Chapter 9

Vanishing theorem I

Let L be a lattice of signature $(2, n)$ with $n \geq 3$. We assume that L has Witt index 2, i.e., has a rank 2 isotropic sublattice. This is always satisfied when $n \geq 5$. Let Γ be a finite-index subgroup of $\mathrm{O}^+(L)$. Let $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n)$ be a partition with ${}^t\lambda_1 + {}^t\lambda_2 \leq n$ which expresses an irreducible representation of $\mathrm{O}(n, \mathbb{C})$. We assume $\lambda \neq 1, \det$. In this chapter, as an application of the J-filtration, we prove the following vanishing theorem.

Theorem 9.1. *Let $\lambda \neq 1, \det$. If $k < \lambda_1 + n/2 - 1$, then $M_{\lambda,k}(\Gamma) = 0$. In particular, we have $M_{\lambda,k}(\Gamma) = 0$ whenever $k < n/2$.*

This generalizes the well-known vanishing theorem $M_k(\Gamma) = 0$ for $0 < k < n/2 - 1$ in the scalar-valued case. This classical fact can be deduced from the vanishing of scalar-valued Jacobi forms (Fourier–Jacobi coefficients) of weight $< n/2 - 1$. Our proof of Theorem 9.1 is a natural generalization of this approach. The outline is as follows.

The first step is to take the projection $\mathcal{E}_{\lambda,k} \to \mathrm{Gr}^{-\lambda_1} \mathcal{E}_{\lambda,k}$ to the first graded quotient of the J-filtration for each 1-dimensional cusp J. Then we apply the classical vanishing theorem of scalar-valued Jacobi forms (Proposition 7.19) to $\mathrm{Gr}^{-\lambda_1} \mathcal{E}_{\lambda,k}$. This tells us that when $k - \lambda_1 < n/2 - 1$, the Fourier coefficients of a modular form at a 0-dimensional cusp $I \subset J$ are contained in a proper subspace of $V(I)_{\lambda,k}$. Finally, running J over all 1-dimensional cusps containing I, we find that the Fourier coefficients are zero.

The second step of this argument (and hence the bound in Theorem 9.1) could be improved for some specific (Γ, L) if a stronger vanishing theorem of classical Jacobi forms is available (cf. Remark 9.4). Theorem 9.1 would be a prototype in this direction.

Let us look at Theorem 9.1 in the cases $n = 3, 4$ under the accidental isomorphisms.

Example 9.2. Let $n = 3$. Recall from Example 3.4 that the orthogonal weight

$$(\lambda, k) = ((d), k)$$

corresponds to the $\mathrm{GL}(2, \mathbb{C})$-weight $(\rho_1, \rho_2) = (k + d, k - d)$ for Siegel modular forms of genus 2. In this case, the bound in Theorem 9.1 is $k < d + 1/2$, namely, $k \leq d$. This is rewritten as $\rho_2 \leq 0$. This is the same bound as the vanishing theorem of Freitag [15] and Weissauer [47] for Siegel modular forms of genus 2.

In the case of Siegel modular forms of genus 2, the idea to use Jacobi forms to derive a vanishing theorem of vector-valued modular forms seems to go back to Ibukiyama. See [25, Section 6] (and also [26, p. 54]). Our proof of Theorem 9.1 can be regarded as a generalization of the argument of Ibukiyama.

Example 9.3. Let $n=4$. Recall from Example 3.5 that the orthogonal weight $(\lambda, k) = ((d), k)$ corresponds to the weight $(r, \rho \boxtimes \rho)$ with $r = k - d$ and $\rho = \mathrm{Sym}^d$ for Hermitian modular forms of degree 2. In this case, the bound in Theorem 9.1 is $k < d+1$, i.e., $k \leq d$. Thus Theorem 9.1 says that there is no nonzero Hermitian modular form of degree 2 and weight $(r, \rho \boxtimes \rho)$ with $\rho = \mathrm{Sym}^d \neq 1$ when $r \leq 0$. Furthermore, our second vanishing theorem (Theorem 11.1 (1)) says that there is no nonzero cusp form when $r \leq 1$.

The rest of this chapter is as follows. In Section 9.1 we prove Theorem 9.1. In Section 9.2 we give an application of Theorem 9.1 to the vanishing of holomorphic tensors of small degree on the modular variety $\mathscr{F}(\Gamma)$.

