Chapter 8

Filtrations associated to 1-dimensional cusps

Let L, Γ, J be as in Section 7. In this chapter we introduce filtrations on the automorphic vector bundles canonically associated to the J-cusp, and study its basic properties. These filtrations will play a fundamental role in the study of the Fourier–Jacobi expansion. Our geometric approach will be effective here. In Section 8.1 we define the filtration on the second Hodge bundle $\mathcal{E}$. This induces filtrations on general automorphic vector bundles $\mathcal{E}_{\lambda,k}$ (Section 8.2). In Section 8.3 we study these filtrations from the viewpoint of representations of a parabolic subgroup. In Section 8.4, as the first application of our filtration, we prove that vector-valued Jacobi forms decompose, in a certain sense, into scalar-valued Jacobi forms of various weights. The second application will be given in Section 9.

8.1 J-filtration on $\mathcal{E}$

In this section we define a filtration on $\mathcal{E}$ canonically associated to J. For $[\omega] \in \mathcal{D}$ we consider the filtration

$$0 \subset \omega^{\perp} \cap J_{\mathbb{C}} \subset \omega^{\perp} \cap J_{\mathbb{C}}^{\perp} \subset \omega^{\perp} \tag{8.1}$$

on $\omega^{\perp} = \omega^{\perp} \cap L_{\mathbb{C}}$.

Lemma 8.1. *Let $p\colon \omega^{\perp} \to \omega^{\perp}/\mathbb{C}\omega$ be the projection. Then $p(\omega^{\perp} \cap J_{\mathbb{C}})$ has dimension 1 and $p(\omega^{\perp} \cap J_{\mathbb{C}}^{\perp}) = p(\omega^{\perp} \cap J_{\mathbb{C}})^{\perp}$ in $\omega^{\perp}/\mathbb{C}\omega$.*

Proof. Since $(\omega, J) \not\equiv 0$, we have $\dim(\omega^{\perp} \cap J_{\mathbb{C}}) = 1$. The fact that $\mathbb{C}\omega \not\subset J_{\mathbb{C}}$ then implies that $p(\omega^{\perp} \cap J_{\mathbb{C}})$ has dimension 1. Next we prove the second assertion. It is clear that $p(\omega^{\perp} \cap J_{\mathbb{C}}^{\perp}) \subset p(\omega^{\perp} \cap J_{\mathbb{C}})^{\perp}$. Since $p(\omega^{\perp} \cap J_{\mathbb{C}})^{\perp}$ is of codimension 1 in $\omega^{\perp}/\mathbb{C}\omega$ by the first assertion, it is sufficient to show that $p(\omega^{\perp} \cap J_{\mathbb{C}}^{\perp})$ is of codimension 1 too. Since $\mathbb{C}\omega \not\subset J_{\mathbb{C}}$, we have $(\omega, J^{\perp}) \not\equiv 0$. This implies that $\omega^{\perp} \cap J_{\mathbb{C}}^{\perp}$ is of codimension 1 in $J_{\mathbb{C}}^{\perp}$, and so of codimension 2 in $\omega^{\perp}$. The fact that $\mathbb{C}\omega \not\subset J_{\mathbb{C}}^{\perp}$ implies that the projection $\omega^{\perp} \cap J_{\mathbb{C}}^{\perp} \to \omega^{\perp}/\mathbb{C}\omega$ is injective. Hence $p(\omega^{\perp} \cap J_{\mathbb{C}}^{\perp})$ is of codimension 1 in $\omega^{\perp}/\mathbb{C}\omega$. ∎

Let $\mathcal{E}_J$ be the sub line bundle of $\mathcal{E}$ whose fiber over $[\omega] \in \mathcal{D}$ is the image of $\omega^{\perp} \cap J_{\mathbb{C}}$ in $\omega^{\perp}/\mathbb{C}\omega$. This is an isotropic sub line bundle of $\mathcal{E}$. Taking the image of (8.1) in $\omega^{\perp}/\mathbb{C}\omega$ and varying $[\omega] \in \mathcal{D}$, we obtain the filtration

$$0 \subset \mathcal{E}_J \subset \mathcal{E}_J^{\perp} \subset \mathcal{E} \tag{8.2}$$

on $\mathcal{E}$. We call it the J*-filtration* on $\mathcal{E}$. By construction, this is $\Gamma(J)_{\mathbb{R}}$-invariant.

We calculate the graded quotients of the J-filtration. Let $\pi\colon \mathcal{D} \to \mathbb{H}_J$ be the projection to the J-cusp and $\mathcal{L}_J$ be the Hodge bundle on $\mathbb{H}_J$. We write $V(J) = (J^\perp/J)_{\mathbb{C}}$ as before.

Proposition 8.2. *We have $\Gamma(J)_{\mathbb{R}}$-equivariant isomorphisms*

$$\mathcal{E}_J \simeq \pi^*\mathcal{L}_J, \quad \mathcal{E}_J^\perp/\mathcal{E}_J \simeq V(J) \otimes \mathcal{O}_{\mathcal{D}}, \quad \mathcal{E}/\mathcal{E}_J^\perp \simeq \pi^*\mathcal{L}_J^{-1}. \tag{8.3}$$

Proof. We begin with $\mathcal{E}_J$. Let $[\omega] \in \mathcal{D}$. The fiber of $\mathcal{E}_J$ over $[\omega]$ is the line $\omega^\perp \cap J_{\mathbb{C}} \subset J_{\mathbb{C}}$, while that of $\pi^*\mathcal{L}_J$ is the image of $\mathbb{C}\omega$ in $(L/J^\perp)_{\mathbb{C}}$. In order to compare these two lines, we consider the canonical isomorphisms

$$(L/J^\perp)_{\mathbb{C}} \to J_{\mathbb{C}}^\vee \leftarrow J_{\mathbb{C}}. \tag{8.4}$$

Here the first map is induced by the pairing on L, and the second map is induced by the canonical symplectic form $J \times J \to \wedge^2 J \simeq \mathbb{Z}$ on J. The second map sends a line in $J_{\mathbb{C}}$ to its annihilator in $J_{\mathbb{C}}^\vee$. In (8.4), the above two lines are both sent to the line $(\mathbb{C}\omega, \cdot)|_{J_{\mathbb{C}}}$ in $J_{\mathbb{C}}^\vee$ (the pairing of $J_{\mathbb{C}}$ with $\mathbb{C}\omega$). This gives the canonical isomorphism

$$(\pi^*\mathcal{L}_J)_{[\omega]} = \mathrm{Im}(\mathbb{C}\omega \to (L/J^\perp)_{\mathbb{C}}) \to \omega^\perp \cap J_{\mathbb{C}} = (\mathcal{E}_J)_{[\omega]}.$$

Varying $[\omega]$, we obtain a $\Gamma(J)_{\mathbb{R}}$-equivariant isomorphism $\pi^*\mathcal{L}_J \simeq \mathcal{E}_J$.

Consequently, we obtain the description of the last graded quotient

$$\mathcal{E}/\mathcal{E}_J^\perp \simeq \mathcal{E}_J^\vee \simeq \pi^*\mathcal{L}_J^{-1},$$

where the first map is induced by the quadratic form on $\mathcal{E}$.

