Chapter 2

The inviscid operator

2.1 Preliminaries

We begin by presenting the notation, frequently used in the sequel

$$\langle f, g\rangle_{L^2(a,b)} = \int_a^b \bar{f}(x)g(x)\,dx.$$

Note that when $(a, b) = (0, 1)$ the abbreviated notation $\langle f, g\rangle$ is used instead of $\langle f, g\rangle_{L^2(0,1)}$.

Next, recall that the differential expression of the inviscid operator (also called the Rayleigh operator) is given by

$$\mathcal{A}_{\lambda,\alpha} \stackrel{\text{def}}{=} (U + i\lambda)\Big(-\frac{d^2}{dx^2} + \alpha^2\Big) + U'', \tag{2.1.1}$$

where $\lambda \in \mathbb{C}$ and $\alpha \in \mathbb{R}$.

Note that, for any $\phi \in D(\mathcal{B}^{\mathfrak{D}}_{\lambda,\alpha,\beta})$, we have

$$\mathcal{A}_{\lambda,\alpha}\phi = \lim_{\beta\to\infty} \beta^{-1}\mathcal{B}^{\mathfrak{D}}_{\lambda,\alpha,\beta}\phi.$$

We consider here only spaces of even functions in $(-1, 1)$, hence, as explained in the introduction, we restrict the operator to $(0, 1)$ and consider the Neumann condition at 0 and the Dirichlet condition at 1. We thus define $\mathcal{A}^{\mathfrak{N},\mathfrak{D}}_{\lambda,\alpha}$ as follows:

- for $\Re\lambda \neq 0$ or when $\Re\lambda = 0$ and $\Im\lambda \notin [0, U(0)]$, on

$$D(\mathcal{A}^{\mathfrak{N},\mathfrak{D}}_{\lambda,\alpha}) = \big\{\phi \in H^2(0, 1) \mid \phi'(0) = 0 \text{ and } \phi(1) = 0\big\}, \tag{2.1.2a}$$

- for $\Re\lambda = 0$ and $\Im\lambda \in [0, U(0)]$

$$D(\mathcal{A}^{\mathfrak{N},\mathfrak{D}}_{0,\alpha}) = \big\{\phi \in H^2((0, 1), (U - \Im\lambda)^2 dx) \mid \phi'(0) = 0 \text{ and } \phi(1) = 0\big\}, \tag{2.1.2b}$$

  which is equipped with the norm

$$\|u\|_{D(\mathcal{A}^{\mathfrak{N},\mathfrak{D}}_{0,\alpha})} = \int_0^1 [|u''|^2] + [|u'|^2 + |u|^2](U - \Im\lambda)^2\,dx.$$

Hence, in the following, we restrict attention to the interval [0, 1] assuming in this section that $U \in C^3([0,1])$ and satisfies the condition:

$$U(1) = 0, \tag{2.1.3a}$$

$$\max_{x\in[0,1]} U''(x) < 0, \tag{2.1.3b}$$

$$U'(0) = 0. \tag{2.1.3c}$$

We normalize U so that

$$U'(1) = -1. \tag{2.1.3d}$$

For convenience of notation we omit the superscripts $\mathfrak{N}$ and $\mathfrak{D}$ in the sequel whenever there is not any fear of ambiguity. Since

$$\mathcal{A}^U_{\lambda,\alpha} = \overline{\mathcal{A}^{-U}_{\bar{\lambda},\alpha}},$$

the analysis in this section applies to the case where $\min_{x\in[-1,1]} U''(x) > 0$ replaces equation (2.1.3b) as well.

In this section, we obtain a variety of estimates for $\mathcal{A}^{-1}_{\lambda,\alpha}$ that are necessary in order to obtain bounds in the same parameter regime for $\mathcal{B}^{-1}_{\lambda,\alpha}$. We note that since

$$\mathcal{A}_{\lambda,\alpha} = \overline{\mathcal{A}_{-\bar{\lambda},\alpha}}$$

the results in this section do not depend on the sign of $\Re\lambda$.

Let $\lambda = \mu + i\nu$. We begin by summarizing the results of this section in Figure 2.1. We map in this figure the regions in the $(|\mu|, \nu)$ plane where the various estimates of $\mathcal{A}^{-1}_{\lambda,\alpha}$ can be found. We refer the reader to Section 1.2 where the consideration that led us to split the λ plane to these regimes are briefly explained. We note that the results of Sections 2.6–2.11 are valid for any $\alpha \geq 0$. Section 2.3 addresses the case $\alpha = \lambda = 0$, Section 2.4 addresses small α values, and Section 2.5 relatively large values of α. The constants determining the boundaries of the various regimes satisfy $0 < \nu_1 < U(0)$, $\nu_0 < 0$, and κ_0 must be sufficiently large while μ_0 and λ_0 must be sufficiently small.

2.2 Some more preliminaries

We begin by defining the following (formally) self-adjoint unbounded operator on

$$H^0_U(0,1) := L^2((0,1), U^2 dx)$$

by

$$\mathcal{M}_U = -U^{-2}\frac{d}{dx}U^2\frac{d}{dx}, \tag{2.2.1a}$$

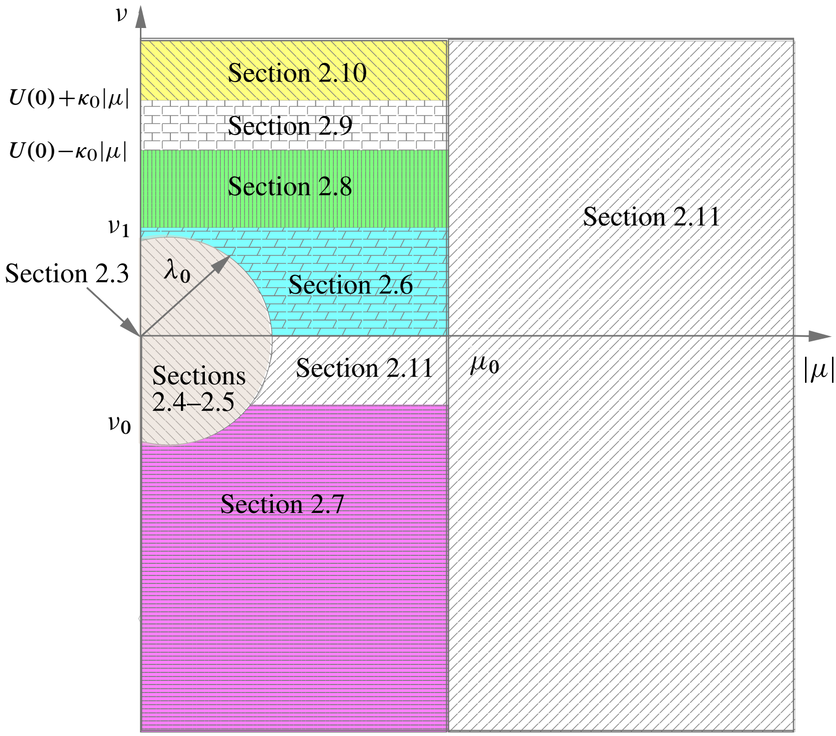

Figure 2.1. Summary of the results in Chapter 2. For each zone in $\mathbb{C}$ for the parameter λ, we indicate the section where the basic inequality is proved. Recall that $\mu = \Re\lambda$, and $\nu = \Im\lambda$, μ_0, ν_0, ν_1, λ_0, κ_0 are positive constants which are introduced in the various sections.

which is naturally associated via the Lax–Migram lemma with the quadratic form $\mathcal{Q}_U$, defined on

$$H^1_U(0,1) = \big\{u \in L^2_{\mathrm{loc}}(0,1) \mid Uu \in L^2(0,1) \text{ and } Uu' \in L^2(0,1)\big\} \tag{2.2.1b}$$

by

$$\mathcal{Q}_U(w) = \|Uw'\|_2^2. \tag{2.2.1c}$$

Recall that U satisfies (2.1.3).

Remark 2.2.1. When U is replaced by $e^{-\phi}$, we arrive at a well-known problem considered in Statistical Mechanics and Morse theory (Witten Laplacians), see, for example, [14] and references therein. As observed in [7], we arrive at a singular case of this theory because U is vanishing at $x = 1$.

We can now state the following lemma.

Lemma 2.2.2. *Suppose that $\mathcal{M}_U$ is a self-adjoint operator with compact resolvent on $L^2((0,1), U^2\,dx)$. If $\{\kappa_n\}_{n=1}^{\infty}$ denotes the non-decreasing sequence of which the* *spectrum $\sigma(\mathcal{M}_U)$ is consisted, then*

$$\kappa_1 = 0$$

and

$$\kappa_2 \geq \lambda_2^N |U(0)|^2, \tag{2.2.2}$$

where λ_2^N denotes the second eigenvalue of the radially symmetric Neumann–Laplacian (i.e., the Laplacian reduced to the radially symmetric functions satisfying the Neumann condition.) in the unit ball in $\mathbb{R}^3$.

Proof. The proof that $\kappa_1 = 0$ is trivial (the associated eigenfunction is the non-vanishing constant function). To prove the lower bound for κ_2 we first observe that, by the variational max-min characterization of the second eigenvalue,

$$\kappa_2 = \sup_{\psi \in H_U^1(0,1)} \inf_{\substack{w \in H_U^1(0,1) \\ \langle w, U^2\psi\rangle = 0}} \frac{\|Uw'\|_2^2}{\|Uw\|_2^2} = \sup_{\phi \in H_U^1(0,1)} \inf_{\substack{w \in H_U^1(0,1) \\ \langle w, (1-x)^2\phi\rangle = 0}} \frac{\|Uw'\|_2^2}{\|Uw\|_2^2}. \tag{2.2.3}$$

The second equality can be proved by writing $\psi = \frac{(1-x)^2}{U^2}\phi$. Then, we use the fact that by the concavity of U

$$U(0)(1-x) \leq U(x) \leq (1-x) \quad \text{(see (2.1.3b))} \tag{2.2.4}$$

to obtain

$$|U(0)|^{-2} \frac{\|(1-x)w'\|_2^2}{\|(1-x)w\|_2^2} \geq \frac{\|Uw'\|_2^2}{\|Uw\|_2^2} \geq |U(0)|^2 \frac{\|(1-x)w'\|_2^2}{\|(1-x)w\|_2^2}. \tag{2.2.5}$$

Let $U_0(x) = 1 - x$ and κ_n^0 denote the nth eigenvalue of $\mathcal{M}_{U_0}$. By (2.2.5) and [15, Theorem 11.12] we have that

$$|U(0)|^2 \kappa_2^0 \leq \kappa_2 \leq |U(0)|^{-2} \kappa_2^0. \tag{2.2.6}$$

Setting $\rho(x) = 1 - x$ we obtain that

$$\mathcal{M}_{U_0} = -\rho^{-2} \frac{d}{d\rho} \rho^2 \frac{d}{d\rho},$$

which is defined on (the Neumann condition is the natural boundary condition associated with Q_{U_0})

$$D(\mathcal{M}_{U_0}) = \{u \in H^2([0,1]; \rho^2 d\rho) \mid u'(1) = 0\}.$$

Hence, $\mathcal{M}_{U_0}$ is the radially symmetric Neumann Laplacian, and we may conclude that

$$\kappa_2^0 = \lambda_2^N.$$

The above, together with (2.2.6) yields (2.2.2). ∎

We next recall Hardy's inequality on finite intervals (see, for example, [18, equation (1.25)]).

Lemma 2.2.3. *Let $w \in H^1(a,b)$ satisfy $w(b) = 0$. Then, we have*

$$\|([x-a]w)'\|_2^2 = \|[x-a]w'\|_2^2 \geq \frac{1}{4}\|w\|_2^2. \tag{2.2.7}$$

Proof. Let $\tilde{w} \in H^1(a,\infty)$ be given by

$$\tilde{w}(x) = \begin{cases} w(x), & x \in (a,b), \\ 0, & x \geq b. \end{cases}$$

Then Hardy's inequality on $\mathbb{R}_+$ applied to $\tilde{w}$ implies that

$$\|([x-a]w)'\|_{L^2(a,b)}^2 = \|([x-a]\tilde{w})'\|_{L^2(a,\infty)}^2 \geq \frac{1}{4}\|\tilde{w}\|_{L^2(a,\infty)}^2 = \frac{1}{4}\|w\|_{L^2(a,b)}^2.$$

To complete the proof we first write

$$\|([x-a]w)'\|_2^2 = \|[x-a]w'\|_2^2 + \|w\|_2^2 + 2\Re\langle w, [x-a]w'\rangle.$$

An integration by parts then yields

$$2\Re\langle w, [x-a]w'\rangle = \Re\langle [x-a], (|w|^2)'\rangle = -\|w\|_2^2.$$

Hence,

$$\|([x-a]w)'\|_2^2 = \|[x-a]w'\|_2^2,$$

which completes the proof of the lemma. ∎

If we drop the requirement that $w(b) = 0$ we can state the following lemma.

Lemma 2.2.4. *Let $w \in H^1(a,b)$. Then, we have*

$$\|([x-a]w)'\|_2^2 \geq \frac{1}{4}\|w\|_2^2. \tag{2.2.8}$$

Proof. We use Hardy's inequality in $\mathbb{R}_+$ for the extension

$$\tilde{w}(x) = \begin{cases} w(x), & x \in (a,b), \\ w(b)\frac{b-a}{x-a}, & x \geq b. \end{cases}$$

∎

2.3 Estimates in the case $\alpha = \lambda = 0$

We begin with the simplest possible case, for which $\alpha = \lambda = 0$. We recall that

$$\mathcal{A}_{0,0} := -U\frac{d^2}{dx^2} + U''$$

is defined on

$$D(\mathcal{A}_{0,0}) = \{u \in H^2((0,1), U^2 dx) \mid u(1) = u'(0) = 0\},$$

corresponding to a Dirichlet–Neumann problem on (0, 1).

Next, let $W^{1,p}(0,1)$ denote the normed space

$$W^{1,p}(0,1) := \{u \in L^p(0,1) \mid u' \in L^p(0,1)\},$$

with its natural norm denoted by $\|\cdot\|_{1,p}$.

Observing that U belongs to the kernel of $\mathcal{A}_{0,0}$, we set

$$\phi = c_\parallel U + \phi_\perp, \tag{2.3.1a}$$

where

$$c_\parallel = \frac{\langle U, \phi\rangle}{\|U\|_2^2}, \tag{2.3.1b}$$

in which $\langle \cdot, \cdot \rangle$ denotes the natural $L^2(0,1)$ inner product.

Lemma 2.3.1. *Let $U \in C^2([0,1])$ satisfy (2.1.3). There exists $C > 0$ such that for any $(\phi, v) \in D(\mathcal{A}_{0,0}) \times W^{1,2}(0,1)$ satisfying*

$$\mathcal{A}_{0,0}\,\phi = v, \tag{2.3.2a}$$

$$\langle 1, v\rangle = 0 \quad \text{and} \quad v(1) = 0 \tag{2.3.2b}$$

and

$$\lim_{\substack{x\to 1\\ x<1}} \phi'(x) = 0, \tag{2.3.2c}$$

we have

$$\|\phi_\perp\|_{1,2} \leq C\,\|v\|_2 \tag{2.3.3a}$$

and

$$|c_\parallel| \leq C\,\|v\|_2^{1/2}\|v\|_{1,2}^{1/2}. \tag{2.3.3b}$$

Proof. Let $w = \phi/U$. Clearly, $w = c_\parallel + w_\perp$ with $w_\perp = \phi_\perp/U$, and

$$\mathcal{A}_{0,0}\,\phi = U^2\mathcal{M}_U\,w = U^2\mathcal{M}_U\,w_\perp = v. \tag{2.3.4}$$

Step 1. Estimate of $\|\phi_\perp'\|_2$. Taking the inner product with $w_\perp$ yields

$$\|Uw_\perp'\|_2^2 = \langle w_\perp, v\rangle. \tag{2.3.5}$$

Since $\langle w_\perp, U^2\rangle = \langle \phi_\perp, U\rangle = 0$, we now use (2.2.2), (2.3.5), and Hardy's inequality (2.2.7) to obtain that

$$\|\phi_\perp\|_2^2 = \|Uw_\perp\|_2^2 \leq \kappa_2\|Uw_\perp'\|_2^2 \leq \kappa_2\left\|\frac{\phi_\perp}{U}\right\|_2\|v\|_2 \leq C\,\|\phi_\perp'\|_2\|v\|_2. \tag{2.3.6}$$

Note that by (2.1.3) and (2.2.7) it holds that

$$\begin{aligned}\|w_\perp\|_2 &\leq \left\|\frac{\phi_\perp}{U(0)(1-x)}\right\|_2 \leq \left\|\frac{\tilde{\phi}_\perp}{U(0)(1-x)}\right\|_{L^2(-\infty,1)} \\ &\leq \frac{2}{U(0)}\|\tilde{\phi}_\perp'\|_{L^2(-\infty,1)} = \frac{2}{U(0)}\|\phi_\perp'\|_2,\end{aligned} \tag{2.3.7}$$

where $\tilde{\phi}_\perp \in H^1_{\mathrm{loc}}((-\infty, 1])$ is given by

$$\tilde{\phi}_\perp(x) = \begin{cases} \phi_\perp(x), & x \in [0,1], \\ \phi_\perp(0), & x < 0. \end{cases}$$

Integration by parts yields

$$\|\phi_\perp'\|_2^2 = \|(Uw_\perp)'\|_2^2 = \|Uw_\perp'\|_2^2 - \langle Uw_\perp, U''w_\perp\rangle \leq \|Uw_\perp'\|_2^2 + C\|\phi_\perp\|_2\|w_\perp\|_2. \tag{2.3.8}$$

Using (2.3.5) and (2.3.7), we obtain

$$\|\phi_\perp'\|_2 \leq C(\|v\|_2 + \|\phi_\perp\|_2).$$

By (2.3.6) we then obtain

$$\|\phi_\perp'\|_2 \leq C\|v\|_2. \tag{2.3.9}$$

Note that to obtain (2.3.9) we have used the mere fact that $v \in L^2(0,1)$.

