Chapter 11

Vanishing theorem II

Let L be a lattice of signature $(2,n)$ with $n \geq 3$ and Γ be a finite-index subgroup of $\mathrm{O}^+(L)$. Let $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n)$ be a partition expressing an irreducible representation of $\mathrm{O}(n,\mathbb{C})$. We assume $\lambda \neq 1, \det$. Therefore $\lambda_1 > 0$ and $\lambda_n = 0$. In this chapter we prove our second type of vanishing theorem. We define the *corank* of λ, denoted by $\mathrm{corank}(\lambda)$, as the maximal index $1 \leq i \leq [n/2]$ such that

$$\lambda_1 = \lambda_2 = \cdots = \lambda_i \quad \text{and} \quad \lambda_n = \lambda_{n-1} = \cdots = \lambda_{n+1-i} = 0.$$

Let

$$\overline{\lambda} = (\overline{\lambda}_1, \dots, \overline{\lambda}_{[n/2]}) = (\lambda_1 - \lambda_n, \lambda_2 - \lambda_{n-1}, \dots, \lambda_{[n/2]} - \lambda_{n+1-[n/2]})$$

be the highest weight for $\mathrm{SO}(n,\mathbb{C})$ associated to λ. Then $\mathrm{corank}(\lambda)$ is the maximal index i such that $\overline{\lambda}_1 = \overline{\lambda}_2 = \cdots = \overline{\lambda}_i$. Let $|\overline{\lambda}| = \sum_i \overline{\lambda}_i$ be as in Section 10.

Our second vanishing theorem is the following.

Theorem 11.1. *Let $\lambda \neq 1, \det$. If $k < n + \lambda_1 - \mathrm{corank}(\lambda) - 1$, there is no nonzero square integrable modular form of weight (λ, k). In particular,*

(1) $S_{\lambda,k}(\Gamma) = 0$ *when* $k < n + \lambda_1 - \mathrm{corank}(\lambda) - 1$.

(2) $M_{\lambda,k}(\Gamma) = 0$ *when* $k < n - |\overline{\lambda}| - 1$.

We compare Theorems 11.1 and 9.1. The bound $n/2 + \lambda_1 - 1$ in Theorem 9.1 is smaller than the main bound $n + \lambda_1 - \mathrm{corank}(\lambda) - 1$ in Theorem 11.1 because $\mathrm{corank}(\lambda) \leq [n/2]$. However, Theorem 11.1 is about square integrable modular forms, while Theorem 9.1 is about the whole $M_{\lambda,k}(\Gamma)$, so Theorem 11.1 does not supersede Theorem 9.1. The comparison of Theorem 11.1 (1) and Theorem 9.1 raises the question if we have convergent Eisenstein series in the range

$$n/2 + \lambda_1 - 1 \leq k < n + \lambda_1 - \mathrm{corank}(\lambda) - 1.$$

As for the comparison of Theorem 11.1 (2) and Theorem 9.1, it depends on λ which $n - |\overline{\lambda}| - 1$ or $n/2 - 1 + \lambda_1$ is larger. Roughly speaking, Theorem 11.1 (2) is stronger when $|\overline{\lambda}|$ is small, while Theorem 9.1 is stronger when λ_1 is large. Thus Theorems 11.1 and 9.1 are rather complementary.

The proof of Theorem 11.1 follows the same strategy as Weissauer's vanishing theorem for vector-valued Siegel modular forms [47]. If we have a square integrable modular form $f \neq 0$, we can construct a unitary highest weight module for the Lie algebra of $\mathrm{SO}^+(L_{\mathbb{R}})$ by a standard procedure (cf. [23, 47] for the Siegel case). By computing its weight and comparing it with the classification of unitarizable highest weight modules [12, 13, 28], we obtain the bound $k \geq n + \lambda_1 - \mathrm{corank}(\lambda) - 1$. The more specific assertions (1), (2) in Theorem 11.1 are derived from combination with Theorem 10.1.