Finally, we consider the middle graded quotient $\mathcal{E}_J^\perp/\mathcal{E}_J$. The fiber of this vector bundle over $[\omega] \in \mathcal{D}$ is $(\omega^\perp \cap J_{\mathbb{C}}^\perp)/(\omega^\perp \cap J_{\mathbb{C}})$. We have a natural map

$$(\omega^\perp \cap J_{\mathbb{C}}^\perp)/(\omega^\perp \cap J_{\mathbb{C}}) \to J_{\mathbb{C}}^\perp/J_{\mathbb{C}} = V(J). \tag{8.5}$$

This is clearly injective. Since the source and the target have the same dimension, this map is an isomorphism. Varying $[\omega]$, we obtain a $\Gamma(J)_{\mathbb{R}}$-equivariant isomorphism $\mathcal{E}_J^\perp/\mathcal{E}_J \to V(J) \otimes \mathcal{O}_{\mathcal{D}}$. ∎

Next we choose a rank 1 primitive sublattice I of J and describe the J-filtration under the I-trivialization.

Proposition 8.3. *The I-trivialization $\mathcal{E} \simeq V(I) \otimes \mathcal{O}_{\mathcal{D}}$ sends the J-filtration* (8.2) *on $\mathcal{E}$ to the filtration*

$$(0 \subset J/I \subset J^\perp/I \subset I^\perp/I)_{\mathbb{C}} \otimes \mathcal{O}_{\mathcal{D}}$$

on $V(I) \otimes \mathcal{O}_{\mathcal{D}}$.

Proof. Since the I-trivialization $V(I) \otimes \mathcal{O}_{\mathcal{D}} \to \mathcal{E}$ preserves the quadratic forms, it suffices to check that this sends $(J/I)_{\mathbb{C}} \otimes \mathcal{O}_{\mathcal{D}}$ to $\mathcal{E}_J$. Recall that the I-trivialization at $[\omega] \in \mathcal{D}$ is the composition map

$$I_{\mathbb{C}}^{\perp}/I_{\mathbb{C}} \to \omega^{\perp} \cap I_{\mathbb{C}}^{\perp} \to \omega^{\perp}/\mathbb{C}\omega. \tag{8.6}$$

The inverse of the first map sends the line $\omega^{\perp} \cap J_{\mathbb{C}}$ in $\omega^{\perp} \cap I_{\mathbb{C}}^{\perp}$ to the line $J_{\mathbb{C}}/I_{\mathbb{C}}$ in $I_{\mathbb{C}}^{\perp}/I_{\mathbb{C}}$, and the second map sends $\omega^{\perp} \cap J_{\mathbb{C}}$ to $(\mathcal{E}_J)_{[\omega]}$ by definition. Therefore (8.6) sends $J_{\mathbb{C}}/I_{\mathbb{C}}$ to $(\mathcal{E}_J)_{[\omega]}$. This proves our assertion. ■

The J-filtration descends to a filtration on the descent of $\mathcal{E}$ to $\mathcal{X}(J) = \mathcal{D}/U(J)_{\mathbb{Z}}$. We consider the canonical extension over the partial toroidal compactification $\overline{\mathcal{X}(J)}$.

Proposition 8.4. *The J-filtration on $\mathcal{E}$ over $\mathcal{X}(J)$ extends to a filtration on the canonical extension of $\mathcal{E}$ over $\overline{\mathcal{X}(J)}$ by $\overline{\Gamma(J)_{\mathbb{R}}}$-invariant sub vector bundles. The isomorphisms* (8.3) *for the graded quotients on $\mathcal{X}(J)$ extend to isomorphisms between the canonical extensions of both sides over $\overline{\mathcal{X}(J)}$.*

Proof. We choose a rank 1 primitive sublattice I of J. Recall that the canonical extension of $\mathcal{E}$ is defined via the I-trivialization $\mathcal{E} \to V(I) \otimes \mathcal{O}_{\mathcal{X}(J)}$. By Proposition 8.3, the I-trivialization sends the sub vector bundles $\mathcal{E}_J$ and $\mathcal{E}_J^{\perp}$ of $\mathcal{E}$ to the sub vector bundles $(J/I)_{\mathbb{C}} \otimes \mathcal{O}_{\mathcal{X}(J)}$ and $(J^{\perp}/I)_{\mathbb{C}} \otimes \mathcal{O}_{\mathcal{X}(J)}$ of $V(I) \otimes \mathcal{O}_{\mathcal{X}(J)}$, respectively. The latter clearly extend to the sub vector bundles $(J/I)_{\mathbb{C}} \otimes \mathcal{O}_{\overline{\mathcal{X}(J)}}$ and $(J^{\perp}/I)_{\mathbb{C}} \otimes \mathcal{O}_{\overline{\mathcal{X}(J)}}$ of $V(I) \otimes \mathcal{O}_{\overline{\mathcal{X}(J)}}$, respectively. This means that $\mathcal{E}_J$ and $\mathcal{E}_J^{\perp}$ extend to sub vector bundles of the canonical extension of $\mathcal{E}$. They are still $\overline{\Gamma(J)_{\mathbb{R}}}$-invariant by continuity.

We prove that the isomorphisms (8.3) extend over $\overline{\mathcal{X}(J)}$. We begin with $\mathcal{E}_J \simeq \pi^*\mathcal{L}_J$. For each $[\omega] \in \mathcal{D}$ we have the following commutative diagram of isomorphisms between 1-dimensional linear spaces:

$$\begin{array}{ccc} \omega^{\perp} \cap J_{\mathbb{C}} & \xrightarrow{p_1} & (\mathbb{C}\omega, \cdot)|_{J_{\mathbb{C}}} \\ {\scriptstyle p_3}\downarrow & & \downarrow{\scriptstyle p_4} \\ J_{\mathbb{C}}/I_{\mathbb{C}} & \xrightarrow[p_2]{} & I_{\mathbb{C}}^{\vee}. \end{array}$$

Here p_1 is restriction of the second isomorphism $J_{\mathbb{C}} \to J_{\mathbb{C}}^{\vee}$ in (8.4), p_2 is the map induced from this $J_{\mathbb{C}} \to J_{\mathbb{C}}^{\vee}$, p_3 is the natural projection, and p_4 is the restriction of the natural map $J_{\mathbb{C}}^{\vee} \to I_{\mathbb{C}}^{\vee}$ to the line $(\mathbb{C}\omega, \cdot)|_{J_{\mathbb{C}}}$ of $J_{\mathbb{C}}^{\vee}$. Recall from the proof of Proposition 8.2 that p_1 is identified with the isomorphism $\mathcal{E}_J \to \pi^*\mathcal{L}_J$ at $[\omega]$ after the canonical isomorphism

$$J_{\mathbb{C}}^{\vee} \simeq (L/J^{\perp})_{\mathbb{C}}.$$

Varying $[\omega]$, we obtain the following commutative diagram of isomorphisms between line bundles on $\mathcal{X}(J)$:

$$\begin{array}{ccc} \mathcal{E}_J & \xrightarrow{p_1} & \pi^*\mathcal{L}_J \\ {\scriptstyle p_3}\downarrow & & \downarrow{\scriptstyle p_4} \\ (J/I)_{\mathbb{C}} \otimes \mathcal{O}_{\mathcal{X}(J)} & \xrightarrow[p_2]{} & I_{\mathbb{C}}^{\vee} \otimes \mathcal{O}_{\mathcal{X}(J)}. \end{array}$$

Here p_1 is the isomorphism we want to extend, p_2 is the constant homomorphism, p_3 is the I-trivialization of $\mathcal{E}_J$, and p_4 is the pullback of the I-trivialization of $\mathcal{L}_J$ (cf. Remark 5.11). By construction, the canonical extension of $\mathcal{E}_J$ is given via p_3. Similarly, by the proof of Proposition 5.10, the canonical extension of $\pi^*\mathcal{L}_J$ is given via p_4. Since p_2 is constant, it extends over $\overline{\mathcal{X}(J)}$. Then this commutative diagram shows that p_1 extends to an isomorphism between the canonical extensions of $\mathcal{E}_J$ and $\pi^*\mathcal{L}_J$.

