Chapter 10

Square integrability

Let L be a lattice of signature $(2, n)$ with $n \geq 3$ and Γ be a finite-index subgroup of $\mathrm{O}^+(L)$. In this chapter we study convergence of the Petersson inner product

$$\int_{\mathcal{F}(\Gamma)} (f, g)_{\lambda,k} \mathrm{vol}_{\mathcal{D}}$$

for $f, g \in M_{\lambda,k}(\Gamma)$, where $(\ ,\ )_{\lambda,k}$ is the Petersson metric on the vector bundle $\mathcal{E}_{\lambda,k}$ and $\mathrm{vol}_{\mathcal{D}}$ is the invariant volume form on $\mathcal{D}$.

For $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n)$ let $\overline{\lambda} = (\lambda_1 - \lambda_n, \ldots, \lambda_{[n/2]} - \lambda_{n+1-[n/2]})$ be the associated highest weight for $\mathrm{SO}(n, \mathbb{C})$ (see Section 3.6.1). We denote by $|\overline{\lambda}|$ the sum of all components of $\overline{\lambda}$. Our results are summarized as follows.

Theorem 10.1. *Let $f, g \in M_{\lambda,k}(\Gamma)$ with $\lambda \neq 1, \det$.*

(1) *If f is a cusp form, then $\int_{\mathcal{F}(\Gamma)} (f, g)_{\lambda,k} \mathrm{vol}_{\mathcal{D}} < \infty$.*

(2) *When $k \geq n + |\overline{\lambda}| - 1$, f is a cusp form if and only if*

$$\int_{\mathcal{F}(\Gamma)} (f, f)_{\lambda,k} \mathrm{vol}_{\mathcal{D}} < \infty.$$

(3) *When $k \leq n - |\overline{\lambda}| - 2$, we always have $\int_{\mathcal{F}(\Gamma)} (f, g)_{\lambda,k} \mathrm{vol}_{\mathcal{D}} < \infty$.*

See Remark 10.13 for the scalar-valued case. The assertion (1) should be more or less standard. The assertions (2) and (3) give a characterization of square integrability except in the range

$$n - |\overline{\lambda}| - 1 \leq k \leq n + |\overline{\lambda}| - 2. \tag{10.1}$$

The assertion (3) is in fact an intermediate step in the proof of our second vanishing theorem (Theorem 11.1), where we eventually prove that $M_{\lambda,k}(\Gamma) = 0$ in the range $k \leq n - |\overline{\lambda}| - 2$.

This chapter starts with defining the Petersson metrics on the Hodge bundles explicitly (Section 10.1) and calculating them over the tube domain (Section 10.2). In Section 10.3 we give some asymptotic estimates needed in the proof of Theorem 10.1. In Section 10.4 we prove Theorem 10.1.

10.1 Petersson metrics

In this section we explicitly define the Petersson metrics on the Hodge bundles $\mathcal{L}$ and $\mathcal{E}$, and hence on the automorphic vector bundles $\mathcal{E}_{\lambda,k}$.

We begin with $\mathcal{L}$. By the definition of $\mathcal{D}$, the Hermitian form $(\cdot, \bar{\cdot})$ on $L_{\mathbb{C}}$ is positive on the lines parametrized by $\mathcal{D}$. Thus restriction of this Hermitian form defines a Hermitian metric on each fiber of $\mathcal{L}$, and hence an $\mathrm{O}^+(L_{\mathbb{R}})$-invariant Hermitian metric on $\mathcal{L}$. We call it the *Petersson metric* on $\mathcal{L}$ and denote it by $(\, , \,)_{\mathcal{L}}$.

Next we consider $\mathcal{E}$. We first define the real part of $\mathcal{E}$. We write $\underline{L_{\mathbb{R}}}$ for the product real vector bundle $L_{\mathbb{R}} \times \mathcal{D}$, which we regard as a sub real vector bundle of $L_{\mathbb{C}} \otimes \mathcal{O}_{\mathcal{D}}$ in the natural way. Then we define a sub real vector bundle of $\underline{L_{\mathbb{R}}}$ by

$$\mathcal{E}_{\mathbb{R}} := \mathcal{L}^{\perp} \cap \underline{L_{\mathbb{R}}} = (\mathcal{L} \oplus \bar{\mathcal{L}})^{\perp} \cap \underline{L_{\mathbb{R}}}.$$

This is a real vector bundle of rank n. By the second expression, the fiber of $\mathcal{E}_{\mathbb{R}}$ over $[\omega] \in \mathcal{D}$ is the negative-definite subspace

$$\langle \mathrm{Re}(\omega), \mathrm{Im}(\omega) \rangle^{\perp} \cap L_{\mathbb{R}} \tag{10.2}$$

of $L_{\mathbb{R}}$ (cf. Section 2.1). The $\mathrm{O}^+(L_{\mathbb{R}})$-action on $\underline{L_{\mathbb{R}}}$ preserves the sub vector bundle $\mathcal{E}_{\mathbb{R}}$. The natural homomorphism

$$\mathcal{E}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} \hookrightarrow \mathcal{L}^{\perp} \to \mathcal{E}$$

gives an $\mathrm{O}^+(L_{\mathbb{R}})$-equivariant C^{∞}-isomorphism between $\mathcal{E}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ and $\mathcal{E}$. This defines a real structure of $\mathcal{E}$.

By the description (10.2) of the fibers, the real vector bundle $\mathcal{E}_{\mathbb{R}}$ is naturally endowed with an $\mathrm{O}^+(L_{\mathbb{R}})$-invariant negative-definite quadratic form. We take the (-1)-scaling to turn it to positive-definite. This is a Riemannian metric on $\mathcal{E}_{\mathbb{R}}$. It extends to a Hermitian metric on $\mathcal{E}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ in the usual way. (Explicitly, the Hermitian pairing between two vectors v, w is the quadratic pairing between v and $\bar{w}$.) Via the C^{∞}-isomorphism $\mathcal{E}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} \to \mathcal{E}$, we obtain an $\mathrm{O}^+(L_{\mathbb{R}})$-invariant Hermitian metric on $\mathcal{E}$. We call it the *Petersson metric* on $\mathcal{E}$ and denote it by $(\, , \,)_{\mathcal{E}}$.