The rest of this chapter is devoted to the proof of Theorem 11.1. The construction of highest weight module occupies Sections 11.1 and 11.2, and the concluding step is done in Section 11.3.

11.1 Lifting to the Lie group

In this section we work with $G = \mathrm{SO}^+(L_{\mathbb{R}})$. We lift a square integrable modular form on $\mathcal{D}$ to a square integrable function on G in a standard way. We choose a base point $[\omega_0] \in \mathcal{D}$. Let $K \simeq \mathrm{SO}(2,\mathbb{R}) \times \mathrm{SO}(n,\mathbb{R})$ be the stabilizer of $[\omega_0]$ in G. We denote by $\mathfrak{g}$, $\mathfrak{k}$ the Lie algebras of G, K, respectively. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$ with respect to $\mathfrak{k}$, and $\mathfrak{p}_{\mathbb{C}} = \mathfrak{p}_+ \oplus \mathfrak{p}_-$ be the eigendecomposition for the adjoint action of $\mathfrak{so}(2,\mathbb{R}) \subset \mathfrak{k}$ on $\mathfrak{p}$. Then $\mathfrak{p}$ is identified with the real tangent space $T_{[\omega_0],\mathbb{R}}\mathcal{D}$ of $\mathcal{D}$ at $[\omega_0]$, and the decomposition $\mathfrak{p}_{\mathbb{C}} = \mathfrak{p}_+ \oplus \mathfrak{p}_-$ corresponds to the decomposition

$$T_{[\omega_0],\mathbb{R}}\mathcal{D} \otimes_{\mathbb{R}} \mathbb{C} = T^{1,0}_{[\omega_0]}\mathcal{D} \oplus T^{0,1}_{[\omega_0]}\mathcal{D}.$$

For each point $[\omega] = g([\omega_0])$ of $\mathcal{D}$, the g-action gives an isomorphism $\mathfrak{p}_- \to T^{0,1}_{[\omega]}\mathcal{D}$. This isomorphism is unique up to the adjoint action of K.

The Lie group $P_- = \exp(\mathfrak{p}_-)$ is abelian and is the unipotent radical of the stabilizer of $[\omega_0]$ in $\mathrm{SO}(L_{\mathbb{C}})$ (see, e.g., [2, pp. 107–108]). Therefore, in view of (1.2), P_- coincides with the group of Eichler transvections of $L_{\mathbb{C}}$ with respect to the isotropic line $\mathbb{C}\omega_0$. In particular, P_- acts trivially on $\mathbb{C}\omega_0 = \mathcal{L}_{[\omega_0]}$ and $\omega_0^{\perp}/\mathbb{C}\omega_0 = \mathcal{E}_{[\omega_0]}$. We will use this property in the proof of Claim 11.3 (3) below.

Now let λ be a partition for $\mathrm{O}(n,\mathbb{C})$ and $\bar{\lambda}$ be the associated highest weight for $\mathrm{SO}(n,\mathbb{C})$. To start with $\mathrm{O}(n,\mathbb{C})$ is somewhat roundabout here, but this is for consistency with the formulation of Theorem 11.1 and eventually with other chapters. We first consider the case when V_λ remains irreducible as a representation of $\mathrm{SO}(n,\mathbb{C})$ (cf. Section 3.6.1). Let $W_{\bar{\lambda},k}$ be the finite-dimensional irreducible $\mathbb{C}$-representation of $K \simeq \mathrm{SO}(n,\mathbb{R}) \times \mathrm{SO}(2,\mathbb{R})$ with highest weight $(\bar{\lambda}, k)$.

Lemma 11.2. *Assume that either n is odd or $n = 2m$ is even with ${}^t\lambda_1 \neq m$. Let $f \neq 0$ be a square integrable modular form of weight (λ, k) for a finite-index subgroup Γ of $\mathrm{SO}^+(L)$. Then there exists a smooth function $\phi_f \neq 0$ on G with the following properties.*

(1) $\phi_f \in L^2(\Gamma \backslash G)$.