Step 2. Estimate of $c_\parallel$. By (2.3.2a) and the fact that $\mathcal{A}_{0,0}\phi_\perp = v$, it holds that

$$\|\phi_\perp''\|_2 \leq C\left(\left\|\frac{\phi_\perp}{U}\right\|_2 + \left\|\frac{v}{U}\right\|_2\right).$$

By Hardy's inequality (2.2.8) we then obtain that

$$\|\phi_\perp''\|_2 \leq C(\|\phi_\perp'\|_2 + \|v'\|_2).$$

Using (2.3.9) we may conclude that

$$\|\phi_\perp''\|_2 \leq C\|v\|_{1,2}. \tag{2.3.10}$$

Using Sobolev embedding, (2.1.3d) and (2.3.2c) yields

$$|c_\parallel| = |\phi_\perp'(1)| \leq \|\phi_\perp'\|_\infty \leq \|\phi_\perp''\|_2^{1/2}\|\phi_\perp'\|_2^{1/2}.$$

We can now conclude (2.3.3b) from (2.3.9) and (2.3.10). ∎

2.4 Estimate of $(\mathcal{A}_{\lambda,\alpha})^{-1}$ for $\Re\lambda \neq 0$ and $\alpha \ll |\lambda|^{1/2}$

We continue with the following estimate of $(\mathcal{A}_{\lambda,0})^{-1}$ when $\Re\lambda \neq 0$. From (2.1.1), we recall that

$$\mathcal{A}_{\lambda,0} \overset{\text{def}}{=} -(U+i\lambda)\frac{d^2}{dx^2} + U'',$$

and that its domain is defined in (2.1.2a):

$$D(\mathcal{A}_{\lambda,0}) = \big\{u \in H^2(0,1) \mid u(1) = u'(0) = 0\big\}.$$

We shall then consider $\mathcal{A}_{\lambda,\alpha}^{-1}$ for α small enough.

Proposition 2.4.1. *Let $p > 1$ and $U \in C^3([0,1])$ satisfy (2.1.3). There exists $C > 0$ such that, for $\lambda \in \mathbb{C}$ satisfying $\Re\lambda \neq 0$ and $|\lambda| < U(3/4)$, it holds for any $(\phi, v) \in D(\mathcal{A}_{\lambda,0}) \times L^p(0,1)$ satisfying $\mathcal{A}_{\lambda,0}\,\phi = v$ that*

$$\|\phi\|_{1,2} \leq C\Big(\frac{1}{|\lambda|}\Big|\int_0^1 v\,dx\Big| + \Big\|\frac{v}{U+i\lambda}\Big\|_1 + \|v\|_p\Big). \tag{2.4.1}$$

Proof. *Step 1.* We prove that

$$\|\phi\|_\infty \leq C\Big(\frac{1}{|\lambda|}\Big|\int_0^1 v\,dx\Big| + \Big\|\frac{v}{U+i\lambda}\Big\|_1 + \|v\|_p\Big). \tag{2.4.2}$$

Step 1.1. The estimate on $[1/2,1]$. As, for any $\phi \in D(\mathcal{A}_{\lambda,0})$,

$$\mathcal{A}_{\lambda,0}\phi = -[(U+i\lambda)\phi' - U'\phi]',$$

we may conclude that

$$\frac{\phi(x)}{U(x)+i\lambda} = -\int_x^1 [K_2(x,\lambda) - K_2(t,\lambda)]v(t)dt + K_2(x,\lambda)\int_0^1 v(t)\,dt, \tag{2.4.3}$$

where

$$K_2(x,\lambda) = \int_x^1 \frac{ds}{(U+i\lambda)^2(s)}.$$

A first integration by parts yields

$$K_2(x,\lambda) = \frac{1}{U'(U+i\lambda)} + \frac{1}{i\lambda} - \int_x^1 \frac{U''ds}{(U')^2(U+i\lambda)}.$$

An additional integration by parts further gives

$$\begin{aligned}&-\int_x^1 \frac{U''ds}{(U')^2(U+i\lambda)}\\ &\quad= \frac{U''}{(U')^3}\log(U+i\lambda) - U''(1)\log(i\lambda) + \int_x^1 \Big(\frac{U''}{(U')^3}\Big)'\log(U+i\lambda)\,ds.\end{aligned}$$

For $|\lambda| < U(3/4)$ (a bounded set in $\mathbb{C}$), there exists $C > 0$ such that for all $x \in [1/2, 1]$ (where $U'(x) \neq 0$)

$$\Big|\int_x^1 \Big(\frac{U''}{(U')^3}\Big)' \log(U + i\lambda)\, ds\Big| \leq C. \tag{2.4.4}$$

To prove (2.4.4), we introduce, for $\nu > 0$, the real value x_ν, which is defined by

$$U(x_\nu) = \nu \text{ for } 0 < \nu < U(0),\ x_\nu = 1 \text{ if } \nu \leq 0 \text{ and } x_\nu = 0 \text{ if } \nu > U(0). \tag{2.4.5}$$

Note that, for $0 < \nu \leq U(0)$, $x_\nu \in [0, 1)$, is indeed uniquely defined by the assumed monotonicity of U (see (2.1.3)).

Then we use the fact that for $|\lambda| < U(3/4)$ (implying $|\nu| < U(3/4)$, $|U'(x_\nu)| > 0$), there exists $C > 0$ such that

$$|\log(U(x) + i\lambda)| \leq C\,[1 + \log|x - x_\nu|^{-1}].$$

Hence,

$$\log(U + i\lambda) \text{ is uniformly bounded in } L^q(0,1) \quad \forall\, 1 \leq q < +\infty, \tag{2.4.6}$$

readily verifying (2.4.4).

Consequently, it holds that

$$\Big|K_2(x,\lambda) - \frac{1}{U'(U+i\lambda)} - \frac{1}{i\lambda} - \frac{U''}{(U')^3}\log(U+i\lambda) + U''(1)\log(i\lambda)\Big| \leq C. \tag{2.4.7}$$

Let

$$\hat{K}_2(x,\lambda) = K_2(x,\lambda) - \frac{1}{i\lambda} + U''(1)\log(i\lambda). \tag{2.4.8}$$

By (2.4.7), we get the existence of $C > 0$ such that, for any $x \in [1/2, 1]$ and $|\lambda| < U(3/4)$ we have

$$\big|(U + i\lambda)(x)\hat{K}_2(x,\lambda)\big| \leq C. \tag{2.4.9}$$

We now write, for any $x \in [1/2, 1]$,

$$\Big|\int_x^1 \frac{v}{U'(U+i\lambda)}dt\Big| \leq C\Big\|\frac{v}{U+i\lambda}\Big\|_1,$$

and hence, by (2.4.6) and (2.4.7), for all $x \in [1/2, 1]$ and $|\lambda| \leq U(3/4)$ we have, for (p, q) satisfying $\frac{1}{p} + \frac{1}{q} = 1$,

$$\begin{aligned}\Big|\int_x^1 \hat{K}_2(t,\lambda)v(t)dt\Big| &\leq C\Big(\Big\|\frac{v}{U+i\lambda}\Big\|_1 + \|\log(U+i\lambda)\|_q\|v\|_p + \|v\|_1\Big)\\ &\leq \hat{C}\Big(\Big\|\frac{v}{U+i\lambda}\Big\|_1 + \|v\|_p\Big).\end{aligned} \tag{2.4.10}$$

Substituting (2.4.8), (2.4.9), and (2.4.10) into (2.4.3) given that $K_2(x,\lambda)-K_2(t,\lambda) = \widehat{K}_2(x,\lambda) - \widehat{K}_2(t,\lambda)$ yields

$$\|\phi\|_{L^\infty(1/2,1)} \le C\Big(\frac{1}{|\lambda|}\Big|\int_0^1 v\,dx\Big| + \Big\|\frac{v}{U+i\lambda}\Big\|_1 + \|v\|_p\Big), \tag{2.4.11}$$

which completes the estimate of ϕ in $[\frac{1}{2},1]$.

Step 1.2. Estimate of $\|\phi\|_{L^\infty(0,1/2)}$. Next, we consider the case where $x\in[0,1/2)$. In this interval we need to address the fact that $U'(0)=0$, as is assumed in (2.1.3a). Here, we use the assumption $|\lambda|<U(3/4)$ to obtain that for $x\in(0,1/2)$,

$$\Big|K_2(x,\lambda) - K_2\Big(\frac{1}{2},\lambda\Big)\Big| = \Big|\widehat{K}_2(x,\lambda) - \widehat{K}_2\Big(\frac{1}{2},\lambda\Big)\Big| \le C. \tag{2.4.12}$$

Since by (2.4.9)

$$\big|(U(1/2)+i\lambda)\widehat{K}_2(1/2,\lambda)\big| \le C,$$

we obtain that

$$\Big|\widehat{K}_2(1/2,\lambda)\Big| \le C. \tag{2.4.13}$$

Combining (2.4.13) with (2.4.12) and (2.4.8) yields

$$\|K_2(\cdot,\lambda)\|_{L^\infty(0,1/2)} \le \frac{C}{|\lambda|}.$$

Hence, for all $x\in[0,1/2]$ it holds that

$$|K_2(x,\lambda)(U+i\lambda)(x)| \le \frac{C}{|\lambda|}. \tag{2.4.14}$$

Furthermore, by (2.4.8), (2.4.12), and (2.4.13) we have that

$$\begin{aligned}
&\Big|\int_x^1 (K_2(t,\lambda)-K_2(x,\lambda))v(t)dt\Big| \\
&\quad\le \Big|\int_x^{1/2} (\widehat{K}_2(t,\lambda)-\widehat{K}_2(x,\lambda))v(t)dt\Big| + \Big|\int_{1/2}^1 (\widehat{K}_2(t,\lambda)-\widehat{K}_2(x,\lambda))v(t)dt\Big| \\
&\quad\le \Big|\int_{1/2}^1 (\widehat{K}_2(t,\lambda)-\widehat{K}_2(x,\lambda))v(t)dt\Big| + C\|v\|_1 \\
&\quad\le \widehat{C}\Big(\frac{1}{|\lambda|}\Big|\int_0^1 v\,dx\Big| + \Big\|\frac{v}{U+i\lambda}\Big\|_1 + \|v\|_p\Big),
\end{aligned}$$

where, to obtain the last inequality, we used (2.4.10).

We can then conclude that

$$\Big|(U+i\lambda)(x)\int_x^1\Big(K_2(t,\lambda)-K_2(x,\lambda)\Big)v(t)dt\Big| \\ \leq C\Big(\frac{1}{|\lambda|}\Big|\int_0^1 v\,dx\Big| + \Big\|\frac{v}{U+i\lambda}\Big\|_1 + \|v\|_p\Big).$$

Substituting the above, together with (2.4.14) into (2.4.3) yields

$$\|\phi\|_{L^\infty(0,1/2)} \leq C\Big(\frac{1}{|\lambda|}\Big|\int_0^1 v\,dx\Big| + \Big\|\frac{v}{U+i\lambda}\Big\|_1 + \|v\|_p\Big).$$

Combined with (2.4.11) the above readily yields (2.4.2).

Step 2. We prove (2.4.1). We begin by rewriting $\mathcal{A}_{\lambda,0}\,\phi = v$ in the form

$$-\phi'' = -\frac{U''}{U+i\lambda}\phi + \frac{v}{U+i\lambda}.$$

Taking the inner product with ϕ in $L^2(0,1)$, integration by parts yields, as $\phi'(0) = \phi(1) = 0$,

$$\|\phi'\|_2^2 = \Re\Big\langle\phi,\frac{U''\phi}{U+i\lambda}\Big\rangle + \Re\Big\langle\phi,\frac{v}{U+i\lambda}\Big\rangle. \tag{2.4.15}$$

Let $\widehat{\chi} \in C^\infty(\mathbb{R},[0,1])$ satisfy

$$\widehat{\chi}(x) = \begin{cases} 0 & |x| < \frac{1}{4}, \\ 1 & |x| > \frac{1}{2}. \end{cases} \tag{2.4.16}$$

Let further $\tilde{\chi} = 1 - \widehat{\chi}$. The first term on the right-hand side of (2.4.12) can be rewritten after an integration by part as

$$-\Re\Big\langle\phi,\frac{U''\phi}{U+i\lambda}\Big\rangle = \Re\Big\langle\Big(\frac{U''|\phi|^2\widehat{\chi}}{U'}\Big)',\log(U+i\lambda)\Big\rangle - \Re\Big\langle\tilde{\chi}\phi,\frac{U''\phi}{U+i\lambda}\Big\rangle. \tag{2.4.17}$$

For the first term on the right-hand side of (2.4.17), we write

$$\Big|\Re\Big\langle\Big(\frac{U''|\phi|^2\widehat{\chi}}{U'}\Big)',\log(U+i\lambda)\Big\rangle\Big| \leq C\|\phi\|_\infty\|[|\phi|+|\phi'|]\log(U+i\lambda)\|_1.$$

Using (2.4.6) together with Poincaré's inequality leads to

$$\Big|\Re\Big\langle\Big(\frac{U''|\phi|^2\widehat{\chi}}{U'}\Big)',\log(U+i\lambda)\Big\rangle\Big| \leq \widehat{C}\|\phi\|_\infty\|\phi'\|_2. \tag{2.4.18}$$

For the second term on the right-hand side of (2.4.17) we have, observing that $U(x) + i\lambda$ does not vanish for x in the support of $\tilde{\chi}$ and $|\lambda| < U(3/4)$,

$$\Big|\Re\Big\langle\tilde{\chi}\phi,\frac{U''\phi}{U+i\lambda}\Big\rangle\Big| \leq C\|\phi\|_2^2. \tag{2.4.19}$$

Substituting (2.4.18) and (2.4.19) into (2.4.17) yields, again with the aid of Poincaré's inequality,

$$\left|\Re\left\langle\phi,\frac{U''\phi}{U+i\lambda}\right\rangle\right| \leq C\|\phi'\|_2\|\phi\|_\infty. \tag{2.4.20}$$

For the second term on the right-hand side of (2.4.15) we use Poincaré's inequality and Sobolev's embeddings to obtain

$$\left|\Re\left\langle\phi,\frac{v}{U+i\lambda}\right\rangle\right| \leq \|\phi\|_\infty\left\|\frac{v}{U+i\lambda}\right\|_1 \leq \|\phi'\|_2\left\|\frac{v}{U+i\lambda}\right\|_1. \tag{2.4.21}$$

Substituting (2.4.21) together with (2.4.20) into (2.4.15) yields

$$\|\phi'\|_2 \leq C\left(\left\|\frac{v}{U+i\lambda}\right\|_1 + \|\phi\|_\infty\right).$$

We can now conclude (2.4.1) from (2.4.2). ∎

We can now state as a corollary, an estimate for $\mathcal{A}_{\lambda,\alpha}^{-1}$ when α is relatively small.