The Petersson metric on $\mathcal{E}$ induces an $\mathrm{O}^+(L_{\mathbb{R}})$-invariant Hermitian metric on $\mathcal{E}^{\otimes d}$, and hence by restriction an $\mathrm{O}^+(L_{\mathbb{R}})$-invariant Hermitian metric on $\mathcal{E}_{\lambda}$ with $|\lambda| = d$. Taking the tensor product with the Petersson metric on $\mathcal{L}^{\otimes k}$, we obtain an $\mathrm{O}^+(L_{\mathbb{R}})$-invariant Hermitian metric on $\mathcal{E}_{\lambda,k}$. We call it the *Petersson metric* on $\mathcal{E}_{\lambda,k}$ and denote it by $(\, , \,)_{\lambda,k}$.

Remark 10.2. When L is the primitive integral cohomology of a lattice-polarized *K3* surface X with period $[\omega] \in \mathcal{D}$, we have the identifications

$$\mathcal{L}_{[\omega]} = H^{2,0}(X), \quad \mathcal{E}_{\mathbb{R},[\omega]} = H^{1,1}_{\mathrm{prim}}(X, \mathbb{R}),$$

and $\mathcal{E}_{\mathbb{R},[\omega]} \otimes_{\mathbb{R}} \mathbb{C} \to \mathcal{E}_{[\omega]}$ is identified with $H^{1,1}_{\mathrm{prim}}(X, \mathbb{C}) \to H^{2,0}(X)^{\perp}/H^{2,0}(X)$. On $H^{2,0}(X)$ and $H^{1,1}_{\mathrm{prim}}(X, \mathbb{C})$ we have the so-called *Hodge metrics* defined by $\int_X \alpha \wedge \overline{\beta}$ and $-\int_X \alpha \wedge \overline{\beta}$, respectively (see [46, Section 6.3.2]). Thus the Petersson metrics on $\mathcal{L}$ and $\mathcal{E}$ are essentially the Hodge metrics in this geometric setting.

Let I be a rank 1 primitive isotropic sublattice of L. For a vector v of $V(I)_{\mathbb{R}} = (I^{\perp}/I)_{\mathbb{R}}$, let s_v be the section of $\mathcal{E}$ which corresponds to the constant section v of $V(I) \otimes \mathcal{O}_{\mathcal{D}}$ by the I-trivialization $V(I) \otimes \mathcal{O}_{\mathcal{D}} \simeq \mathcal{E}$. We compute the Hermitian pairing between these distinguished sections. We choose and fix a lift $V(I)_{\mathbb{R}} \hookrightarrow I_{\mathbb{R}}^{\perp}$ of $V(I)_{\mathbb{R}}$ and regard vectors of $V(I)_{\mathbb{R}}$ as vectors of $I_{\mathbb{R}}^{\perp} \subset L_{\mathbb{R}}$ in this way.

Lemma 10.3. *Let $v_1, v_2 \in V(I)_{\mathbb{R}}$. The pairing of the sections s_{v_1}, s_{v_2} of $\mathcal{E}$ with respect to the Petersson metric $(\ ,\ )_{\mathcal{E}}$ is given by*

$$(s_{v_1}([\omega]), s_{v_2}([\omega]))_{\mathcal{E}} = -(v_1, v_2) + \frac{2 \cdot (v_1, \mathrm{Im}(\omega)) \cdot (v_2, \mathrm{Im}(\omega))}{(\mathrm{Im}(\omega), \mathrm{Im}(\omega))}$$

for $[\omega] \in \mathcal{D}$. In the right-hand side, $(\ ,\ )$ is the quadratic form on $L_{\mathbb{R}}$, and ω is normalized so as to have real pairing with $I_{\mathbb{R}}$. In particular, $(s_{v_1}, s_{v_2})_{\mathcal{E}}$ is $\mathbb{R}$-valued.

Proof. Let $[\omega] \in \mathcal{D}$. We choose a nonzero vector $l \in I$. We may normalize ω so that $(l, \omega) = 1$. For $v \in V(I)_{\mathbb{R}} \subset I_{\mathbb{R}}^{\perp}$ we write

$$\alpha(v) = \frac{(v, \mathrm{Im}(\omega))}{(\mathrm{Im}(\omega), \mathrm{Im}(\omega))} = \frac{(v, \mathrm{Im}(\omega))}{(\mathrm{Re}(\omega), \mathrm{Re}(\omega))} \in \mathbb{R}$$

and define a vector of $L_{\mathbb{C}}$ by

$$s'_v([\omega]) = v - (v, \omega)l + \sqrt{-1}\alpha(v)\omega. \tag{10.3}$$

Claim 10.4. *s'_v is a section of $\mathcal{E}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ and is the image of s_v under the C^{∞}-isomorphism $\mathcal{E} \to \mathcal{E}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$.*

We prove Claim 10.4. The conditions to be checked are

$$(\mathrm{Re}(s'_v([\omega])), \omega) = 0, \quad (\mathrm{Im}(s'_v([\omega])), \omega) = 0, \quad s'_v([\omega]) \in s_v([\omega]) + \mathbb{C}\omega.$$

Since $s_v([\omega]) = v - (v, \omega)l + \mathbb{C}\omega$ by Lemma 2.6, the last condition follows from the definition of s'_v. We check the first equality. Since

$$\mathrm{Re}(s'_v([\omega])) = v - (v, \mathrm{Re}(\omega))l - \alpha(v) \cdot \mathrm{Im}(\omega),$$

we see that

$$\begin{aligned}(\mathrm{Re}(s'_v([\omega])), \omega) &= (v, \omega) - (v, \mathrm{Re}(\omega)) - \sqrt{-1}\alpha(v)(\mathrm{Im}(\omega), \mathrm{Im}(\omega)) \\ &= (v, \omega) - (v, \mathrm{Re}(\omega)) - \sqrt{-1}(v, \mathrm{Im}(\omega)) \\ &= 0.\end{aligned}$$

In the first equality we used $(\mathrm{Re}(\omega), \mathrm{Im}(\omega)) = 0$. The equality $(\mathrm{Im}(s'_v([\omega])), \omega) = 0$ can be verified similarly. This proves Claim 10.4.