(2) $\mathfrak{p}_- \cdot \phi_f = 0$. *(Here* $\mathfrak{g}$ *acts on* ϕ_f *as the derivative of the right* G*-translations.)*

(3) *The linear subspace of* $L^2(\Gamma\backslash G)$ *spanned by the right* K*-translations of* ϕ_f *is finite dimensional and is isomorphic to* $W_{\bar{\lambda},k}^{\vee}$ *as a* K*-representation.*

Proof. We choose a rank 1 primitive isotropic sublattice I of L and let $j(g, [\omega])$ be the factor of automorphy associated to the I-trivialization $\mathcal{E}_{\lambda,k} \simeq V(I)_{\lambda,k} \otimes \mathcal{O}_{\mathcal{D}}$. The homomorphism

$$K \to \mathrm{End}(V(I)_{\lambda,k}), \quad k \mapsto j(k, [\omega_0]), \tag{11.1}$$

defines a representation of K on $V(I)_{\lambda,k} \simeq (\mathcal{E}_{\lambda,k})_{[\omega_0]}$. This is irreducible of highest weight $(\overline{\lambda}, k)$ by our assumption on λ. The Petersson metric on $(\mathcal{E}_{\lambda,k})_{[\omega_0]}$ is K-invariant. Via the I-trivialization at $[\omega_0]$, this defines a K-invariant Hermitian metric on $V(I)_{\lambda,k}$. The induced constant Hermitian metric on the product vector bundle $G \times V(I)_{\lambda,k}$ over G corresponds to the Petersson metric on $\mathcal{E}_{\lambda,k}$ through the isomorphism

$$\mathcal{E}_{\lambda,k} \simeq G \times_K (\mathcal{E}_{\lambda,k})_{[\omega_0]} \simeq G \times_K V(I)_{\lambda,k}. \tag{11.2}$$

Via the I-trivialization we regard the modular form f as a $V(I)_{\lambda,k}$-valued holomorphic function on $\mathcal{D}$. We define a $V(I)_{\lambda,k}$-valued smooth function $\tilde{f}$ on G by

$$\tilde{f}(g) = j(g, [\omega_0])^{-1} \cdot f(g([\omega_0])), \quad g \in G.$$

This is the $V(I)_{\lambda,k}$-valued function on G that corresponds to the section f of $\mathcal{E}_{\lambda,k}$ via the G-equivariant isomorphism (11.2).

Claim 11.3. *The* $V(I)_{\lambda,k}$*-valued function* $\tilde{f}$ *satisfies the following.*

(1) $\tilde{f}(\gamma g) = \tilde{f}(g)$ *for* $\gamma \in \Gamma$.

(2) $\tilde{f}(gk) = k^{-1}(\tilde{f}(g))$ *for* $k \in K$, *where* k^{-1} *acts on* $V(I)_{\lambda,k}$ *by* (11.1).

(3) $\mathfrak{p}_- \cdot \tilde{f} = 0$.

(4) $\tilde{f}$ *is square integrable over* $\Gamma\backslash G$ *with respect to the Haar measure on* G *and the Hermitian metric on* $V(I)_{\lambda,k}$.

All these properties should be standard. We supply an argument for the sake of completeness (cf. [23] for the Siegel modular case). The property (1) follows from the Γ-invariance of f, and the property (2) is just the invariance of $\tilde{f}$ under the K-action on $G \times V(I)_{\lambda,k}$. Both (1) and (2) can also be checked directly by using the cocycle condition for $j(g, [\omega])$.