Corollary 2.4.2. *Under the assumptions of Proposition 2.4.1, there exist $C > 0$ and $\delta_0 > 0$ such that, for all $\lambda \in \mathbb{C}$ satisfying $\Re\lambda \neq 0$ and $|\lambda| < U(3/4)$, for all α such that*

$$|\alpha| \leq \delta_0 \min(|\lambda|^{1/2}, \log^{-1/2}|\mu|^{-1}),$$

and for any pair $(\phi, v) \in D(\mathcal{A}_{\lambda,\alpha}) \times L^p(0,1)$ satisfying

$$\mathcal{A}_{\lambda,\alpha}\phi = v, \tag{2.4.22}$$

it holds that

$$\|\phi\|_{1,2} \leq C\left(\frac{1}{|\lambda|}\left|\int_0^1 v\,dx\right| + \left\|\frac{v}{U+i\lambda}\right\|_1 + \|v\|_p\right).$$

The proof is obtained by rewriting (2.4.22) in the manner

$$\mathcal{A}_{\lambda,0}\phi = v + \alpha^2\phi,$$

or

$$\phi = \mathcal{A}_{\lambda,0}^{-1}v + \alpha^2\mathcal{A}_{\lambda,0}^{-1}\phi.$$

We now use (2.4.1) to obtain

$$\begin{aligned}\|\phi\|_{1,2} \leq C&\left(\frac{1}{|\lambda|}\left|\int_0^1 v\,dx\right| + \left\|\frac{v}{U+i\lambda}\right\|_1 + \|v\|_p\right)\\ &+ C\alpha^2\left(\frac{1}{|\lambda|}\|\phi\|_1 + \left\|\frac{\phi}{U+i\lambda}\right\|_1 + \|\phi\|_p\right).\end{aligned}$$

Since $|\nu| \leq U(3/4) < 1$,

$$\Big\|\frac{1}{U + i\lambda}\Big\|_1 \leq C \log|\mu|^{-1}, \tag{2.4.23}$$

to obtain that

$$\Big\|\frac{\phi}{U + i\lambda}\Big\|_1 \leq \Big\|\frac{1}{U + i\lambda}\Big\|_1 \|\phi\|_\infty \leq C \log|\mu|^{-1}\|\phi\|_\infty .$$

Sobolev embeddings, given that $|\alpha| < \delta_0 \min(|\lambda|^{1/2}, \log^{-1/2}|\mu|^{-1})$, complete the proof of the corollary when $p < \infty$ for sufficiently small δ_0.

2.5 Small $|\lambda|$ and $\alpha > \|U\|_2^{-1}(\Im\lambda)_+^{1/2}$

Set for any $\lambda \in \mathbb{C}$ and $\delta > 0$

$$\alpha_{\lambda,\delta} = \|U\|_2^{-1}((\Im\lambda)_+(1 + 2\delta))^{1/2}. \tag{2.5.1}$$

In this section, we attempt to prove the invertibility of $\mathcal{A}_{\lambda,\alpha}$ as defined in (2.1.2) for sufficiently small $|\lambda|$ and $\alpha \geq \alpha_{\lambda,\delta}$.

To be able to state the result of this section we define, for $p > 1$ and $\Re\lambda \neq 0$, on $W^{1,p}(0,1)$ the maps

$$v \mapsto N_0(v,\lambda) := \min\Big(\Big\|(1-x)^{1/2}\frac{v}{U + i\lambda}\Big\|_1, \|v\|_{1,p}\Big) \tag{2.5.2a}$$

and

$$N_1(v,\lambda) = \Big|\Big\langle 1, \frac{v}{U + i\lambda}\Big\rangle\Big|. \tag{2.5.2b}$$

We can now state and prove the following proposition.

Proposition 2.5.1. *Let $r > 1$, $p > 1$, and $\delta > 0$ and $U \in C^3([0,1])$ satisfy (2.1.3). There exist $\lambda_0 > 0$ and $C > 0$ such that for $0 < |\lambda| < \lambda_0$, $\alpha \geq \alpha_{\lambda,\delta}$ and $(\phi, v) \in D(\mathcal{A}_{\lambda,\alpha}) \times W^{1,p}(0,1)$ satisfying $\mathcal{A}_{\lambda,\alpha}\phi = v$, we have, with $c_\parallel = \langle U, \phi\rangle/\|U\|_2^2$ and $\phi_\perp = \phi - c_\parallel U$,*

$$|c_\parallel| \leq \frac{1 + C|\lambda|^2\log|\lambda|^{-1}}{|\alpha^2\|U\|_2^2 + i\lambda|}\big[\|v\|_1 + C|\lambda|N_1(v,\lambda)\big] \tag{2.5.3a}$$

and

$$\|\phi_\perp\|_{1,2} \leq C\Big[N_0(v,\lambda) + \frac{|\lambda|}{|\alpha^2\|U\|_2^2 + i\lambda|}(\|v\|_1 + |\lambda|N_1(v,\lambda))\Big]. \tag{2.5.3b}$$

Proof. Step 1. We prove the existence of $\lambda_0 > 0$ and $C > 0$ such that, for $|\lambda| \leq \lambda_0$ and $\alpha \geq \alpha_{\lambda,\delta}$ it holds that

$$|c_\parallel| \leq \frac{1 + C|\lambda|^2 \log|\lambda|^{-1}}{|\alpha^2\|U\|_2^2 + i\lambda|}\big(\|v\|_1 + C|\lambda|\|\phi_\perp\|_{1,2} + |\lambda| N_1(v,\lambda)\big). \tag{2.5.4}$$

As

$$U(-\phi'' + \alpha^2\phi) - U''\phi = v - i\lambda(-\phi'' + \alpha^2\phi) \tag{2.5.5}$$

or equivalently, by (2.3.2) and (2.3.4)

$$U^2(\mathcal{M}_U + \alpha^2)w = \frac{Uv}{U + i\lambda} + i\lambda\frac{U''\phi}{U + i\lambda}, \tag{2.5.6}$$

where $\mathcal{M}_U$ is given by (2.2.1) and $w = U^{-1}\phi$.

Taking the inner product with 1 and integrating by parts then yields

$$\alpha^2\|U\|_2^2\, c_\parallel = \Big\langle 1, \frac{Uv}{U + i\lambda}\Big\rangle + i\lambda\Big\langle 1, \frac{U''\phi}{U + i\lambda}\Big\rangle. \tag{2.5.7}$$

We now write

$$\Big\langle 1, \frac{U''\phi}{U + i\lambda}\Big\rangle = c_\parallel\Big\langle 1, \frac{U''U}{U + i\lambda}\Big\rangle + \Big\langle 1, \frac{U''\phi_\perp}{U + i\lambda}\Big\rangle. \tag{2.5.8}$$

For the first term on the right-hand side we write, as $U'(0) = 0$ and $U'(1) = -1$,

$$c_\parallel\Big\langle 1, \frac{U''U}{U + i\lambda}\Big\rangle = -c_\parallel\Big(1 + i\lambda\Big\langle 1, \frac{U''}{U + i\lambda}\Big\rangle\Big). \tag{2.5.9}$$

Since, for $\Im\lambda < U(1/2)$,

$$\begin{aligned}\Big\langle 1, \frac{U''}{U + i\lambda}\Big\rangle_{L^2(1/2,1)} &= -U''(1)\log(i\lambda) - \frac{U''(1/2)}{U'(1/2)}\log(U(1/2) + i\lambda) \\ &\quad - \Big\langle\Big(\frac{U''}{U'}\Big)', \log(U + i\lambda)\Big\rangle_{L^2(1/2,1)},\end{aligned}$$

we may conclude the existence of $C > 0$ and $0 < \lambda_0 < U(1/2)$, such that, for $|\lambda| \leq \lambda_0$,

$$\Big|\Big\langle 1, \frac{U''}{U + i\lambda}\Big\rangle_{L^2(1/2,1)}\Big| \leq C\log|\lambda|^{-1}.$$

As

$$\Big|\Big\langle 1, \frac{U''}{U + i\lambda}\Big\rangle_{L^2(0,1/2)}\Big| \leq C,$$

we obtain

$$\Big|\Big\langle 1, \frac{U''}{U + i\lambda}\Big\rangle\Big| \leq C\log|\lambda|^{-1}. \tag{2.5.10}$$

Next, we consider the second term on the right-hand side of (2.5.8) (note that $\phi_\perp(1) = 0$)

$$\begin{aligned}&\Big|\Big\langle 1, \frac{U''\phi_\perp}{U+i\lambda}\Big\rangle_{L^2(1/2,1)}\Big| \\ &\quad = \Big| - \frac{U''\phi_\perp}{U'}\log((U(1/2)+i\lambda))_{|x=1/2} + \Big\langle\Big(\frac{U''\phi_\perp}{U'}\Big)', \log(U+i\lambda)\Big\rangle\Big| \\ &\quad \leq C\|\phi_\perp\|_{1,2}.\end{aligned}$$

For $\Im\lambda < U(1/2)$, we can write

$$\Big|\Big\langle 1, \frac{U''\phi_\perp}{U+i\lambda}\Big\rangle_{L^2(0,1/2)}\Big| \leq C\|\phi_\perp\|_1,$$

and hence we may conclude that

$$\Big|\Big\langle 1, \frac{U''\phi_\perp}{U+i\lambda}\Big\rangle\Big| \leq C\|\phi_\perp\|_{1,2}. \tag{2.5.11}$$

Substituting the above, together with (2.5.9) and (2.5.10) into (2.5.8) yields

$$\Big|\Big\langle 1, \frac{U''\phi}{U+i\lambda}\Big\rangle + c_\parallel\Big| \leq C|\lambda|\log|\lambda|^{-1}c_\parallel + C\|\phi_\perp\|_{1,2}. \tag{2.5.12}$$

We next rewrite (2.5.7) in the form

$$(\alpha^2\|U\|_2^2 + i\lambda)\, c_\parallel = \Big\langle 1, \frac{Uv}{U+i\lambda}\Big\rangle + i\lambda\Big(\Big\langle 1, \frac{U''\phi}{U+i\lambda}\Big\rangle + c_\parallel\Big), \tag{2.5.13}$$

and then observe that

$$\Big\langle 1, \frac{Uv}{U+i\lambda}\Big\rangle = \Big|\langle 1, v\rangle - i\lambda\Big\langle 1, \frac{v}{U+i\lambda}\Big\rangle\Big| \leq \|v\|_1 + |\lambda|N_1(v,\lambda).$$

Substituting the above together with (2.5.12) into (2.5.13) yields

$$|c_\parallel| \leq \frac{1}{|\alpha^2\|U\|_2^2 + i\lambda|}\big(\|v\|_1 + C|\lambda|^2\log|\lambda|^{-1}c_\parallel + C|\lambda|\|\phi_\perp\|_{1,2} + |\lambda|N_1(v,\lambda)\big).$$

Given the fact that when $\Im\lambda > 0$, $\alpha \geq \alpha_{\lambda,\delta}$, we have

$$\frac{1}{|\alpha^2\|U\|_2^2 + i\lambda|} \leq \frac{1}{\delta|\lambda|},$$

and that when $\Im\lambda \leq 0$ we have

$$\frac{1}{|\alpha^2\|U\|_2^2 + i\lambda|} \leq \frac{1}{|\lambda|},$$

we obtain (2.5.4) for sufficiently small $\lambda_0 > 0$.

Step 2. We prove (2.5.3).

Step 2.1. For $w_\perp = U^{-1}\phi_\perp$, we prove that

$$\|Uw_\perp'\|_2^2 + \alpha^2\|\phi_\perp\|_2^2 \leq C\|\phi_\perp'\|_2(N_0(v,\lambda) + |\lambda|\,|c_\||) + \Big|i\lambda\Big\langle\frac{\phi_\perp}{U}, \frac{U''\phi_\perp}{U+i\lambda}\Big\rangle\Big|. \tag{2.5.14}$$

Taking the inner product in (2.5.5) with $w_\perp$ yields (note that $w' = w_\perp'$)

$$\|Uw_\perp'\|_2^2 + \alpha^2\|\phi_\perp\|_2^2 = \Big\langle\phi_\perp, \frac{v}{U+i\lambda}\Big\rangle + i\lambda\Big\langle w_\perp, \frac{U''\phi}{U+i\lambda}\Big\rangle. \tag{2.5.15}$$

We now turn to estimate the right-hand side of (2.5.15). For the first term we have, using the fact that $|\phi_\perp(x)| \leq (1-x)^{1/2}\|\phi_\perp'\|_2$,

$$\Big|\Big\langle\phi_\perp, \frac{v}{U+i\lambda}\Big\rangle\Big| \leq \|\phi_\perp'\|_2\Big\|(1-x)^{1/2}\frac{v}{U+i\lambda}\Big\|_1. \tag{2.5.16}$$

Furthermore, splitting the domain of integration $(0,1)$ into two intervals: $(0,1/4)$ and $(1/4,1)$ and then integrating by parts on $(1/4,1)$ yield, for $0<|\lambda|<\lambda_0\leq U(1/2)$,

$$\begin{aligned}\Big\langle\phi_\perp, \frac{v}{U+i\lambda}\Big\rangle &= -\overline{\phi_\perp}\,v\log(U+i\lambda)\big|_{x=\frac14}\\ &\quad - \int_{1/4}^1 \log(U+i\lambda)\Big(\frac{\overline{\phi_\perp}\,v}{U'}\Big)'\,dx + \int_0^{1/4}\overline{\phi_\perp}\frac{v}{U+i\lambda}\,dx.\end{aligned} \tag{2.5.17}$$

Using a Sobolev embedding and Poincaré's inequality yields for $|\lambda|\leq U(1/2)$

$$\big|\overline{\phi_\perp}\,v\log(U+i\lambda)\big|_{x=\frac14}\big| \leq C\|\phi_\perp\|_\infty\|v\|_\infty \leq C\|\phi_\perp'\|_2\|v\|_{1,p}. \tag{2.5.18}$$

Furthermore, it holds that

$$\Big|\int_0^{1/4}\overline{\phi_\perp}\frac{v}{U+i\lambda}\,dx\Big| \leq C\,\|v\|_2\|\phi_\perp\|_2, \tag{2.5.19}$$

and, as in the proof of (2.4.20),

$$\Big|\int_{1/4}^1\log(U+i\lambda)\Big(\frac{\overline{\phi_\perp}v}{U'}\Big)'\,dx\Big| \leq C(\|v\|_{1,p}\|\phi_\perp\|_\infty + \|v\|_\infty\|\phi_\perp'\|_2).$$

Substituting the above, together with (2.5.19) and (2.5.18) into (2.5.17) we can conclude, with the aid of Poincaré's inequality and Sobolev's embeddings, that

$$\Big|\Big\langle\phi_\perp, \frac{v}{U+i\lambda}\Big\rangle\Big| \leq C\|\phi_\perp'\|_2\|v\|_{1,p}. \tag{2.5.20}$$

Combining (2.5.20) with (2.5.16) and (2.5.2) yields

$$\Big|\Big\langle\phi_\perp, \frac{v}{U+i\lambda}\Big\rangle\Big| \leq C\|\phi_\perp'\|_2 N_0(v,\lambda). \tag{2.5.21}$$

For the second term we have, using the decomposition (2.3.1)

$$i\lambda\Big\langle w_\perp, \frac{U''\phi}{U+i\lambda}\Big\rangle = ic_\parallel\lambda\Big\langle \phi_\perp, \frac{U''}{U+i\lambda}\Big\rangle + i\lambda\Big\langle w_\perp, \frac{U''\phi_\perp}{U+i\lambda}\Big\rangle. \tag{2.5.22}$$

To estimate the first term in (2.5.22) we use (2.5.11) to obtain

$$\Big|c_\parallel\lambda\Big\langle \phi_\perp, \frac{U''}{U+i\lambda}\Big\rangle\Big| \le C|\lambda|\,|c_\parallel|\|\phi_\perp\|_{1,2}. \tag{2.5.23}$$

Substituting the above together with (2.5.22) and (2.5.21) into (2.5.15) yields (2.5.14). The estimate of the last term in (2.5.14) is the object of the next step.

Step 2.2. We prove that for every $\epsilon > 0$ there exists $\lambda_0 > 0$ such that for all $|\lambda| \le \lambda_0$ it holds that

$$\Big|i\lambda\Big\langle \frac{\phi_\perp}{U}, \frac{U''\phi_\perp}{U+i\lambda}\Big\rangle\Big| \le \epsilon\|\phi_\perp'\|_2^2. \tag{2.5.24}$$

Clearly,

$$i\lambda\Big\langle \frac{\phi_\perp}{U}, \frac{U''\phi_\perp}{U+i\lambda}\Big\rangle = i\lambda\int_0^1 \frac{U''|\phi_\perp|^2}{U(U+i\lambda)}\,dx. \tag{2.5.25}$$

Recall the definition of x_ν in (2.4.5), and let $1-x_\nu \le d < 1/2$.