We return to the proof of Lemma 10.3. We take two vectors $v_1, v_2 \in V(I)_{\mathbb{R}}$. By definition, $(s_{v_1}([\omega]), s_{v_2}([\omega]))_{\mathcal{E}}$ is the pairing of $s'_{v_1}([\omega])$ and $s'_{v_2}([\omega])$ with respect to the Hermitian form on $\mathcal{E}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$. This in turn is the pairing of the vectors $s'_{v_1}([\omega])$ and $\overline{s'_{v_2}([\omega])}$ of $L_{\mathbb{C}}$ with respect to the (-1)-scaling of the quadratic form on $L_{\mathbb{C}}$. By the expression (10.3) of $s'_v([\omega])$, we can calculate

$$\begin{aligned}
&-(s_{v_1}([\omega]), s_{v_2}([\omega]))_{\mathcal{E}} \\
&\quad = (v_1 - (v_1, \omega)l + \sqrt{-1}\alpha(v_1)\omega, v_2 - (v_2, \bar{\omega})l - \sqrt{-1}\alpha(v_2)\bar{\omega}) \\
&\quad = (v_1, v_2) + \alpha(v_1)\alpha(v_2)(\omega, \bar{\omega}) - 2\alpha(v_1)(\mathrm{Im}(\omega), v_2) - 2\alpha(v_2)(\mathrm{Im}(\omega), v_1).
\end{aligned}$$

Since we have

$$\begin{aligned}
\alpha(v_1)\alpha(v_2)(\omega, \bar{\omega}) &= 2\alpha(v_1)(\mathrm{Im}(\omega), v_2) = 2\alpha(v_2)(\mathrm{Im}(\omega), v_1) \\
&= \frac{2(v_1, \mathrm{Im}(\omega))(v_2, \mathrm{Im}(\omega))}{(\mathrm{Im}(\omega), \mathrm{Im}(\omega))},
\end{aligned}$$

this proves Lemma 10.3. ■

Remark 10.5. By the expression (10.3), the imaginary part of $s'_v([\omega])$ is nonzero for general $[\omega]$. This shows that the real structure on $\mathcal{E} \simeq V(I) \otimes \mathcal{O}_{\mathcal{D}}$ given by $\mathcal{E}_{\mathbb{R}}$ is different from that given by $V(I)_{\mathbb{R}}$. Nevertheless, the Petersson metric on the real part given by $V(I)_{\mathbb{R}}$ is $\mathbb{R}$-valued by Lemma 10.3.

Let $\mathrm{vol}_{\mathcal{D}}$ be the invariant volume form on $\mathcal{D}$. The Petersson metric $(\ ,\ )_{\det,n}$ of weight $(\det, n)$ gives an invariant metric on the canonical bundle $K_{\mathcal{D}} \simeq \mathcal{L}^{\otimes n} \otimes \det$, where det stands for the determinant character (cf. Example 2.2). This can be used to express $\mathrm{vol}_{\mathcal{D}}$ as follows. If Ω is an arbitrary nonzero vector of $(K_{\mathcal{D}})_{[\omega]}$ over a point $[\omega]$ of $\mathcal{D}$, the volume form $\mathrm{vol}_{\mathcal{D}}$ at $[\omega]$ is written as

$$\mathrm{vol}_{\mathcal{D}}([\omega]) = \frac{\Omega \wedge \bar{\Omega}}{(\Omega, \Omega)_{\det,n}} \tag{10.4}$$

up to a constant independent of $[\omega]$. Indeed, the right-hand side does not depend on the choice of Ω, and the differential form of degree (n, n) on $\mathcal{D}$ defined by the right-hand side is clearly $\mathrm{O}^+(L_{\mathbb{R}})$-invariant, so it should coincide with $\mathrm{vol}_{\mathcal{D}}$ up to constant.

10.2 Petersson metrics on the tube domain

Let I be a rank 1 primitive isotropic sublattice of L. We calculate the Petersson metrics on $\mathcal{L}$, $\mathcal{E}$ over the tube domain $\mathcal{D}_I \subset U(I)_{\mathbb{C}}$. We choose a rank 1 isotropic sublattice $I' \subset L$ with $(I, I') \neq 0$. Recall that the choice of I' determines a tube domain realization $\mathcal{D} \to \mathcal{D}_I$. We take a generator l of I and identify $U(I)_{\mathbb{Q}} \simeq V(I)_{\mathbb{Q}}$ accordingly.

Lemma 10.6. *On the tube domain $\mathcal{D}_I$ we have*

$$(s_l(Z), s_l(Z))_{\mathcal{L}} = 2(\mathrm{Im}(Z), \mathrm{Im}(Z)), \tag{10.5}$$

$$(s_{v_1}(Z), s_{v_2}(Z))_{\mathcal{E}} = -(v_1, v_2) + \frac{2 \cdot (v_1, \mathrm{Im}(Z)) \cdot (v_2, \mathrm{Im}(Z))}{(\mathrm{Im}(Z), \mathrm{Im}(Z))}, \tag{10.6}$$

for $Z \in \mathcal{D}_I$. Here s_l is the section of $\mathcal{L}$ corresponding to the dual vector of l, v_1, v_2 are vectors of $V(I)_{\mathbb{R}}$, and $(\ ,\ )$ in the right-hand sides are the natural quadratic form on $V(I)_{\mathbb{R}} \simeq U(I)_{\mathbb{R}}$.