The property (4) holds because we have

$$\begin{aligned}\int_{\Gamma\backslash G} (\tilde{f}(g), \tilde{f}(g)) d\mu_G &= \int_{\Gamma\backslash\mathcal{D}} \mathrm{vol}_{\mathcal{D}} \int_K (\tilde{f}(g), \tilde{f}(g)) d\mu_K \\ &= \int_{\Gamma\backslash\mathcal{D}} (f, f)_{\lambda,k} \mathrm{vol}_{\mathcal{D}}\end{aligned}$$

up to constant, where $d\mu_G$, $d\mu_K$ are the Haar measures on G, K, respectively, and $(\ ,\ )$ in the first line is the Hermitian metric on $V(I)_{\lambda,k}$.

Finally, we check the property (3). We have

$$X \cdot \tilde{f}(g) = (X \cdot j(g, [\omega_0])^{-1}) f(g([\omega_0])) + j(g, [\omega_0])^{-1}(X \cdot f(g([\omega_0])))$$

for $X \in \mathfrak{p}_-$. Then $X \cdot f(g([\omega_0])) = 0$ by the holomorphicity of f. As for the first term, since P_- fixes $[\omega_0]$ and acts trivially on $(\mathcal{E}_{\lambda,k})_{[\omega_0]}$ as noticed before, we have

$$j(g \exp(tX), [\omega_0]) = j(g, \exp(tX)([\omega_0])) \circ j(\exp(tX), [\omega_0]) = j(g, [\omega_0]).$$

This shows that $X \cdot j(g, [\omega_0]) = 0$, and so,

$$X \cdot j(g, [\omega_0])^{-1} = -j(g, \omega_0)^{-1} \circ (X \cdot j(g, [\omega_0])) \circ j(g, [\omega_0])^{-1} = 0.$$

Therefore $X \cdot \tilde{f} = 0$. Claim 11.3 is thus verified.

We go back to the proof of Lemma 11.2. The property (2) in Claim 11.3 means that $\tilde{f}$ as a vector of the K-representation

$$L^2(\Gamma\backslash G, V(I)_{\lambda,k}) \simeq L^2(\Gamma\backslash G) \otimes V(I)_{\lambda,k} \simeq L^2(\Gamma\backslash G) \otimes W_{\bar{\lambda},k}$$

is K-invariant. Therefore it corresponds to a nonzero K-homomorphism

$$\Phi_f : W_{\bar{\lambda},k}^{\vee} \to L^2(\Gamma\backslash G),$$

which must be injective by the irreducibility of $W_{\bar{\lambda},k}^{\vee}$. The image of Φ_f consists of the scalar-valued functions $L \circ \tilde{f}$ for $L \in V(I)_{\lambda,k}^{\vee}$. By the irreducibility, the K-orbit of any such nonzero vector generates the image of Φ_f. Then we put $\phi_f = L \circ \tilde{f}$ for an arbitrary $L \neq 0$. The property (3) in Claim 11.3 implies the property (2) in Lemma 11.2. This finishes the proof of Lemma 11.2. ■

11.2 Highest weight modules

In this section we construct from ϕ_f a unitary highest weight module of $\mathfrak{g}$. The result is summarized in Propositions 11.4 and 11.5.

First we recall the theory of highest weight modules following [12, 13, 24] and specialized to $G = \mathrm{SO}^+(L_{\mathbb{R}})$. Let $\mathfrak{k}_0 = \mathfrak{so}(2, \mathbb{R})$ and $\mathfrak{k}_1 = \mathfrak{so}(n, \mathbb{R})$. Then $\mathfrak{k} = \mathfrak{k}_0 \oplus \mathfrak{k}_1$, $\mathfrak{k}_0$ is the centre of $\mathfrak{k}$, and $\mathfrak{k}_1 = [\mathfrak{k}, \mathfrak{k}]$ is the semi-simple part of $\mathfrak{k}$. We take a maximal abelian subalgebra $\mathfrak{h}$ of $\mathfrak{k}$. Then $\mathfrak{h}_{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$. We have $\mathfrak{h} = \mathfrak{k}_0 \oplus \mathfrak{h}_1$ with $\mathfrak{h}_1 = \mathfrak{h} \cap \mathfrak{k}_1$ being a maximal abelian subalgebra of $\mathfrak{k}_1$. We may take a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}_{\mathbb{C}}$ constructed from the root data in $\mathfrak{h}_{\mathbb{C}}$ which is the direct sum of a Borel subalgebra of $\mathfrak{k}_{\mathbb{C}}$ and $\mathfrak{p}_-$ (rather than $\mathfrak{p}_+$).