The integral over $(1-2d,1)$. We attempt, see Step 2 of the proof of [3, Proposition 4.14], to prove that for $d < \frac{1}{2}$ there exists $\widehat{C} > 0$ and λ_0 such that for $|\lambda| \le \lambda_0$

$$\Big|i\lambda\int_{1-2d}^1 \frac{U''|\phi_\perp|^2}{U(U+i\lambda)}\,dx\Big| \le \widehat{C}\,d\log|d|^{-1}\|\phi_\perp'\|_2^2. \tag{2.5.26}$$

To estimate this integral we use the identity

$$\frac{1}{U(U+i\lambda)} = \frac{1}{i\lambda}\Big[\frac{1}{U} - \frac{1}{U+i\lambda}\Big],$$

to obtain that

$$i\lambda\int_{1-2d}^1 \frac{U''|\phi_\perp|^2}{U(U+i\lambda)}\,dx = \int_{1-2d}^1 \frac{U''|\phi_\perp|^2}{U}\,dx - \int_{1-2d}^1 \frac{U''|\phi_\perp|^2}{U+i\lambda}\,dx. \tag{2.5.27}$$

Integration by parts yields

$$\int_{1-2d}^1 \frac{U''|\phi_\perp|^2}{U+i\lambda}\,dx = \Big(\frac{U''}{U'}|\phi_\perp|^2|\log(U+i\lambda)|\Big)\Big|_{x=1-2d} - \int_{1-2d}^1 \Big(\frac{U''}{U'}|\phi_\perp|^2\Big)'|\log(U+i\lambda)|\,dx. \tag{2.5.28}$$

Next, we observe that

$$|\phi_\perp(x)|^2 \le 2d\|\phi_\perp'\|_2^2, \quad 0 \le U(x) \le 2d,\ \forall x \in [1-2d,1]. \tag{2.5.29}$$

We also note that, for λ_0 small enough and $|\lambda| \leq \lambda_0$,

$$|\log(U(1-2d)+i\lambda)| \leq C \log|d|^{-1}. \tag{2.5.30}$$

Hence, the first term on the right-hand side of (2.5.28) can be estimated as follows:

$$\Big|\Big(\frac{U''}{U'}|\phi_\perp|^2|\log(U+i\lambda)|\Big)\Big|_{x=1-2d} \leq C\, d \log|d|^{-1}\, \|\phi_\perp'\|_2^2. \tag{2.5.31}$$

For the second term, we write

$$\begin{aligned}
&\Big|\int_{1-2d}^1 \Big(\frac{U''}{U'}|\phi_\perp|^2\Big)'|\log(U+i\lambda)|\,dx\Big| \\
&\quad\leq \Big\|\frac{U''}{U'}\Big\|_{W^{1,\infty}(1-2d,1)}\int_{1-2d}^1 (|\phi_\perp|^2+2|\phi_\perp|\,|\phi_\perp'|)|\log(U+i\lambda)|\,dx \\
&\quad\leq C\int_{1-2d}^1 (|\phi_\perp|^2+|\phi_\perp|\,|\phi_\perp'|)\log(U+i\lambda)|\,dx.
\end{aligned} \tag{2.5.32}$$

As

$$\|\log(U+i\lambda)\|_{L^p(1-2d,1)} \leq Cd^{1/p}\log|d|^{-1} \quad \text{for } p\in\{1,2\},$$

we obtain, using (2.5.29), from (2.5.32) that

$$\Big|\int_{1-2d}^1 \Big(\frac{U''}{U'}|\phi_\perp|^2\Big)'|\log(U+i\lambda)|\,dx\Big| \leq Cd\log|d|^{-1}\|\phi_\perp'\|_2^2.$$

Substituting the above, together with (2.5.29) into (2.5.28) then yields

$$\Big|\int_{1-2d}^1 \frac{U''|\phi_\perp|^2}{U+i\lambda}\,dx\Big| \leq Cd\log|d|^{-1}\|\phi_\perp'\|_2^2. \tag{2.5.33}$$

We now estimate the first term on the right-hand side of (2.5.27). Employing (2.3.7) and Poincaré's inequality yields

$$\Big|\int_{1-2d}^1 \frac{U''|\phi_\perp|^2}{U}\,dx\Big| \leq C\|\phi_\perp\|_{L^2(1-2d,1)}\|\phi_\perp/U\|_{L^2(1-2d,1)} \leq \widetilde{C}\, d\|\phi_\perp'\|_2^2.$$

Substituting the above together with (2.5.33) into (2.5.27) yields (2.5.26).

The integral over $[0,1-2d]$. By (2.1.3) there exists $C>0$ such that for all $1-x_\nu \leq d < 1/2$,

$$\Big\|\frac{1}{U+i\lambda}\Big\|_{L^\infty(0,1-2d)} \leq \frac{1}{U(1-2d)-\nu} \leq \frac{C}{d}.$$

Hence, given that $U'(1)<0$,

$$\Big|\lambda\int_0^{1-2d}\frac{U''|\phi_\perp|^2}{U(U+i\lambda)}\,dx\Big| \leq C\frac{|\lambda|}{d^2}\|\phi_\perp\|_2^2.$$

Combining the above with (2.5.26) yields that for all $1 - x_\nu \leq d$, with the aid of Poincaré's inequality

$$\left|\lambda \int_0^1 \frac{U''|\phi_\perp|^2}{U(U+i\lambda)}\,dx\right| \leq C\Big(\frac{|\lambda|}{d^2} + d\,\log|d|^{-1}\Big)\|\phi_\perp'\|_2^2. \tag{2.5.34}$$

Let $\epsilon > 0$. Clearly, there exists $d(\epsilon) > 0$ such that for $d \in (0, d(\epsilon))$

$$d\,\log|d|^{-1} \leq \frac{\epsilon}{2C}.$$

Furthermore, for (ϵ, d) as above we add the condition

$$\lambda_0 \leq \min\Big(\frac{\epsilon d^2}{2C}, d\,U(0)\Big).$$

As

$$U(0)(1 - x_\nu) \leq |\nu| < |\lambda| < \lambda_0 \leq d\,U(0),$$

we obtain that $d \geq 1 - x_\nu$, therefore, (2.5.24) can now be verified with the aid of (2.5.34).

Step 2.3. We complete the proof of (2.5.3). Substituting (2.5.24) into (2.5.14) yields for any $\epsilon > 0$, the existence of $C > 0$ and $\lambda_\epsilon > 0$, such that for $|\lambda| < \lambda_\epsilon$, it holds that

$$\|Uw_\perp'\|_2^2 + \alpha^2\|\phi_\perp\|_2^2 \leq C\|\phi_\perp'\|_2(N_0(v,\lambda) + |\lambda||c_\parallel|) + \epsilon\|\phi_\perp'\|_2^2. \tag{2.5.35}$$

We now attempt to bound $\|\phi_\perp'\|_2$. As

$$\phi_\perp' = Uw_\perp' + U'w_\perp,$$

and since $\|U'\|_\infty \leq 1$ we may use (2.3.7) to obtain

$$\|\phi_\perp'\|_2^2 \leq 2\|Uw_\perp'\|_2^2 + C\|\phi_\perp\|_2^2. \tag{2.5.36}$$

On the other hand, by (2.2.2) and (2.2.3) we have that

$$\|\phi_\perp\|_2^2 \leq C\|Uw_\perp'\|_2^2,$$

and hence, combining with (2.5.36), we obtain

$$\|\phi_\perp'\|_2 \leq C\|Uw_\perp'\|_2.$$

Substituting the above into (2.5.35) yields, with the aid of Poincaré's inequality and a suitable choice of ϵ, the existence of λ_0 and $C > 0$ such that for $|\lambda| \leq \lambda_0$,

$$\|\phi_\perp'\|_2 \leq C(N_0(v,\lambda) + |\lambda|\,|c_\parallel|). \tag{2.5.37}$$

We now combine (2.5.37) with (2.5.4) to obtain (2.5.3a) and (2.5.3b). ∎

While a direct use of (2.5.3) will be made in the proof of (5.2.2), we shall also need to transform $N_1(v,\lambda)$ into a more conventional bound, which is precisely what we achieve in the next lemma.

Lemma 2.5.2. *Let $U \in C^2([0,1])$ satisfy (2.1.3), $p>1$, $0<\nu_0<U(1/2)$, and $\mu_0>0$. There exist $C>0$ such that for all $|\mu|<\mu_0$, and $|\nu|\le\nu_0$ it holds that*

$$N_1(v,\lambda) \le |v(1)|\log|\lambda| + C\|v\|_{1,p}, \tag{2.5.38}$$

where N_1 is introduced in (2.5.2b).

Proof. Clearly,

$$\Big\langle 1, \frac{v}{U+i\lambda}\Big\rangle = \Big\langle 1, \frac{v}{U+i\lambda}\Big\rangle_{L^2(0,1/2)} + \Big\langle 1, \frac{v}{U+i\lambda}\Big\rangle_{L^2(1/2,1)}.$$

Integration by parts yields

$$\begin{aligned}\Big\langle 1, \frac{v}{U+i\lambda}\Big\rangle_{L^2(1/2,1)} &= v(1)\log(i\lambda) - \frac{v(1/2)}{U'(1/2)}\log(U(1/2)+i\lambda) \\ &\quad - \int_{1/2}^1 \Big(\frac{v}{U'}\Big)'\log(U+i\lambda)\,dx.\end{aligned}$$

Furthermore, as $\nu<U(1/2)$ it holds that

$$\Big|\Big\langle 1, \frac{v}{U+i\lambda}\Big\rangle_{L^2(0,1/2)}\Big| \le \frac{\|v\|_1}{|U(1/2)+i\lambda|}.$$

Consequently, by (2.5.2b), we can conclude (2.5.38) for any $p>1$. ∎

2.6 The case $0 < \Im\lambda < U(0)$

Lemma 2.3.1 and Propositions 2.4.1, 2.5.1 address some of the cases where $|\lambda|$ is small. We now consider the case $0<\nu<\nu_1$ for some $\nu_1<U(0)$, (recall that $\nu=\Im\lambda$). For later reference (see Lemma 5.2.1 and Proposition 5.4.1), we also consider the case where ν is small using a different approach than that of the previous section. We set, for $p>1$ and $v\in W^{1,p}(0,1)$,

$$\begin{aligned}N(v,\lambda) := \min\Big(&\Big\|[(1-x)^{1/2}+\nu^{-1/2}(1-x)]\frac{v}{U+i\lambda}\Big\|_1, \\ &|\nu|^{1/2}(\|v'\|_p + |v(1)|\log|\nu|^{-1}) + \|v\|_2 + \nu^{-1/2}\|v\|_1\Big).\end{aligned} \tag{2.6.1}$$

For small values of $|\lambda|$, since $\mathcal{A}_{i\nu,0}(U-\nu)=0$ and since $U(1)-\nu=\nu\ll 1$, we expect $\mathcal{A}_{\lambda,0}$ to be almost singular, and that the norm of $\phi=\mathcal{A}_{\lambda,0}^{-1}v$ would be much greater in the space spanned by $U-\nu$. We thus use the decomposition

$$\phi = c_\parallel^\nu(U-\nu) + [\phi - c_\parallel^\nu(U-\nu)],$$

where $c_\parallel^\nu$ is defined by (2.6.3).

The proof of the following proposition is somewhat similar to the proof of [3, Proposition 4.15]. In addition to the above-mentioned difference, resulting from the non-invertibility of $\mathcal{A}_{0,0}$, we need to address the Neumann condition at $x = 0$ here, which complicates the estimate of $\|\phi'\|_{L^2(0,x_\nu)}$, where x_ν is given by (2.4.5). This estimate, which is addressed in step 3 of the proof, significantly contributes to its length and complexity.

Proposition 2.6.1. *Let $p > 1$, $U \in C^3([0,1])$ satisfy (2.1.3), and $0 < \nu_1 < U(0)$. There exist $\mu_0 > 0$ and $C > 0$ such that for all $\lambda = \mu + i\nu$ with $0 < \nu < \nu_1$, and $0 < |\mu| \le \mu_0$ and $\alpha \ge 0$, we have, for all $(\phi, v) \in D(\mathcal{A}_{\lambda,\alpha}) \times W^{1,p}(0,1)$ satisfying $\mathcal{A}_{\lambda,\alpha}\phi = v$ (where $\mathcal{A}_{\lambda,\alpha}$ is defined in (2.1.1)),*

$$\|\phi - c_\|^\nu (U-\nu)\|_{1,2} + \nu^{1/2}|c_\|^\nu| \le \frac{C}{\nu} N(v,\lambda), \tag{2.6.2}$$

where

$$c_\|^\nu = \frac{\langle \phi - \phi(x_\nu), U-\nu\rangle_{L^2(0,x_\nu)}}{\|U-\nu\|^2_{L^2(0,x_\nu)}}, \tag{2.6.3}$$

in which x_ν is defined by (2.4.5).

Proof. *Step 1.* We prove that there exists $C > 0$ such that, for all $0 < |\mu| \le 1$ it holds that

$$|\phi(x_\nu)| \le C\Big|\Big\langle \phi, \frac{v}{U+i\lambda}\Big\rangle\Big|^{1/2} \tag{2.6.4}$$

for all pairs $(\phi, v) \in D(\mathcal{A}_{\lambda,\alpha}) \times W^{1,p}(0,1)$ satisfying (2.4.22).

It can be easily verified (since $\mathcal{A}_{\lambda,\alpha}\phi = v$) that

$$\Im\Big\langle \phi, \frac{v}{U-\nu+i\mu}\Big\rangle = -\mu\Big\langle \frac{U''\phi}{(U-\nu)^2+\mu^2}, \phi\Big\rangle. \tag{2.6.5}$$

As

$$|\phi(x)|^2 \ge \frac{1}{2}|\phi(x_\nu)|^2 - |\phi(x)-\phi(x_\nu)|^2,$$

we may use (2.6.5) to obtain, observing that $-U'' > 0$,

$$\Big|\Im\Big\langle \phi, \frac{v}{U-\nu+i\mu}\Big\rangle\Big| \ge |\mu|\Big\langle \frac{|U''|}{(U-\nu)^2+\mu^2}, \frac{1}{2}|\phi(x_\nu)|^2 - |\phi(x)-\phi(x_\nu)|^2\Big\rangle. \tag{2.6.6}$$

Hence,

$$\frac{|\mu|}{2}|\phi(x_\nu)|^2\Big\|\frac{|U''|^{1/2}}{U+i\lambda}\Big\|_2^2 \le |\mu|\sup|U''|\Big\|\frac{\phi-\phi(x_\nu)}{U+i\lambda}\Big\|_2^2 + \Big|\Big\langle \phi, \frac{v}{U-\nu+i\mu}\Big\rangle\Big|. \tag{2.6.7}$$

Since $|U(x)-\nu| \le |x-x_\nu|$, and $|U''|>0$ we can conclude, that for some positive C

$$\Big\|\frac{|U''|^{1/2}}{U+i\lambda}\Big\|_2^2 \ge \frac{1}{C}\int_0^1 \frac{1}{(x-x_\nu)^2+\mu^2}dx.$$

As $|\mu|\le 1$ and $x_\nu \in [x_{\nu_1},1]$, we obtain after the change of variable $y=(x_\nu - x)/|\mu|$

$$\int_0^1 \frac{1}{(x-x_\nu)^2+\mu^2}dx = \frac{1}{|\mu|}\int_{\frac{x_\nu-1}{|\mu|}}^{\frac{x_\nu}{|\mu|}} \frac{1}{1+y^2}dy \ge \frac{1}{|\mu|}\int_0^{\frac{x_{\nu_1}}{|\mu|}} \frac{1}{1+y^2}dy.$$

Consequently, there exists $\hat{C}>0$, such that for all $|\mu|\le 1$ and $\nu\in(0,\nu_1)$,

$$\Big\|\frac{|U''|^{1/2}}{U+i\lambda}\Big\|_2^2 \ge \frac{1}{\hat{C}}|\mu|^{-1}. \tag{2.6.8}$$

A similar argument is employed in the proof of [3, Proposition 4.14] (see between equations (4.59) and (4.60) there). By (2.6.7) we then have

$$|\phi(x_\nu)|^2 \le C\Big[|\mu|\Big\|\frac{\phi-\phi(x_\nu)}{U+i\lambda}\Big\|_2^2 + \Big|\Big\langle\phi,\frac{v}{U-\nu+i\mu}\Big\rangle\Big|\Big]. \tag{2.6.9}$$

To estimate the first term on the right-hand side of (2.6.9) we first observe that for some $C=C(\nu_1)>0$ we have, for all $\lambda=\mu+i\nu$ such that $0<\nu<\nu_1$

$$\Big|\frac{1}{U(x)+i\lambda}\Big| \le \frac{C}{|x-x_\nu|} \quad \forall x\in(x_{\nu_1}/4,1),$$

where $x_{\nu_1}=x_\nu\big|_{\nu=\nu_1}$, and

$$\Big|\frac{1}{U(x)+i\lambda}\Big| \le C \quad \forall x\in(0,x_{\nu_1}/4).$$

Consequently, we may write

$$\Big|\frac{1}{U(x)+i\lambda}\Big| \le \frac{C}{|x-x_\nu|} \quad \forall x\in(0,1). \tag{2.6.10}$$

We now apply Hardy's inequality (2.2.8) to $w=(x-x_\nu)^{-1}(\phi-\phi(x_\nu))$ in $(x_\nu,1)$. It follows that

$$\Big\|\frac{\phi-\phi(x_\nu)}{x-x_\nu}\Big\|_{L^2(x_\nu,1)}^2 \le \frac{1}{4}\|\phi'\|_{L^2(x_\nu,1)}^2. \tag{2.6.11}$$

