lambda,k} \otimes \Theta_J^{\otimes m}$ over Δ_J is holomorphic at the $\gamma(I)$-cusp of $\mathbb{H}_J$ if and only if the $\gamma^{-1}\overline{\Gamma(J)_{\mathbb{Z}}}\gamma$-invariant section $\gamma^*\phi$ of $\mathcal{E}_{\lambda,k} \otimes \Theta_J^{\otimes m}$ is holomorphic at the I-cusp of $\mathbb{H}_J$.*

Proof. This holds because the pullback of a Fourier expansion of ϕ at the $\gamma(I)$-cusp by the γ-action

$$\gamma : U(I)_{\mathbb{C}}/U(J)_{\mathbb{C}} \to U(\gamma I)_{\mathbb{C}}/U(J)_{\mathbb{C}}$$

and the isomorphism $\gamma\colon V(I)_{\lambda,k} \to V(\gamma I)_{\lambda,k}$ gives a Fourier expansion of $\gamma^*\phi$ at the I-cusp. ■

Now we go back to modular forms on $\mathcal{D}$ and refine Proposition 7.3 for $M_{\lambda,k}(\Gamma)$. Recall that the m-th Fourier–Jacobi coefficient of a modular form was initially defined as a $V(I)_{\lambda,k}$-valued function on Δ_J by (7.3), and then regarded as a section of the vector bundle $\mathcal{E}_{\lambda,k} \otimes \Theta_J^{\otimes m}$ by the (I, ω_J)-trivialization. By Corollary 7.5, this section is independent of I.

Proposition 7.12. *For $m > 0$ the m-th Fourier–Jacobi coefficient of a modular form $f \in M_{\lambda,k}(\Gamma)$ as a section of $\mathcal{E}_{\lambda,k} \otimes \Theta_J^{\otimes m}$ is a Jacobi form of weight (λ, k) and index m in the sense of Definition 7.10. When f is a cusp form, the Fourier–Jacobi coefficient is a Jacobi cusp form.*

Proof. In what follows, $\widetilde{\phi}_m$ stands for the m-th Fourier–Jacobi coefficient of f as a section of $\mathcal{E}_{\lambda,k} \otimes \Theta_J^{\otimes m}$. What has to be shown is that $\widetilde{\phi}_m$ is $\overline{\Gamma(J)}_{\mathbb{Z}}$-invariant and is holomorphic at every cusp of $\mathbb{H}_J$. We first check the cusp condition. Let $I \subset J$ be an arbitrary cusp (not necessarily the initial one). Corollary 7.5 ensures that the Fourier expansion of $\widetilde{\phi}_m$ at the I-cusp of $\mathbb{H}_J$ is given by the series (7.3) obtained from the Fourier expansion of f at the I-cusp of $\mathcal{D}$. Then the holomorphicity condition for $\widetilde{\phi}_m$ at I, written in the form (7.11), follows from the cusp condition in the Fourier expansion of f at I. The assertion for cusp forms follows similarly.

It remains to check the $\overline{\Gamma(J)}_{\mathbb{Z}}$-invariance of $\widetilde{\phi}_m$. Let $\phi_m = \widetilde{\phi}_m \otimes \omega_J^{\otimes -m}$. This is a section of $\mathcal{E}_{\lambda,k}|_{\Delta_J}$ whose I-trivialization is the (I, ω_J)-trivialized form (7.3) of $\widetilde{\phi}_m$. By Proposition 7.3, we have the expansion

$$f = \sum_m (\pi_1^* \phi_m) \omega_J^m \tag{7.12}$$

as a section of $\mathcal{E}_{\lambda,k}$, where we view ω_J as a generator of the ideal sheaf $\mathcal{I}$ of Δ_J. We let $\gamma \in \overline{\Gamma(J)}_{\mathbb{Z}}$ act on this equality. Then we have

$$\gamma^* f = \sum_m \gamma^*(\pi_1^* \phi_m)(\gamma^* \omega_J)^m = \sum_m \pi_1^*(\gamma^* \phi_m)(\gamma^* \omega_J)^m$$

by Proposition 5.6. By Lemma 7.2, we have $\gamma^* \omega_J = (\pi_1^* j_\gamma) \cdot \omega_J$ for a holomorphic function j_γ on Δ_J. Therefore we have

$$\gamma^* f = \sum_m \pi_1^*(j_\gamma^m \cdot \gamma^* \phi_m) \omega_J^m. \tag{7.13}$$

Since f is Γ-invariant, we have $\gamma^* f = f$. Comparing (7.12) and (7.13), we obtain $\phi_m = j_\gamma^m \cdot \gamma^* \phi_m$ for every m. This means that $\widetilde{\phi}_m = \phi_m \otimes \omega_J^{\otimes m}$ is γ-invariant. This proves Proposition 7.12. ■