Let $\widetilde{\rho} \in \mathfrak{h}_{\mathbb{C}}^{\vee}$ be a weight which is dominant and integral with respect to $\mathfrak{k}_{\mathbb{C}}$ (rather than $\mathfrak{g}_{\mathbb{C}}$). According to the decomposition $\mathfrak{h} = \mathfrak{k}_0 \oplus \mathfrak{h}_1$, we can write

$$\widetilde{\rho} = (\rho, \alpha), \quad \rho \in (\mathfrak{h}_1)_{\mathbb{C}}^{\vee}, \quad \alpha \in (\mathfrak{k}_0)_{\mathbb{C}}^{\vee} \simeq \mathbb{C},$$

with ρ a dominant and integral weight for $(\mathfrak{k}_1)_{\mathbb{C}} = \mathfrak{so}(n, \mathbb{C})$. Here we identify

$$(\mathfrak{k}_0)_{\mathbb{C}}^{\vee} \simeq \mathbb{C}$$

by the pairing with the unique maximal non-compact positive root. (In the notation of [13, Section 4], $\rho = (m_2, \dots, m_n)$ and $\alpha = m_1$; in the notation of [12, Sections 10 and 11], $\rho = (\lambda_2, \dots, \lambda_n)$ and $\alpha = \lambda_1 + z$.) We denote by $\mathbb{C}_{\rho,\alpha}$ the 1-dimensional module of $\mathfrak{h}_{\mathbb{C}}$ of weight (ρ, α). We can regard $\mathbb{C}_{\rho,\alpha}$ as a module of $\mathfrak{b}$ naturally. We also denote by $W_{\rho,\alpha}$ the finite-dimensional irreducible module of $\mathfrak{k}_{\mathbb{C}}$ of highest weight (ρ, α). This is compatible with the notation in Section 11.1.

Let $\mathfrak{U}(\mathfrak{g}_{\mathbb{C}})$ and $\mathfrak{U}(\mathfrak{b})$ be the universal enveloping algebras of $\mathfrak{g}_{\mathbb{C}}$ and $\mathfrak{b}$, respectively. Let

$$M(\rho, \alpha) = \mathfrak{U}(\mathfrak{g}_{\mathbb{C}}) \otimes_{\mathfrak{U}(\mathfrak{b})} \mathbb{C}_{\rho,\alpha}$$

be the Verma module of $\mathfrak{g}_{\mathbb{C}}$ with highest weight (ρ, α). The module $M(\rho, \alpha)$ has a unique irreducible quotient $L(\rho, \alpha)$ (see [24, Section 1.3]). This is called the *irreducible highest weight module* of $\mathfrak{g}_{\mathbb{C}}$ with highest weight (ρ, α). The module $L(\rho, \alpha)$ is also a unique irreducible quotient of the generalized (or parabolic) Verma module

$$N(\rho, \alpha) = \mathfrak{U}(\mathfrak{g}_{\mathbb{C}}) \otimes_{\mathfrak{U}(\mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}_-)} W_{\rho,\alpha},$$

because $N(\rho, \alpha)$ is also a quotient of $M(\rho, \alpha)$ (see [24, Section 9.4]). The highest weight module $L(\rho, \alpha)$ is said to be *unitarizable* if it is isomorphic as a $\mathfrak{g}_{\mathbb{C}}$-module to the K-finite part of a unitary representation of G.

