Chapter 1

Introduction

1.1 Main results

Consider the incompressible Navier–Stokes equations in the two-dimensional pipe $D = \mathbb{R} \times (-1, 1)$

$$\begin{cases} \partial_t \mathbf{v} - \epsilon \Delta \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla p & \text{in } \mathbb{R}_+ \times D, \\ \mathbf{v} = v_b \hat{i}_1 & \text{on } \mathbb{R}_+ \times \partial D, \end{cases} \tag{1.1.1}$$

where $\hat{i}_1 = (1, 0)$, $\mathbf{v} = (v_1, v_2)$ is the fluid velocity, and p is the pressure.

The parameter

$$R := \frac{1}{\epsilon} \tag{1.1.2}$$

is the Reynolds number of the flow and

$$v_b : \partial D \to \mathbb{R}$$

is the boundary velocity.

Since the flow is incompressible we must have

$$\operatorname{div} \mathbf{v} = 0.$$

We linearize (1.1.1) near the laminar flow (cf. [3])

$$\mathbf{v} = U(x_2)\hat{i}_1,$$

to obtain the linearized equation

$$\mathbf{u}_t - \mathcal{T}_0(\mathbf{u}, q) = 0,$$

where $\mathbf{u} = (u_1, u_2)$ and q are defined on $\mathbb{R}_+ \times D$, and $\mathcal{T}_0$ is the map

$$(\mathbf{u}, q) \mapsto \mathcal{T}_0(\mathbf{u}, q) := -\epsilon\, \Delta \mathbf{u} + U\, \frac{\partial \mathbf{u}}{\partial x_1} + u_2\, U'\, \hat{i}_1 - \nabla q. \tag{1.1.3}$$

We proceed with a formal derivation of the Orr–Sommerfeld equation, intentionally skipping the definitions of $\mathbf{v}$, p, $\mathbf{u}$, and q. Interested readers can read the entire derivation in [3]. The associated resolvent equation for $\mathcal{T}_0$ assumes the form

$$\mathcal{T}_0(\mathbf{u}, q) - \Lambda \mathbf{u} = \mathbf{f}, \tag{1.1.4}$$

where $\operatorname{div} \mathbf{u} = 0$ and $\Lambda \in \mathbb{C}$ is the spectral parameter.

Hence, we may define a stream function

$$\mathbf{u} = \nabla_\perp \psi = (-\psi_{x_2}, \psi_{x_1}).$$

Substituting the above into (1.1.4) and then taking the curl of the ensuing equation for ψ yields

$$\Big(-\epsilon\Delta^2 + U\frac{\partial}{\partial x_1}\Delta - U''\frac{\partial}{\partial x_1} - \Lambda\,\Delta \Big)\psi = F, \tag{1.1.5}$$

where $F = \text{curl}\, f$.

We consider $U \in C^2([-1,1])$ (we later restrict ourselves to $U \in C^4([-1,1])$) satisfying the following:

$$U(\pm 1) = 0, \tag{1.1.6a}$$

$$\max_{x\in[-1,1]} U''(x) < 0, \tag{1.1.6b}$$

$$U(-x) = U(x). \tag{1.1.6c}$$

We normalize U so that

$$U'(\pm 1) = \mp 1. \tag{1.1.6d}$$

Substituting $\psi(x_1, x_2) = \phi(x_2)\, e^{i\alpha x_1}$ into (1.1.5) yields for $\phi : (-1, 1) \to \mathbb{C}$ the equation

$$\mathcal{B}^{\mathfrak{D}}_{\lambda,\alpha,\beta}\phi = f, \tag{1.1.7a}$$

where (setting $x_2 = x$)

$$\mathcal{B}^{\mathfrak{D}}_{\lambda,\alpha,\beta} = (\mathcal{L}_\beta - \beta\lambda)\Big(\frac{d^2}{dx^2} - \alpha^2\Big) - i\beta U'', \tag{1.1.7b}$$

where

$$\mathcal{L}_\beta = -\frac{d^2}{dx^2} + i\beta U. \tag{1.1.7c}$$

In the above

$$\beta = \alpha\epsilon^{-1} = \alpha R \tag{1.1.8}$$

(R being the Reynolds number introduced in (1.1.2)), and, for $\beta \neq 0$,

$$\lambda = \beta^{-1}\Big(\frac{\Lambda}{\epsilon} - \alpha^2\Big). \tag{1.1.9}$$

We refer to [3, Section 3] for the details of the derivation. We use the pair of parameters (α, β) instead of (α, R) since the asymptotic limit we consider in the sequel is $\beta \to \infty$.