When $m = 0$ and $\lambda \neq \det$, let us denote by $J_{\lambda,k,0}(\Gamma(J)_{\mathbb{Z}})$ the space of $\overline{\Gamma(J)}_{\mathbb{Z}}$-invariant sections of

$$\mathcal{E}_\lambda^J \otimes \mathcal{L}^{\otimes k}|_{\Delta_J} \simeq \pi_2^* \mathcal{L}_J^{\otimes k+\lambda_1} \otimes V(J)_{\lambda'}$$

over Δ_J which is holomorphic at every cusp of $\mathbb{H}_J$. By the result of Section 6, the 0-th Fourier–Jacobi coefficient $\phi_0 = f|_{\Delta_J}$ of a modular form $f \in M_{\lambda,k}(\Gamma)$ belongs to this space (cuspidal when $\lambda \neq 1$). Then, as a refinement of Proposition 7.3 for $M_{\lambda,k}(\Gamma)$, we see that the Fourier–Jacobi expansion gives the embedding

$$M_{\lambda,k}(\Gamma) \hookrightarrow \prod_{m \geq 0} J_{\lambda,k,m}(\Gamma(J)_{\mathbb{Z}}), \quad f = \sum_m (\pi_1^* \phi_m)\omega_J^m \mapsto (\phi_m \otimes \omega_J^m)_m,$$

which is canonically determined by J.

7.4 Classical Jacobi forms

In this section we introduce coordinates and translate Jacobi forms with $\lambda = 0$ in the sense of Definition 7.10 to classical scalar-valued Jacobi forms à la [20, 43]. The result is stated in Proposition 7.18. Our purpose is to deduce a vanishing theorem in the present setting (Proposition 7.19) from the one for classical Jacobi forms.

7.4.1 Coordinates

We begin by setting some notations. In $U(J)_{\mathbb{Q}} \simeq \wedge^2 J_{\mathbb{Q}}$ we have two natural lattices: $\wedge^2 J$ and $U(J)_{\mathbb{Z}}$. The former depends on L, and the latter depends on Γ. Recall that the positive generator of $U(J)_{\mathbb{Z}}$ is denoted by $v_{J,\Gamma}$ (Section 7.1), and the positive generator of $\wedge^2 J$ is denoted by v_J (Section 5.1.2). Then $v_J = \beta_0 v_{J,\Gamma}$ for some rational number $\beta_0 > 0$. This constant β_0 depends only on L and Γ. We choose an isotropic plane in $L_{\mathbb{Q}}$ whose pairing with $J_{\mathbb{Q}}$ is nondegenerate, and denote it by $J_{\mathbb{Q}}^{\vee}$ for the obvious reason. This is fixed throughout Section 7.4. We identify $V(J)_{\mathbb{Q}} = (J^{\perp}/J)_{\mathbb{Q}}$ with the subspace $(J_{\mathbb{Q}} \oplus J_{\mathbb{Q}}^{\vee})^{\perp}$ of $L_{\mathbb{Q}}$.

Next we choose a rank 1 primitive sublattice I of J. Let e_1, f_1, e_2, f_2 be the standard hyperbolic basis of $2U$. We take an embedding $2U_{\mathbb{Q}} \hookrightarrow L_{\mathbb{Q}}$ which sends

$$\mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \to J, \quad \mathbb{Z}e_1 \to I, \quad \mathbb{Q}f_1 \oplus \mathbb{Q}f_2 \to J_{\mathbb{Q}}^{\vee}$$

isomorphically. Thus it is compatible with $I \subset J$ in the sense of Section 5. We identify e_1, f_1, e_2, f_2 with their image in $L_{\mathbb{Q}}$. Then $v_J = e_2 \otimes e_1$. We define vectors $l_J, l_{J,\Gamma} \in U(I)_{\mathbb{Q}}$ (as in Sections 5.1.2 and 7.1) by $l_J = f_2 \otimes e_1$ and $l_{J,\Gamma} = \beta_0 l_J$. We also put $I' = \mathbb{Z}f_1$. The choice of these data has two effects: it introduces coordinates on $\mathcal{D}$ and on the Jacobi group.