Next we consider $\mathcal{E}_J^{\perp}/\mathcal{E}_J \to V(J) \otimes \mathcal{O}_{\mathcal{X}(J)}$. We observe that for each $[\omega] \in \mathcal{D}$, the natural composition

$$(\omega^{\perp} \cap J_{\mathbb{C}}^{\perp})/(\omega^{\perp} \cap J_{\mathbb{C}}) \to (J_{\mathbb{C}}^{\perp}/I_{\mathbb{C}})/(J_{\mathbb{C}}/I_{\mathbb{C}}) \to J_{\mathbb{C}}^{\perp}/J_{\mathbb{C}},$$

where the first isomorphism comes from $\omega^{\perp} \cap I_{\mathbb{C}}^{\perp} \to I_{\mathbb{C}}^{\perp}/I_{\mathbb{C}}$, coincides with the isomorphism (8.5) defining $\mathcal{E}_J^{\perp}/\mathcal{E}_J \to V(J) \otimes \mathcal{O}_{\mathcal{X}(J)}$ at $[\omega]$. Therefore the isomorphism $\mathcal{E}_J^{\perp}/\mathcal{E}_J \to V(J) \otimes \mathcal{O}_{\mathcal{X}(J)}$ in (8.3) factorizes as

$$\mathcal{E}_J^{\perp}/\mathcal{E}_J \to (J^{\perp}/I)_{\mathbb{C}} \otimes \mathcal{O}_{\mathcal{X}(J)}/(J/I)_{\mathbb{C}} \otimes \mathcal{O}_{\mathcal{X}(J)} \to V(J) \otimes \mathcal{O}_{\mathcal{X}(J)},$$

where the first isomorphism is induced by the I-trivialization and hence gives the canonical extension of $\mathcal{E}_J^{\perp}/\mathcal{E}_J$, and the second isomorphism is the constant homomorphism. The constancy of the second isomorphism ensures that it extends over $\overline{\mathcal{X}(J)}$. This shows that the isomorphism $\mathcal{E}_J^{\perp}/\mathcal{E}_J \to V(J) \otimes \mathcal{O}_{\mathcal{X}(J)}$ in (8.3) extends to an isomorphism between the canonical extensions.

Finally, the extendability of $\mathcal{E}/\mathcal{E}_J^{\perp} \simeq \pi^*\mathcal{L}_J^{-1}$ follows from the extendability of $\mathcal{E}_J \simeq \pi^*\mathcal{L}_J$ and the fact that the quadratic form on $\mathcal{E}$ extends over the canonical extension (by construction). ■

8.2 J-filtration on $\mathcal{E}_{\lambda,k}$

In this section we use the J-filtration on $\mathcal{E}$ to define a filtration on a general automorphic vector bundle $\mathcal{E}_{\lambda,k}$.

We begin with a recollection from linear algebra. Let V be a $\mathbb{C}$-linear space of finite dimension endowed with a decreasing filtration of length 3:

$$0 \subset F^1V \subset F^0V \subset F^{-1}V = V.$$

We denote by

$$\mathrm{Gr}^r\, V = F^r V / F^{r+1} V$$

the r-th graded quotient. (By convention, $F^2 V = 0$.) Let $d > 0$. On the tensor product $V^{\otimes d}$ we have a decreasing filtration of length $2d+1$ defined by

$$F^r V^{\otimes d} = \sum_{|\vec{i}|=r} F^{i_1} V \otimes F^{i_2} V \otimes \cdots \otimes F^{i_d} V, \quad -d \leq r \leq d, \tag{8.7}$$

where $\vec{i} = (i_1, \ldots, i_d)$ run over all multi-indices such that $|\vec{i}| = i_1 + \cdots + i_d$ is equal to r. The graded quotient $\mathrm{Gr}^r\, V^{\otimes d} = F^r V^{\otimes d} / F^{r+1} V^{\otimes d}$ is canonically isomorphic to

$$\mathrm{Gr}^r\, V^{\otimes d} \simeq \bigoplus_{|\vec{i}|=r} \mathrm{Gr}^{i_1}\, V \otimes \mathrm{Gr}^{i_2}\, V \otimes \cdots \otimes \mathrm{Gr}^{i_d}\, V. \tag{8.8}$$

This construction of filtration is well known in the case $d = 2$; the construction for general d is obtained inductively.

We apply this construction relatively to the J-filtration on the second Hodge bundle $\mathcal{E}$. We put

$$F^1 \mathcal{E} = \mathcal{E}_J, \quad F^0 \mathcal{E} = \mathcal{E}_J^{\perp}, \quad F^{-1} \mathcal{E} = \mathcal{E},$$

and define a decreasing filtration

$$0 \subset F^d \mathcal{E}^{\otimes d} \subset F^{d-1} \mathcal{E}^{\otimes d} \subset \cdots \subset F^{-d} \mathcal{E}^{\otimes d} = \mathcal{E}^{\otimes d}$$

of length $2d+1$ on $\mathcal{E}^{\otimes d}$ by

$$F^r \mathcal{E}^{\otimes d} = \sum_{|\vec{i}|=r} F^{i_1} \mathcal{E} \otimes F^{i_2} \mathcal{E} \otimes \cdots \otimes F^{i_d} \mathcal{E}, \quad -d \leq r \leq d.$$

This is a filtration by $\Gamma(J)_{\mathbb{R}}$-invariant sub vector bundles.

Lemma 8.5. *We have a $\Gamma(J)_{\mathbb{R}}$-equivariant isomorphism*

$$\mathrm{Gr}^r\, \mathcal{E}^{\otimes d} \simeq \pi^* \mathcal{L}_J^{\otimes r} \otimes \bigoplus_{|\vec{i}|=r} V(J)^{\otimes b(\vec{i})},$$

where $b(\vec{i}) \geq 0$ is the number of components i_* of $\vec{i} = (i_1, \ldots, i_d)$ equal to 0.