We define $\mathcal{B}^{\mathfrak{D}}_{\lambda,\alpha,\beta}$ on

$$D(\mathcal{B}^{\mathfrak{D}}_{\lambda,\alpha,\beta}) = \big\{u \in H^4(-1,1), u(1) = u'(1) = u(-1) = u'(-1) = 0\big\}. \tag{1.1.10}$$

We focus our interest on the restriction $\mathcal{B}^{\mathfrak{D},\text{sym}}_{\lambda,\alpha,\beta}$ of the operator $\mathcal{B}^{\mathfrak{D}}_{\lambda,\alpha,\beta}$ to functions that are symmetric with respect to the reflection $x \mapsto -x$. Hence, we are led to consider the equivalent restricted operator $\mathcal{B}^{\mathfrak{N},\mathfrak{D}}_{\lambda,\alpha,\beta}$ on $(0,1)$ whose domain is

$$D(\mathcal{B}^{\mathfrak{N},\mathfrak{D}}_{\lambda,\alpha,\beta}) = \{u \in H^4(0,1), u'(0) = u^{(3)}(0) = 0 \text{ and } u(1) = u'(1) = 0\}. \quad (1.1.11)$$

We leave the discussion of anti-symmetric modes to future research. Note that since

$$\mathcal{B}^{U}_{\lambda,\alpha,\beta} = \overline{\mathcal{B}^{-U}_{\bar{\lambda},\alpha,\beta}}$$

(where $\bar{\mathcal{B}}$ denotes the complex conjugate of $\mathcal{B}$) the analysis in the sequel applies to the case where $\min_{x\in[-1,1]} U''(x) > 0$ as well. Clearly, the Poiseuille flow associated with $U(x) = (1-x^2)/2$ meets all the criteria in (1.1.6). Another example mentioned in [13] is given by $U(x) = (2/\pi)\cos \pi x/2$. Note that U is decreasing on $(0,1)$.

In [13, Theorem 1.1], it has been established by Grenier–Guo–Nguyen that for sufficiently large R and for each α satisfying

$$C_L\, R^{-1/7} \le \alpha \le C_R\, R^{-1/11},$$

or equivalently when β is large and

$$C_L^{7/6}\, \beta^{-1/6} \le \alpha \le C_R^{11/10}\, \beta^{-1/10},$$

there exists $\lambda \in \mathbb{C}$ with negative real part such that $\mathcal{B}^{\mathfrak{N},\mathfrak{D}}_{\lambda,\alpha,\beta}$ is not invertible. For the case of a Poiseuille flow, the positive constants C_L and C_R have been determined from well-known formal asymptotic calculations (cf. the book [12] by P. G. Drazin and W. H. Reid).

In the present contribution, we consider the converse problem, i.e., we attempt to show that, for any $\delta > 0$, $(\mathcal{B}^{\mathfrak{D},\text{sym}}_{\lambda,\alpha,\beta})^{-1}$ is bounded for $\Re\lambda \le 0$ when

$$\alpha \ge \beta^{-1/10+\delta} \quad \text{or} \quad 0 \le \alpha \le \alpha_L\, \beta^{-1/6}.$$

Note that unlike [13] we do not provide the precise estimate derived by formal asymptotics, i.e., $\alpha_L < C_L$ and $\beta^{-1/10+\delta} > \beta^{1/10}$. The determination of the precise curves is left to future research.

Recall from equation (1.1.8) that $\beta = \alpha/\epsilon$. Our main results are the following two theorems.