Proof. We begin with $(\ ,\ )_{\mathcal{L}}$. We can view the section s_l over $\mathcal{D}_I$ as a function $\mathcal{D}_I \to L_{\mathbb{C}}$ which lifts the inverse $\mathcal{D}_I \to \mathcal{D}$ of the tube domain realization and satisfies $(s_l, l) \equiv 1$. Let l' be the vector of $I'_{\mathbb{Q}}$ with $(l, l') = 1$, and we identify $V(I)_{\mathbb{Q}}$ with $(I_{\mathbb{Q}} \oplus I'_{\mathbb{Q}})^{\perp}$. Then we can explicitly write s_l as

$$s_l(Z) = l' + Z - 2^{-1}(Z, Z)l \in L_{\mathbb{C}}$$

for $Z \in \mathcal{D}_I \subset V(I)$. Thus we have

$$\begin{aligned}(s_l(Z), s_l(Z))_{\mathcal{L}} &= (s_l(Z), \overline{s_l(Z)}) = (Z, \bar{Z}) - (Z, Z)/2 - \overline{(Z, Z)}/2 \\ &= 2(\mathrm{Im}(Z), \mathrm{Im}(Z)).\end{aligned}$$

Next we calculate $(\ ,\ )_{\mathcal{E}}$. By Lemma 10.3, we have

$$(s_{v_1}(Z), s_{v_2}(Z))_{\mathcal{E}} = -(v_1, v_2) + \frac{2 \cdot (v_1, \mathrm{Im}(s_l(Z))) \cdot (v_2, \mathrm{Im}(s_l(Z)))}{(\mathrm{Im}(s_l(Z)), \mathrm{Im}(s_l(Z)))}.$$

Since

$$\mathrm{Im}(s_l(Z)) = \mathrm{Im}(Z) - 2^{-1}\,\mathrm{Im}((Z, Z))l,$$

we see that

$$(\mathrm{Im}(s_l(Z)), \mathrm{Im}(s_l(Z))) = (\mathrm{Im}(Z), \mathrm{Im}(Z)), \quad (v_i, \mathrm{Im}(s_l(Z))) = (v_i, \mathrm{Im}(Z)).$$

This proves (10.6). ∎

At each point $Z \in \mathcal{D}_I$, the Petersson metric on $\mathcal{E}$ can be understood as follows. We take an $\mathbb{R}$-basis $v_1, \dots, v_n$ of $V(I)_{\mathbb{R}}$ such that $v_1 \in \mathbb{R}\,\mathrm{Im}(Z)$ and $(v_i, \mathrm{Im}(Z)) = 0$ for $i > 1$. Then, by (10.6), we have

$$(s_{v_i}(Z), s_{v_j}(Z))_{\mathcal{E}} = \begin{cases} (v_1, v_1), & i = j = 1, \\ -(v_i, v_j), & i, j > 1, \\ 0, & i = 1,\ j > 1. \end{cases}$$

The right-hand side can be seen as the positive-definite modification of the hyperbolic quadratic form on $V(I)_{\mathbb{R}}$ given by taking the (-1)-scaling of the negative-definite subspace $\mathrm{Im}(Z)^{\perp}$. The Petersson metric on $\mathcal{E}_Z \simeq V(I)$ is the Hermitian extension of this modified real metric on $V(I)_{\mathbb{R}}$ to $V(I)$.

Finally, we recall the expression of $\mathrm{vol}_{\mathcal{D}}$ over $\mathcal{D}_I$. Let vol_I be a flat volume form on $\mathcal{D}_I \subset U(I)_{\mathbb{C}}$. Then, as it is well known, we have

$$\mathrm{vol}_{\mathcal{D}} = (\mathrm{Im}(Z), \mathrm{Im}(Z))^{-n} \, \mathrm{vol}_I . \tag{10.7}$$

This can be seen by substituting $\Omega = s_l^{\otimes n} \otimes v_0$ in (10.4) and using (10.5), where v_0 is a nonzero vector of det. The section $s_l^{\otimes n} \otimes v_0$ of $\mathcal{L}^{\otimes n} \otimes \det$ corresponds to a flat canonical form on $\mathcal{D}_I \subset U(I)_{\mathbb{C}}$ by its $U(I)_{\mathbb{C}}$-invariance.

10.3 Asymptotic estimates on the tube domain

In this section we prepare some estimates of the Petersson metrics on $\mathcal{E}_{\lambda,k}$ over the tube domain $\mathcal{D}_I$. This will be a main ingredient in the proof of Theorem 10.1. We keep the setting of Section 10.2.

We choose an $\mathbb{R}$-basis $\{v_i\}_i$ of the real part $(V(I)_{\mathbb{R}})_{\lambda}$ of $V(I)_{\lambda}$. Then $\{v_i\}_i$ is also a $\mathbb{C}$-basis of $V(I)_{\lambda}$. Let s_i' be the section of $\mathcal{E}_{\lambda}$ corresponding to v_i via the I-trivialization $\mathcal{E}_{\lambda} \simeq V(I)_{\lambda} \otimes \mathcal{O}_{\mathcal{D}}$ and let $s_i = s_i' \otimes s_l^{\otimes k}$. Then $\{s_i\}_i$ is a frame of $\mathcal{E}_{\lambda,k}$ corresponding to a basis of $V(I)_{\lambda,k}$ by the I-trivialization. Accordingly, we express a section f of $\mathcal{E}_{\lambda,k}$ over $\mathcal{D} \simeq \mathcal{D}_I$ as $f = \sum_i f_i s_i$ with f_i a scalar-valued holomorphic function on $\mathcal{D}_I$.

Lemma 10.7. *There exist real homogeneous polynomials $\{P_{ij}\}_{i,j}$ on $U(I)_{\mathbb{R}}$ of degree $\leq 2|\lambda|$ determined by the basis $\{v_i\}_i$ of $(V(I)_{\mathbb{R}})_{\lambda}$ such that*

$$(f, g)_{\lambda,k} \mathrm{vol}_{\mathcal{D}} = \sum_{i,j} f_i \overline{g_j} \cdot P_{ij}(\mathrm{Im}(Z)) \cdot (\mathrm{Im}(Z), \mathrm{Im}(Z))^{k-n-|\lambda|} \, \mathrm{vol}_I \tag{10.8}$$

for all sections $f = \sum_i f_i s_i$, $g = \sum_i g_i s_i$ of $\mathcal{E}_{\lambda,k}$ over $\mathcal{D}_I$. The matrix $(P_{ij}(\mathrm{Im}(Z)))_{i,j}$ is symmetric and positive-definite for $Z \in \mathcal{D}_I$.