Proof. By (8.8) we have

$$\mathrm{Gr}^r\, \mathcal{E}^{\otimes d} \simeq \bigoplus_{|\vec{i}|=r} \mathrm{Gr}^{i_1}\, \mathcal{E} \otimes \cdots \otimes \mathrm{Gr}^{i_d}\, \mathcal{E}. \tag{8.9}$$

By Proposition 8.2, each factor $\mathrm{Gr}^{i_*}\,\mathcal{E}$ is isomorphic to $\pi^*\mathcal{L}_J$, $V(J)\otimes\mathcal{O}_{\mathcal{D}}$, $\pi^*\mathcal{L}_J^{-1}$ according to $i_* = 1, 0, -1$, respectively. Let $a(\vec{i})$, $b(\vec{i})$, $c(\vec{i})$ be the number of components i_* of $\vec{i} = (i_1, \dots, i_d)$ equal to $1, 0, -1$, respectively. Then (8.9) can be written more explicitly as

$$\mathrm{Gr}^r\,\mathcal{E}^{\otimes d} \simeq \bigoplus_{|\vec{i}|=r} V(J)^{\otimes b(\vec{i})} \otimes \pi^*\mathcal{L}_J^{\otimes a(\vec{i})-c(\vec{i})}.$$

We have $a(\vec{i}) - c(\vec{i}) = |\vec{i}| = r$. ■

Since $\mathrm{Gr}^{-i}\,\mathcal{E} \simeq (\mathrm{Gr}^i\,\mathcal{E})^\vee$, the expression (8.9) shows that we have the duality

$$\mathrm{Gr}^{-r}\,\mathcal{E}^{\otimes d} \simeq (\mathrm{Gr}^r\,\mathcal{E}^{\otimes d})^\vee,$$

by sending an index $\vec{i} = (i_1, \dots, i_d)$ to its dual index $(-i_1, \dots, -i_d)$.

By Proposition 8.3, the I-trivialization

$$\mathcal{E}^{\otimes d} \simeq V(I)^{\otimes d} \otimes \mathcal{O}_{\mathcal{D}}$$

sends the sub vector bundle $F^r\mathcal{E}^{\otimes d}$ of $\mathcal{E}^{\otimes d}$ to the sub vector bundle $F^rV(I)^{\otimes d} \otimes \mathcal{O}_{\mathcal{D}}$ of $V(I)^{\otimes d} \otimes \mathcal{O}_{\mathcal{D}}$, where $F^rV(I)^{\otimes d}$ is the filtration (8.7) applied to $V = V(I)$, $F^1V = (J/I)_{\mathbb{C}}$ and $F^0V = (J^\perp/I)_{\mathbb{C}}$. This implies that the filtration $F^\bullet\mathcal{E}^{\otimes d}$ on $\mathcal{E}^{\otimes d}$ over $\mathcal{X}(J)$ extends to a filtration on the canonical extension of $\mathcal{E}^{\otimes d}$ over $\overline{\mathcal{X}(J)}$ by sub vector bundles. (We use the same notation.)

Now we consider a general automorphic vector bundle $\mathcal{E}_{\lambda,k} = \mathcal{E}_\lambda \otimes \mathcal{L}^{\otimes k}$. Let $d = |\lambda|$. Recall from Section 3.2 that $\mathcal{E}_\lambda = c_\lambda \cdot \mathcal{E}^{[d]}$ is defined as an $\mathrm{O}^+(L_{\mathbb{R}})$-invariant sub vector bundle of $\mathcal{E}^{\otimes d}$, where $c_\lambda = b_\lambda a_\lambda$ is the Young symmetrizer for λ. We define a decreasing filtration on $\mathcal{E}_\lambda$ by taking the intersection with $F^r\mathcal{E}^{\otimes d}$ inside $\mathcal{E}^{\otimes d}$:

$$F^r\mathcal{E}_\lambda = \mathcal{E}_\lambda \cap F^r\mathcal{E}^{\otimes d}, \quad -d \le r \le d.$$

Then we take the twist by $\mathcal{L}^{\otimes k}$:

$$F^r\mathcal{E}_{\lambda,k} = F^r\mathcal{E}_\lambda \otimes \mathcal{L}^{\otimes k}.$$

This is a $\Gamma(J)_{\mathbb{R}}$-invariant filtration on $\mathcal{E}_{\lambda,k}$. We call it the *J-filtration* on $\mathcal{E}_{\lambda,k}$. This is a standard filtration on $\mathcal{E}_{\lambda,k}$ that can be induced from the J-filtration on $\mathcal{E}$. In Proposition 8.13, we will prove that the range of the level r reduces to $-\lambda_1 \le r \le \lambda_1$.

Remark 8.6. We also have the following natural expressions of $F^r\mathcal{E}_\lambda$:

$$F^r\mathcal{E}_\lambda = c_\lambda(\mathcal{E}^{[d]} \cap F^r\mathcal{E}^{\otimes d}) = \mathcal{E}^{[d]} \cap c_\lambda(F^r\mathcal{E}^{\otimes d}).$$

These equalities hold because we have $c_\lambda(F^r\mathcal{E}^{\otimes d}) \subset F^r\mathcal{E}^{\otimes d}$ by the $\mathfrak{S}_d$-invariance of $F^r\mathcal{E}^{\otimes d}$ and c_λ is an idempotent up to scalar multiplication.

Let

$$F^r V(I)_\lambda = V(I)_\lambda \cap F^r V(I)^{\otimes d}, \quad -d \le r \le d, \tag{8.10}$$

be the similar filtration on $V(I)_\lambda$. The I-trivialization $\mathcal{E}_\lambda \simeq V(I)_\lambda \otimes \mathcal{O}_{\mathcal{D}}$ sends the J-filtration $F^\bullet \mathcal{E}_\lambda$ on $\mathcal{E}_\lambda$ to the filtration $F^\bullet V(I)_\lambda \otimes \mathcal{O}_{\mathcal{D}}$ on $V(I)_\lambda \otimes \mathcal{O}_{\mathcal{D}}$. This implies that the J-filtration on $\mathcal{E}_{\lambda,k}$, after descending to $\mathcal{X}(J)$, extends to a filtration on the canonical extension of $\mathcal{E}_{\lambda,k}$ over $\overline{\mathcal{X}(J)}$ by $\overline{\Gamma(J)}_{\mathbb{R}}$-invariant sub vector bundles.

Proposition 8.7. *At the boundary divisor Δ_J of $\overline{\mathcal{X}(J)}$, we have a $\overline{\Gamma(J)}_{\mathbb{R}}$-equivariant isomorphism*

$$\mathrm{Gr}^r(\mathcal{E}_{\lambda,k}|_{\Delta_J}) \simeq (\pi_2^* \mathcal{L}_J^{\otimes r+k})^{\oplus \alpha(r)}, \tag{8.11}$$

where $\alpha(r) \ge 0$ is the rank of $\mathrm{Gr}^r \mathcal{E}_\lambda$ and π_2 is the projection $\Delta_J \to \mathbb{H}_J$.