Theorem 1.1.1. *Let $U \in C^4([-1,1])$ satisfy (1.1.6). Then there exist positive α_L, C, Υ, and $\beta_0 > 1$ such that for all $\beta > \beta_0$ it holds that*

$$\sup_{\substack{0\le\alpha\le\alpha_L\beta^{-1/6} \\ \Re\lambda<\Upsilon\beta^{-1/2}}} \left\|(\mathcal{B}^{\mathfrak{D},\text{sym}}_{\lambda,\alpha,\beta})^{-1}\right\| + \left\|\frac{d}{dx}(\mathcal{B}^{\mathfrak{D},\text{sym}}_{\lambda,\alpha,\beta})^{-1}\right\| \le C\,\beta^{-1/2}\log\beta. \quad (1.1.12)$$

The condition $\alpha \in [0, \alpha_L \beta^{-1/6}]$ can be rephrased in terms of the pair (α, R) as $\alpha \in [0, \alpha_L^{6/7} R^{-1/7}]$.

Theorem 1.1.2. *Let $U \in C^4([-1,1])$ satisfy (1.1.6). Let further $\hat{\mu}_m > 0$ be given by* [3, equation (6.57)]. *Then for any $\delta > 0$ and any $\hat{\Upsilon} > 0$, there exist positive C, Υ, and β_0 such that for all $\beta > \beta_0$ we have*

$$\sup_{\substack{\beta^{-1/10+\delta} \leq \alpha \\ \Re\lambda \leq \min(\Upsilon \beta^{-1/2}, \beta^{-1/3}[\hat{\mu}_m - \hat{\Upsilon} - \alpha^2 \beta^{-2/3}/2])}} \left\| (\mathcal{B}_{\lambda,\alpha,\beta}^{\mathfrak{D},\mathrm{sym}})^{-1} \right\| + \left\| \frac{d}{dx} (\mathcal{B}_{\lambda,\alpha,\beta}^{\mathfrak{D},\mathrm{sym}})^{-1} \right\| \leq C \beta^{-1/2+\delta}. \tag{1.1.13}$$

The condition $\alpha \geq \beta^{-1/10+\delta}$ can be rephrased in terms of the pair (α, R) as $\alpha \geq R^{(-1+10\delta)/(11-10\delta)}$. Note that the condition $\Re\lambda \leq \beta^{-1/3}([\hat{\mu}_m - \hat{\Upsilon} - \alpha^2 \beta^{-2/3}/2])$ guarantees, by (1.1.9) that $\mathcal{B}_{\lambda,\alpha,\beta}^{\mathfrak{D},\mathrm{sym}}$ is invertible for $\Lambda \leq \alpha\beta^{-1/3}[\hat{\mu}_m + \alpha^2\beta^{-2/3}/2 - \hat{\Upsilon}]$ and hence the stability of the laminar flow even for $\alpha \gtrsim \beta^{1/3}$.

In the above

$$\left\| (\mathcal{B}_{\lambda,\alpha,\beta}^{\mathfrak{D},\mathrm{sym}})^{-1} \right\| = \sup_{\substack{f \in L^2(0,1) \\ \|f\|_2 = 1}} \| (\mathcal{B}_{\lambda,\alpha,\beta}^{\mathfrak{D},\mathrm{sym}})^{-1} f \|_2$$

and

$$\left\| \frac{d}{dx} (\mathcal{B}_{\lambda,\alpha,\beta}^{\mathfrak{D},\mathrm{sym}})^{-1} \right\| = \sup_{\substack{f \in L^2(0,1) \\ \|f\|_2 = 1}} \left\| \frac{d}{dx} (\mathcal{B}_{\lambda,\alpha,\beta}^{\mathfrak{D},\mathrm{sym}})^{-1} f \right\|_2,$$

where $\| \cdot \|_2$ denotes the standard $L^2(0,1)$ norm.

In recent years, hydrodynamic stability of shear flows has attracted significant attention. For the case of a Couette flow we mention only a partial list of rigorous analytical results [4–6, 8]. In [3], we have established similar estimates for the Orr–Sommerfeld operator, together with semigroup estimates for the linearized Navier–Stokes operator in the case where $|U'| > 0$ in $[-1, 1]$ (see also the works of Chen, Wei, and Zhang [9] and of Jia [17] for recent generalizations). In contrast with the present case the Orr–Sommerfeld operator has, when $|U''| > 0$, a bounded resolvent in the half-plane $\Re\lambda \leq 0$.