The coordinates on $\mathcal{D}$ are introduced following Section 5.1.2. The choice of $I' = \mathbb{Z}f_1$ defines the tube domain realization $\mathcal{D} \to \mathcal{D}_I \subset U(I)_{\mathbb{C}}$. According to the decomposition

$$U(I)_{\mathbb{C}} = (U_{\mathbb{C}} \oplus V(J)_{\mathbb{C}}) \otimes I_{\mathbb{C}} = \mathbb{C}l_J \times (V(J) \otimes \mathbb{C}e_1) \times \mathbb{C}v_J,$$

we express a point of $U(I)_{\mathbb{C}}$ as

$$Z = \tau l_J + z \otimes e_1 + w v_J = (\tau, z, w), \quad \tau, w \in \mathbb{C}, \ z \in V(J)_{\mathbb{C}}.$$

These are the same coordinates as in (5.5) except that z in (5.5) is $z \otimes e_1$ here. When $Z \in \mathcal{D}_I$, the corresponding point of $\mathcal{D}$ is $\mathbb{C}\omega(Z)$, where

$$\omega(Z) = f_1 + \tau f_2 + z + w e_2 - ((z, z)/2 + \tau w) e_1 \in L_{\mathbb{C}}. \tag{7.14}$$

Note that this vector is normalized so as to have pairing 1 with e_1. In these coordinates, the Siegel domain realization $\mathcal{D} \to \mathcal{V}_J \to \mathbb{H}_J$ with respect to J is the restriction of the projection

$$\mathbb{C} l_J \times V(J) \times \mathbb{C} v_J \to \mathbb{C} l_J \times V(J) \to \mathbb{C} l_J, \quad (\tau, z, w) \mapsto (\tau, z) \mapsto \tau$$

to the tube domain $\mathcal{D}_I$. The coordinates introduced on $\mathbb{H}_J \subset \mathbb{P}(L/J^{\perp})_{\mathbb{C}}$ and $\mathcal{V}_J \subset \mathbb{P}(L/J)_{\mathbb{C}}$ are written as

$$\mathbb{H} \xrightarrow{\simeq} \mathbb{H}_J, \quad \tau \mapsto \tau l_J = \mathbb{C}(f_1 + \tau f_2), \tag{7.15}$$

$$\mathbb{H} \times V(J) \xrightarrow{\simeq} \mathcal{V}_J, \quad (\tau, z) \mapsto \tau l_J + z \otimes e_1 = \mathbb{C}(f_1 + \tau f_2 + z). \tag{7.16}$$

Note that the isomorphism (7.15) maps the cusps $\mathbb{P}^1_{\mathbb{Q}} = \{i\infty\} \cup \mathbb{Q}$ of $\mathbb{H} \subset \mathbb{P}^1$ to the cusps $\mathbb{P} J^{\vee}_{\mathbb{Q}}$ of $\mathbb{H}_J \subset \mathbb{P} J^{\vee}_{\mathbb{C}}$, and especially maps the cusp $i\infty$ to the I-cusp $I^{\perp} \cap \mathbb{P} J^{\vee}_{\mathbb{C}}$ of $\mathbb{H}_J$.

Next we consider the Jacobi group $\Gamma(J)_F$, $F = \mathbb{Q}, \mathbb{R}$. Recall from (5.7) that the splitting $L_F = (J_F \oplus J^{\vee}_F) \oplus V(J)_F$ defines an isomorphism

$$\Gamma(J)_F \simeq \mathrm{SL}(J_F) \ltimes W(J)_F, \tag{7.17}$$

which we fix below. (This splitting depends on $J^{\vee}_F$, but not on I.) We identify

$$\mathrm{SL}(J_F) = \mathrm{SL}(J^{\vee}_F) = \mathrm{SL}(2, F)$$

by the basis f_2, f_1 of $J^{\vee}_F$, or equivalently, by the basis $e_1, -e_2$ of J_F. Thus an element $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2, F)$ acts on $J_F \oplus J^{\vee}_F$ by

$$e_1 \mapsto a e_1 - c e_2, \quad e_2 \mapsto -b e_1 + d e_2, \quad f_1 \mapsto d f_1 + b f_2, \quad f_2 \mapsto c f_1 + a f_2.$$

Finally, we have a splitting of the Heisenberg group $W(J)_F$ *as a set*:

$$\begin{aligned} W(J)_F &\simeq U(J)_F \times (V(J)_F \otimes F e_1) \times (V(J)_F \otimes F e_2) \\ &\simeq F \times V(J)_F \times V(J)_F, \end{aligned} \tag{7.18}$$

where we take v_J as the basis of $U(J)_F$. Accordingly, we write an element of $W(J)_F$ as (α, v_1, v_2), where $\alpha \in F$ and $v_1, v_2 \in V(J)_F \subset L_F$. In this expression,

$(\alpha, 0, 0) = \alpha v_J$ corresponds to $E_{\alpha e_2 \wedge e_1} \in U(J)_F$, $(0, v_1, 0)$ to $E_{v_1 \otimes e_1}$, and $(0, 0, v_2)$ to $E_{v_2 \otimes e_2}$. Note that each $V(J)_F \otimes Fe_1$ and $V(J)_F \otimes Fe_2$ are, respectively, subgroups of $W(J)_F$, but they do not commute.