Proof. Since $\mathcal{L}|_{\Delta_J} \simeq \pi_2^* \mathcal{L}_J$ by Proposition 5.10, it suffices to prove this assertion in the case $k = 0$. By Lemma 8.5, we have a $\overline{\Gamma(J)}_{\mathbb{R}}$-equivariant embedding

$$\mathrm{Gr}^r \mathcal{E}_\lambda \hookrightarrow \mathrm{Gr}^r \mathcal{E}^{\otimes |\lambda|} \simeq (\pi^* \mathcal{L}_J^{\otimes r})^{\oplus b}$$

over $\mathcal{X}(J)$ for some $b > 0$. By Proposition 8.4, this embedding extends over $\overline{\mathcal{X}(J)}$. By restricting it to Δ_J, we obtain a $\overline{\Gamma(J)}_{\mathbb{R}}$-equivariant embedding

$$\mathrm{Gr}^r(\mathcal{E}_\lambda|_{\Delta_J}) \hookrightarrow (\pi_2^* \mathcal{L}_J^{\otimes r})^{\oplus b}.$$

The image of this embedding is a $\overline{\Gamma(J)}_{\mathbb{R}}$-invariant sub vector bundle of $(\pi_2^* \mathcal{L}_J^{\otimes r})^{\oplus b}$. Since the Heisenberg group $W(J)_{\mathbb{R}} \subset \Gamma(J)_{\mathbb{R}}$ acts on each fiber of $\pi_2 \colon \Delta_J \to \mathbb{H}_J$ transitively, this image can be written as $\pi_2^* \mathcal{F}$ for some $\mathrm{SL}(J_{\mathbb{R}})$-invariant sub vector bundle $\mathcal{F}$ of $(\mathcal{L}_J^{\otimes r})^{\oplus b}$. By the $\mathrm{SL}(J_{\mathbb{R}})$-invariance, $\mathcal{F}$ is isomorphic to a direct sum of copies of $\mathcal{L}_J^{\otimes r}$. ■

Before finishing this section, we look at two typical examples.

Example 8.8. Let $\lambda = (1^d)$ with $0 < d < n$, namely, $V_\lambda = \wedge^d V$. We have $\wedge^i \mathcal{E}_J = 0$ if $i > 1$ and $(\wedge^i \mathcal{E}_J^\perp) \wedge (\wedge^j \mathcal{E}) = \wedge^{i+j} \mathcal{E}$ if $j > 0$. This shows that the J-filtration on $\wedge^d \mathcal{E}$ reduces to the following filtration of length 3:

$$0 \subset \mathcal{E}_J \wedge \left(\wedge^{d-1} \mathcal{E}_J^\perp\right) \subset \wedge^d \mathcal{E}_J^\perp + \mathcal{E}_J \wedge \left(\wedge^{d-1} \mathcal{E}\right) \subset \wedge^d \mathcal{E}.$$

These three subspaces have level $1, 0, -1$, respectively. (Note that we have $\wedge^{d-1} \mathcal{E} = (\wedge^{d-2} \mathcal{E}_J^\perp) \wedge \mathcal{E}$ in the second term and $\wedge^d \mathcal{E} = (\wedge^{d-1} \mathcal{E}_J^\perp) \wedge \mathcal{E}$ in the last term.) The three graded quotients are, respectively, isomorphic to

$$\mathcal{E}_J \otimes \wedge^{d-1}(\mathcal{E}_J^\perp / \mathcal{E}_J) \simeq \wedge^{d-1} V(J) \otimes \pi^* \mathcal{L}_J,$$
$$\wedge^d (\mathcal{E}_J^\perp / \mathcal{E}_J) \oplus \wedge^{d-2}(\mathcal{E}_J^\perp / \mathcal{E}_J) \simeq \left(\wedge^d V(J) \oplus \wedge^{d-2} V(J)\right) \otimes \mathcal{O}_{\mathcal{D}},$$
$$(\mathcal{E}/\mathcal{E}_J^\perp) \otimes \wedge^{d-1}(\mathcal{E}_J^\perp / \mathcal{E}_J) \simeq \wedge^{d-1} V(J) \otimes \pi^* \mathcal{L}_J^{-1}.$$

Here $\wedge^{d-2} V(J) = 0$ when $d = 1$, and $\wedge^d V(J) = 0$ when $d = n - 1$.

Example 8.9. The J-filtration on $\mathrm{Sym}^d \mathcal{E}$ has length $2d+1$, with subspaces

$$F^r \mathrm{Sym}^d \mathcal{E} = \sum_{\substack{a+b+c=d \\ a-c=r}} \mathrm{Sym}^a \mathcal{E}_J \cdot \mathrm{Sym}^b \mathcal{E}_J^{\perp} \cdot \mathrm{Sym}^c \mathcal{E}, \quad -d \leq r \leq d.$$

The graded quotient $\mathrm{Gr}^r \mathrm{Sym}^d \mathcal{E}$ is isomorphic to

$$\pi^* \mathcal{L}_J^{\otimes r} \otimes (\mathrm{Sym}^{d-|r|} V(J) \oplus \mathrm{Sym}^{d-|r|-2} V(J) \oplus \cdots \oplus \mathrm{Sym}^{0 \text{ or } 1} V(J)).$$

This shows that the J-filtration on the main irreducible component $\mathcal{E}_{(d)}$ of $\mathrm{Sym}^d \mathcal{E}$ has length $2d+1$ with graded quotient

$$\mathrm{Gr}^r \mathcal{E}_{(d)} \simeq \pi^* \mathcal{L}_J^{\otimes r} \otimes \mathrm{Sym}^{d-|r|} V(J), \quad -d \leq r \leq d. \tag{8.12}$$

8.3 J-filtration and representations

In this section we study the J-filtration, in its I-trivialized form, from the viewpoint of representations of a parabolic subgroup. As consequences, we determine the range of possible levels, and also relate the Siegel operator (Section 6) to the J-filtration.

We choose a rank 1 primitive sublattice $I \subset J$. Let $P(J/I)_{\mathbb{C}}$ be the stabilizer of the isotropic line $(J/I)_{\mathbb{C}} \subset V(I)$ in $\mathrm{O}(V(I))$. As in (6.4), $P(J/I)_{\mathbb{C}}$ sits in the exact sequence

$$0 \to U(J/I)_{\mathbb{C}} \to P(J/I)_{\mathbb{C}} \to \mathrm{GL}((J/I)_{\mathbb{C}}) \times \mathrm{O}(V(J)) \to 1, \tag{8.13}$$

where $U(J/I)_{\mathbb{C}} \simeq V(J) \otimes (J/I)_{\mathbb{C}}$ is the unipotent radical of $P(J/I)_{\mathbb{C}}$ consisting of the Eichler transvections of $V(I)$ with respect to $(J/I)_{\mathbb{C}}$. The filtration

$$(F^r V(I))_{-1 \leq r \leq 1} = (0 \subset (J/I)_{\mathbb{C}} \subset (J^{\perp}/I)_{\mathbb{C}} \subset V(I))$$

on $V(I)$ is $P(J/I)_{\mathbb{C}}$-invariant. The unipotent radical $U(J/I)_{\mathbb{C}}$ acts on the graded quotients trivially, so they are representations of

$$\mathrm{GL}((J/I)_{\mathbb{C}}) \times \mathrm{O}(V(J)) \simeq \mathbb{C}^* \times \mathrm{O}(n-2, \mathbb{C}).$$

Specifically,

- $\mathrm{Gr}^1 V(I) = (J/I)_{\mathbb{C}}$ is the weight 1 character of $\mathbb{C}^*$.
- $\mathrm{Gr}^0 V(I) = V(J)$ is the standard representation of $\mathrm{O}(V(J))$.
- $\mathrm{Gr}^{-1} V(I) = (J/I)_{\mathbb{C}}^{\vee}$ is the weight -1 character of $\mathbb{C}^*$.