Now we go back to modular forms on $\mathcal{D}$.

Proposition 11.4. *Assume that either n is odd or $n = 2m$ is even with ${}^t\lambda_1 \neq m$. If we have a square integrable modular form $f \neq 0 \in M_{\lambda,k}(\Gamma)$, then the irreducible highest weight module $L(\overline{\lambda}^{\vee}, -k)$ is unitarizable.*

Proof. Let V_f be the minimal Hilbert subspace of $L^2(\Gamma \backslash G)$ which contains the right G-translations of the function ϕ_f in Lemma 11.2. This is a sub unitary representation of $L^2(\Gamma \backslash G)$. The K-finite part $(V_f)_K$ of V_f is a $(\mathfrak{g}, K)$-module. Let V_0 be the subspace of $(V_f)_K$ generated by the right K-translations of ϕ_f. By Lemma 11.2 (3), V_0 is isomorphic to $W_{\overline{\lambda},k}^{\vee} = W_{\overline{\lambda}^{\vee},-k}$ as a K-representation. By Lemma 11.2 (2), V_0 is annihilated by $\mathfrak{p}_-$. Indeed, for $X \in \mathfrak{p}_-$ and $k \in K$, we have

$$k^{-1} \cdot (X \cdot (k \cdot \phi_f)) = \mathrm{Ad}_{k^{-1}}(X) \cdot \phi_f = 0$$

because the adjoint action of K preserves $\mathfrak{p}_-$. Therefore the natural homomorphism $\mathfrak{U}(\mathfrak{g}_{\mathbb{C}}) \otimes_{\mathbb{C}} V_0 \twoheadrightarrow (V_f)_K$ descends to a surjective homomorphism

$$N(\overline{\lambda}^\vee, -k) \simeq \mathfrak{U}(\mathfrak{g}_{\mathbb{C}}) \otimes_{\mathfrak{U}(\mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}_-)} V_0 \twoheadrightarrow (V_f)_K$$

from the generalized Verma module $N(\overline{\lambda}^\vee, -k)$. By the minimality of the quotient $L(\overline{\lambda}^\vee, -k)$, this in turn implies that there exists a surjective homomorphism

$$(V_f)_K \twoheadrightarrow L(\overline{\lambda}^\vee, -k).$$

Since $(V_f)_K$ is unitarizable, so is $L(\overline{\lambda}^\vee, -k)$. ∎

So far we have considered the case when V_λ remains irreducible as an $\mathrm{SO}(n, \mathbb{C})$-representation. It remains to consider the exceptional case $n = 2m, {}^t\lambda_1 = m$, where V_λ gets reducible. In that case, Proposition 11.4 is modified as follows. For a highest weight $\rho = (\rho_1, \dots, \rho_m)$ for $\mathrm{SO}(2m, \mathbb{C})$, we write $\rho^\dagger = (\rho_1, \dots, \rho_{m-1}, -\rho_m)$ as in Section 3.6.1.

Proposition 11.5. *Let $n = 2m$ be even and ${}^t\lambda_1 = m$. Suppose that we have a square integrable modular form $f \neq 0 \in M_{\lambda,k}(\Gamma)$. Then either $L(\overline{\lambda}^\vee, -k)$ or $L((\overline{\lambda}^\dagger)^\vee, -k)$ is unitarizable.*

Proof. According to the decomposition of $\mathcal{E}_\lambda$ in Proposition 3.12 (2), we can write $f = (f_+, f_-)$ with f_+ of weight $(\overline{\lambda}, k)$ and f_- of weight $(\overline{\lambda}^\dagger, k)$ with respect to $\mathrm{SO}(n, \mathbb{R}) \times \mathrm{SO}(2, \mathbb{R})$. We have either $f_+ \neq 0$ or $f_- \neq 0$. Then we can do the same construction for the nonzero component $f_\pm$ as before, by using the component-wise I-trivialization (3.22). ∎

