Preface

In this book, we study the behavior of moduli spaces of holomorphic disks with Lagrangian boundary condition under a multiple symplectic cut. From the point of view of almost complex manifolds, such a multiple symplectic cut is equivalent to multi-directional neck-stretching. The behavior of pseudoholomorphic maps under such neck-stretching was studied by Eleny Ionel in [47], and continued from the symplectic point of view in a series of papers by Brett Parker [65–73]. In parallel, a theory of logarithmic Gromov–Witten invariants have been developed by Abramovich–Chen–Gross–Siebert [2], among others. In the neck-stretching limit, limits of holomorphic maps are equipped with *tropical graphs*, meaning a graph in some affine manifold in which the edges are geodesics. The main result in the works mentioned above is an expression for counts of holomorphic curves as a sum over tropical graphs.

The purpose of this monograph is to develop a similar theory for the counts of holomorphic disks with Lagrangian boundary condition that appear in Floer theory. Because the structure constants in Floer theory are not themselves invariants, such a theory must be developed at the chain level. We prefer to work in a Morse model for Lagrangian Floer theory, which means that we develop a theory from scratch rather than depending on the previous work of, for example, Parker. We use the opportunity to give an explicit treatment. The transversality scheme using stabilizing divisors is built into the exposition. We believe that these features will make future generalizations and modifications easier to carry out.

The applications are both computational and theoretical. By way of computations, we give in the book various examples of computations of disk potentials for Lagrangians in del Pezzo manifolds and flag varieties. These computations were known before by various methods, but the approach here allows simple combinatorial calculations.

Our initial motivation for this work was a theoretical application: the *weak unobstructedness* of moment fibers of almost toric manifolds. However, this result ended up being split off into a separate paper for reasons of space. Weak unobstructedness is the condition necessary for Floer theory of a Lagrangian submanifold to be well defined. Our result is a generalization of weak unobstructedness of toric moment fibers proved by Fukaya–Oh–Ohta–Ono [37].

We do not treat in this book Lagrangians that intersect the neck regions. This generalization can be used to develop tropical disk counts for tropical Lagrangians, in the sense of Mikhalkin [61], and should be considered another possible chapter in a sequel, whether to be written by ourselves or other authors.

We thank Jonny Evans, Mohammad F. Tehrani, and Nick Sheridan for discussions and helpful emails. We thank various institutions for hosting us in the course of working on this project. We started working on this project during a visit by C. W. to

the Institute for Mathematical Sciences in Chennai. Rutgers University, New Brunswick hosted S. V. on multiple occasions, and both of us were hosted by Tata Institute of Fundamental Research in Mumbai for a visit. S. V. acknowledges support from the Matric grant MTR/2017/000920 administered by the Science and Engineering Research Board. C. W. thanks the National Science Foundation for support via grant DMS 2105417. Of course, any opinions, findings, and conclusions or recommendations expressed in this material are those of the author(s) and do not necessarily reflect the views of the National Science Foundation.

We gratefully thank the referees for their careful reading and detailed suggestions, which dramatically improved the exposition. We also thank Paul Seidel and the rest of the EMS Monographs in Mathematics editorial board, whose advice at various stages was extremely useful. Finally, we thank the editorial staff at EMS for their diligent work on the book.

We especially thank our families, Vijay and Uma and Venu and Anu, for the first author, and Neepa and Romola Maitra and Julia and Sophie Woodward, for the second author, for their continued support.