The hydrodynamic stability of symmetric flows in a channel has been considered extensively in physics (cf. for instance [12,19–21]). These works, just like that of [13], all attempt to determine as function of β the region in the $(\alpha, \Im\lambda)$ plane where the Orr–Sommerfeld is unstable. In a recent work [11], the stability of Poiseuille flow has been established in the case of a Navier-slip boundary condition. This means that the boundary condition $u'(\pm 1) = 0$ in (1.1.11) is replaced by $u''(\pm 1) = 0$. The stability of a pipe Poiseuille flow has also been addressed in [10].

1.2 Proof strategy

In the following informal discussion, we present the main ingredients of the proofs of Theorems 1.1.1 and 1.1.2. Some of the definitions appearing in the discussion will remain slightly vague and will be reformulated more precisely in the next sections. The reader may be interested in reviewing the relevant part of this presentation before diving into the technical details of any part of the analysis in the sequel.

We begin with a rather heuristic discussion. Consider the equation

$$\mathcal{B}_{\lambda,\alpha,\beta}\phi = f, \tag{1.2.1}$$

where $(\phi, f) \in D(\mathcal{B}^{\mathfrak{N},\mathfrak{D}}_{\lambda,\alpha,\beta}) \times L^2(0,1)$ and $\mathcal{B}_{\lambda,\alpha,\beta} = \mathcal{B}^{\mathfrak{N},\mathfrak{D}}_{\lambda,\alpha,\beta}$.

Our goal is to estimate the operator $\mathcal{B}^{-1}_{\lambda,\alpha,\beta}$ under specific conditions on the parameters $(\lambda, \alpha, \beta) \in \mathbb{C} \times \mathbb{R}_+ \times \mathbb{R}_+$. We may rewrite the above equation in the form

$$\mathcal{A}_{\lambda,\alpha}\phi = v, \tag{1.2.2a}$$

where

$$v = \beta^{-1}[f + \phi^{(4)} - \alpha^2\phi''] \tag{1.2.2b}$$

and

$$\mathcal{A}_{\lambda,\alpha} = (U + i\lambda)\Big(-\frac{d^2}{dx^2} + \alpha^2\Big) + U'' \tag{1.2.2c}$$

is the inviscid (Rayleigh) operator whose study will be the main object of Section 1.2. We define $\mathcal{A}_{\lambda,\alpha}$ for $\Re\lambda \neq 0$ or when $\Re\lambda = 0$ and $\Im\lambda \notin [0, U(0)]$, on

$$D(\mathcal{A}^{\mathfrak{N},\mathfrak{D}}_{\lambda,\alpha}) = \big\{\phi \in H^2(0,1) \mid \phi'(0) = 0 \text{ and } \phi(1) = 0\big\}. \tag{1.2.2d}$$

It intuitively appears that v should tend to 0 as $\beta \to \infty$, and thus, we adopt the following proof strategy.

(1) We prove that v becomes small as $\beta \to \infty$.

(2) We obtain a bound for $\|\mathcal{A}^{-1}_{\lambda,\alpha}v\|_{1,2}$ (where $\|\cdot\|_{1,2}$ denotes the standard $H^1(0,1)$ norm).

After successfully completing the above stages we expect to obtain an inequality of the form

$$\|\phi'\|_2 \leq \delta_1(\beta)\|f\|_2 + \delta_2(\beta)\|\phi'\|_2.$$

If for sufficiently large β it holds that $\delta_2(\beta) < 1/2$, we can conclude from here an estimate for $\|\mathcal{B}^{-1}_{\lambda,\alpha,\beta}\| + \|\frac{d}{dx}\mathcal{B}^{-1}_{\lambda,\alpha,\beta}\|$.

Estimation of $\mathcal{A}_{\lambda,\alpha}^{-1}$. We use in Chapter 2 a similar procedure to the one used in [3]. Let $\lambda = \mu + i\nu$. Given that $|U''| > 0$ in $[0, 1]$ and since

$$\Im\Big\langle \phi, \frac{\mathcal{A}_{\lambda,\alpha}\phi}{U + i\lambda} \Big\rangle = -\mu \Big\| \frac{|U''|^{1/2}\phi}{U + i\lambda} \Big\|_2^2, \tag{1.2.3}$$

we easily obtain that

$$\Big\| \frac{\phi}{U + i\lambda} \Big\|_2 \leq \frac{C}{|\mu|} \|v\|_2. \tag{1.2.4}$$