Proof. The section s_i' is an $\mathbb{R}$-linear combination of $|\lambda|$-fold tensor products of the distinguished sections s_v of $\mathcal{E}$ associated to $v \in V(I)_{\mathbb{R}}$. (Recall that $V_{\lambda} \subset V^{\otimes |\lambda|}$.) The equation (10.6) can be written as

$$(s_{v_1}(Z), s_{v_2}(Z))_{\mathcal{E}} = \frac{-(v_1, v_2)(\mathrm{Im}(Z), \mathrm{Im}(Z)) + 2(v_1, \mathrm{Im}(Z))(v_2, \mathrm{Im}(Z))}{(\mathrm{Im}(Z), \mathrm{Im}(Z))}.$$

The numerator is a real homogeneous polynomial of $\mathrm{Im}(Z)$ of degree ≤ 2. Therefore the Petersson paring between s_i' and s_j' can be written as

$$(s_i'(Z), s_j'(Z))_{\lambda} = P_{ij}(\mathrm{Im}(Z)) \cdot (\mathrm{Im}(Z), \mathrm{Im}(Z))^{-|\lambda|} \tag{10.9}$$

for a real homogeneous polynomial P_{ij} of $\mathrm{Im}(Z)$ of degree $\leq 2|\lambda|$. Together with (10.5) and (10.7), we obtain

$$(s_i(Z), s_j(Z))_{\lambda,k}\mathrm{vol}_{\mathcal{D}} = P_{ij}(\mathrm{Im}(Z)) \cdot (\mathrm{Im}(Z), \mathrm{Im}(Z))^{k-n-|\lambda|}\,\mathrm{vol}_I\,.$$

This proves the equality (10.8). Since the matrix $((s_i'(Z), s_j'(Z))_\lambda)_{i,j}$ is real symmetric and positive-definite, so is $(P_{ij}(\mathrm{Im}(Z)))_{i,j}$ by (10.9). ■

Let Γ be a finite-index subgroup of $\mathrm{O}^+(L)$ and let $\mathcal{X}(I) = \mathcal{D}_I/U(I)_{\mathbb{Z}}$. We take a regular $\Gamma(I)_{\mathbb{Z}}$-admissible cone decomposition Σ_I of $\mathcal{C}_I^+ \subset U(I)_{\mathbb{R}}$ in the sense of Section 3.5.1. Let $\mathcal{X}(I)^{\Sigma_I}$ be the associated partial toroidal compactification of $\mathcal{X}(I)$. Let σ be a cone in Σ_I of dimension c. By the regularity of Σ_I, we can write $\sigma = \mathbb{R}_{\geq 0}v_1 + \cdots + \mathbb{R}_{\geq 0}v_c$ such that $v_1, \ldots, v_c$ is a part of a $\mathbb{Z}$-basis of $U(I)_{\mathbb{Z}}$, say $v_1, \ldots, v_n$. Let $l_1, \ldots, l_n \in U(I)_{\mathbb{Z}}^\vee$ be the dual basis of $v_1, \ldots, v_n$. Then $z_i = (l_i, Z)$, $1 \leq i \leq n$, are flat coordinates on $U(I)_{\mathbb{C}}$. We have

$$\mathrm{vol}_I = dz_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_n$$

up to constant. We write $q_i = e(z_i)$ for $1 \leq i \leq c$. Let Δ_σ be the boundary stratum of $\mathcal{X}(I)^{\Sigma_I}$ corresponding to the cone σ, and $\Delta_i = \Delta_{v_i}$ be the boundary divisor corresponding to the ray $\mathbb{R}_{\geq 0}v_i$. Then $q_1, \ldots, q_c, z_{c+1}, \ldots, z_n$ give local coordinates around Δ_σ. The divisor Δ_i is defined by $q_i = 0$, and Δ_σ is defined by $q_1 = \cdots = q_c = 0$. We write $q_i = r_i e(\theta_i)$ with $r_i = |q_i|$ and $0 \leq \theta_i < 1$. Then

$$\begin{aligned}\mathrm{vol}_I &= \frac{dq_1}{q_1} \wedge \frac{d\bar{q}_1}{\bar{q}_1} \wedge \cdots \wedge \frac{dq_c}{q_c} \wedge \frac{d\bar{q}_c}{\bar{q}_c} \wedge dz_{c+1} \wedge \cdots \wedge d\bar{z}_n \\ &= \frac{1}{r_1 \cdots r_c} dr_1 \wedge d\theta_1 \wedge \cdots \wedge dr_c \wedge d\theta_c \wedge dz_{c+1} \wedge \cdots \wedge d\bar{z}_n \end{aligned} \tag{10.10}$$

up to constant.

We give an asymptotic estimate of the right-hand side of (10.8) as $q_1, \ldots, q_c$ approach to 0. We take an arbitrary base point $Z_0 \in \mathcal{D}_I$ and consider a flow of points of the form

$$Z = Z(t_1, \ldots, t_c) = Z_0 + \sqrt{-1}(t_1 v_1 + \cdots + t_c v_c), \quad t_1, \ldots, t_c \to \infty. \tag{10.11}$$

This flow converges to a point of Δ_σ as $t_1, \ldots, t_c \to \infty$, and every point of Δ_σ can be obtained in this way. Let $v_0 = \mathrm{Im}(Z_0)$. This is a vector in the positive cone $\mathcal{C}_I$.

Lemma 10.8. *The following asymptotic estimates hold as $t_1, \ldots, t_c \to \infty$.*

$$P_{ij}(\mathrm{Im}(Z)) = O((t_1 + \cdots + t_c)^{2|\lambda|}), \tag{10.12}$$

$$(\mathrm{Im}(Z), \mathrm{Im}(Z)) = O((t_1 + \cdots + t_c)^2), \tag{10.13}$$

$$(\mathrm{Im}(Z), \mathrm{Im}(Z))^{-1} = O((t_1 + \cdots + t_c)^{-1}). \tag{10.14}$$

Proof. We have $\mathrm{Im}(Z) = v_0 + \sum_i t_i v_i$. Since P_{ij} is a real homogeneous polynomial of degree $\leq 2|\lambda|$ on $U(I)_{\mathbb{R}}$, we see that $P_{ij}(v_0 + \sum_i t_i v_i)$ is a real inhomogeneous polynomial of $t_1, \dots, t_c$ of degree $\leq 2|\lambda|$. This implies (10.12). Next we have

$$(\mathrm{Im}(Z), \mathrm{Im}(Z)) = (v_0, v_0) + 2\sum_i (v_0, v_i)t_i + 2\sum_{i \neq j}(v_i, v_j)t_i t_j + \sum_i (v_i, v_i)t_i^2.$$