Let $d > 0$. As in (8.7), let

$$F^r V(I)^{\otimes d} = \sum_{|\vec{i}|=r} F^{i_1} V(I) \otimes \cdots \otimes F^{i_d} V(I), \quad -d \leq r \leq d,$$

be the induced filtration on $V(I)^{\otimes d}$. This is $P(J/I)_{\mathbb{C}}$-invariant. By (8.8), the unipotent radical $U(J/I)_{\mathbb{C}}$ acts on the graded quotients $\mathrm{Gr}^r\, V(I)^{\otimes d}$ trivially. Hence $\mathrm{Gr}^r\, V(I)^{\otimes d}$ is a representation of $\mathbb{C}^* \times \mathrm{O}(V(J))$. Specifically, by the same calculation as in Lemma 8.5, we have

$$\mathrm{Gr}^r\, V(I)^{\otimes d} \simeq \chi_r \boxtimes \bigoplus_{|\vec{i}|=r} V(J)^{\otimes b(\vec{i})}, \tag{8.14}$$

where χ_r is the weight r character of $\mathbb{C}^*$. If we take a lift of $\mathbb{C}^* \times \mathrm{O}(V(J))$ in (8.13), we have a decomposition

$$V(I)^{\otimes d} \simeq \bigoplus_{r=-d}^{d} \mathrm{Gr}^r\, V(I)^{\otimes d}$$

as a representation of $\mathbb{C}^* \times \mathrm{O}(V(J))$ because $\mathbb{C}^* \times \mathrm{O}(V(J))$ is reductive. By (8.14), this is the weight decomposition with respect to $\mathbb{C}^*$.

Now let $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n)$ be a partition expressing an irreducible representation of $\mathrm{O}(V(I)) \simeq \mathrm{O}(n, \mathbb{C})$. As in (8.10), let

$$F^r V(I)_\lambda = V(I)_\lambda \cap F^r V(I)^{\otimes|\lambda|}$$

be the filtration induced on the space $V(I)_\lambda$. This is a $P(J/I)_{\mathbb{C}}$-invariant filtration, and $U(J/I)_{\mathbb{C}}$ acts on the graded quotients trivially. By the above argument, if we take a lift of $\mathbb{C}^* \times \mathrm{O}(V(J))$ in (8.13), we have a decomposition

$$V(I)_\lambda \simeq \bigoplus_r \mathrm{Gr}^r\, V(I)_\lambda \tag{8.15}$$

as a representation of $\mathbb{C}^* \times \mathrm{O}(V(J))$, and this agrees with the weight decomposition for $\mathbb{C}^*$ with $\mathrm{Gr}^r\, V(I)_\lambda$ being the weight r subspace.

Proposition 8.10. *Let* $\lambda \neq \det$. *We have*

$$F^{\lambda_1+1} V(I)_\lambda = 0, \quad F^{-\lambda_1} V(I)_\lambda = V(I)_\lambda. \tag{8.16}$$

Thus the filtration $F^\bullet V(I)_\lambda$ *has length* $\leq 2\lambda_1 + 1$, *from level* $-\lambda_1$ *to* λ_1. *Moreover, we have*

$$F^{\lambda_1} V(I)_\lambda = V(I)_\lambda^{U(J/I)_{\mathbb{C}}}. \tag{8.17}$$

Proof. This is purely a representation-theoretic calculation. We write $V = V(I)$ and take a basis $e_1, \ldots, e_n$ of V such that $(J/I)_{\mathbb{C}} = \mathbb{C}e_1$, $(e_i, e_j) = 1$ if $i + j = n + 1$, and $(e_i, e_j) = 0$ otherwise. We also write $P = P(J/I)_{\mathbb{C}}$ and $U = U(J/I)_{\mathbb{C}}$. (The same notation as in the proof of Proposition 6.3.) We identify $V(J)$ with $V' = \langle e_2, \ldots, e_{n-1} \rangle$. This defines a lift $\mathbb{C}^* \times \mathrm{O}(V') \hookrightarrow P$. Then $\mathbb{C}^*$ acts on $\mathbb{C}e_1$ by weight 1, on V' by weight 0, and on $\mathbb{C}e_n$ by weight -1.

We first prove (8.16). Recall from (3.1) that

$$V_\lambda \subset \wedge^{{}^t\lambda_1} V \otimes \cdots \otimes \wedge^{{}^t\lambda_{\lambda_1}} V. \tag{8.18}$$

Since the weights of $\mathbb{C}^*$ on each space $\wedge^i V$ are only -1, 0, 1, the weights of $\mathbb{C}^*$ on the right-hand side of (8.18) are contained in the range $[-\lambda_1, \lambda_1]$. Therefore the weights of $\mathbb{C}^*$ on V_λ are contained in $[-\lambda_1, \lambda_1]$. Since $\mathrm{Gr}^r\, V_\lambda$ is the weight r subspace for the action of $\mathbb{C}^*$, this shows that $\mathrm{Gr}^r\, V_\lambda \neq 0$ only when $-\lambda_1 \leq r \leq \lambda_1$. This implies (8.16).

Next we prove (8.17). In Proposition 6.3, we proved that $V_\lambda^U \simeq \chi_{\lambda_1} \boxtimes W$ as a representation of $\mathbb{C}^* \times \mathrm{O}(V')$, where W is a representation of

$$\mathrm{O}(V') \simeq \mathrm{O}(n-2, \mathbb{C}).$$

(We do not use precise information on W.) In particular, $\mathbb{C}^*$ acts on V_λ^U by weight λ_1. This means that $V_\lambda^U \subset F^{\lambda_1} V_\lambda$. On the other hand, since U acts trivially on

$$\mathrm{Gr}^{\lambda_1}\, V_\lambda = F^{\lambda_1} V_\lambda / F^{\lambda_1+1} V_\lambda = F^{\lambda_1} V_\lambda,$$

we also see that $F^{\lambda_1} V_\lambda \subset V_\lambda^U$. Therefore $F^{\lambda_1} V_\lambda = V_\lambda^U$. ■

We have the following duality between the graded quotients.

Lemma 8.11. *We have* $\mathrm{Gr}^r\, V(I)_\lambda \simeq \mathrm{Gr}^{-r}\, V(I)_\lambda$ *as representations of* $\mathrm{O}(V(J))$.

Proof. We keep the notation as in the proof of Proposition 8.10 and take the $\mathbb{C}^* \times \mathrm{O}(V')$-decomposition (8.15) of V_λ. Let ι be the involution of V which exchanges e_1 and e_n and acts on $V' = \langle e_2, \dots, e_{n-1} \rangle$ trivially. Thus ι and $\mathbb{C}^* = \mathrm{SO}(\langle e_1, e_n \rangle)$ generate $\mathrm{O}(\langle e_1, e_n \rangle)$. The involution ι normalizes $\mathbb{C}^* \times \mathrm{O}(V')$. Its adjoint action acts on $\mathbb{C}^*$ by $\alpha \mapsto \alpha^{-1}$, and acts on $\mathrm{O}(V')$ trivially. Therefore the action of ι on V_λ maps the weight r subspace $\mathrm{Gr}^r\, V_\lambda$ to the weight $-r$ subspace $\mathrm{Gr}^{-r}\, V_\lambda$, and this map is $\mathrm{O}(V')$-equivariant. ■

It will be useful to know that the graded quotients in level $-\lambda_1$ and λ_1 are indeed nontrivial.