From the above (accompanied by a rather straightforward integration by parts) it is not difficult to show that

$$\|\mathcal{A}_{\lambda,\alpha}^{-1}\| + \Big\| \frac{d}{dx} \mathcal{A}_{\lambda,\alpha}^{-1} \Big\| \leq \frac{C}{|\mu|}.$$

The above estimate is unsatisfactory in the limit $\mu \to 0$, and hence finer estimates need to be established. We use the fact that $\mathcal{A}_{i\nu,\alpha}$ is self-adjoint. Thus, for $\nu \notin (0, U(0)]$ (see Sections 2.7 and 2.10) we may write

$$\Big\langle \frac{\phi}{U - \nu}, \mathcal{A}_{i\nu,\alpha}\phi \Big\rangle = \Big\| (U - \nu)\Big(\frac{\phi}{U - \nu}\Big)' \Big\|_2^2 + \alpha^2 \|\phi\|_2^2,$$

and obtain from it an estimate for $\|\phi'\|_2$ in the case where $|\mu|$ is small.

For $\nu \in (0, U(0)]$, we have to address the singularity where $U = \nu$. Given the fact that U is increasing in $[0, 1]$ there exists a unique $x_\nu \in [0, 1)$ where $U(x_\nu) = \nu$. Let $\chi \in C^\infty([0, 1], [0, 1])$ denote a cutoff function supported on $[0, (1 + x_\nu/2)]$. Setting $\varphi = \phi - \phi(x_\nu)\chi$ we may write

$$\begin{aligned} &\Big\| (U - \nu)\Big(\frac{\varphi}{U - \nu}\Big)' \Big\|_2^2 + \alpha^2 \|\phi\|_2^2 \\ &\quad = \Big\langle \frac{\varphi}{U - \nu}, \mathcal{A}_{i\nu,\alpha}\varphi \Big\rangle - \phi(x_\nu)\Big[\Big\langle \frac{\varphi}{U - \nu}, U''\chi \Big\rangle - \langle \varphi, \chi'' - \alpha^2\chi \rangle\Big]. \end{aligned} \tag{1.2.5}$$

For the above balance to become useful for the purpose of obtaining estimates for $\|\phi'\|_2$, we need to obtain an estimate for $\phi(x_\nu)$. To this end we use (1.2.3) to obtain (for $\mu \neq 0$)

$$\mu|\phi(x_\nu)|^2 \Big\| \frac{1}{U + i\lambda} \|_2^2 \leq \|\phi\|_\infty \Big\| \frac{v}{U + i\lambda} \Big\|_1 + C|\mu| \Big\| \frac{\phi - \phi(x_\nu)}{U - \nu} \Big\|_2^2.$$

Under the condition in [3] on U, which is assumed to be strictly monotone, the above estimates leads to

$$|\phi(x_\nu)| \leq \|\phi\|_\infty \Big\| \frac{v}{U + i\lambda} \Big\|_1 + C|\mu| \, \|\phi'\|_2.$$

Substituting the above into (1.2.5) (properly amended to account for small values of $|\mu|$) yields an estimate for $\|\phi'\|_2$.

To adapt the above method to the present context we need to overcome several difficulties.

(1) It holds that $\mathcal{A}_{0,0}U = 0$ and since $U \in D(\mathcal{A}_{0,0})$, $(\mathcal{A}_{\lambda,\alpha})^{-1}$ becomes strongly singular in the limit $(\lambda, \alpha) \to (0, 0)$.

(2) The boundary condition at $x = 0$ is a Neumann condition in contrast with the Dirichlet condition in [3]. Thus, we have to write (1.2.5) separately on the intervals $(0, x_\nu)$ and $(x_\nu, 1)$. On $(x_\nu, 1)$ we may use the same method used in [3]. However, on $(0, x_\nu)$, considering $\phi/(U - \nu)$, we obtain bounds of this quotient for small values of α that are significantly greater than those obtained for larger values of α.

(3) The quadratic behavior of $U - U(0)$ as opposed to the linear behavior considered in [3].

The first pair of difficulties is addressed by the same techniques.

- For small values of α we use the fact that we can consider $(\mathcal{A}_{0,\lambda})^{-1}$ as an integral operator to obtain satisfactory estimates for its norm (see Proposition 2.4.1).
- For larger values of α we use again (1.2.5) (see Section 2.5).