9.1 Proof of Theorem 9.1

In this section we prove Theorem 9.1. Let $\lambda \neq 1, \det$ and assume that $k - \lambda_1 < n/2 - 1$. For a rank 2 primitive isotropic sublattice J of L, we denote by $F_J \mathcal{E}_{\lambda,k} = F_J^{-\lambda_1+1}\mathcal{E}_{\lambda,k}$ the level $-\lambda_1 + 1$ (= the first) sub vector bundle of $\mathcal{E}_{\lambda,k}$ in the J-filtration. Here we add J in the notation in order to indicate the cusp.

Step 1. Every Jacobi form in $J_{\lambda,k,m}(\Gamma(J)_{\mathbb{Z}})$ takes values in the sub vector bundle $F_J \mathcal{E}_{\lambda,k} \otimes \Theta_J^{\otimes m}$ of $\mathcal{E}_{\lambda,k} \otimes \Theta_J^{\otimes m}$.

Proof. Recall from (8.20) that we have an embedding

$$\mathrm{Gr}^{-\lambda_1}(J_{\lambda,k,m}(\Gamma(J)_{\mathbb{Z}})) \hookrightarrow J_{k-\lambda_1,m}(\Gamma(J)_{\mathbb{Z}})^{\oplus \alpha(-\lambda_1)}.$$

Since $k - \lambda_1 < n/2 - 1$, we have

$$J_{k-\lambda_1,m}(\Gamma(J)_{\mathbb{Z}}) = 0$$

by Proposition 7.19. Therefore $\mathrm{Gr}^{-\lambda_1}(J_{\lambda,k,m}(\Gamma(J)_{\mathbb{Z}})) = 0$, which means that every Jacobi form in $J_{\lambda,k,m}(\Gamma(J)_{\mathbb{Z}})$ takes values in $F_J \mathcal{E}_{\lambda,k} \otimes \Theta_J^{\otimes m}$. ■

Now let $f \in M_{\lambda,k}(\Gamma)$. We want to prove that $f = 0$. We fix a rank 1 primitive isotropic sublattice I of L and let $f = \sum_l a(l) q^l$ be the Fourier expansion of f at the I-cusp, where $a(l) \in V(I)_{\lambda,k}$. For a rank 2 primitive isotropic sublattice J of L containing I, we denote by $F_J V(I)_\lambda = F_J^{-\lambda_1+1} V(I)_\lambda$ the level $-\lambda_1 + 1$ subspace in the J-filtration (8.10) on $V(I)_\lambda$ and write

$$F_J V(I)_{\lambda,k} = F_J V(I)_\lambda \otimes (I_{\mathbb{C}}^{\vee})^{\otimes k} \subset V(I)_{\lambda,k}.$$

Step 2. Every Fourier coefficient $a(l)$ is contained in the subspace $F_J V(I)_{\lambda,k}$ of $V(I)_{\lambda,k}$.

Proof. Let σ_J be the isotropic ray in $U(I)_{\mathbb{R}}$ corresponding to J. If $l \in \sigma_J$, then $a(l)$ appears as a Fourier coefficient of the restriction $f|_{\Delta_J}$ of f to Δ_J. By Lemma 6.2 and Proposition 8.10, we see that $a(l)$ is contained in $F_J^{\lambda_1} V(I)_{\lambda,k} \subset F_J V(I)_{\lambda,k}$.

Next let $l \notin \sigma_J$. Then $a(l)$ appears as a Fourier coefficient of the m-th Fourier–Jacobi coefficient ϕ_m of f for some $m > 0$ along the J-cusp (see Section 7.1). By Proposition 7.12, ϕ_m is a Jacobi form of weight (λ, k) and index m. By Step 1, ϕ_m as a section of $\mathcal{E}_{\lambda,k} \otimes \Theta_J^{\otimes m}$ takes values in the sub vector bundle $F_J \mathcal{E}_{\lambda,k} \otimes \Theta_J^{\otimes m}$. Since the I-trivialization over $\overline{\mathcal{X}(J)}$ sends $F_J \mathcal{E}_{\lambda,k}$ to $F_J V(I)_{\lambda,k} \otimes \mathcal{O}_{\overline{\mathcal{X}(J)}}$, this implies that the Jacobi form ϕ_m, regarded as a $V(I)_{\lambda,k}$-valued function on Δ_J via the (I, ω_J)-trivialization, takes values in the subspace $F_J V(I)_{\lambda,k}$ of $V(I)_{\lambda,k}$. It follows that its Fourier coefficients $a(l)$ are contained in $F_J V(I)_{\lambda,k}$. ■

Step 3. Every Fourier coefficient $a(l)$ is zero.