A similar bound can be the interval $(0,x_\nu)$:

$$\Big\|\frac{\phi-\phi(x_\nu)}{x-x_\nu}\Big\|_{L^2(0,x_\nu)}^2 \le \frac{1}{4}\|\phi'\|_{L^2(0,x_\nu)}^2. \tag{2.6.12}$$

Consequently,

$$\left\| \frac{\phi - \phi(x_\nu)}{x - x_\nu} \right\|_{L^2(0,1)}^2 \le \frac{1}{4} \|\phi'\|_{L^2(0,1)}^2 \tag{2.6.13}$$

from which we easily conclude, in view of (2.6.10),

$$\left\| \frac{\phi - \phi(x_\nu)}{U + i\lambda} \right\|_2^2 \le C \|\phi'\|_2^2. \tag{2.6.14}$$

Substituting (2.6.14) into (2.6.9) readily yields

$$|\phi(x_\nu)|^2 \le C\left(\left| \left\langle \phi, \frac{v}{U + i\lambda} \right\rangle \right| + |\mu| \, \|\phi'\|_2^2 \right). \tag{2.6.15}$$

Using the positivity of $-U''$ on $[0, 1]$ and (2.6.5) for the second inequality we then obtain that

$$\left\| \frac{\phi}{U + i\lambda} \right\|_2^2 \le C \int_0^1 \frac{(-U'')}{(U - \nu)^2 + \mu^2} |\phi|^2 dx \le \frac{\widehat{C}}{|\mu|} \left| \left\langle \phi, \frac{v}{U - \nu + i\mu} \right\rangle \right|. \tag{2.6.16}$$

Since

$$\|\phi'\|_2^2 + \alpha^2 \|\phi\|_2^2 = \Re\left\langle \phi, \frac{\mathcal{A}_{\lambda,\alpha}\phi}{U + i\lambda} \right\rangle - \Re\left\langle U''\phi, \frac{\phi}{U + i\lambda} \right\rangle,$$

we may conclude from (2.6.16), Poincaré's inequality, and (2.4.22) that

$$\begin{aligned} \|\phi'\|_2^2 + \alpha^2 \|\phi\|_2^2 &\le \left| \left\langle \phi, \frac{v}{U + i\lambda} \right\rangle \right| + C \left\| \frac{\phi}{U + i\lambda} \right\|_2 \|\phi\|_2 \\ &\le \left| \left\langle \phi, \frac{v}{U + i\lambda} \right\rangle \right| + \frac{\widehat{C}}{|\mu|^{1/2}} \left| \left\langle \phi, \frac{v}{U + i\lambda} \right\rangle \right|^{1/2} \|\phi'\|_2, \end{aligned} \tag{2.6.17}$$

from which we conclude, given that $|\mu| \le 1$,

$$\|\phi'\|_2^2 \le \frac{C}{|\mu|} \left| \left\langle \phi, \frac{v}{U + i\lambda} \right\rangle \right|. \tag{2.6.18}$$

Substituting (2.6.18) into (2.6.15) yields (2.6.4).

Step 2. We prove that for any $A > 0$, there exists C and μ_A such that, for $\alpha^2 \le A$ and λ such that $|\mu| \le \mu_A$ and $\nu \in (0, \nu_1)$

$$\|\phi\|_{H^1(x_\nu,1)} \le C\left[\nu^{-1/2} \left| \left\langle \phi, \frac{v}{U + i\lambda} \right\rangle \right|^{1/2} + N(v, \lambda) \right] \tag{2.6.19}$$

holds for any pair $(\phi, v) \in D(\mathcal{A}_{\lambda,\alpha}) \times W^{1,p}(0, 1)$ satisfying (2.4.22).

Let $\chi \in C^\infty(\mathbb{R}, [0, 1])$ be given by

$$\chi(x) = \begin{cases} 1 & x < 1/2, \\ 0 & x > 3/4. \end{cases} \tag{2.6.20}$$

With $d = 1 - x_\nu$, let $\chi_d(x) = \chi((x - x_\nu)/d)$ and set

$$\phi = \varphi + \phi(x_\nu)\chi_d. \tag{2.6.21}$$

Note that by the choice of d, φ satisfies also the boundary conditions at $x \in \{0, 1\}$.

It can be easily verified that

$$\mathcal{A}_{\lambda,\alpha}\varphi = v + \phi(x_\nu)\big((U + i\lambda)(\chi_d'' - \alpha^2\chi_d) - U''\chi_d\big).$$

By (2.6.21) we have that

$$w := (U - \nu)^{-1}\varphi \in H^2(0, 1), \tag{2.6.22}$$

and hence we can rewrite the above equality (using (2.4.22) twice) in the form

$$\begin{aligned}
&-\Big((U-\nu)^2\Big(\frac{\varphi}{U-\nu}\Big)'\Big)' + \alpha^2(U-\nu)\varphi \\
&\quad = v + \phi(x_\nu)\big((U-\nu)(\chi_d'' - \alpha^2\chi_d) - U''\chi_d\big) + i\mu(\phi'' - \alpha^2\phi) \\
&\quad = \frac{(U-\nu)v}{U+i\lambda} + \phi(x_\nu)\big((U-\nu)(\chi_d'' - \alpha^2\chi_d) - U''\chi_d\big) + i\mu\frac{U''\phi}{U+i\lambda}.
\end{aligned} \tag{2.6.23}$$

Taking the inner product with w and integrating by parts, exploiting the fact that $\varphi(x_\nu) = 0$, then yields

$$\begin{aligned}
&\|(U-\nu)w'\|^2_{L^2(x_\nu,1)} + \alpha^2\|\varphi\|^2_{L^2(x_\nu,1)} \\
&\quad = \Big\langle\varphi, \frac{v}{U+i\lambda}\Big\rangle_{L^2(x_\nu,1)} - \langle w, \phi(x_\nu)U''\chi_d\rangle_{L^2(x_\nu,1)} \\
&\qquad + \phi(x_\nu)\langle\varphi, \chi_d'' - \alpha^2\chi_d\rangle_{L^2(x_\nu,1)} + i\mu\Big\langle w, \frac{U''\phi}{U+i\lambda}\Big\rangle_{L^2(x_\nu,1)}.
\end{aligned} \tag{2.6.24}$$

We now estimate the four terms appearing in the right-hand side of (2.6.24), using precisely the same procedure as in the proof of [3, Proposition 4.13]. For the first term on the right-hand side of (2.6.24) we obtain with the aid of (2.6.21)

$$\begin{aligned}
\Big|\Big\langle\varphi, \frac{v}{U+i\lambda}\Big\rangle_{L^2(x_\nu,1)}\Big| &\leq \Big|\Big\langle\phi - \phi(x_\nu), \frac{v}{U+i\lambda}\Big\rangle_{L^2(x_\nu,1)}\Big| \\
&\quad + |\phi(x_\nu)|\Big|\Big\langle 1 - \chi_d, \frac{v}{U+i\lambda}\Big\rangle_{L^2(x_\nu,1)}\Big|.
\end{aligned} \tag{2.6.25}$$

Since the integration is carried over $(x_\nu, 1)$ we can estimate the first term on the right-hand side of (2.6.25) by using Hardy's inequality (2.2.8) and (2.6.10)

$$\begin{aligned}
\Big|\Big\langle\phi - \phi(x_\nu), \frac{v}{U+i\lambda}\Big\rangle_{L^2(x_\nu,1)}\Big| &\leq \Big\|\frac{\phi - \phi(x_\nu)}{x - x_\nu}\Big\|_{L^2(x_\nu,1)}\Big\|\frac{(x-x_\nu)v}{U+i\lambda}\Big\|_{L^2(x_\nu,1)} \\
&\leq C\|\phi'\|_{L^2(x_\nu,1)}\|v\|_2.
\end{aligned}$$

For the second term on the right-hand side of (2.6.25) we first note that since

$$\phi(x_\nu) = -\int_{x_\nu}^1 \phi'(x)\,dx,$$

we may conclude that

$$|\phi(x_\nu)| \leq d^{1/2}\|\phi'\|_2. \tag{2.6.26}$$

By the definition of χ_d,

$$\Big\|\frac{1-\chi_d}{U+i\lambda}\Big\|_\infty \leq \frac{C}{d},$$

and hence,

$$\Big\|\frac{1-\chi_d}{U+i\lambda}\Big\|_{L^2(x_\nu,1)} = \Big\|\frac{1-\chi_d}{U+i\lambda}\Big\|_2 \leq \frac{C}{d^{1/2}}.$$

Hence, by the above and (2.6.26),

$$|\phi(x_\nu)|\Big|\Big\langle 1-\chi_d, \frac{v}{U+i\lambda}\Big\rangle_{L^2(x_\nu,1)}\Big| \leq d^{1/2}\|\phi'\|_2\Big\|\frac{1-\chi_d}{U+i\lambda}\Big\|_2\|v\|_2 \leq C\|\phi'\|_2\|v\|_2.$$

Hence,

$$\Big|\Big\langle \varphi, \frac{v}{U+i\lambda}\Big\rangle_{L^2(x_\nu,1)}\Big| \leq C\|\phi'\|_2\|v\|_2. \tag{2.6.27}$$

In addition, we may write, observing that $\varphi(1)=0$,

$$\Big|\Big\langle \varphi, \frac{v}{U+i\lambda}\Big\rangle_{L^2(x_\nu,1)}\Big| \leq \|\varphi'\|_{L^2(x_\nu,1)}\Big\|(1-x)^{1/2}\frac{v}{U+i\lambda}\Big\|_1.$$

Using (2.6.26), yields

$$\|\varphi'\|_{L^2(x_\nu,1)} \leq \|\phi'\|_{L^2(x_\nu,1)} + |\phi(x_\nu)|\,\|\chi_d'\|_2 \leq C\|\phi'\|_2,$$

which leads to

$$\Big|\Big\langle \varphi, \frac{v}{U+i\lambda}\Big\rangle_{L^2(x_\nu,1)}\Big| \leq C\|\phi'\|_{L^2(x_\nu,1)}\Big\|(1-x)^{1/2}\frac{v}{U+i\lambda}\Big\|_1. \tag{2.6.28}$$

Combining (2.6.27) and (2.6.28) yields the existence of $C>0$ such that

$$\Big|\Big\langle \varphi, \frac{v}{U+i\lambda}\Big\rangle_{L^2(x_\nu,1)}\Big| \leq C\|\phi'\|_{L^2(x_\nu,1)}N(v,\lambda). \tag{2.6.29}$$

To estimate the second term $\langle w, \phi(x_\nu)U''\chi_d\rangle_{L^2(x_\nu,1)}$ on the right-hand side of (2.6.24), we note that by Hardy's inequality (2.2.8) and (2.6.26), we have

$$\begin{aligned}\|w\|_{L^2(x_\nu,1)} &\leq C\|\varphi'\|_{L^2(x_\nu,1)} \leq \widehat{C}\Big(\|\phi'\|_{L^2(x_\nu,1)} + \frac{1}{d^{1/2}}|\phi(x_\nu)|\Big)\\ &\leq \widetilde{C}\|\phi'\|_{L^2(x_\nu,1)}.\end{aligned} \tag{2.6.30}$$

From (2.6.30) we then get

$$|\langle w, \phi(x_\nu)U''\chi_d\rangle_{L^2(x_\nu,1)}| \leq C\,|\phi(x_\nu)|\|\phi'\|_{L^2(x_\nu,1)}. \tag{2.6.31}$$

Next, we write for the third term $(\phi(x_\nu)\langle\varphi, \chi_d'' - \alpha^2\chi_d\rangle_{L^2(x_\nu,1)})$ on the right-hand side of (2.6.24), using integration by parts (note that $\chi_d'(x_\nu) = 0 = \chi_d'(1)$) and the fact that $\alpha^2 \leq A$

$$\begin{aligned}|\langle\varphi, \chi_d'' - \alpha^2\chi_d\rangle_{L^2(x_\nu,1)}| &\leq \|\varphi'\|_{L^2(x_\nu,1)}\|\chi_d'\|_2 + C_A\,\|\varphi\|_{L^2(x_\nu,1)} \\ &\leq \widehat{C}_A\left(\frac{1}{d^{1/2}}\|\varphi'\|_{L^2(x_\nu,1)} + \|\varphi\|_{L^2(x_\nu,1)}\right).\end{aligned}$$

(For convenience we drop the notation referring to the dependence on A in the sequel.) Consequently, by (2.6.21),

$$\begin{aligned}&|\phi(x_\nu)|\,|\langle\varphi, \chi_d'' - \alpha^2\chi_d\rangle_{L^2(x_\nu,1)}| \\ &\leq C\,|\phi(x_\nu)|\Big(\frac{1}{d^{1/2}}\Big(\|\phi'\|_{L^2(x_\nu,1)} + \frac{1}{d^{1/2}}|\phi(x_\nu)|\Big) + (\|\phi\|_{L^2(x_\nu,1)} + d^{1/2}|\phi(x_\nu)|)\Big).\end{aligned}$$

Hence, using Poincaré's inequality and (2.6.26), yields

$$|\phi(x_\nu)|\,|\langle\varphi, \chi_d'' - \alpha^2\chi_d\rangle_{L^2(x_\nu,1)}| \leq C\,\frac{|\phi(x_\nu)|}{d^{1/2}}\|\phi'\|_{L^2(x_\nu,1)}. \tag{2.6.32}$$

To estimate the last term $\big(i\mu\langle w, \frac{U''\phi}{U+i\lambda}\rangle_{(x_\nu,1)}\big)$ on the right-hand side of (2.6.24), we first write

$$\begin{aligned}&\Big|\Big\langle w, U''\frac{\phi}{U+i\lambda}\Big\rangle_{L^2(x_\nu,1)}\Big| \\ &\qquad\leq \Big|\Big\langle w, U''\frac{\phi - \phi(x_\nu)}{U+i\lambda}\Big\rangle_{L^2(x_\nu,1)}\Big| + \Big|\Big\langle w, U''\frac{\phi(x_\nu)}{U+i\lambda}\Big\rangle_{L^2(x_\nu,1)}\Big|.\end{aligned}$$

We then use (2.6.30), (2.6.26), and [3, equation (4.38)], which reads, for $\nu \in [0, \nu_1]$, with $\nu_1 < U(0)$,

$$\Big\|\frac{1}{U+i\lambda}\Big\|_2 \leq \check{C}_{\nu_1}|\mu|^{-1/2}, \tag{2.6.33}$$

to obtain that

$$\begin{aligned}\Big|\Big\langle w, U''\frac{\phi(x_\nu)}{U+i\lambda}\Big\rangle_{L^2(x_\nu,1)}\Big| &\leq C\,\|w\|_{L^2(x_\nu,1)}|\phi(x_\nu)|\Big\|\frac{1}{U+i\lambda}\Big\|_2 \\ &\leq \widehat{C}\,|\mu|^{-1/2}d^{1/2}\|\phi'\|^2_{L^2(x_\nu,1)}.\end{aligned} \tag{2.6.34}$$

Furthermore, we have, by (2.6.14) and (2.6.30),

$$\Big|\Big\langle w, U''\frac{\phi - \phi(x_\nu)}{U+i\lambda}\Big\rangle_{L^2(x_\nu,1)}\Big| \leq C\,\|\phi'\|^2_{L^2(x_\nu,1)}.$$

Substituting the above and (2.6.34) together with (2.6.29), (2.6.31), and (2.6.32) into (2.6.24) yields that there exists $C > 0$ such that

$$\begin{aligned}&\|(U-\nu)w'\|^2_{L^2(x_\nu,1)} + \alpha^2\|\varphi\|^2_{L^2(x_\nu,1)} \\ &\leq C\Big([|\mu|^{1/2}d^{1/2} + |\mu|]\|\phi'\|^2_{L^2(x_\nu,1)} + \Big[\frac{|\phi(x_\nu)|}{d^{1/2}} + N(v,\lambda)\Big]\|\phi'\|_{L^2(x_\nu,1)}\Big).\end{aligned} \tag{2.6.35}$$

As $|U(x)-\nu| \geq \frac{1}{C}|x-x_\nu|$ for all $x \in (x_\nu,1)$, we can apply Hardy's inequality (2.2.7) to $(U-\nu)w'$ on $(x_\nu,1)$ to obtain

$$\begin{aligned}&\|w\|^2_{L^2(x_\nu,1)} \leq \hat{C}\|(U-\nu)w'\|^2_{L^2(x_\nu,1)} \\ &\leq \tilde{C}\Big(\big[|\mu|^{1/2}d^{1/2} + |\mu|\big]\|\phi'\|^2_{L^2(x_\nu,1)} + \Big[\frac{|\phi(x_\nu)|}{d^{1/2}} + N(v,\lambda)\Big]\|\phi'\|_{L^2(x_\nu,1)}\Big).\end{aligned} \tag{2.6.36}$$

Continuing as in Step 2 of the proof of [3, Proposition 4.14] we write, using the definition of w and φ,

$$\|\phi'\|_{L^2(x_\nu,1)} \leq \|(U-\nu)w'\|_{L^2(x_\nu,1)} + \|U'w\|_{L^2(x_\nu,1)} + C\,d^{-1/2}|\phi(x_\nu)|, \tag{2.6.37}$$

from which we conclude with the aid of (2.6.35) and (2.6.36) that, for sufficiently small μ_A,

$$\|\phi'\|_{L^2(x_\nu,1)} \leq C\Big(\frac{|\phi(x_\nu)|}{d^{1/2}} + N(v,\lambda)\Big). \tag{2.6.38}$$

From (2.6.38) we can conclude (2.6.19) with the aid of Poincaré's inequality, the fact that $d \geq \frac{1}{C}\nu$, and (2.6.4).