The estimate (10.13) is obvious from this expression. Since $v_0 \in \mathcal{C}_I$ and $v_1, \dots, v_c \in \overline{\mathcal{C}_I}$, all coefficients in the right-hand side are nonnegative; possibly except for (v_i, v_i) with $i \geq 1$, they are furthermore positive. Therefore we have

$$(\mathrm{Im}(Z), \mathrm{Im}(Z)) > 2\sum_i (v_0, v_i)t_i > C \cdot \sum_i t_i$$

for some constant $C > 0$. This implies (10.14). ∎

Lemma 10.9. *In a small neighbourhood of an arbitrary point of Δ_σ, we have*

$$P_{ij}(\mathrm{Im}(Z)) = O((-\log r_1 \cdots r_c)^{2|\lambda|}), \tag{10.15}$$
$$(\mathrm{Im}(Z), \mathrm{Im}(Z)) = O((-\log r_1 \cdots r_c)^2), \tag{10.16}$$
$$(\mathrm{Im}(Z), \mathrm{Im}(Z))^{-1} = O((-\log r_1 \cdots r_c)^{-1}), \tag{10.17}$$

as $q_1, \dots, q_c \to 0$.

Proof. We consider the flow (10.11) with Z_0 varying over the range where $\mathrm{Re}(Z_0)$ is in a fundamental neighbourhood of $U(I)_{\mathbb{R}}/U(I)_{\mathbb{Z}}$ and $v_0 = \mathrm{Im}(Z_0)$ is in a small neighbourhood of an arbitrary point of $\mathcal{C}_I$. Since

$$r_i = |q_i| = \exp(-2\pi(l_i, \mathrm{Im}(Z))) = \exp(-2\pi(l_i, v_0) - 2\pi t_i),$$

we have

$$t_i = -(2\pi)^{-1}\log r_i - (l_i, v_0). \tag{10.18}$$

The constant term $-(l_i, v_0)$ depends on $v_0 = \mathrm{Im}(Z_0)$ continuously. Therefore our assertions follow by substituting $t_i \sim -(2\pi)^{-1}\log r_i$ in the estimates in Lemma 10.8 and using $\log r_1 + \cdots + \log r_c = \log r_1 \cdots r_c$. ∎

Summing up the calculations so far, we obtain the following asymptotic estimate of $(f, g)_{\lambda,k}\mathrm{vol}_{\mathcal{D}}$.

Proposition 10.10. *Let $f = \sum_i f_i s_i$ and $g = \sum_i g_i s_i$ be as in Lemma 10.7. In a small neighbourhood of an arbitrary point of Δ_σ, we have*

$$(f, g)_{\lambda,k}\mathrm{vol}_{\mathcal{D}} = \sum_{i,j} f_i \bar{g}_j \cdot O((-\log r_1 \cdots r_c)^{\alpha}) \cdot (r_1 \cdots r_c)^{-1}$$
$$\times\, dr_1 \wedge \cdots \wedge dr_c \wedge d\theta_1 \wedge \cdots \wedge d\theta_c \wedge dz_{c+1} \wedge \cdots \wedge d\bar{z}_n$$

as $q_1, \ldots, q_c \to 0$, *where*

$$\alpha = \begin{cases} 2k - 2n, & k \geq n + |\lambda|, \\ k - n + |\lambda|, & k < n + |\lambda|. \end{cases}$$

Proof. By substituting (10.15) and (10.10) in the right-hand side of (10.8), we obtain

$$\begin{aligned} &(f, g)_{\lambda,k} \mathrm{vol}_{\mathcal{D}} \\ &\quad = \sum_{i,j} f_i \bar{g}_j \cdot O((-\log r_1 \cdots r_c)^{2|\lambda|}) \cdot (\mathrm{Im}(Z), \mathrm{Im}(Z))^{k-n-|\lambda|} \\ &\qquad \times (r_1 \cdots r_c)^{-1} \cdot dr_1 \wedge \cdots \wedge dr_c \wedge d\theta_1 \wedge \cdots \wedge d\theta_c \wedge dz_{c+1} \wedge \cdots \wedge d\bar{z}_n. \end{aligned}$$

Then, according to whether the power degree $k - n - |\lambda|$ of $(\mathrm{Im}(Z), \mathrm{Im}(Z))$ is non-negative or negative, we use (10.16) and (10.17), respectively. ■

Before going to Section 10.4, we recall the following exercise in calculus.

Lemma 10.11. *Let* $m \in \mathbb{Z}$. *The integral*

$$\lim_{\varepsilon \to 0} \int_{\varepsilon}^{1/2} \frac{1}{(\log r)^m \cdot r} dr$$

converges if $m \geq 2$, *and diverges if* $m \leq 1$.

Proof. This can be seen from

$$\left(\frac{1}{(\log r)^{m-1}} \right)' = \frac{1 - m}{(\log r)^m \cdot r}$$

when $m \neq 1$, and $(\log(-\log r))' = ((\log r) \cdot r)^{-1}$ when $m = 1$. ■

10.4 Proof of Theorem 10.1

Now we prove Theorem 10.1. Let us begin with some reductions. For the proof of Theorem 10.1, there is no loss of generality even if we replace the given group Γ by a subgroup of finite index. Thus we may assume that Γ is neat. In particular, Γ is contained in $\mathrm{SO}^+(L)$. By Proposition 3.12 (1), when ${}^t\lambda_1 > n/2$, we have $\mathcal{E}_\lambda \simeq \mathcal{E}_{\bar{\lambda}}$ as $\mathrm{SO}^+(L_{\mathbb{R}})$-equivariant vector bundles. This isomorphism preserves the Petersson metrics up to constant by their uniqueness as $\mathrm{SO}^+(L_{\mathbb{R}})$-invariant Hermitian metrics. Thus we have a natural isomorphism

$$M_{\lambda,k}(\Gamma) \simeq M_{\bar{\lambda},k}(\Gamma)$$

which preserves the Petersson inner product up to constant. Since the highest weight for the partition $\overline{\lambda}$ is $\overline{\lambda}$ itself, the assertions of Theorem 10.1 for weight (λ, k) follow from those for weight $(\overline{\lambda}, k)$. Therefore, for the proof of Theorem 10.1, we may assume that ${}^t\lambda_1 \leq n/2$.