Proof. Let $W = \bigcap_{J \supset I} F_J V(I)_\lambda$. By applying Step 2 to all $J \supset I$, we find that $a(l)$ is contained in $W \otimes (I_{\mathbb{C}}^{\vee})^{\otimes k}$. We shall prove that $W = 0$. Since $(J/I)_{\mathbb{Q}}$ runs over all isotropic lines in $V(I)_{\mathbb{Q}}$ in the definition of W and

$$F_{\gamma J} V(I)_\lambda = \gamma(F_J V(I)_\lambda)$$

for $\gamma \in \mathrm{O}(V(I)_{\mathbb{Q}})$, we see that W is an $\mathrm{O}(V(I)_{\mathbb{Q}})$-invariant subspace of $V(I)_\lambda$. Since $\mathrm{O}(V(I)_{\mathbb{Q}})$ is Zariski dense in $\mathrm{O}(V(I))$, we find that W is $\mathrm{O}(V(I))$-invariant. But $V(I)_\lambda$ is irreducible as a representation of $\mathrm{O}(V(I))$, so we have either $W = 0$ or $W = V(I)_\lambda$. Since $F_J V(I)_\lambda \neq V(I)_\lambda$ by Lemma 8.12, we have $W \neq V(I)_\lambda$. Therefore $W = 0$. This finishes the proof of Theorem 9.1. ■

Remark 9.4. At least when V_λ remains irreducible as a representation of $\mathrm{SO}(n, \mathbb{C})$, it is also possible to replace the argument in Step 3 by an argument using the symmetry of the Fourier coefficients in Proposition 3.6 and the Zariski density of $\overline{\Gamma(I)_{\mathbb{Z}}}$ as in the proof of Proposition 3.7. This approach allows improvement of Theorem 9.1 when a stronger vanishing theorem of scalar-valued Jacobi forms holds for $\Gamma(J)_{\mathbb{Z}}$.

9.2 Vanishing of holomorphic tensors

In this section, as an application of Theorem 9.1, we deduce vanishing of holomorphic tensors of small degree on the modular variety $\mathcal{F}(\Gamma) = \Gamma \backslash \mathcal{D}$. To be more precise, let X be the regular locus of $\mathcal{F}(\Gamma)$. Sections of $(\Omega_X^1)^{\otimes k}$ are called *holomorphic tensors* on X. Among them, those which extend holomorphically over a smooth projective compactification of X are a birational invariant of $\mathcal{F}(\Gamma)$.

Theorem 9.5. *When* $0 < k < n/2 - 1$, *we have* $H^0(X, (\Omega_X^1)^{\otimes k}) = 0$. *In particular,* $H^0(\widetilde{X}, (\Omega_{\widetilde{X}}^1)^{\otimes k}) = 0$ *for any smooth projective model* $\widetilde{X}$ *of* $\mathcal{F}(\Gamma)$.

Proof. Let $\pi: \mathcal{D} \to \mathcal{F}(\Gamma)$ be the projection. We can pull back sections of $(\Omega_X^1)^{\otimes k}$ to Γ-invariant sections of $(\Omega_{\pi^{-1}(X)}^1)^{\otimes k}$. They extend holomorphically over $\mathcal{D}$ because the complement of $\pi^{-1}(X)$ in $\mathcal{D}$ is of codimension ≥ 2. Hence we have an embedding

$$H^0(X, (\Omega_X^1)^{\otimes k}) \hookrightarrow H^0(\mathcal{D}, (\Omega_{\mathcal{D}}^1)^{\otimes k})^{\Gamma}. \tag{9.1}$$

Recall from (2.4) that $\Omega_{\mathcal{D}}^1 \simeq \mathcal{E} \otimes \mathcal{L}$. If we denote by $\mathrm{St}^{\otimes k} = \bigoplus_\alpha V_{\lambda(\alpha)}$ the irreducible decomposition of $\mathrm{St}^{\otimes k}$, we thus obtain an embedding

$$H^0(X, (\Omega_X^1)^{\otimes k}) \hookrightarrow \bigoplus_\alpha M_{\lambda(\alpha),k}(\Gamma). \tag{9.2}$$