In Section 2.6, we present a different analysis, which is valid for any $\alpha \geq 0$ with stronger singularity in the limit $\lambda \to 0$. In all cases, we use the orthogonal decomposition $\phi = C_\parallel(U - \nu) + \phi_\perp$ to obtain separate estimates for $C_\parallel$ and $\phi_\perp$, estimates of the latter being significantly smaller than the former. To overcome the last difficulty we simply use (1.2.4) in Section 2.9. In Section 2.8, we consider the transition between the quadratic behavior of U near x_ν and the linear behavior considered in Section 2.6.

Estimate of $\mathcal{B}_{\lambda,\alpha,\beta}^{-1}$. To obtain an estimate of v (see (1.2.2b)) we set

$$v_{\mathfrak{D}} := \mathcal{A}_{\lambda,\alpha}\phi + (U + i\lambda)\phi''(1)\hat{\psi}, \tag{1.2.6}$$

where

$$\hat{\psi}(x) = \frac{\mathrm{Ai}\big(\beta^{1/3}e^{-i\pi/6}[(1-x) - i\lambda]\big)}{\mathrm{Ai}\big(e^{-i2\pi/3}\beta^{1/3}\lambda\big)}\eta(x).$$

Here, Ai is the Airy function and $\eta \in C^\infty([0,1],[0,1])$ is supported on $(1/4, 1]$, and satisfies $\eta \equiv 1$ on $[1/2, 1]$. Note that $v_{\mathfrak{D}}(1) = v'_{\mathfrak{D}}(0) = 0$ and that $\hat{\psi}$ is a good approximation for the $L^2(-\infty, 1)$ solution of

$$\begin{cases} \big(-\frac{d^2}{dx^2} + i\beta[(1-x) + i\lambda]\big)u = 0 & \text{in } (-\infty, 1), \\ u(1) = 1. \end{cases} \tag{1.2.7}$$

We can now rewrite (1.2.1) in the form

$$(\mathcal{L}_\beta - \beta\lambda)v_{\mathfrak{D}} = g_{\mathfrak{D}},$$

where

$$\mathcal{L}_\beta = -\frac{d^2}{dx^2} + i\beta U$$

is defined on

$$D(\mathcal{L}_\beta) = \{u \in H^2(0,1) \mid u(1) = u'(0) = 0\}.$$

While the precise form of $g_{\mathfrak{D}}$ need not concern us in this brief summary of the proof we still need to obtain an estimate of its $L^2(0,1)$ norm. Thus, we get an estimate of v by working through the following steps.

(1) Estimate of $\phi''(1)$.

(2) Estimate of $g_{\mathfrak{D}}$.

(3) Estimate of $v_{\mathfrak{D}}$.

(4) Estimate of $\mathcal{A}_{\lambda,\alpha}^{-1} v_{\mathfrak{D}}$ and of $\phi''(1)\mathcal{A}_{\lambda,\alpha}^{-1}(U + i\lambda)\hat{\psi}$.

We use two different methods for the estimation of $\phi''(1)$.

For α values that are not too small. We rewrite (1.2.1) in the form

$$\Big(-\frac{d^2}{dx^2} + i\beta[U + i\lambda]\Big)(\phi'' - \alpha^2\phi) = i\beta U''\phi + f.$$

Given $\phi(1) = \phi'(1) = \phi'(0) = 0$, we may conclude that for $\mathfrak{z}(x) = \cosh(\alpha x)/\cosh\alpha$ it holds that

$$\langle \mathfrak{z}, \phi'' - \alpha^2\phi \rangle = 0. \tag{1.2.8a}$$

Consequently, we define the Schrödinger operator $\mathcal{L}_\beta^{\mathfrak{z}}$ with the same differential operator as for $\mathcal{L}_\beta$ but with the following domain

$$D(\mathcal{L}_\beta^{\mathfrak{z}}) = \big\{u \in H^2(0,1) \mid u'(0) = 0, \langle \mathfrak{z}, u\rangle = 0\big\}. \tag{1.2.8b}$$

Let $(v, g) \in D(\mathcal{L}_\beta^{\mathfrak{z}}) \times L^2(0,1)$ satisfy

$$(\mathcal{L}_\beta^{\mathfrak{z}} - \beta\lambda)v = g. \tag{1.2.9}$$

In Chapter 4, we obtain estimates for $v(1)$ that we later use in Chapter 5 (except for Sections 5.8 and 5.7) to obtain an estimate for $\phi''(1)$. Again, we have to distinguish between the quadratic case (Section 4.3) and the linear case (Section 4.2).