Proposition 7.13. *The action of $\Gamma(J)_F$ on $\mathcal{D}$ is described as follows.*

(1) $(\alpha, 0, 0) \in U(J)_F$ *acts by*

$$(\tau, z, w) \mapsto (\tau, z, w + \alpha).$$

(2) $(0, v_1, 0) \in W(J)_F$ *acts by*

$$(\tau, z, w) \mapsto (\tau, z + v_1, w).$$

(3) $(0, 0, v_2) \in W(J)_F$ *acts by*

$$(\tau, z, w) \mapsto (\tau, z + \tau v_2, w - (v_2, z) - 2^{-1}(v_2, v_2)\tau).$$

(4) $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2, F)$ *acts by*

$$(\tau, z, w) \mapsto \left(\frac{a\tau + b}{c\tau + b}, \frac{z}{c\tau + d}, w + \frac{c(z, z)}{2(c\tau + d)}\right).$$

Proof. Let $\omega(Z) \in L_{\mathbb{C}}$ be as in (7.14). By direct calculation using the definition (1.3) of Eichler transvections, we see that

$$\begin{aligned}
E_{\alpha e_2 \wedge e_1}(\omega(Z)) &= f_1 + \tau f_2 + z + (w + \alpha)e_2 + Ae_1 \\
&= \omega(Z + (0, 0, \alpha)), \\
E_{v_1 \otimes e_1}(\omega(Z)) &= f_1 + \tau f_2 + (z + v_1) + we_2 + Ae_1 \\
&= \omega(Z + (0, v_1, 0)), \\
E_{v_2 \otimes e_2}(\omega(Z)) &= f_1 + \tau f_2 + (z + \tau v_2) + (w - (z, v_2) - (\tau/2)(v_2, v_2))e_2 + Ae_1 \\
&= \omega(Z + (0, \tau v_2, -(z, v_2) - (\tau/2)(v_2, v_2))), \\
\begin{pmatrix} a & b \\ c & d \end{pmatrix}(\omega(Z)) &= (c\tau + d) f_1 + (a\tau + b) f_2 \\
&\quad + z + ((c\tau + d)w + (c/2)(z, z))e_2 + Ae_1.
\end{aligned}$$

Here the constant A in each equation is an unspecified constant determined by the isotropicity condition. This proves (1)–(4). ∎

Proposition 7.13 agrees with the classical description of the action of Jacobi group in [20, p. 1185]. (α, v_1, v_2 correspond to r, y, x in [20], respectively.) We note two consequences of the calculation in Proposition 7.13.

Corollary 7.14. *Let $\gamma \in \Gamma(J)_{\mathbb{R}}$ and $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ be its image in* $\mathrm{SL}(2,\mathbb{R})$. *The factor of automorphy of the γ-action on $\mathcal{L}$ with respect to the I-trivialization $\mathcal{L} \simeq I_{\mathbb{C}}^{\vee} \otimes \mathcal{O}_{\mathfrak{D}}$ is $c\tau + d$.*

Proof. In view of (2.2), this follows by looking at the coefficients of f_1 in the equations in the proof of Proposition 7.13. ∎

This gives a computational explanation of the $\Gamma(J)_{\mathbb{R}}$-equivariant isomorphism $\mathcal{L} \simeq \pi^* \mathcal{L}_J$ in Lemma 5.9. We also provide a computational proof of Lemma 7.2.

Corollary 7.15 (cf. Lemma 7.2). *Let $\gamma \in \Gamma(J)_{\mathbb{R}}$ and $\omega_J = e((l_{J,\Gamma}, Z))$ be as in Section 7.1. Then $\gamma^* \omega_J = j_\gamma(\tau, z)\omega_J$ for a function $j_\gamma(\tau, z)$ of (τ, z) which does not depend on the w-component.*

Proof. Since $l_{J,\Gamma} = \beta_0 l_J$, if we express $Z = (\tau, z, w)$, we have

$$\omega_J = e((l_{J,\Gamma}, Z)) = e((\beta_0 l_J, w v_J)) = e(\beta_0 w).$$

Therefore, if we denote by $\gamma^* w$ the w-component of $\gamma(Z)$, we have

$$(\gamma^* \omega_J)/\omega_J = e(\beta_0(\gamma^* w - w)). \tag{7.19}$$

It remains to observe from Proposition 7.13 that $\gamma^* w - w$ depends only on (τ, z). ∎