Finally, we recall the classification of unitarizable irreducible highest weight modules [12, 13, 28]. For our purpose, we restrict ourselves to those weights (ρ, α) such that $\alpha \in \mathbb{Z}$ and ρ is a highest weight for $\mathrm{SO}(n, \mathbb{C})$ (rather than $\mathfrak{so}(n, \mathbb{C})$). In this situation, the version in [13] is convenient to use. For such a weight $\rho = (\rho_1, \dots, \rho_{[n/2]})$, we denote by $\mathrm{corank}(\rho)$ the maximal index i such that

$$\rho_1 = \rho_2 = \cdots = \rho_{i-1} = |\rho_i|.$$

Theorem 11.6 ([12,13,28]). *Let $\rho = (\rho_1, \dots, \rho_{[n/2]})$ be a highest weight for $\mathrm{SO}(n, \mathbb{C})$. Assume that $\rho_1 \neq 0$, i.e., ρ nontrivial. Let $\alpha \in \mathbb{Z}$. Then the irreducible highest weight module $L(\rho, \alpha)$ is unitarizable if and only if $-\alpha \geq n + \rho_1 - \mathrm{corank}(\rho) - 1$.*

Here we follow [13, Theorems 4.2 and 4.3], with $\alpha = m_1$, $\rho = (m_2, \dots, m_n)$ and $\mathrm{corank}(\rho) = i - 1$ in the notation there. A complete classification of unitary irreducible highest weight modules for general (ρ, α) (and also for other Lie groups) is given in [12, 28]. For the proof of Theorem 11.1, we just use the "only if" part of Theorem 11.6.

Remark 11.7. In fact, the result of [12] tells us more than unitarizability. Let $\rho_1 > 0$. By the calculation of "the first reduction point" in [12, Lemmas 10.3 and 11.3], we see that the generalized Verma module $N(\rho, \alpha)$ is already irreducible when $-\alpha > n + \rho_1 - \mathrm{corank}(\rho) - 1$. Thus $L(\rho, \alpha) = N(\rho, \alpha)$ in that case. Furthermore, according to [12, Theorem 2.4 (b)], $L(\rho, \alpha)$ belongs to the holomorphic discrete series when $-\alpha > n + \rho_1 - 1$, and to the limit of holomorphic discrete series when $-\alpha = n + \rho_1 - 1$. Note that $\alpha = \lambda_1 + z$ in the notation of [12, Sections 10 and 11], and this λ_1 corresponds to $-\rho_1 - n + 1$ in our notation, so z in [12] is $\alpha + n + \rho_1 - 1$ here.

11.3 Proof of Theorem 11.1

With the preliminaries in Sections 11.1 and 11.2, we can now complete the proof of Theorem 11.1. Let $n \geq 3$ and $\lambda \neq 1, \det$. We first consider the case when either n is odd or $n = 2m$ is even with ${}^t\lambda_1 \neq m$. Suppose that we have a square integrable modular form $f \neq 0 \in M_{\lambda,k}(\Gamma)$. Then the highest weight module $L(\overline{\lambda}^\vee, -k)$ is unitarizable by Proposition 11.4. By applying Theorem 11.6 to $(\rho, \alpha) = (\overline{\lambda}^\vee, -k)$, we see that (λ, k) must satisfy

$$k \geq n + (\overline{\lambda}^\vee)_1 - \mathrm{corank}(\overline{\lambda}^\vee) - 1.$$