Lemma 8.12. *Let* $\lambda \neq \det$. *We have* $\mathrm{Gr}^{\lambda_1}\, V(I)_\lambda \neq 0$ *and* $\mathrm{Gr}^{-\lambda_1}\, V(I)_\lambda \neq 0$.

Proof. We keep the notation as in the proof of Proposition 8.10. Recall from (3.2) that V_λ contains the vector

$$(e_1 \wedge \cdots \wedge e_{{}^t\lambda_1}) \otimes (e_1 \wedge \cdots \wedge e_{{}^t\lambda_2}) \otimes \cdots \otimes (e_1 \wedge \cdots \wedge e_{{}^t\lambda_{\lambda_1}}).$$

Since ${}^t\lambda_1 < n$ by $\lambda \neq \det$, this vector is contained in the weight λ_1 subspace for the $\mathbb{C}^*$-action. Therefore $\mathrm{Gr}^{\lambda_1}\, V_\lambda \neq 0$. The nontriviality of $\mathrm{Gr}^{-\lambda_1}\, V_\lambda$ then follows from Lemma 8.11. ■

Since (8.15) is the weight decomposition for $\mathbb{C}^*$, we can write

$$\mathrm{Gr}^r\, V(I)_\lambda \simeq \chi_r \boxtimes V(J)_{\lambda'(r)}$$

as a representation of $\mathbb{C}^* \times \mathrm{O}(V(J))$, where $V(J)_{\lambda'(r)}$ is some (in general reducible) representation of $\mathrm{O}(V(J)) \simeq \mathrm{O}(n-2, \mathbb{C})$. The representation $V(J)_{\lambda'(r)}$ can be understood through the restriction rule of V_λ for $\mathrm{SO}(2, \mathbb{C}) \times \mathrm{O}(n-2, \mathbb{C}) \subset \mathrm{O}(n, \mathbb{C})$. See [30, 32] for a description of this restriction rule in terms of the Littlewood–Richardson numbers.

By translating the conclusions of Proposition 8.10 and Lemmas 8.11 and 8.12 by the I-trivialization, we obtain the following consequence for the J-filtration on $\mathcal{E}_\lambda$.

Proposition 8.13. *Let* $\lambda \neq \det$. *The J-filtration $F^\bullet \mathcal{E}_\lambda$ on $\mathcal{E}_\lambda$ satisfies*

$$F^{\lambda_1+1}\mathcal{E}_\lambda = 0, \quad F^{-\lambda_1}\mathcal{E}_\lambda = \mathcal{E}_\lambda$$

and

$$F^{\lambda_1}\mathcal{E}_\lambda = \mathrm{Gr}^{\lambda_1}\, \mathcal{E}_\lambda \neq 0, \quad \mathrm{Gr}^{-\lambda_1}\, \mathcal{E}_\lambda \neq 0.$$

Thus $F^\bullet \mathcal{E}_\lambda$ has length $\leq 2\lambda_1 + 1$, from level $-\lambda_1$ to λ_1. The graded quotients $\mathrm{Gr}^r\, \mathcal{E}_\lambda$ *and* $\mathrm{Gr}^{-r}\, \mathcal{E}_\lambda$ *have the same rank. Moreover, $F^{\lambda_1}\mathcal{E}_\lambda$ coincides with the sub vector bundle $\mathcal{E}_\lambda^J$ of $\mathcal{E}_\lambda$ defined in Section 6.2.*

Remark 8.14. (1) By this description of $\mathcal{E}_\lambda^J$, some of the results of Section 6.2 also follow from the results of Section 8.2.

(2) The isomorphism (8.11) can be written better as

$$\mathrm{Gr}^r(\mathcal{E}_{\lambda,k}|_{\Delta_J}) \simeq \pi_2^* \mathcal{L}_J^{\otimes r+k} \otimes V(J)_{\lambda'(r)}.$$

8.4 Decomposition of Jacobi forms

In this section we use the J-filtration on $\mathcal{E}_{\lambda,k}$ to show that vector-valued Jacobi forms decompose, in a sense, into some tuples of scalar-valued Jacobi forms.

Proposition 8.15. *Let* $\lambda \neq \det$. *There exists an injective map*

$$J_{\lambda,k,m}(\Gamma(J)_{\mathbb{Z}}) \hookrightarrow \bigoplus_{r=-\lambda_1}^{\lambda_1} J_{k+r,m}(\Gamma(J)_{\mathbb{Z}})^{\oplus \alpha(r)}, \tag{8.19}$$

where $\alpha(r)$ is the rank of $\mathrm{Gr}^r\, \mathcal{E}_\lambda$.

Proof. We use the notation in Section 7. Let $F^r J_{\lambda,k,m}(\Gamma(J)_{\mathbb{Z}})$ be the subspace of $J_{\lambda,k,m}(\Gamma(J)_{\mathbb{Z}})$ consisting of Jacobi forms which take values in the sub vector bundle

$F^r\mathcal{E}_{\lambda,k} \otimes \Theta_J^{\otimes m}$ of $\mathcal{E}_{\lambda,k} \otimes \Theta_J^{\otimes m}$. This defines a filtration on $J_{\lambda,k,m}(\Gamma(J)_{\mathbb{Z}})$ from level $r = -\lambda_1$ to λ_1. By the exact sequence

$$0 \to F^{r+1}\mathcal{E}_{\lambda,k} \otimes \Theta_J^{\otimes m} \to F^r\mathcal{E}_{\lambda,k} \otimes \Theta_J^{\otimes m} \to \mathrm{Gr}^r\,\mathcal{E}_{\lambda,k} \otimes \Theta_J^{\otimes m} \to 0$$

and Proposition 8.7, we obtain an embedding

$$\begin{aligned}\mathrm{Gr}^r(J_{\lambda,k,m}(\Gamma(J)_{\mathbb{Z}})) &\hookrightarrow H^0(\Delta_J, \mathrm{Gr}^r\,\mathcal{E}_{\lambda,k} \otimes \Theta_J^{\otimes m})^{\overline{\Gamma(J)}_{\mathbb{Z}}} \\ &\simeq H^0(\Delta_J, (\pi_2^*\mathcal{L}_J^{\otimes r+k})^{\oplus\alpha(r)} \otimes \Theta_J^{\otimes m})^{\overline{\Gamma(J)}_{\mathbb{Z}}}.\end{aligned}$$

The image of this embedding is contained in $J_{r+k,m}(\Gamma(J)_{\mathbb{Z}})^{\oplus\alpha(r)}$, namely, holomorphic at the cusps of $\mathbb{H}_J$. Indeed, if we take the (I, ω_J)-trivialization at $I \subset J$, the quotient homomorphism $F^r\mathcal{E}_{\lambda,k} \otimes \Theta_J^{\otimes m} \to \mathrm{Gr}^r\,\mathcal{E}_{\lambda,k} \otimes \Theta_J^{\otimes m}$ is identified with the quotient homomorphism

$$F^rV(I)_\lambda \otimes (I_{\mathbb{C}}^\vee)^{\otimes k} \otimes \mathcal{O}_{\Delta_J} \to \mathrm{Gr}^r\,V(I)_\lambda \otimes (I_{\mathbb{C}}^\vee)^{\otimes k} \otimes \mathcal{O}_{\Delta_J}.$$

Since this is constant over Δ_J, its effect on the Fourier expansion of a Jacobi form is just reducing each Fourier coefficient from $F^rV(I)_\lambda \otimes (I_{\mathbb{C}}^\vee)^{\otimes k}$ to $\mathrm{Gr}^r\,V(I)_\lambda \otimes (I_{\mathbb{C}}^\vee)^{\otimes k}$, so the Fourier coefficients still satisfy the holomorphicity condition at the I-cusp.