The function $j_\gamma(\tau, z)$ is the inverse of the factor of automorphy of the γ-action ($=$ pullback by γ^{-1}) on the conormal bundle Θ_J of Δ_J with respect to the ω_J-trivialization. Thus $j_\gamma(\tau, z)$ is the multiplier in the slash operator by γ on Θ_J with respect to the ω_J-trivialization. By (7.19), $j_\gamma(\tau, z)$ is explicitly written as follows.

$$j_\gamma(\tau, z) = \begin{cases} e(\beta_0 \alpha), & \gamma = (\alpha, 0, 0), \\ 1, & \gamma = (0, v_1, 0), \\ e(-\beta_0(v_2, z) - 2^{-1}\beta_0(v_2, v_2)\tau), & \gamma = (0, 0, v_2), \\ e\left(\frac{\beta_0 c(z,z)}{2(c\tau + d)}\right), & \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right). \end{cases} \tag{7.20}$$

If we divide $\Gamma(J)_{\mathbb{R}}$ by $U(J)_{\mathbb{R}}$, these coincide with the multipliers in the slash operator in [43, p. 248] with $k = 0$ and the quadratic space $V(J)_{\mathbb{Q}}(-\beta_0)$. (We identify the half-integral matrix F in [43] with the even lattice with Gram matrix $2F$, and this lattice tensored with $\mathbb{Q}$ corresponds to our $V(J)_{\mathbb{Q}}(-\beta_0)$.)

7.4.2 Translation to classical Jacobi forms

Now, using the coordinates prepared in Section 7.4.1, we describe Jacobi forms with $\lambda = 0$ in a more classical manner. We identify

$$\Delta_J \simeq \mathbb{H} \times V(J)$$

by (7.16) and accordingly use the coordinates (τ, z) on Δ_J. We put $q_J = e(\tau) = e((v_J, Z))$ and $q_{J,\Gamma} = e((v_{J,\Gamma}, Z))$ (as in (7.10)) for $Z = (\tau, z) \in \Delta_J$. Since $v_{J,\Gamma} = \beta_0^{-1} v_J$, then $q_{J,\Gamma} = e(\beta_0^{-1}\tau) = (q_J)^{\beta_0^{-1}}$. We also write $\beta_2 = \beta_0^{-1}\beta_1$.

Let $\phi \in J_{k,m}(\Gamma(J)_{\mathbb{Z}})$ be a Jacobi form of weight $(0,k)$ and index m in the sense of Definition 7.10. Via the (I, ω_J)-trivialization and the basis e_1 of I,

$$\mathcal{L}^{\otimes k} \otimes \Theta_J^{\otimes m} \simeq (I_{\mathbb{C}}^{\vee})^{\otimes k} \otimes \mathcal{O}_{\Delta_J} \simeq \mathcal{O}_{\Delta_J},$$

we regard ϕ as a scalar-valued function on Δ_J. Let $V(J)(\beta_0 m)$ be the scaling of the quadratic space $V(J)$ by $\beta_0 m$.

Lemma 7.16. *We identify $V(J) = V(J)(\beta_0 m)$ as a $\mathbb{C}$-linear space naturally and regard ϕ as a function on $\Delta_J \simeq \mathbb{H} \times V(J)(\beta_0 m)$. Then ϕ has a Fourier expansion of the form*

$$\phi(\tau, z) = \sum_{n \in \beta_2\mathbb{Z}} \sum_{l \in K_I(\beta_0 m)^{\vee}} a(n,l) q^l q_J^n, \quad \tau \in \mathbb{H},\ z \in V(J)(\beta_0 m).$$

Here $q^l = e((l,z))$ with (l,z) being the pairing in $V(J)(\beta_0 m)$, and K_I is some full lattice in $V(J)_{\mathbb{Q}}$ such that $K_I(\beta_0)$ is an even lattice. The holomorphicity condition at the I-cusp is $2n \geq |(l,l)|$.

Proof. Recall from (7.10) that ϕ as a function on $\mathbb{H} \times V(J)$ has a Fourier expansion of the form

$$\phi(\tau, z) = \sum_{n \in \beta_1\mathbb{Z}} \sum_{l \in K'} a(n,l) q^l q_{J,\Gamma}^n, \quad \tau \in \mathbb{H},\ z \in V(J),$$

where K' is some full lattice in $V(J)_{\mathbb{Q}}$ and $q^l = e((l,z))$. (The vectors l in (7.10) are $l \otimes e_1$ here.) The I-cusp condition is $2nm \geq |(l,l)|$. We substitute $q_{J,\Gamma} = (q_J)^{\beta_0^{-1}}$ and rewrite $\beta_0^{-1} n$ as n. Then this expression is rewritten as

$$\phi(\tau, z) = \sum_{n \in \beta_2\mathbb{Z}} \sum_{l \in K'} a(n,l) q^l q_J^n, \quad \tau \in \mathbb{H},\ z \in V(J),$$

with the I-cusp condition being $2n\beta_0 m \geq |(l,l)|$. By enlarging K', we may assume that $K' = K_I^{\vee}$ for a lattice $K_I \subset V(J)_{\mathbb{Q}}$ such that $K_I(\beta_0)$ is even.

