Nina Nikolaevna Uraltseva: A journey of lifelong achievements in mathematics

1 Introduction

Nina Uraltseva has significantly impacted mathematics through her innovative research in analysis and partial differential equations (PDEs), as well as her development of refined analytical methods. She is especially acclaimed for her early collaborative work with Olga Ladyzhenskaya on linear and quasilinear elliptic and parabolic equations, which are now considered foundational. Her profound contributions also extend to areas such as degenerate and geometric equations, variational inequalities, and free boundary problems.

Nina Uraltseva has authored over 100 publications and garnered more than 10 thousand citations on MathSciNet. Her renowned book "Linear and Quasilinear Equations of Parabolic Type", co-authored with Ladyzhenskaya and Solonnikov, boasts over 6100 citations.[1] The elliptic counterpart of this book, written with Ladyzhenskaya, has received over 2900 citations.[2] This highlights her immense influence in the field of partial differential equations. Despite the emergence of numerous new books on PDEs, these classic works remain indispensable and highly valued by Ph.D. students and early-career researchers.

Over the years, Nina Uraltseva has significantly contributed to the mathematical community by serving on numerous influential committees. Notably, she chaired the PDE Panel of the International Congress of Mathematicians in Berlin in 1998 and the Prize Committee of the European Congress of Mathematics in Stockholm in 2004. She has also served as an expert for research foundations, including the European Research Council, the Russian Foundation for Basic Research, and the Russian Science Foundation.

Uraltseva has held editorial positions for several journals, such as Editor-in-Chief for the Proceedings of the St. Petersburg Mathematical Society and the Journal of Problems in Mathematical Analysis. She has also been a member of the editorial committees for Algebra and Analysis (translated into the St. Petersburg Mathematical Journal), Vestnik St. Petersburg State University, and the Lithuanian Mathematical Journal. As a frequent visitor to universities worldwide, she has delivered talks at various international conferences and schools.

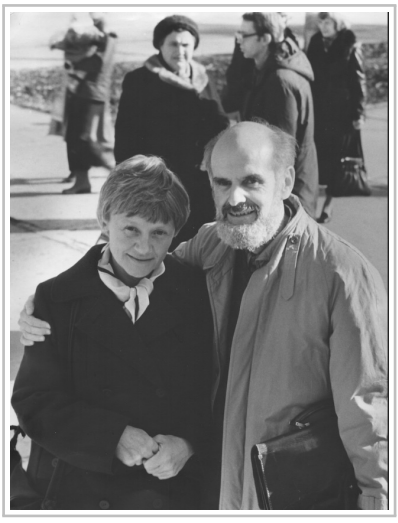
[1] Russian edition and English translation.

[2] Two Russian editions, English and French translations.

As a leading expert in the analysis of PDEs, Uraltseva has inspired many female students across different areas of mathematics, encouraging them to pursue research and careers in the field. Her motivational talks, especially at conferences focusing on women in mathematics, have played a crucial role in attracting numerous women to the discipline.

Nina Uraltseva is renowned for her dedication to educating Ph.D. students and her commitment to fostering their development. She has supervised 13 Ph.D. students, four of whom have attained habilitation. Her willingness to engage in discussions and brainstorming sessions with students and colleagues highlights her as one of the most devoted and influential figures in the mathematical community.

Nina's interests are not limited to scientific activities only. In her youth, she used to be a very good basketball player and an active member of the university basketball team (despite her small height of 156 cm). She enjoyed mountain hiking, canoeing, and car driving. In the 1980s, Nina took part in five archaeological expeditions in the north of Russia and excavated Paleolithic ceramics. She is also a passionate lover of classical music.



Left: in archaelogical expedition, Cape Kanin, 1987;
Right: with Prof. Robert Finn, Novosibirsk, 1978

Throughout her career, Nina Uraltseva has made numerous friends worldwide, thanks to her kind personality and utmost politeness. Her unbiased approach and open-mindedness towards various mathematical problems have earned her great respect and popularity among colleagues and students alike. She is celebrated not only for her contributions to mathematics but also for her exceptional character as a person.

2 Mathematical contributions

Nina Uraltseva has made groundbreaking contributions to analysis of partial differential equations (PDEs), particularly in the fields of elliptic and parabolic equations. She is widely recognized for her influential work with Olga Ladyzhenskaya on linear and quasilinear equations. However, her research on degenerate and geometric equations, variational inequalities, and free boundary problems has been equally impactful.

In the early stages of her career, Uraltseva concentrated on Hilbert's 19th and 20th problems, which address the existence and regularity of solutions to variational problems. She demonstrated that minimizers of energy integrals, under natural conditions, possess $C^{2,\alpha}$ regularity, extending existing results from two dimensions to higher dimensions. Her approach, based on energy inequalities, was built on the foundational ideas of De Giorgi and Nash. Moreover, this approach covered also general divergence type uniformly elliptic equations. These results were developed and transferred to uniformly parabolic equations in her joint papers with Ladyzhenskaya. In particular, this gave a complete solution of Hilbert's 19th and 20th problems for the second order equations. Their findings were compiled in classic monographs [3] (nine years later, the second edition [5], essentially enlarged, was published) and [2].