Recall from Section 3.6.1 that $\overline{\lambda}^\vee = \overline{\lambda}$ in the case $n \not\equiv 2 \bmod 4$ and $\overline{\lambda}^\vee = \overline{\lambda}^\dagger$ in the case $n \equiv 2 \bmod 4$. Since $n \geq 3$, we have $\overline{\lambda}_1 = (\overline{\lambda}^\dagger)_1$, and so,

$$(\overline{\lambda}^\vee)_1 = \overline{\lambda}_1 = \lambda_1 \tag{11.3}$$

in both cases. Since $\mathrm{corank}(\overline{\lambda}^\dagger) = \mathrm{corank}(\overline{\lambda})$, we also have

$$\mathrm{corank}(\overline{\lambda}^\vee) = \mathrm{corank}(\overline{\lambda}) = \mathrm{corank}(\lambda) \tag{11.4}$$

by the definition of $\mathrm{corank}(\lambda)$. (Note that all components of $\overline{\lambda}$ are nonnegative.) Hence (λ, k) satisfies the bound

$$k \geq n + \lambda_1 - \mathrm{corank}(\lambda) - 1. \tag{11.5}$$

This proves the main assertion of Theorem 11.1. The assertion (1) for $S_{\lambda,k}(\Gamma)$ is then a consequence of Theorem 10.1 (1). As for the assertion (2), we note that the inequality

$$n - |\overline{\lambda}| - 1 < n + \lambda_1 - \mathrm{corank}(\lambda) - 1$$

holds, because $\mathrm{corank}(\overline{\lambda}) \leq |\overline{\lambda}|$ and $\lambda_1 > 0$. Therefore, when $k < n - |\overline{\lambda}| - 1$, any modular form of weight (λ, k) is square integrable by Theorem 10.1 (3), but at the same time its weight violates the bound (11.5). This implies that $M_{\lambda,k}(\Gamma) = 0$ in this case.

Next we consider the exceptional case when $n = 2m$ is even and ${}^t\lambda_1 = m$. Note that $\overline{\lambda} = \lambda$ in this case. If we have a square integrable modular form $f \neq 0 \in M_{\lambda,k}(\Gamma)$, then either $L(\overline{\lambda}^\vee, -k)$ or $L((\overline{\lambda}^\dagger)^\vee, -k)$ is unitarizable by Proposition 11.5. Since $\lambda^\vee = \lambda$ or $\lambda^\dagger$, this means that either $L(\lambda, -k)$ or $L(\lambda^\dagger, -k)$ is unitarizable. By Theorem 11.6, we obtain the bound

$$k \geq \min(n + \lambda_1 - \mathrm{corank}(\lambda) - 1, n + (\lambda^\dagger)_1 - \mathrm{corank}(\lambda^\dagger) - 1).$$

Since $(\lambda^\dagger)_1 = \lambda_1$ and $\mathrm{corank}(\lambda^\dagger) = \mathrm{corank}(\lambda)$ as before, this reduces to the same bound as (11.5). The rest of the argument is similar to the non-exceptional case. This completes the proof of Theorem 11.1. ∎

Remark 11.8. Since Theorem 11.1 (2) is derived from Theorem 10.1 (3), this part could be improved if we could improve the characterization of square integrability in the remaining range (10.1).

Remark 11.9. Let $V_f \subset L^2(\Gamma\backslash G)$ be the unitary representation attached to a square integrable modular form $f \in M_{\lambda,k}(\Gamma)$, say in the non-exceptional case. Recall from the proof of Proposition 11.4 that

$$N(\overline{\lambda}^\vee, -k) \twoheadrightarrow (V_f)_K \twoheadrightarrow L(\overline{\lambda}^\vee, -k).$$

If we apply Remark 11.7 to $(\rho, \alpha) = (\overline{\lambda}^\vee, -k)$ and use (11.3) and (11.4), we find that

$$(V_f)_K \simeq L(\overline{\lambda}^\vee, -k) \simeq N(\overline{\lambda}^\vee, -k)$$

when $k \geq n + \lambda_1 - \mathrm{corank}(\lambda)$. The unitary representation V_f belongs to the holomorphic discrete series when $k \geq n + \lambda_1$, and to the limit of holomorphic discrete series when $k = n + \lambda_1 - 1$.