Next we identify $V(J) = V(J)(\beta_0 m)$ as a $\mathbb{C}$-linear space, which multiplies the quadratic form by $\beta_0 m$. This identification maps the lattice $K_I^{\vee} \subset V(J)$ to the lattice $\beta_0 m K_I(\beta_0 m)^{\vee} \subset V(J)(\beta_0 m)$. Then, by multiplying the index lattice $K_I^{\vee}$ by $(\beta_0 m)^{-1}$ and identifying it with $K_I(\beta_0 m)^{\vee}$ by this scaling, the Fourier expansion of ϕ as a function on $\mathbb{H} \times V(J)(m\beta_0)$ is written as

$$\phi(\tau, z) = \sum_{n \in \beta_2\mathbb{Z}} \sum_{l \in K_I(\beta_0 m)^{\vee}} a(n,l) q^l q_J^n, \quad \tau \in \mathbb{H},\ z \in V(J)(\beta_0 m),$$

where (l,z) in $q^l = e((l,z))$ is the pairing in $V(J)(\beta_0 m)$. The I-cusp condition is then rewritten as $2n \geq |(l,l)|$ for $l \in K_I(\beta_0 m)^\vee$. ■

Here we passed from $q_{J,\Gamma}$ to q_J because the latter does not depend on Γ, and passed from $V(J)$ to $V(J)(\beta_0 m)$ in order to match our holomorphicity condition at the I-cusp to the holomorphicity condition at $i\infty$ of Skoruppa [43, p. 249].

Next we shrink the integral Jacobi group $\overline{\Gamma(J)_{\mathbb{Z}}}$ to a subgroup of simpler form. We let $\Gamma_J \subset \mathrm{SL}(J)$ be the intersection of $\Gamma(J)_{\mathbb{Z}}$ with the lifted group $\mathrm{SL}(J_{\mathbb{Q}}) \subset \Gamma(J)_{\mathbb{Q}}$. (This is different from the notation in Section 6.3 in general.) Note that Γ_J does not depend on I (but on $J_{\mathbb{Q}}^\vee \subset L_{\mathbb{Q}}$). The splitting (7.17) defines an isomorphism

$$\Gamma(J)_{\mathbb{Q}}/U(J)_{\mathbb{Q}} \simeq \mathrm{SL}(J_{\mathbb{Q}}) \ltimes (V(J)_{\mathbb{Q}} \otimes J_{\mathbb{Q}}),$$

where $\mathrm{SL}(J_{\mathbb{Q}})$ acts on $V(J)_{\mathbb{Q}} \otimes J_{\mathbb{Q}}$ by its natural action on $J_{\mathbb{Q}}$. We fix this splitting of $\Gamma(J)_{\mathbb{Q}}/U(J)_{\mathbb{Q}}$. The inclusion $\Gamma(J)_{\mathbb{Z}} \subset \Gamma(J)_{\mathbb{Q}}$ defines a canonical injective map $\overline{\Gamma(J)_{\mathbb{Z}}} \to \Gamma(J)_{\mathbb{Q}}/U(J)_{\mathbb{Q}}$. Its image is not necessarily a semi-product. Elements in the intersection $\overline{\Gamma(J)_{\mathbb{Z}}} \cap (V(J)_{\mathbb{Q}} \otimes \mathbb{Q}e_i)$ are images of elements of $\Gamma(J)_{\mathbb{Z}}$ of the form $E_{\alpha e_1 \wedge e_2} \circ E_{v \otimes e_i}$, $v \in V(J)_{\mathbb{Q}}$, but $\alpha e_1 \wedge e_2 \in U(J)_{\mathbb{Q}}$ is not necessarily contained in $U(J)_{\mathbb{Z}}$ in general. We remedy these two subtle problems by passing to a subgroup of $\overline{\Gamma(J)_{\mathbb{Z}}}$ as follows.