In more recent work with Alexander Nazarov [7], she explored conditions for classical results to divergence type equations, such as the strong maximum principle and Harnack's inequality, achieving optimal outcomes under specific constraints on lower-order coefficients.

Between 1979 and 1985, Uraltseva expanded her research to uniformly elliptic quasilinear equations of nondivergence type, where coefficients and their derivatives could be unbounded. In this field, she established $C^{1,\alpha}$ a priori estimates for strong solutions to the Dirichlet problem for such equations by adapting methods developed by Ladyzhenskaya, Krylov, and Safonov, along with utilizing the Aleksandrov–Bakelman, and Krylov maximum estimates (see the survey paper [6]). Later in a series of papers she transferred these results to the oblique derivative problem.

Nina Uraltseva was also a pioneer in regularity theory for degenerate quasilinear equations. Notably, she proved $C^{1,\alpha}$ regularity for p-harmonic functions in 1968, a result that remains central to the analysis of p-Laplace type equations. Unfortunately, the paper [10] remained unknown outside of Soviet Union, and 9 years later this result was rediscovered by Karen Uhlenbeck.

In the paper [13] joint with her graduate student Anarkul Urdaletova, Uraltseva obtained the regularity results for solutions of a certain class of elliptic equations with anisotropic degeneration.

Nina's collaboration with Ladyzhenskaya on geometric equations, including the capillarity problem, led to a development of boundary estimates for minimal surface-

type equations, where the smoothness of the domain, rather than its convexity, was required.

In a series of works of the 1990s (partially joint with Vladimir Oliker; see the article [8] and references therein), Uraltseva studied the evolution of surfaces $S(t)$, which are graphs of functions defined in a bounded domain. The boundary of the surface is considered fixed, and the speed of movement depends on the average curvature of $S(t)$. To avoid geometrical assumptions on the domain, authors introduced the notion of generalized solution to corresponding nonlinear parabolic problem and proved that it converges as $t \to \infty$ to the generalized solution of corresponding elliptic problem.

Nina's contributions to variational inequalities have been particularly significant, especially in the context of problems involving convex constraints and free boundaries, such as the Signorini problem in elasticity. Together with Arina Arkhipova, she investigated elliptic and parabolic variational inequalities with unilateral and bilateral boundary constraints, known as boundary obstacle problems. This research was instrumental in establishing key regularity results for such problems, including $C^{1,\alpha}$ regularity under mild assumptions on the coefficients and obstacles. Uraltseva's methods employed De Giorgi-type energy estimates and the Signorini complementarity condition to derive geometric improvements in Dirichlet energy. These ideas were further extended to problems involving two obstacles and quasilinear elliptic systems. A partial review of results on the problem of regularity of solutions of variational inequalities is contained in the article [12].

For the last 30 years, Nina Uraltseva actively works in the field of free boundary problems. She achieved optimal regularity results for both the solutions and the free boundaries. Her research included pioneering analyzing how free boundaries approach fixed boundaries and developing tools for weakly coupled systems and two-phase problems. In particular, she demonstrated that in obstacle problems near fixed boundaries, the free boundary touches tangentially, extending C^1 regularity to the free boundary near the point of contact. For two-phase obstacle problems, she showed that free boundaries touch tangentially at branch points and are locally C^1 surfaces, using advanced geometric tools such as the balanced energy functional. In the context of weakly coupled systems, she further developed the epiperimetric inequality to establish $C^{1,\beta}$ regularity.

Some of Uraltseva's results on free boundary problems were included in the monograph [9] (jointly with Arshak Petrosyan and Henrik Shahgholian), which was highly valued by experts.

The technique developed for two-phase free boundary problems was later applied by Uraltseva and Darya Apushkinskaya in the study of models with spatially distributed hysteresis. This line of research is continued in the fresh paper [1].

Alexander von Humboldt Research Prize Award Ceremony, 2006

Scientific achievements of Nina Uraltseva are highly regarded by international mathematical community. Jointly with Olga Ladyzhenskaya, she was awarded the Chebyshev Prize of the Academy of Sciences of USSR (1966) and the USSR State Prize (1969). She was acknowledged by the titles of Honorary Scientist of the Russian Federation (2000), Honorary Professor of St. Petersburg State University (2003), and Honorary Doctor of Royal Institute of Technology, Stockholm, Sweden (2006). In 2006, she also received Alexander von Humboldt Research Prize in recognition of her academic record. For her works on free boundary problems, she was awarded the Chebyshev Prize by Government of St. Petersburg (2017). Twice, in 1970 and 1986, she was an invited speaker of the International Congresses of Mathematicians. In 2005, she was chosen as the Lecturer of the European Mathematical Society.

February 2025

Darya Apushkinskaya
Ari Laptev
Alexander Nazarov
Henrik Shahgholian

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