Step 3. We prove that for any $A > 0$, there exist C and μ_A such that, for $\alpha^2 \leq A$, $|\mu| \leq \mu_A$, and $\nu \in [0,\nu_1)$

$$|c^\nu_\parallel| + \|\phi\|_{H^1(0,x_\nu)} \leq C\Big(\nu^{-1}\Big|\Big\langle\phi, \frac{v}{U+i\lambda}\Big\rangle\Big|^{1/2} + \nu^{-1/2}N(v,\lambda)\Big), \tag{2.6.39}$$

holds for any pair $(\phi, v) \in D(\mathcal{A}_{\lambda,\alpha}) \times W^{1,p}(0,1)$ satisfying (2.4.22).

Here, we need to obtain an estimate for $\|w\|_{L^2(0,x_\nu)}$, where we recall from (2.6.22) that $w := (U-\nu)^{-1}\varphi$.

To this end we need an estimate for $w(\hat{x}_0)$ for some $\hat{x}_0 \in (x_\nu/2, x_\nu)$, to be determined at later stage. Clearly, there exists $\hat{x}_1 \in ((1+x_\nu)/2, 1)$ such that

$$|\phi'(\hat{x}_1)| \leq \frac{\sqrt{2}}{d^{1/2}}\|\phi'\|_{L^2(x_\nu,1)}$$

and

$$|\phi(\hat{x}_1)| \leq d^{1/2}\|\phi'\|_{L^2(x_\nu,1)}.$$

Furthermore, it holds for all $x \in (x_\nu/2, x_\nu)$ that

$$|\phi'(x)| \leq |\phi'(\hat{x}_1)| + \Big|\int_x^{\hat{x}_1} \phi''(t)\,dt\Big|.$$

Consequently, as

$$|w(\hat{x}_0)| = \Big|\frac{\phi(\hat{x}_0) - \phi(x_\nu)}{U(\hat{x}_0) - \nu}\Big| \leq \frac{1}{|U(\hat{x}_0) - \nu|}\int_{\hat{x}_0}^{x_\nu} |\phi'(x)|\,dx,$$

we may conclude that

$$|w(\hat{x}_0)| \leq \frac{1}{|U(\hat{x}_0) - \nu|}\int_{\hat{x}_0}^{x_\nu} \Big[|\phi'(\hat{x}_1)| + \Big|\int_x^{\hat{x}_1} \phi''(t)\,dt\Big|\Big]\,dx.$$

With the aid of (2.4.22) we then have

$$|w(\hat{x}_0)| \leq \frac{1}{|U(\hat{x}_0) - \nu|}\int_{\hat{x}_0}^{x_\nu} \Big(|\phi'(\hat{x}_1)| + \Big|\int_x^{\hat{x}_1} \Big[\frac{U''\phi - v}{U + i\lambda} + \alpha^2\phi\Big]\,dt\Big|\Big)\,dx. \tag{2.6.40}$$

We now write

$$\int_x^{\hat{x}_1} \frac{U''\phi}{U + i\lambda}\,dt = \phi(x_\nu)\int_x^{\hat{x}_1} \frac{U''}{U + i\lambda}\,dt + \int_x^{\hat{x}_1} \frac{U''[\phi - \phi(x_\nu)]}{U + i\lambda}\,dt. \tag{2.6.41}$$

To facilitate the estimate of the integral appearing in the first term on the right-hand side of (2.6.41) we use an integration by parts to obtain

$$\int_x^{\hat{x}_1} \frac{U''}{U + i\lambda}\,dt = \Big(\frac{U''}{U'}\log(U + i\lambda)\Big)\Big|_{\hat{x}_1}^{x} - \int_x^{\hat{x}_1} \Big(\frac{U''}{U'}\Big)'\log(U + i\lambda)\,dt.$$

As $0 < \nu < \nu_1$, U''/U' and $(\frac{U''}{U'})'$ are uniformly bounded in $(x_\nu/2, 1)$ and in view of the inequality

$$|\log(U + i\lambda)| \leq \log|x - x_\nu|^{-1} + C,$$

we observe that the L^1-norm of $\log(U(x) + i\lambda)$ is bounded and that

$$|\log(U + i\lambda)(\hat{x}_1)| \leq C(|\log(d^2 + \mu^2)| + 1). \tag{2.6.42}$$

Hence, we have

$$\Big|\phi(x_\nu)\int_x^{\hat{x}_1} \frac{U''(t)}{U(t) + i\lambda}\,dt\Big| \leq C|\phi(x_\nu)|[1 + |\log(d^2 + \mu^2)| + |\log(U + i\lambda)(\hat{x})|].$$

For the second term on the right-hand side of (2.6.41), we have by (2.6.14)

$$\begin{aligned}\Big|\int_{x_\nu}^{\hat{x}_1} \frac{U''(t)[\phi(t) - \phi(x_\nu)]}{U(t) + i\lambda}\,dt\Big| &\leq C|\hat{x}_1 - x_\nu|^{1/2}\Big[\int_{x_\nu}^{\hat{x}_1} \Big|\frac{\phi(t) - \phi(x_\nu)}{U(t) + i\lambda}\Big|^2\,dt\Big]^{1/2} \\ &\leq \widetilde{C}\,d^{1/2}\|\phi'\|_{L^2(x_\nu,\hat{x}_1)}.\end{aligned}$$

In a similar manner we obtain that

$$\Big|\int_x^{x_\nu} \frac{U''[\phi(t)-\phi(x_\nu)]}{U(t)+i\lambda}\,dt\Big| \leq C(x_\nu - x)^{1/2}\|\phi'\|_{L^2(x,x_\nu)}.$$

Consequently,

$$\Big|\int_x^{\hat{x}_1} \frac{U''[\phi(t)-\phi(x_\nu)]}{U(t)+i\lambda}\,dt\Big| \leq C\big(\|\phi'\|_{L^2(x_\nu,\hat{x}_1)} + (x_\nu - x)^{1/2}\|\phi'\|_{L^2(x,x_\nu)}\big).$$

Hence,

$$\begin{aligned}\Big|\int_x^{\hat{x}_1} &\frac{U''\phi}{U+i\lambda}\,dt\Big| \\ &\leq C\Big(\|\phi'\|_{L^2(x_\nu,\hat{x}_1)} + (x_\nu - x)^{1/2}\|\phi'\|_{L^2(x,x_\nu)} \\ &\qquad + |\phi(x_\nu)|[1 + |\log(d^2+\mu^2)| + |\log(U+i\lambda)(x)|]\Big).\end{aligned} \tag{2.6.43}$$

Next, we write

$$\Big|\int_x^{\hat{x}_1} \phi(t)\,dt\Big| \leq C\,d^{1/2}\,\|\phi\|_{L^2(x_\nu,1)} + \Big|\int_x^{x_\nu}\phi(t)\,dt\Big|.$$

Then, with the aid of Poincaré's inequality we obtain

$$\Big|\int_x^{\hat{x}_1} \phi(t)\,dt\Big| \leq C\,d\,\|\phi'\|_{L^2(x_\nu,1)} + |\phi(x_\nu)|(x_\nu - x) + C(x_\nu - x)^{3/2}\|\phi'\|_{L^2(x,x_\nu)}.$$

Substituting the above, together with (2.6.43), into (2.6.40) yields

$$\begin{aligned}|w(\hat{x}_0)| \leq \frac{C(1+\alpha^2)}{|U(\hat{x}_0)-\nu|}\int_{\hat{x}_0}^{x_\nu} &\big[d^{-1/2}\|\phi'\|_{L^2(x_\nu,1)} + (x_\nu - x)^{1/2}\|\phi'\|_{L^2(x,x_\nu)} \\ &+ |\phi(x_\nu)|[1 + |\log(U+i\lambda)(x)| + |\log(d^2+\mu^2)|]\big]\,dx.\end{aligned}$$

We now write, using (2.6.42),

$$\begin{aligned}\frac{1}{|U(\hat{x}_0)-\nu|}\int_{\hat{x}_0}^{x_\nu} |\log(U+i\lambda)(x)|\,dx &\leq \frac{C}{x_\nu - \hat{x}_0}\int_{\hat{x}_0}^{x_\nu}[1 + |\log(x_\nu - x)|]\,dx \\ &\leq \widehat{C}[1 + \log(|\hat{x}_0 - x_\nu|^{-1})].\end{aligned}$$

Consequently, since $|(\hat{x}_0 - x_\nu)(U(\hat{x}) - \nu)^{-1}|$ is uniformly bounded for $\hat{x}_0 \in (x_\nu, x_\nu/2)$ and $\alpha^2 \leq A$,

$$\begin{aligned}|w(\hat{x}_0)| \leq C\big[&|\phi(x_\nu)|(1 + \log|\hat{x}_0 - x_\nu|^{-1} + |\log(d^2+\mu^2)|) \\ &+ (x_\nu - \hat{x}_0)^{1/2}\|\phi'\|_{L^2(\hat{x}_0,x_\nu)} + d^{-1/2}\|\phi'\|_{L^2(x_\nu,1)}\big].\end{aligned} \tag{2.6.44}$$

We can now apply Hardy's inequality (2.2.7) to $w - w(\hat{x}_0)$ on the interval $(\hat{x}_0, x_\nu)$ to obtain

$$\begin{aligned}\|w - w(\hat{x}_0)\|^2_{L^2(\hat{x}_0,x_\nu)} &\leq 4\|([x - x_\nu][w - w(\hat{x}_0)])'\|^2_{L^2(\hat{x}_0,x_\nu)} \\ &\leq C(\nu_1)\|(U-\nu)w'\|^2_{L^2(\hat{x}_0,x_\nu)}.\end{aligned}$$

By (2.6.44) and (2.6.4) we then have

$$\begin{aligned}&\|w\|_{L^2(\hat{x}_0,x_\nu)} \\ &\leq C\big(\|(U-\nu)w'\|_{L^2(\hat{x}_0,x_\nu)} + d^{-1/2}\|\phi'\|_{L^2(x_\nu,1)} \\ &\qquad + [\|\phi'\|_2^{1/2}N(v,\lambda)^{\frac{1}{2}} + |\mu|^{1/2}\|\phi'\|_2](1 + \log(|\hat{x}_0 - x_\nu|^{-1}) + |\log(d^2+\mu^2)|) \\ &\qquad + (x_\nu - \hat{x}_0)^{1/2}\|\phi'\|_{L^2(\hat{x}_0,x_\nu)}\big).\end{aligned}$$

Note that C is independent of $\hat{x}_0 \in (x_\nu/2, x_\nu)$.

On the other hand by Poincaré's inequality, we have

$$\|w - w(\hat{x}_0)\|^2_{L^2(0,\hat{x}_0)} \leq C\|\,w'\|^2_{L^2(0,\hat{x}_0)}.$$

Observing that $\frac{1}{2}x_{\nu_1} \leq \hat{x}_0 \leq x_\nu \leq 1$ we obtain for all $x \in (0, \hat{x}_0)$

$$U(x) - \nu \geq U(\hat{x}_0) - U(x_\nu) \geq \left|U'\Big(\frac{1}{2}x_{\nu_1}\Big)\right| |\hat{x}_0 - x_\nu|.$$

Consequently,

$$\|w - w(\hat{x}_0)\|^2_{L^2(0,\hat{x}_0)} \leq C(x_\nu - \hat{x}_0)^{-2}\|(U-\nu)w'\|^2_{L^2(0,\hat{x}_0)}, \tag{2.6.45}$$

and hence, as $\hat{x}_0 \leq x_\nu$,

$$\begin{aligned}&\|w\|_{L^2(0,x_\nu)} \\ &\leq C\big((x_\nu - \hat{x}_0)^{-1}\|(U-\nu)w'\|_{L^2(0,x_\nu)} + d^{-1/2}\|\phi'\|_{L^2(x_\nu,1)} \\ &\qquad + [\|\phi'\|_2^{1/2}N(v,\lambda)^{\frac{1}{2}} + |\mu|^{1/2}\|\phi'\|_2](1 + \log|\hat{x}_0 - x_\nu|^{-1} + |\log(d^2+\mu^2)|) \\ &\qquad + (x_\nu - \hat{x}_0)^{1/2}\|\phi'\|_{L^2(\hat{x}_0,x_\nu)}\big).\end{aligned}$$

Continuing as in Step 2 of the proof of [3, Proposition 4.14] we establish that

$$\begin{aligned}&\|\phi'\|_{L^2(0,x_\nu)} \\ &\leq C\big((x_\nu - \hat{x}_0)^{-1}\|(U-\nu)w'\|_{L^2(0,x_\nu)} + d^{-1/2}\|\phi'\|_{L^2(x_\nu,1)} \\ &\qquad + [\|\phi'\|_2^{1/2}N(v,\lambda)^{\frac{1}{2}} + |\mu|^{1/2}\|\phi'\|_2](1 + \log(|\hat{x}_0 - x_\nu|^{-1}) + |\log(d^2+\mu^2)|) \\ &\qquad + (x_\nu - \hat{x}_0)^{1/2}\|\phi'\|_{L^2(\hat{x}_0,x_\nu)}\big).\end{aligned} \tag{2.6.46}$$

Taking the inner product in $L^2(0, x_\nu)$ of (2.6.23) with w yields, as in (2.6.24) (note that $\chi_d \equiv 1$ in $(0, x_\nu)$)

$$\begin{aligned}
&\|(U-\nu)w'\|^2_{L^2(0,x_\nu)} + \alpha^2\|\varphi\|^2_{L^2(0,x_\nu)} \\
&\quad = \Big\langle \varphi, \frac{v}{U+i\lambda}\Big\rangle_{L^2(0,x_\nu)} - \langle w, \phi(x_\nu)U''\rangle_{L^2(0,x_\nu)} \\
&\qquad - \alpha^2\phi(x_\nu)\langle\varphi, 1\rangle_{L^2(0,x_\nu)} + i\mu\Big\langle w, \frac{U''\phi}{U+i\lambda}\Big\rangle_{L^2(0,x_\nu)}.
\end{aligned} \tag{2.6.47}$$

As in the proof of (2.6.29) we obtain below for the first term on the right-hand side of (2.6.47)

$$\Big|\Big\langle \varphi, \frac{v}{U+i\lambda}\Big\rangle_{L^2(0,x_\nu)}\Big| \leq C\|\phi'\|_{L^2(0,x_\nu)}\, N(v,\lambda). \tag{2.6.48}$$

Indeed, as $\varphi' \equiv \phi'$ in $(0, x_\nu)$ and $\varphi(x_\nu) = 0$, we get for $x \in (0, x_\nu)$

$$|\varphi(x)| \leq \|\varphi'\|_{L^2(0,x_\nu)}(x_\nu - x)^{1/2} \leq \|\phi'\|_{L^2(0,x_\nu)}(1-x)^{1/2},$$

from which we conclude that

$$\Big|\Big\langle \varphi, \frac{v}{U+i\lambda}\Big\rangle_{L^2(0,x_\nu)}\Big| \leq \|\phi'\|_{L^2(0,x_\nu)} \Big\|(1-x)^{1/2}\frac{v}{U+i\lambda}\Big\|_1. \tag{2.6.49}$$

In addition, we can write

$$\begin{aligned}
\Big|\Big\langle \varphi, \frac{v}{U+i\lambda}\Big\rangle_{L^2(0,x_\nu)}\Big| &= \Big|\Big\langle \phi - \phi(x_\nu), \frac{v}{U+i\lambda}\Big\rangle_{L^2(0,x_\nu)}\Big| \\
&= \Big|\Big\langle \frac{(\phi-\phi(x_\nu))}{(x-x_\nu)}, \frac{x-x_\nu}{U+i\lambda}v\Big\rangle_{L^2(0,x_\nu)}\Big|
\end{aligned}$$

and then use (2.6.12) and (2.6.10) to obtain

$$\Big|\Big\langle \varphi, \frac{v}{U+i\lambda}\Big\rangle_{L^2(0,x_\nu)}\Big| \leq C\|\phi'\|_2\|v\|_2. \tag{2.6.50}$$

Using the definition of $N(v,\lambda)$ in (2.6.1) we can now conclude (2.6.48) from (2.6.49) and (2.6.50).