Lemma 7.17. *There exists a full lattice K_I' in $V(J)_{\mathbb{Q}}$ such that*

$$\Gamma_J \ltimes (K_I' \otimes_{\mathbb{Z}} J) \subset \overline{\Gamma(J)_{\mathbb{Z}}} \tag{7.21}$$

as subgroups of $\Gamma(J)_{\mathbb{Q}}/U(J)_{\mathbb{Q}}$, and for each $i = 1, 2$, the subgroup $K_I' \otimes_{\mathbb{Z}} \mathbb{Z}e_i$ of this semi-product is contained in the image of $W(J)_{\mathbb{Z}} \cap (V(J)_{\mathbb{Q}} \otimes \mathbb{Q}e_i)$ in $\Gamma(J)_{\mathbb{Q}}/U(J)_{\mathbb{Q}}$, where $V(J)_{\mathbb{Q}} \otimes \mathbb{Q}e_i$ is the component of $W(J)_{\mathbb{Q}}$ in (7.18).

Proof. The intersection of $W(J)_{\mathbb{Z}}$ with the component $V(J)_{\mathbb{Q}} \otimes \mathbb{Q}e_i$ in (7.18) is a full lattice in $V(J)_{\mathbb{Q}} \otimes \mathbb{Q}e_i$ and hence can be written as $K_i \otimes_{\mathbb{Z}} \mathbb{Z}e_i$ for some full lattice K_i in $V(J)_{\mathbb{Q}}$. We put $K_I' = K_1 \cap K_2$. Then the second property holds by construction. Since $J = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$, it follows that

$$K_I' \otimes_{\mathbb{Z}} J \subset \overline{\Gamma(J)_{\mathbb{Z}}} \cap (V(J)_{\mathbb{Q}} \otimes J_{\mathbb{Q}}).$$

Since we also have $\Gamma_J \subset \overline{\Gamma(J)_{\mathbb{Z}}} \cap \mathrm{SL}(J_{\mathbb{Q}})$ by construction, the inclusion (7.21) is verified. ■

The second property in Lemma 7.17 means that $E_{v\otimes e_1}, E_{v\otimes e_2} \in W(J)_{\mathbb{Z}}$ for $v \in K_I'$, and their images in $\Gamma(J)_{\mathbb{Q}}/U(J)_{\mathbb{Q}}$ form the subgroups $K_I' \otimes_{\mathbb{Z}} \mathbb{Z}e_1$, $K_I' \otimes_{\mathbb{Z}} \mathbb{Z}e_2$ in (7.21), respectively. Their factors of automorphy on Θ_J are given by the second and the third line in (7.20), respectively. This is why we require the second property in Lemma 7.17.

We can now state the translation of Jacobi forms in a precise form. For an even negative-definite lattice K', let $J_{k,K'}(\Gamma_J)$ be the space of Jacobi forms of weight k and index lattice $K'(-1)$ for the group

$$\Gamma_J < \mathrm{SL}(J) \simeq \mathrm{SL}(2,\mathbb{Z})$$

in the sense of Skoruppa [43, p. 249]. (In the notation of [43], $K'(-1)$ is the positive-definite even lattice with Gram matrix $2F$, and corresponds to the $\mathbb{Z}^n$ in the Heisenberg group in [43, p. 248]. The dual lattice of $K'(-1)$ corresponds to the index $\mathbb{Z}^n$ in the Fourier expansion in [43, p. 249].)

Proposition 7.18. *There exists a full lattice K in $V(J)_{\mathbb{Q}}$ such that $K(\beta_0)$ is an even lattice and we have an embedding*

$$J_{k,m}(\Gamma(J)_{\mathbb{Z}}) \hookrightarrow J_{k,K(\beta_0 m)}(\Gamma_J)$$

for every $m > 0$ and $k \in \mathbb{Z}$.

Proof. The correspondence is summarized as follows.

(1) Start from a section ϕ of $\mathcal{L}^{\otimes k} \otimes \Theta_J^{\otimes m}$ over Δ_J.

(2) Choose a rank 1 primitive sublattice $I \subset J$ and identify ϕ with a holomorphic function on Δ_J by the (I,ω_J)-trivialization of $\mathcal{L}^{\otimes k} \otimes \Theta_J^{\otimes m}$.

(3) Identify $\Delta_J \simeq \mathbb{H} \times V(J)(\beta_0 m)$ by the coordinates in Section 7.4.1 and the scaling
$$V(J) \simeq V(J)(\beta_0 m).$$

(4) In this way ϕ is identified with a holomorphic function on $\mathbb{H} \times V(J)(\beta_0 m)$.

We shall show that this correspondence defines a well-defined map from $J_{k,m}(\Gamma(J)_{\mathbb{Z}})$ to $J_{k,K(\beta_0 m)}(\Gamma_J)$ for a suitable lattice $K \subset V(J)_{\mathbb{Q}}$.