We take a smooth toroidal compactification $\mathcal{F}(\Gamma)^{\Sigma}$ of $\mathcal{F}(\Gamma)$, where the fans Σ_I are regular. We take a subdivision of Σ_I as follows.

Lemma 10.12. *There exists a $\Gamma(I)_{\mathbb{Z}}$-admissible and regular subdivision Σ_I' of Σ_I such that every cone in Σ_I' contains at most one isotropic ray.*

Proof. We take representatives $\tau_1, \dots, \tau_N$ of $\overline{\Gamma(I)_{\mathbb{Z}}}$-equivalence classes of 2-dimensional cones spanned by two isotropic rays. For each τ_a, we choose a rational vector from the interior of τ_a. This vector has positive norm, and the ray it generates divides τ_a. By letting $\overline{\Gamma(I)_{\mathbb{Z}}}$ act, we obtain a division of every 2-dimensional cone τ spanned by two isotropic rays. This is well defined because $\overline{\Gamma(I)_{\mathbb{Z}}}$ is torsion-free, and so, acts on the set of such cones freely. The collection of these divisions is $\overline{\Gamma(I)_{\mathbb{Z}}}$-invariant by construction.

The division of τ uniquely induces a division of every cone σ having τ as a face, because σ is simplicial. Explicitly, if $\sigma = \mathbb{R}_{\geq 0}v_1 + \cdots + \mathbb{R}_{\geq 0}v_c$, $\tau = \mathbb{R}_{\geq 0}v_1 + \mathbb{R}_{\geq 0}v_2$ and $v_0 \in \tau$ is the division vector, we add the wall $\mathbb{R}_{\geq 0}v_0 + \mathbb{R}_{\geq 0}v_3 + \cdots + \mathbb{R}_{\geq 0}v_c$. The collection of these new walls defines a $\overline{\Gamma(I)_{\mathbb{Z}}}$-invariant subdivision of the fan Σ_I such that every cone contains at most one isotropic ray. Taking its regular subdivision [2, p. 186], we obtain a desired subdivision. ∎

Thus our reduced situation is: Γ is neat, ${}^t\lambda_1 \leq n/2$ so that $\overline{\lambda} = \lambda$, and every cone in Σ_I contains at most one isotropic ray. (The last property will be used only in the proof of the assertion (3).)

Now, the integral $\int_{\mathcal{F}(\Gamma)}(f, g)_{\lambda,k}\mathrm{vol}_{\mathcal{D}}$ converges if for every boundary point x of $\mathcal{F}(\Gamma)^{\Sigma}$ there exists a neighbourhood $U = U_x$ of x such that $\int_U (f, g)_{\lambda,k}\mathrm{vol}_{\mathcal{D}}$ converges. Therefore, for the proof of (1) and (3) of Theorem 10.1, it suffices to verify the convergence of the integral over U. Conversely, when $f = g$, if $\int_U (f, f)_{\lambda,k}\mathrm{vol}_{\mathcal{D}}$ diverges around some boundary point x, then $\int_{\mathcal{F}(\Gamma)}(f, f)_{\lambda,k}\mathrm{vol}_{\mathcal{D}}$ diverges because $(f, f)_{\lambda,k}$ is nonnegative, real-valued. Therefore, for the proof of (2) of Theorem 10.1, it suffices to show that the integral $\int_U (f, f)_{\lambda,k}\mathrm{vol}_{\mathcal{D}}$ diverges at some U when f is not a cusp form.

Recall that we have étale maps $\mathcal{X}(I)^{\Sigma_I} \to \mathcal{F}(\Gamma)^{\Sigma}$ and $\overline{\mathcal{X}(J)} \to \mathcal{F}(\Gamma)^{\Sigma}$ which give local charts around the boundary points of $\mathcal{F}(\Gamma)^{\Sigma}$. Moreover, we have an étale gluing map $\overline{\mathcal{X}(J)} \to \mathcal{X}(I)^{\Sigma_I}$ for $I \subset J$. Therefore the problem is reduced to estimating $\int_U (f, g)_{\lambda,k}\mathrm{vol}_{\mathcal{D}}$ for a small neighbourhood U of a boundary point of the partial toroidal compactification $\mathcal{X}(I)^{\Sigma_I}$ over a 0-dimensional cusp I. We are thus in the situation of Section 10.3. In what follows, we use the same notation as in Section 10.3.

(1) We first prove the assertion (1) of Theorem 10.1. By Proposition 10.10, the local integral $\int_U (f,g)_{\lambda,k}\mathrm{vol}_{\mathcal{D}}$ can be bounded from above by

$$\lim_{\varepsilon_1,\dots,\varepsilon_c\to 0}\int_{\varepsilon_1}^{a_1}\cdots\int_{\varepsilon_c}^{a_c}\int_0^1\cdots\int_0^1\int_{U'}$$
$$\sum_{i,j} f_i\bar{g}_j\cdot O((-\log r_1\cdots r_c)^N)\cdot(r_1\cdots r_c)^{-1}$$
$$\times dr_1\wedge\cdots\wedge dr_c\wedge d\theta_1\wedge\cdots\wedge d\theta_c\wedge dz_{c+1}\wedge\cdots\wedge d\bar{z}_n$$

for some integer $N>0$, where $a_1,\dots,a_c>0$ are small constants and U' is a small open set in Δ_σ with coordinates $z_{c+1},\dots,z_n$. If f is a cusp form, its components f_i vanish at the boundary divisors $\Delta_1,\dots,\Delta_c$ by Lemma 3.9. Hence

$$f_i = q_1\cdots q_c\cdot O(1).$$

Similarly we have $g_j=O(1)$. We also have $-\log r_1\cdots r_c\le\prod_{l=1}^c(-\log r_l)$. Then the above integral can be bounded from above by

$$\lim_{\varepsilon_1,\dots,\varepsilon_c\to 0}\int_{\varepsilon_1}^{a_1}\cdots\int_{\varepsilon_c}^{a_c}\prod_{l=1}^{c} O((-\log r_l)^N)dr_1\wedge\cdots\wedge dr_c.$$

This integral converges because $\int_\varepsilon^a(\log r)^N dr$ converges in $\varepsilon\to 0$. Thus the integral $\int_U (f,g)_{\lambda,k}\mathrm{vol}_{\mathcal{D}}$ converges if f is a cusp form. This proves the assertion (1) of Theorem 10.1.