By (2.2.8) which reads in this case

$$\|w\|_{L^2(0,x_\nu)} \leq C\|\phi'\|_{L^2(0,x_\nu)}, \tag{2.6.51}$$

we have for the second term on the right-hand side of (2.6.47) that

$$|\langle w, \phi(x_\nu)U''\rangle_{L^2(0,x_\nu)}| \leq C\,|\phi(x_\nu)|\|\phi'\|_{L^2(0,x_\nu)}.$$

For the third term we have, using the fact that $\alpha^2 \leq A$, (2.6.21) and Poincaré's inequality,

$$|\alpha^2\phi(x_\nu)\langle\varphi,1\rangle_{L^2(0,x_\nu)}| \leq C\,|\phi(x_\nu)|\big(\|\phi\|_{L^2(0,x_\nu)} + |\phi(x_\nu)|\big) \leq \widehat{C}\,|\phi(x_\nu)|\,\|\phi'\|_{L^2(0,x_\nu)}.$$

Finally, we obtain for the last term of (2.6.47) using (2.6.10), (2.6.12) (as in the proof of (2.6.14) but on the interval $(0, x_\nu)$), and (2.6.51)

$$\Big|\mu\Big\langle w, \frac{U''(\phi-\phi(x_\nu))}{U+i\lambda}\Big\rangle_{L^2(0,x_\nu)}\Big| \leq C|\mu|\|\phi'\|^2_{L^2(0,x_\nu)}.$$

Hence, with the aid of (2.6.33) we conclude, as in the proof of (2.6.34),

$$\begin{aligned}\Big|\mu\Big\langle w, \frac{U''\phi(x_\nu)}{U+i\lambda}\Big\rangle_{L^2(0,x_\nu)}\Big| &\leq C|\mu|\,\|(U+i\lambda)^{-1}\|_{L^2(0,x_\nu)}\,|\phi(x_\nu)|\,\|\phi'\|_{L^2(0,x_\nu)}\\ &\leq \widehat{C}\,|\mu|^{\frac{1}{2}}\,|\phi(x_\nu)|\|\phi'\|_{L^2(0,x_\nu)}\\ &\leq \widehat{C}\,\big(|\mu|\|\phi'\|^2_{L^2(0,x_\nu)}+|\phi(x_\nu)|^2\big)\\ &\leq \widetilde{C}\,\big(|\mu|\|\phi'\|_{L^2(0,x_\nu)}+|\phi(x_\nu)|\big)\|\phi'\|_{L^2(0,x_\nu)}.\end{aligned}$$

Combining the above starting from (2.6.47) yields

$$\begin{aligned}&\|(U-\nu)w'\|^2_{L^2(0,x_\nu)}+\alpha^2\|\varphi\|^2_{L^2(0,x_\nu)}\\ &\quad\leq C\big(|\mu|\,\|\phi'\|^2_{L^2(0,x_\nu)}+[|\phi(x_\nu)|+N(v,\lambda)]\|\phi'\|_{L^2(0,x_\nu)}\big).\end{aligned} \tag{2.6.52}$$

Hence, by (2.6.46) and (2.6.4), there exists $C>0$ such that for any $\hat{x}_0\in(x_\nu/2, x_\nu)$ we have, with $\varepsilon = x_\nu - \hat{x}_0$,

$$\begin{aligned}\|\phi'\|_2 &\leq C(N(v,\lambda)+[1+|\mu|^{1/2}+d^{-1/2}]\|\phi'\|_{L^2(x_\nu,1)})\\ &\quad + C[|\mu|^{1/2}(\varepsilon^{-1}+|\log(d^2+\mu^2)|)+\varepsilon^{1/2}]\|\phi'\|_2.\end{aligned}$$

We can now choose $\hat{x}_0$ such that $\varepsilon = \inf((\frac{1}{4C})^2, x_{\nu_1}/4)$. Then under the condition $|\mu|^{1/2}(\varepsilon^{-1}+|\log(d^2+\mu^2)|)\leq\frac{1}{4C}$, which is valid for μ_A which is small enough, we obtain

$$\|\phi'\|_2 \leq 3C\big(N(v,\lambda)+d^{-1/2}\|\phi'\|_{L^2(x_\nu,1)}\big)\leq\widehat{C}\big(N(v,\lambda)+\nu^{-1/2}\|\phi'\|_{L^2(x_\nu,1)}\big). \tag{2.6.53}$$

Combining (2.6.53) with (2.6.38) yields,

$$\|\phi'\|_2 \leq C\Big(\frac{|\phi(x_\nu)|}{d}+d^{-1/2}N(v,\lambda)\Big), \tag{2.6.54}$$

from which we can conclude a bound on ϕ in $H^1(0,x_\nu)$ as stated in (2.6.39), upon use of Poincaré's inequality and (2.6.4).

To complete the proof of (2.6.39), we need to estimate $c_\parallel^\nu$, which is defined in (2.6.3) by

$$c_\parallel^\nu = \frac{\langle\phi-\phi(x_\nu), U-\nu\rangle_{L^2(0,x_\nu)}}{\|U-\nu\|^2_{L^2(0,x_\nu)}}.$$

We note that for $0 < \nu < \nu_1$ the denominator in the definition of $c_\parallel^\nu$ satisfies

$$\|(U-\nu)\|^2_{L^2(0,x_\nu)} \geq \|(U-\nu)\|^2_{L^2(x_{\nu_1/4},x_{\nu_1/2})} \geq \frac{1}{C}. \tag{2.6.55}$$

Hence,

$$|c_\parallel^\nu| \leq C \left\|\frac{\phi-\phi(x_\nu)}{U-\nu}\right\|_{L^2(0,x_\nu)} \leq \hat{C}\,\|\phi'\|_{L^2(0,x_\nu)}, \tag{2.6.56}$$

which together with (2.6.54) yields (2.6.39).

Step 4. We prove that for any $A > 0$, there exists C and μ_A such that, for $\alpha^2 \leq A$, $|\mu| \leq \mu_A$, and $\nu \in (0,\nu_1)$ such that

$$\|\phi - c_\parallel^\nu(U-\nu)\|_{1,2} \leq C\nu^{-1/2}\Big(\Big|\Big\langle\phi, \frac{v}{U+i\lambda}\Big\rangle\Big|^{1/2} + N(v,\lambda)\Big) \tag{2.6.57}$$

holds for any pair $(\phi, v) \in D(\mathcal{A}_{\lambda,\alpha}) \times W^{1,p}(0,1)$ satisfying (2.4.22).

Since (2.2.2) remains valid if we replace U by $U-\nu$, $1-x$ by $x_\nu - x$, and $(0,1)$ by $(0,x_\nu)$ we may conclude from (2.2.2) that there exists $C > 0$ such that

$$\|(U-\nu)w'\|^2_{L^2(0,x_\nu)} \geq \frac{1}{C}\|\varphi_\perp^\nu\|^2_{L^2(0,x_\nu)}, \tag{2.6.58}$$

where, in the interval $(0,x_\nu)$, $\varphi_\perp^\nu$ is defined by

$$\varphi_\perp^\nu = \phi - \phi(x_\nu) - c_\parallel^\nu(U-\nu) = \varphi - c_\parallel^\nu(U-\nu) \tag{2.6.59}$$

and $c_\parallel^\nu$ is defined in (2.6.3). Note that by construction

$$\langle \varphi_\perp^\nu, U-\nu\rangle_{(0,x_\nu)} = 0. \tag{2.6.60}$$

Furthermore, by (2.6.60) and (2.6.55)

$$\frac{1}{C}|c_\parallel^\nu|^2 \leq |c_\parallel^\nu|^2\|(U-\nu)\|^2_{L^2(0,x_\nu)} \leq \|\varphi\|^2_{L^2(0,x_\nu)}. \tag{2.6.61}$$

Substituting (2.6.61) and (2.6.58) into (2.6.52) (recall again that $\varphi' \equiv \phi'$ in $(0,x_\nu)$) yields, with the aid of (2.6.18) and (2.6.59), for a new constant C

$$\begin{aligned}&\|\varphi_\perp^\nu\|^2_{L^2(0,x_\nu)} + \alpha^2|c_\parallel^\nu|^2 \\ &\leq C\Big(\Big|\Big\langle\phi, \frac{v}{U+i\lambda}\Big\rangle\Big| + \big[|\phi(x_\nu)| + N(v,\lambda)\big]\big(\|(\varphi_\perp^\nu)'\|_{L^2(0,x_\nu)} + |c_\parallel^\nu|\big)\Big).\end{aligned} \tag{2.6.62}$$

Let $w_\perp^\nu := (U-\nu)^{-1}\varphi_\perp^\nu$. As in (2.3.8) we obtain that

$$\|(\varphi_\perp^\nu)'\|^2_{L^2(0,x_\nu)} \leq C\big(\|(U-\nu)(w_\perp^\nu)'\|^2_{L^2(0,x_\nu)} + \|\varphi_\perp^\nu\|_{L^2(0,x_\nu)}\|w_\perp^\nu\|_{L^2(0,x_\nu)}\big). \tag{2.6.63}$$

By Hardy's inequality (2.2.8) applied to $w_\perp^\nu$, and since $(w_\perp^\nu)' = w'$, we may conclude from (2.6.63), with the aid of (2.6.58), that

$$\begin{aligned}\|(\varphi_\perp^\nu)'\|_{L^2(0,x_\nu)} &\leq C(\|(U-\nu)w'\|_{L^2(0,x_\nu)} + \|\varphi_\perp^\nu\|_{L^2(0,x_\nu)})\\ &\leq \widehat{C}\,\|(U-\nu)w'\|_{L^2(0,x_\nu)}.\end{aligned} \tag{2.6.64}$$

We now rewrite (2.6.52) with the aid of (2.6.18) in the form

$$\|(U-\nu)w'\|^2_{L^2(0,x_\nu)} \leq C\left(\Big|\Big\langle\phi, \frac{v}{U+i\lambda}\Big\rangle\Big| + [|\phi(x_\nu)| + N(v,\lambda)]\|\phi'\|_{L^2(0,x_\nu)}\right).$$

Combining the above with (2.6.64) and (2.6.59) yields

$$\begin{aligned}\|(\varphi_\perp^\nu)'\|^2_{L^2(0,x_\nu)} \leq C\bigg(&\Big|\Big\langle\phi, \frac{v}{U+i\lambda}\Big\rangle\Big|\\ &+ [|\phi(x_\nu)| + N(v,\lambda)](\|(\varphi_\perp^\nu)'\|_{L^2(0,x_\nu)} + |c_\parallel^\nu|)\bigg).\end{aligned} \tag{2.6.65}$$

This yields by (2.6.39) and (2.6.4) that

$$\|(\varphi_\perp^\nu)'\|^2_{L^2(0,x_\nu)} \leq C\,\nu^{-1}\left(\Big|\Big\langle\phi, \frac{v}{U+i\lambda}\Big\rangle\Big| + N(v,\lambda)^2\right). \tag{2.6.66}$$

Clearly, by Poincaré's inequality

$$\begin{aligned}&\|\phi - c_\parallel^\nu(U-\nu)\|^2_{L^2(0,x_\nu)}\\ &\quad\leq 2(\|\varphi_\perp^\nu\|^2_{L^2(0,x_\nu)} + |\phi(x_\nu)|^2) \leq C(\|(\varphi_\perp^\nu)'\|^2_{L^2(0,x_\nu)} + |\phi(x_\nu)|^2),\end{aligned}$$

which implies

$$\|\phi - c_\parallel^\nu(U-\nu)\|^2_{H^1(0,x_\nu)} \leq C(\|(\varphi_\perp^\nu)'\|^2_{L^2(0,x_\nu)} + |\phi(x_\nu)|^2). \tag{2.6.67}$$

Combining (2.6.67) with (2.6.39), (2.6.4), and (2.6.66) then yields

$$\|\phi - c_\parallel^\nu(U-\nu)\|_{H^1(0,x_\nu)} \leq C\nu^{-1/2}\left(\Big|\Big\langle\phi, \frac{v}{U+i\lambda}\Big\rangle\Big|^{1/2} + N(v,\lambda)\right). \tag{2.6.68}$$

Next, we write with the aid of (2.6.19) and (2.6.39) (the bound on $|c_\parallel^\nu|$),

$$\begin{aligned}&\|\phi - c_\parallel^\nu(U-\nu)\|_{H^1(x_\nu,1)}\\ &\leq \|\phi\|_{H^1(x_\nu,1)} + d^{1/2}|c_\parallel^\nu| \leq C\left(\nu^{-1/2}\Big|\Big\langle\phi, \frac{v}{U+i\lambda}\Big\rangle\Big|^{1/2} + N(v,\lambda)\right).\end{aligned} \tag{2.6.69}$$

Combining (2.6.69) with (2.6.68) and (2.6.4) yields,

$$\|\phi - c_\parallel^\nu(U-\nu)\|_{1,2} \leq C\nu^{-1/2}\left(\Big|\Big\langle\phi, \frac{v}{U+i\lambda}\Big\rangle\Big|^{1/2} + N(v,\lambda)\right),$$

verifying (2.6.57).

Step 5. We prove that for any $0 < \nu_1 < U(0)$ and $A > 0$, there exists $\mu_A > 0$ and $C_A > 0$ such that (2.6.2) holds for $\alpha^2 \leq A$ and $|\mu| \leq \mu_A$.

For $\nu \in (0, \nu_1)$, let $\zeta_\nu \in C^\infty(\mathbb{R}_+, [0,1])$ satisfy

$$\zeta_\nu(x) = \begin{cases} 0 & x < \frac{x_\nu}{4}, \\ 1 & x > \frac{x_\nu}{2} \end{cases} \tag{2.6.70}$$

and

$$|\zeta_\nu'(x)| \leq C(\nu_1) \quad \forall \nu \in (0, \nu_1),\ \forall x \in (0,1).$$

Let further

$$\tilde{\zeta}_\nu = 1 - \zeta_\nu. \tag{2.6.71}$$

We may now write, omitting the reference to ν for ζ_ν and $\tilde{\zeta}_\nu$,

$$\Big|\Big\langle \zeta\phi, \frac{v}{U+i\lambda}\Big\rangle\Big| \leq |\phi(x_\nu)|\Big|\Big\langle \zeta, \frac{v}{U+i\lambda}\Big\rangle\Big| + \Big|\Big\langle \zeta\big(\phi - \phi(x_\nu)\big), \frac{v}{U+i\lambda}\Big\rangle\Big|. \tag{2.6.72}$$

For the first term on the right-hand side of (2.6.72) we begin by writing that

$$\Big\langle \zeta, \frac{v}{U+i\lambda}\Big\rangle = -\Big\langle \Big(\frac{\zeta\bar{v}}{U'}\Big)', \log(U+i\lambda)\Big\rangle + \frac{\bar{v}(1)\log(U(1)+i\lambda)}{U'(1)}.$$

Then we observe that

$$|\log(U(1)+i\lambda)| = |\log(U(1) - U(x_\nu) + i\mu)| \leq C(\log(d^{-1}+1)),$$

and since $d \leq 1 - x_{\nu_1}$ we obtain

$$|\log(U(1)+i\lambda)| = |\log(U(1) - U(x_\nu) + i\mu)| \leq \hat{C}(\log(d^{-1})).$$

We can then conclude that

$$\Big|\Big\langle \zeta, \frac{v}{U+i\lambda}\Big\rangle\Big| \leq \Big|\Big\langle \Big(\frac{\zeta\bar{v}}{U'}\Big)', \log(U+i\lambda)\Big\rangle\Big| + C|v(1)|\log|d|^{-1}.$$

In view of (2.4.6) we can use Hölder's inequality and the fact that

$$\|v\|_p \leq \|v\|_\infty \leq |v(1)| + \|v'\|_p$$

to obtain, with the aid of (2.4.6), that for any $p > 1$, we have

$$\Big|\Big\langle \Big(\frac{\zeta\bar{v}}{U'}\Big)', \log(U+i\lambda)\Big\rangle\Big| \leq C(|v(1)| + \|v'\|_p).$$

Consequently, there exist $C > 0$ such that

$$\Big|\Big\langle \zeta, \frac{v}{U+i\lambda}\Big\rangle\Big| \leq C\left(\|v'\|_p + |v(1)|\log|d|^{-1}\right). \tag{2.6.73}$$