We replace K_I in Lemma 7.16 and K_I' in Lemma 7.17 by their intersection $K_I \cap K_I'$ and rewrite it as K_I. Then Lemma 7.16 says that our Jacobi form ϕ viewed as a function on $\mathbb{H} \times V(J)(\beta_0 m)$ by the above procedure has the same shape of Fourier expansion as that of Jacobi forms of weight k and index lattice $K_I(\beta_0 m)$ at $i\infty$ in the sense of [43, p. 249]. Our I-cusp condition $2n \geq |(l,l)|$ agrees with the holomorphicity condition at $i\infty$ in [43]. By Corollary 7.14 and (7.20), we see that the factor of automorphy for the action of $\Gamma_J \ltimes (K_I \otimes J)$ on $\mathcal{L}^{\otimes k} \otimes \Theta_J^{\otimes m}$ with respect to the (I,ω_J)-trivialization agrees with the factor of automorphy for the slash operator $|_{k,V(J)(\beta_0 m)}$ in [43, p. 248]. Therefore the function ϕ satisfies the transformation rule of [43, p. 249] (Definition (i)) for the group $\Gamma_J < \mathrm{SL}(J)$ with weight k and index lattice $K_I(\beta_0 m)$. In particular, the function ϕ is also holomorphic (in the sense of [43]) at the cusps equivalent to I under Γ_J.

It remains to cover all cusps. The coincidence of the automorphy factors on $\mathrm{SL}(J_{\mathbb{R}})$ implies that the function $\phi|_{k,V(J)(\beta_0 m)}\gamma$ for $\gamma \in \mathrm{SL}(J)$ is identified with the section $\gamma^*\phi$ via the (I, ω_J)-trivialization. Then we have

$$
\begin{aligned}
& \text{the section } \phi \text{ is holomorphic at the } \gamma I\text{-cusp in our sense} \\
\Leftrightarrow \quad & \text{the section } \gamma^*\phi \text{ is holomorphic at the } I\text{-cusp in our sense} \\
\Leftrightarrow \quad & \text{the function } \phi|_{k,V(J)(\beta_0 m)}\gamma \text{ is holomorphic at } i\infty \text{ in the sense of [43].}
\end{aligned}
$$

The first equivalence follows from Lemma 7.11, and the second equivalence follows by applying the argument so far to the Jacobi form $\gamma^*\phi$ for $\gamma^{-1}\Gamma(J)_{\mathbb{Z}}\gamma \subset \Gamma(J)_{\mathbb{Q}}$ (with $J = \gamma(J)$ and $U(J)_{\mathbb{Z}}$ unchanged). Here the index lattice for $\gamma^*\phi$ is determined from the Fourier expansion of $\gamma^*\phi$ at the I-cusp with the group $\gamma^{-1}\Gamma(J)_{\mathbb{Z}}\gamma$, by the procedure in Lemma 7.16. We denote it by $K_{\gamma I}(\beta_0 m)$, with $K_{\gamma I}$ a full lattice in $V(J)_{\mathbb{Q}}$. This may be in general different from K_I.

Then we take representatives $I_1 = I, I_2, \dots, I_N$ of Γ_J-equivalence classes of rank 1 primitive sublattices of J and put

$$K = \bigcap_i K_{I_i} \subset V(J)_{\mathbb{Q}}.$$

As a function on $\mathbb{H} \times V(J)(\beta_0 m)$, ϕ satisfies the transformation rule of Jacobi forms of weight k and index lattice $K(\beta_0 m)$ for $\Gamma_J < \mathrm{SL}(J)$, and is holomorphic at the cusps $I_1 = i\infty, I_2, \dots, I_N$ of $\mathbb{H}_J \simeq \mathbb{H}$ in the sense of [43]. If $\gamma(I_i)$, $\gamma \in \Gamma_J$, is an arbitrary cusp of $\mathbb{H}_J$, the holomorphicity of $\phi = \phi|_{k,V(J)(\beta_0 m)}\gamma$ at I_i implies that of ϕ at $\gamma(I_i)$. Thus ϕ is holomorphic at all cusps, namely, $\phi \in J_{k,K(\beta_0 m)}(\Gamma_J)$. ■

Proposition 7.18 implies the following.

Proposition 7.19. *We have* $J_{k,m}(\Gamma(J)_{\mathbb{Z}}) = 0$ *when* $k < n/2 - 1$.

Proof. This holds because $J_{k,K'}(\Gamma_J) = 0$ when $k < \mathrm{rk}(K')/2 = n/2 - 1$ (see [43, p. 251]). ■