For smaller values of α and $|\lambda|$. The estimate of $\phi''(1)$, obtained by the above technique becomes deficient, given the singularity of $\mathcal{A}_{0,0}$. We thus integrate (1.2.1) for $\alpha = 0$ to obtain

$$\phi^{(3)}(1) = -\int_0^1 f(x)\,dx. \tag{1.2.10}$$

Then we use the identity

$$\|(U'')^{-1/2}\phi^{(3)}\|_2^2 = -\Re\langle (U'')^{-1}\phi'', \mathcal{B}_{\lambda,0,\beta}\phi\rangle - \frac{1}{U''(1)}\Re(\bar{\phi}''(1)\phi^{(3)}(1))$$
$$- \Re\langle [(U'')^{-1}]'\phi'', \phi^{(3)}\rangle + \mu\beta\|(U'')^{-1/2}\phi''\|_2^2$$

to obtain a proper bound for $\|\phi^{(3)}\|_2$, which together with Sobolev embedding (skipping, of course, some of the details) leads to a satisfactory bound for $|\phi''(1)|$. Note that the effectiveness of this technique is lost when α is not small since (1.2.10) is no longer valid. We use it only in Section 5.8.

Finally, we note that for $\alpha \gtrsim \beta^{-1/3}$ $\mathfrak{z}$ undergoes significant changes through an $\mathcal{O}(\beta^{-1/3})$ boundary layer near $x = 1$. Hence, we can no longer make any good use of (1.2.7) as an estimate for the behavior near $x = 1$ of the solution of (1.2.9). Instead, we need to use the same method developed in [3] for this case. Note that, since $\mathfrak{z}$ is localized near $x = 1$ for large values of α, the effect of the different boundary conditions at $x = 0$ here and in [3] is exponentially small. Resolvent estimates for $\mathcal{L}_\beta^{\mathfrak{z}}$ in this case are brought in Section 4.7 whereas estimates for the inverse of $\mathcal{B}_{\lambda,\alpha,\beta}$ are given in Section 5.7.

Remark. We note an error in the derivation of [3, equation (8.96)] where the term $\Theta_+'\psi_+'/\psi_+(1)$ (Θ_+ is analogous to χ in the present contribution) was overlooked. This error does not affect at all the validity of [3, equation (8.96)] given that the error generated by the missing term can be estimated using (4.2.17), and is much smaller than the right-hand side of [3, equation (8.96)] (which is greater or equal than $C\beta^{-1/4}$ for some positive C).

We skip the rather technical stage of estimating $g_{\mathfrak{D}}$. Once it is done, we need to estimate $(\mathcal{L}_\beta - \beta\lambda)^{-1}$ in $\mathcal{L}(L^2, L^p)$ for $1 \le p \le \infty$ in order to obtain an appropriate estimate for $v_{\mathfrak{D}}$. These estimates are obtained in Chapter 3 for various ranges of λ values. Again we need to distinguish between the linear behavior of $U - U(x_\nu)$ near $x = x_\nu$ (Section 3.1) and the quadratic behavior near $x = 0$ for $\nu = U(0)$ (Section 3.2). Special attention is also devoted to $\mathcal{L}(L^2, L^1)$ and $\mathcal{L}(H^1, L^1)$ estimates (Section 3.3).

Next, we estimate $\mathcal{A}_{\lambda,\alpha}^{-1} v_{\mathfrak{D}}$ using the aforementioned techniques of Chapter 2. For $\mathcal{A}_{\lambda,\alpha}^{-1}(U + i\lambda)\hat{\psi}$ we use the exponential decay of $\hat{\psi}$ away from $x = 1$ to obtain the desired estimate in a rather straightforward manner, except in the case $(\lambda, \alpha) \to (0, 0)$. These estimates are addressed in Section 5.2.

Finally, in Chapter 6 we summarize the results of Chapter 5 and prove the main theorems.