Therefore we obtain a canonical embedding

$$\mathrm{Gr}^r(J_{\lambda,k,m}(\Gamma(J)_{\mathbb{Z}})) \hookrightarrow J_{r+k,m}(\Gamma(J)_{\mathbb{Z}})^{\oplus\alpha(r)}. \tag{8.20}$$

Finally, if we choose a splitting of the filtration $F^\bullet J_{k,\lambda,m}(\Gamma(J)_{\mathbb{Z}})$, we obtain a (non-canonical) isomorphism

$$J_{k,\lambda,m}(\Gamma(J)_{\mathbb{Z}}) \simeq \bigoplus_{r=-\lambda_1}^{\lambda_1} \mathrm{Gr}^r(J_{k,\lambda,m}(\Gamma(J)_{\mathbb{Z}})).$$

This defines an embedding as claimed. ■

As the proof shows, the embedding (8.19) is not canonical: it requires a choice of a splitting of the filtration $F^\bullet J_{\lambda,k,m}(\Gamma(J)_{\mathbb{Z}})$. But at least the last subspace is canonically determined.

Corollary 8.16. *Let* $\lambda \neq \det$. *We have a canonical embedding*

$$J_{k+\lambda_1,m}(\Gamma(J)_{\mathbb{Z}}) \otimes V(J)_{\lambda'} \hookrightarrow J_{\lambda,k,m}(\Gamma(J)_{\mathbb{Z}}),$$

where $\lambda' = (\lambda_2 \geq \cdots \geq \lambda_{n-1})$.

Proof. The last (= level λ_1) subspace $F^{\lambda_1}J_{\lambda,k,m}(\Gamma(J)_{\mathbb{Z}})$ is the space of Jacobi forms with values in $F^{\lambda_1}\mathcal{E}_{\lambda,k} \otimes \Theta_J^{\otimes m}$. By Proposition 8.13 and Theorem 6.1, this sub vector bundle is isomorphic to $\pi_2^*\mathcal{L}_J^{\otimes k+\lambda_1} \otimes V(J)_{\lambda'} \otimes \Theta_J^{\otimes m}$. ■

Example 8.17. Let $n = 3$ and $\lambda = (d)$. In this case, in view of (8.12), the embedding (8.19) takes the form

$$J_{(d),k,m}(\Gamma(J)_{\mathbb{Z}}) \hookrightarrow \bigoplus_{r=-d}^{d} J_{k+r,m}(\Gamma(J)_{\mathbb{Z}}).$$

In the context of Siegel modular forms of genus 2, Ibukiyama–Kyomura [27] found an *isomorphism* of the same shape for a certain type of integral Jacobi groups. (In our notation, $L = 2U \oplus \langle -2 \rangle$, $K = \langle -2 \rangle$, $J \subset 2U$ the standard one, $U(J)_{\mathbb{Z}} = \wedge^2 J$, and $\overline{\Gamma(J)}_{\mathbb{Z}} = \Gamma_J \ltimes (K \otimes J)$.) The method of Ibukiyama and Kyomura is different, based on differential operators. It might be plausible that their decomposition essentially agrees with that of us.

Propositions 8.15 and 7.18 enable us to deduce some basic results for vector-valued Jacobi forms from those for scalar-valued Jacobi forms. We present two such consequences.

Corollary 8.18. *Let* $\lambda \neq \det$. *We have* $J_{\lambda,k,m}(\Gamma(J)_{\mathbb{Z}}) = 0$ *when* $k + \lambda_1 < n/2 - 1$.

Proof. In this case, all weights $k + r$ in (8.19) satisfy $k + r \leq k + \lambda_1 < n/2 - 1$. Then we have $J_{k+r,m}(\Gamma(J)_{\mathbb{Z}}) = 0$ by Proposition 7.19. ■

Corollary 8.19. $J_{\lambda,k,m}(\Gamma(J)_{\mathbb{Z}})$ *has finite dimension. Moreover, we have the following asymptotic estimates:*

$$\dim J_{\lambda,k,m}(\Gamma(J)_{\mathbb{Z}}) = O(k) \quad (k \to \infty),$$
$$\dim J_{\lambda,k,m}(\Gamma(J)_{\mathbb{Z}}) = O(m^{n-2}) \quad (m \to \infty).$$

Proof. By Propositions 8.15 and 7.18, we have

$$\dim J_{\lambda,k,m}(\Gamma(J)_{\mathbb{Z}}) \leq \sum_{r=-\lambda_1}^{\lambda_1} \alpha(r) \cdot \dim J_{k+r,m}(\Gamma(J)_{\mathbb{Z}})$$
$$\leq \sum_{r=-\lambda_1}^{\lambda_1} \alpha(r) \cdot \dim J_{k+r,K(\beta_0 m)}(\Gamma_J),$$

where K, β_0, Γ_J do not depend on λ, k, m. By the dimension formula of Skoruppa [43, Theorem 6], we see that each $J_{k+r,K(\beta_0 m)}(\Gamma_J)$ is finite dimensional and

$$\dim J_{k+r,K(\beta_0 m)}(\Gamma_J) = O(k) \quad (k \to \infty),$$
$$\dim J_{k+r,K(\beta_0 m)}(\Gamma_J) = O(\det K(\beta_0 m)) = O(m^{n-2}) \quad (m \to \infty).$$

These imply the asymptotic estimates for $\dim J_{\lambda,k,m}(\Gamma(J)_{\mathbb{Z}})$. ■

Remark 8.20. From Proposition 8.10, we have imposed the assumption $\lambda \neq \det$. This was necessary in our representation-theoretic calculation. Indeed, (8.17) and Lemma 8.12 do not hold for $\lambda = \det$. On the other hand, since $\Gamma(J)_{\mathbb{Z}} \subset \mathrm{SO}^{+}(L)$, Jacobi forms with $\lambda = \det$ are the same as those with $\lambda = 1$ (scalar-valued Jacobi forms) as far as $\Gamma(J)_{\mathbb{Z}}$ is concerned. The difference arises when we consider the action by the full stabilizer $\Gamma(J)_{\mathbb{Z}}^{*}$, which may contain an element of determinant -1.