(3) Next we prove the assertion (3) of Theorem 10.1. Let $k\le n-|\lambda|-2$. When σ has no isotropic ray, f and g vanish at the boundary divisors $\Delta_1,\dots,\Delta_c$ by Lemma 3.9. (Recall our assumption $\lambda\ne 1,\det$.) Therefore we can give a similar (actually stronger) estimate as in the case (1) above, which implies that $\int_U (f,g)_{\lambda,k}\mathrm{vol}_{\mathcal{D}}$ converges. We consider the case when σ has an isotropic ray, say $\mathbb{R}_{\ge 0}v_1$. Since other rays $\mathbb{R}_{\ge 0}v_2,\dots,\mathbb{R}_{\ge 0}v_c$ are not isotropic by our assumption, we see from Lemma 3.9 that f and g vanish at $\Delta_2,\dots,\Delta_c$. Therefore we have $f=q_2\cdots q_c\cdot O(1)$ and $g=q_2\cdots q_c\cdot O(1)$. By substituting these estimates in the second case of Proposition 10.10, we see that

$$(f,g)_{\lambda,k}\mathrm{vol}_{\mathcal{D}} = (r_2\cdots r_c)\cdot O(1)\cdot O((-\log r_1\cdots r_c)^{k-n+|\lambda|})\cdot r_1^{-1}$$
$$\times dr_1\wedge\cdots\wedge dr_c\wedge d\theta_1\wedge\cdots\wedge d\theta_c\wedge dz_{c+1}\wedge\cdots\wedge d\bar{z}_n.$$

We have $(-\log r_1\cdots r_c)^{-1}\le(-\log r_1)^{-1}$. Therefore $\int_U (f,g)_{\lambda,k}\mathrm{vol}_{\mathcal{D}}$ can be bounded from above by

$$\lim_{\varepsilon_1\to 0}\int_{\varepsilon_1}^{a_1} O((-\log r_1)^{k-n+|\lambda|}\cdot r_1^{-1})dr_1.$$

Since $k-n+|\lambda|\le -2$ by the assumption, this integral converges by Lemma 10.11.

(2) Finally, we prove the assertion (2) of Theorem 10.1. When L has Witt index ≤ 1, we have $S_{\lambda,k}(\Gamma) = M_{\lambda,k}(\Gamma)$ by Proposition 3.7. Thus we may assume that L has Witt index 2. Let $k \geq n + |\lambda| - 1$ and assume that f is not a cusp form. Then f does not vanish identically at a boundary divisor $\Delta = \Delta_\sigma$ corresponding to an isotropic ray $\sigma = \mathbb{R}_{\geq 0} v$ for some 0-dimensional cusp I. We shall show that $\int_U (f, f)_{\lambda,k} \mathrm{vol}_{\mathcal{D}}$ diverges for a general point x of Δ. Thus we consider the case $c = 1$. We rewrite $q_1 = r_1 e(\theta_1)$ as $q = re(\theta)$, and also denote $Z' = (z_2, \dots, z_c)$ which give local charts on Δ.

We go back to the flow $Z = Z_0 + \sqrt{-1} t v$ in Section 10.3. Then $P_{ij}(\mathrm{Im}(Z))$ is a real polynomial of t whose coefficients depend continuously on $v_0 = \mathrm{Im}(Z_0)$. Therefore, by substituting (10.18), we see that in a neighbourhood of x,

$$P_{ij}(\mathrm{Im}(Z)) = Q_{ij}(\log r)$$

for a real polynomial Q_{ij} of one variable whose coefficients depend continuously on Z'. Moreover, as in the proof of Lemma 10.8, we have

$$(\mathrm{Im}(Z), \mathrm{Im}(Z)) = (v_0, v_0) + 2(v_0, v)t \sim -C \log r$$

for some constant $C = C(Z') > 0$ depending continuously on Z'. Therefore, by the same calculation as in Section 10.3, we see that

$$(f, f)_{\lambda,k} \mathrm{vol}_{\mathcal{D}} \succcurlyeq \sum_{i,j} f_i \overline{f_j} Q_{ij}(\log r)(-\log r)^{k-n-|\lambda|} r^{-1} dr \wedge d\theta \wedge \cdots$$

as $r \to 0$.

We take the base change of the frame $(s_i)_i$ by a $\mathrm{GL}_N(\mathbb{C})$-valued holomorphic function $A = A(Z')$ of Z' around x so that $f_1 \to 1$ and $f_i \to 0$ for $i > 1$ as $r \to 0$. This is possible because $f \neq 0 \in V(I)_{\lambda,k}$ around x. Then the real symmetric matrix $Q = (Q_{ij})_{i,j}$ is replaced by the Hermitian matrix ${}^t\overline{A} Q A$, which we denote by $Q' = (Q'_{ij})_{i,j}$. Each Q'_{ij} is a $\mathbb{C}$-polynomial of $\log r$ whose coefficients depend continuously on Z'. Since the Hermitian matrix Q' is positive-definite when r is small, we have in particular $Q'_{11} \neq 0$. Then

$$(f, f)_{\lambda,k} \mathrm{vol}_{\mathcal{D}} \succcurlyeq Q'_{11}(\log r)(-\log r)^{k-n-|\lambda|} r^{-1} dr \wedge d\theta \wedge \cdots$$

as $r \to 0$. Since Q'_{11} is a nonzero real polynomial and $k - n - |\lambda| \geq -1$ by our assumption, we obtain

$$(f, f)_{\lambda,k} \mathrm{vol}_{\mathcal{D}} \succcurlyeq (-\log r)^{-1} r^{-1} dr \wedge d\theta \wedge \cdots$$

as $r \to 0$. By Lemma 10.11, this implies that the integral $\int_U (f, f)_{\lambda,k} \mathrm{vol}_{\mathcal{D}}$ diverges. This completes the proof of Theorem 10.1. ∎

Remark 10.13. As the proof shows, the assertion (1) of Theorem 10.1 holds even when $\lambda = 1, \det$. Similarly, the assertion (2) holds also for $\lambda = 1, \det$ at least when L has Witt index 2. On the other hand, the proof of (3) makes use of Proposition 3.7, which requires $\lambda \neq 1, \det$.