When $\lambda(\alpha) \neq 1, \det$, we have $M_{\lambda(\alpha),k}(\Gamma) = 0$ for $k < n/2$ by Theorem 9.1. The determinant character does not appear in the irreducible decomposition of $\mathrm{St}^{\otimes k}$ if $k < n$ [38, Theorem 8.21]. Finally, when $\lambda(\alpha) = 1$, we have $M_k(\Gamma) = 0$ for $0 < k < n/2 - 1$ as it is classically known. Therefore $H^0(X, (\Omega_X^1)^{\otimes k}) = 0$ when $0 < k < n/2 - 1$. ■

We can also classify possible types of holomorphic tensors on X in the next few degrees $n/2 - 1 \leq k \leq n/2$.

Proposition 9.6. *We write* $N(k) = k!/2^{k/2}(k/2)!$ *when* k *is even.*

(1) *Let* $k = [(n-1)/2]$. *Then we have an embedding*

$$H^0(X, (\Omega_X^1)^{\otimes k}) \hookrightarrow \begin{cases} 0 & n \equiv 0, 3 \mod 4, \\ M_k(\Gamma)^{\oplus N(k)} & n \equiv 1, 2 \mod 4. \end{cases}$$

(2) *Let* $k = n/2$ *with* n *even. Then we have an embedding*

$$H^0(X, (\Omega_X^1)^{\otimes k}) \hookrightarrow \begin{cases} M_{\wedge^k,k}(\Gamma) & n \equiv 2 \mod 4, \\ M_{\wedge^k,k}(\Gamma) \oplus M_k(\Gamma)^{\oplus N(k)} & n \equiv 0 \mod 4. \end{cases}$$

The component $M_{\wedge^k,k}(\Gamma)$ *in* (2) *gives the holomorphic differential forms of degree* $k = n/2$. *The component* $M_k(\Gamma)^{\oplus N(k)}$ *in both* (1) *and* (2) *corresponds to the trivial summands in* $\mathrm{St}^{\otimes k}$. *In both* (1) *and* (2), *the embedding is an isomorphism when* $\langle \Gamma, -\mathrm{id} \rangle$ *contains no reflection.*

Proof. We keep the same notation as in the proof of Theorem 9.5.

(1) When $\lambda(\alpha) \neq 1, \det$, we still have

$$M_{\lambda(\alpha),k}(\Gamma) = 0$$

for $k < n/2$ by Theorem 9.1. The determinant character does not appear too. By [38, Exercise 12.2], $\mathrm{St}^{\otimes k}$ does not contain the trivial representation when k is odd, while it occurs with multiplicity $N(k)$ when k is even.

(2) When $\lambda(\alpha) \neq \wedge^d$ with $0 \leq d \leq n$, we have $\lambda_1 \geq 2$, and so, $M_{\lambda(\alpha),n/2}(\Gamma) = 0$ by Theorem 9.1. By [38, Theorem 8.21], the representations $\wedge^d$ with $d > n/2$ or $d \not\equiv n/2 \bmod 2$ do not appear in $\mathrm{St}^{\otimes n/2}$, and $\wedge^{n/2}$ occurs with multiplicity 1. The multiplicity of the trivial summand is as before. It remains to consider $\wedge^d$ with $0 < d < n/2$ and $d \equiv n/2 \bmod 2$. We apply our second vanishing theorem (Theorem 11.1 (2)). This says that $M_{\wedge^d,n/2}(\Gamma) = 0$ when $n/2 \leq n - d - 2$, namely, $d \leq n/2 - 2$.

Finally, when $\langle \Gamma, -\mathrm{id} \rangle$ contains no reflection, the projection $\mathcal{D} \to \mathcal{F}(\Gamma)$ is unramified in codimension 1 by [21]. Then (9.1) and (9.2) are isomorphisms, and so, the above embeddings are isomorphisms. ∎

Remark 9.7. (1) The weight $k = [(n-1)/2]$ in Proposition 9.6 (1) is the so-called *singular weight* when n is even, and the *critical weight* when n is odd, for scalar-valued modular forms. Since $M_k(\Gamma) \neq 0$ in general for these weights, the bound in Theorem 9.5 is optimal as a general bound.

(2) Theorem 9.5 and Proposition 9.6 imply in particular vanishing of holomorphic differential forms of degree $< n/2$ on X. Via the extension theorem of Pommerening [39], this can also be deduced from the vanishing of the corresponding Hodge components in the L^2-cohomology (cf. [4]).