For the second term on the right-hand side of (2.6.72) we have, using (2.6.59) and Hardy's inequality (2.6.13)

$$\begin{aligned}\Big|\Big\langle\zeta(\phi-\phi(x_\nu)),\frac{v}{U+i\lambda}\Big\rangle\Big| &\leq |c_\parallel^\nu|\Big|\Big\langle\zeta(U-\nu),\frac{v}{U+i\lambda}\Big\rangle\Big| + \Big|\Big\langle\zeta\varphi_\perp^\nu,\frac{v}{U+i\lambda}\Big\rangle\Big| \\ &\leq C|c_\parallel^\nu|\,\|v\|_1 + \Big\|\frac{\varphi_\perp^\nu}{x-x_\nu}\Big\|_2\Big\|\frac{(x-x_\nu)\,v\,\zeta}{U+i\lambda}\Big\|_2 \\ &\leq \hat{C}\big(|c_\parallel^\nu|\,\|v\|_1 + \|(\varphi_\perp^\nu)'\|_2\|v\|_2\big). \end{aligned} \tag{2.6.74}$$

Substituting (2.6.73) and (2.6.74) into (2.6.72) yields

$$\begin{aligned}&\Big|\Big\langle\zeta\phi,\frac{v}{U+i\lambda}\Big\rangle\Big| \\ &\leq C\Big[|\phi(x_\nu)|(\|v'\|_p + |v(1)|\log|d|^{-1}) + |c_\parallel^\nu|\,\|v\|_1 + \|(\varphi_\perp^\nu)'\|_2\|v\|_2\Big].\end{aligned} \tag{2.6.75}$$

As $0<\nu<\nu_1<U(0)$ and $\operatorname{supp}\tilde{\zeta}\subset[0,x_\nu/4]$, it holds that $|\tilde{\zeta}(U+i\lambda)^{-1}|\leq C$ and hence,

$$\Big|\Big\langle\tilde{\zeta}\phi,\frac{v}{U+i\lambda}\Big\rangle\Big| \leq C\|\phi\|_\infty\|v\|_1 \leq C\|\phi'\|_2\|v\|_1. \tag{2.6.76}$$

Combining (2.6.76) with (2.6.75) and (2.6.59) then yields

$$\Big|\Big\langle\phi,\frac{v}{U+i\lambda}\Big\rangle\Big| \leq C\left[|\phi(x_\nu)|\big(\|v'\|_p + |v(1)|\log|d|^{-1}\big) + |c_\parallel^\nu|\,\|v\|_1 + \|(\varphi_\perp^\nu)'\|_2\|v\|_2\right].$$

With the aid of (2.6.4) we then obtain that

$$\Big|\Big\langle\phi,\frac{v}{U+i\lambda}\Big\rangle\Big| \leq C\left[\|v'\|_p^2 + |v(1)|^2\log^2|d|^{-1} + |c_\parallel^\nu|\,\|v\|_1 + \|(\varphi_\perp^\nu)'\|_2\|v\|_2\right]. \tag{2.6.77}$$

Substituting (2.6.77) into (2.6.39) leads to

$$|c_\parallel^\nu| \leq C\big(\nu^{-1}[\|v'\|_p + |v(1)|\log|d|^{-1} + \|v\|_1 + \|(\varphi_\perp^\nu)'\|_2^{1/2}\|v\|_2^{1/2}] + \nu^{-1/2}N(v,\lambda)\big). \tag{2.6.78}$$

Next, we substitute (2.6.77) into (2.6.57) to obtain, in view of (2.6.59)

$$\begin{aligned}\|(\varphi_\perp^\nu)'\|_2 \leq C\big(&\nu^{-1/2}[\|v'\|_p + |v(1)|\log\nu^{-1} + |c_\parallel^\nu|^{1/2}\,\|v\|_1^{1/2} \\ &+ \nu^{-1/2}\|v\|_2] + \nu^{-1/2}N(v,\lambda)\big).\end{aligned} \tag{2.6.79}$$

Combining (2.6.79) with (2.6.78) yields

$$\nu^{1/2}\|(\varphi_\perp^\nu)'\|_2 + \nu|c_\parallel^\nu| \leq C\big(\|v'\|_p + |v(1)|\log\nu^{-1} + \nu^{-1}\|v\|_1 + \nu^{-1/2}\|v\|_2 + N(v,\lambda)\big). \tag{2.6.80}$$

Substituting (2.6.80) and (2.6.4) into (2.6.77) leads to

$$\begin{aligned}|\phi(x_\nu)| &\leq \Big|\Big\langle\phi,\frac{v}{U+i\lambda}\Big\rangle\Big|^{1/2} \\ &\leq C\big(\|v'\|_p + |v(1)|\log\nu^{-1} + \nu^{-1}\|v\|_1 + \nu^{-1/2}\|v\|_2 + N(v,\lambda)\big).\end{aligned} \tag{2.6.81}$$

On the other hand, given the fact that $\phi(1) = 0$ and in view of (2.6.59), it holds that

$$|\phi(x)| = \Big|\int_x^1 \phi'(t)\,dt\Big| \le |c_\parallel^\nu|(1-x) + \|(\varphi_\perp^\nu)'\|_2(1-x)^{1/2}.$$

Consequently,

$$\Big|\Big\langle\phi, \frac{v}{U+i\lambda}\Big\rangle\Big| \le |c_\parallel^\nu|\Big\|(1-x)\,\frac{v}{U+i\lambda}\Big\|_1 + \|(\varphi_\perp^\nu)'\|_2\Big\|(1-x)^{1/2}\,\frac{v}{U+i\lambda}\Big\|_1. \tag{2.6.82}$$

Substituting (2.6.82) into (2.6.39) yields

$$|c_\parallel^\nu| \le \nu^{-2}\Big\|(1-x)\,\frac{v}{U+i\lambda}\Big\|_1 + \nu^{-1}\|(\varphi_\perp^\nu)'\|_2^{1/2}\Big\|(1-x)^{1/2}\,\frac{v}{U+i\lambda}\Big\|_1^{1/2} + \nu^{-1/2}N(v,\lambda). \tag{2.6.83}$$

Then, substituting (2.6.82) into (2.6.57) leads to

$$\|(\varphi_\perp^\nu)'\|_2 \le \nu^{-1/2}|c_\parallel^\nu|^{1/2}\Big\|(1-x)\,\frac{v}{U+i\lambda}\Big\|_1^{1/2} + \nu^{-1}\Big\|(1-x)^{1/2}\,\frac{v}{U+i\lambda}\Big\|_1 + \nu^{-1/2}N(v,\lambda). \tag{2.6.84}$$

Combining (2.6.83) and (2.6.84) we may conclude

$$\nu^{1/2}\|(\varphi_\perp^\nu)'\|_2 + \nu|c_\parallel^\nu| \le C\Big(\nu^{-1}\Big\|(1-x)\,\frac{v}{U+i\lambda}\Big\|_1^{1/2} + \nu^{-1/2}\Big\|(1-x)^{1/2}\frac{v}{U+i\lambda}\Big\|_1 + N(v,\lambda)\Big). \tag{2.6.85}$$

Combining (2.6.85) with (2.6.81) and (2.6.1) yields (2.6.2) for $\alpha^2 \le A$.

Step 6. There exists $A_0 \ge 0$ and $\widehat{C}$ such that if $\alpha^2 \ge A_0$, $|\mu| \le 1$, $\nu \in (0,\nu_1)$, then

$$\|\phi\|_{H^1(0,1)} \le \widehat{C}\,N(v,\lambda) \tag{2.6.86}$$

for any pair $(\phi, v) \in D(\mathcal{A}_{\lambda,\alpha}) \times W^{1,p}(0,1)$ satisfying (2.4.22).

We preliminarily observe that

$$\delta_2 = \sup_{\nu\in(0,\nu_1)}\Big\|\zeta_\nu\frac{U''}{U'}\Big\|_{1,\infty} < +\infty,$$

$$\widehat{C}_0 = \sup_{\substack{|\mu|\le 1\\ \nu\in(0,\nu_1)}}\|\log(U+i\lambda)\|_2 < +\infty$$

and

$$\widehat{C}_1 = \sup_{\substack{|\mu|\le 1\\ \nu\in(0,\nu_1)}}\Big\|\tilde{\zeta}_\nu\frac{U''}{U+i\lambda}\Big\|_\infty < +\infty,$$

where ζ_ν and $\tilde{\zeta}_\nu$ are defined in (2.6.70)–(2.6.71).

Taking (as in (2.4.15) for the case when $\alpha = 0$) the inner product of (2.3.2) with ϕ yields for the real part

$$\|\phi'\|_2^2 + \alpha^2\|\phi\|_2^2 = \Re\Big\langle \phi, \frac{v}{U+i\lambda}\Big\rangle - \Re\Big\langle \zeta U''\phi, \frac{\phi}{U+i\lambda}\Big\rangle - \Re\Big\langle \tilde{\zeta} U''\phi, \frac{\phi}{U+i\lambda}\Big\rangle. \tag{2.6.87}$$

For the first term on the right-hand side, we can use (5.2.16) to obtain

$$\Big|\Re\Big\langle \phi, \frac{v}{U+i\lambda}\Big\rangle\Big| \le C\,\|\phi'\|_2 N(v,\lambda).$$

For the second term we apply Poincaré's inequality, the finiteness of $\widehat{C}_0$ and δ_2, and the Sobolev embedding

$$\|\phi\|_\infty^2 \le 2\|\phi\|_2\|\phi'\|_2,$$

to conclude that for some new constant $\widehat{C}$

$$\begin{aligned}
&\Big|\Big\langle \zeta U''\phi, \frac{\phi}{U+i\lambda}\Big\rangle\Big| \\
&\quad \le \Big|\Big\langle \Big(\frac{U''}{U'}\zeta|\phi|^2\Big)', \log(U+i\lambda)\Big\rangle\Big| \\
&\quad \le \|\log(U+i\lambda)\|_2\|\phi\|_\infty\Big(2\Big\|\zeta\frac{U''}{U'}\Big\|_\infty\|\phi'\|_2 + \Big\|\Big(\zeta\frac{U''}{U'}\Big)'\Big\|_\infty\|\phi\|_2\Big) \\
&\quad \le \widehat{C}\,\|\phi'\|_2^{3/2}\|\phi\|_2^{1/2}.
\end{aligned}$$

For the last term on the right-hand side of (2.6.87), we have

$$\Big|\Big\langle \tilde{\zeta} U''\phi, \frac{\phi}{U+i\lambda}\Big\rangle\Big| \le \|\phi\|_2^2\Big\|\tilde{\zeta}\frac{U''}{U+i\lambda}\Big\|_\infty \le \widehat{C}_1\,\|\phi\|_2^2.$$

Consequently,

$$\|\phi'\|_2^2 + \alpha^2\|\phi\|_2^2 \le \widehat{C}\big(\|\phi'\|_2^{3/2}\|\phi\|_2^{1/2} + \|\phi\|_2^2 + \|\phi'\|_2 N(v,\lambda)\big).$$

Using Young's inequality we obtain, for some $A_0 \ge 0$ and $\widehat{C} > 0$

$$\frac{1}{2}\|\phi'\|_2^2 \le (A_0 - \alpha^2)\|\phi\|_2^2 + \widehat{C}\,N(v,\lambda)^2. \tag{2.6.88}$$

Hence, for $\alpha^2 \ge A_0$, (2.6.86) follows immediately from the above inequality in conjunction with Poincaré's inequality.

Conclusion. Observing that $\frac{1}{C}\nu \le d \le C\nu$, the proof of (2.6.2) follows from (2.6.39), (2.6.57), and (2.6.86). ■

2.7 The case $\Im\lambda < 0$

We now consider the case where $\Im\lambda$ is negative. Due to the non-invertibility of $\mathcal{A}_{0,0}$ the estimates of $\mathcal{A}_{\lambda,\alpha}^{-1}$ become challenging in the limit $\Im\lambda \to 0$ and the bounds necessarily include negative powers of $\nu = \Im\lambda$.

Proposition 2.7.1. *Let $U \in C^3([0,1])$ satisfy (2.1.3). Then there exist $C > 0$ such that for any $\alpha \geq 0$, and $(\phi, v) \in D(\mathcal{A}_{\lambda,\alpha}) \times W^{1,p}(0,1)$ (where $\mathcal{A}_{\lambda,\alpha}$ is defined in (2.1.1)) satisfying (2.4.22), we have for all $\nu < 0$,*

$$\|\phi\|_{1,2} \leq C(1 + |\nu|^{-1}) \left\|(1-x)^{1/2}\frac{v}{U+i\lambda}\right\|_1, \tag{2.7.1a}$$

and, for $-1/2 < \nu < 0$

$$\|\phi'\|_{L^2(1-|\nu|^{1/2},1)} \leq C\,|\nu|^{-3/4}\left\|(1-x)^{1/2}\frac{v}{U+i\lambda}\right\|_1. \tag{2.7.1b}$$

Furthermore, it holds that for all $\nu < 0$

$$|\phi(x)| \leq C(1-x)^{1/2}[1 + |\nu|^{-1/2}(1-x)^{1/2}]\left|\left\langle \phi, \frac{v}{U+i\lambda}\right\rangle\right|^{1/2}. \tag{2.7.2}$$

Proof. We begin by rewriting $\mathcal{A}_{\lambda,\alpha}\phi = v$ in the form

$$-((U-\nu)^2 w')' + \alpha^2 (U-\nu)^2 w = v - i\mu\frac{v - U''\phi}{U+i\lambda} = \frac{(U-\nu)v + i\mu U''\phi}{U+i\lambda},$$

where

$$w = \phi/(U-\nu).$$

Taking the inner product with w on the left yields (see (2.6.24) and (2.6.47))

$$\|(U-\nu)w'\|_2^2 + \alpha^2\|\phi\|_2^2 = \left\langle \phi, \frac{v}{U+i\lambda}\right\rangle + i\mu\left\langle w, \frac{U''\phi}{U+i\lambda}\right\rangle.$$

For the last term on the right-hand side we have, since $U'' < 0$ and $U - \nu > 0$,

$$\Re\left(i\mu\left\langle w, \frac{U''\phi}{U+i\lambda}\right\rangle\right) = |\mu|^2\left\langle \frac{\phi}{U-\nu}, \frac{U''\phi}{|U+i\lambda|^2}\right\rangle < 0.$$

Hence,

$$\|(U-\nu)w'\|_2^2 + \alpha^2\|\phi\|_2^2 \leq \Re\left\langle \phi, \frac{v}{U+i\lambda}\right\rangle. \tag{2.7.3}$$

We now write, using the fact that $U(x) - \nu \geq C^{-1}(1-x-\nu)$,

$$\begin{aligned}|w(x)| &\leq \left|\int_x^1 w'(t)\,dt\right| \leq \left[\int_x^1 \frac{1}{(U-\nu)^2}\,dt\right]^{1/2}\|(U-\nu)w'\|_2 \\ &\leq C\frac{(1-x)^{1/2}}{|\nu|^{1/2}(1-x-\nu)^{1/2}}\|(U-\nu)w'\|_2.\end{aligned} \tag{2.7.4}$$

Using this time the bound $U(x) - \nu \le 1 - x - \nu$ together with (2.7.3) and (2.7.4), we obtain (2.7.2). Integrating (2.7.4) squared over $[0, 1]$ yields

$$\|w\|_2^2 \le C\,|\nu|^{-1}\,\|(U-\nu)w'\|_2^2. \tag{2.7.5}$$

By writing $\phi = (U - \nu)w$, we obtain that

$$\|\phi'\|_2^2 \le 2\,\|(U-\nu)w'\|_2^2 + 2\,\|U'w\|_2^2,$$

which leads, together with (2.7.5), to

$$\|\phi'\|_2^2 \le C(|\nu|^{-1}+1)\|(U-\nu)w'\|_2^2. \tag{2.7.6}$$

We can now establish (2.7.1a) by combining (2.7.6) with (2.7.3) and the fact that $|\phi(x)| \le \|\phi'\|_2(1-x)^{1/2}$.

To obtain (2.7.1b) we write for $-1/2 < \nu < 0$,

$$\|\phi'\|^2_{L^2(1-|\nu|^{1/2},1)} \le 2\,\|(U-\nu)w'\|^2_{L^2(1-|\nu|^{1/2},1)} + 2\,\|U'w\|^2_{L^2(1-|\nu|^{1/2},1)}. \tag{2.7.7}$$

By (2.7.4) and the fact that for all $x \in [0, 1]$

$$0 \le \frac{1-x}{|\nu|(1-x-\nu)} \le \frac{1}{|\nu|},$$

we obtain via integration over $(1 - |\nu|^{1/2}, 1)$ that

$$\|U'w\|^2_{L^2(1-|\nu|^{1/2},1)} \le C\,\|w\|^2_{L^2(1-|\nu|^{1/2},1)} \le \hat{C}\,|\nu|^{-1/2}\|(U-\nu)w'\|_2^2.$$

Substituting the above into (2.7.7) yields

$$\|\phi'\|^2_{L^2(1-|\nu|^{1/2},1)} \le C\,|\nu|^{-1/2}\|(U-\nu)w'\|_2^2. \tag{2.7.8}$$

By (2.7.3) and (2.7.1a) we then obtain that for $-1/2 < \nu < 0$,

$$\begin{aligned}\|\phi'\|^2_{L^2(1-|\nu|^{1/2},1)} &\le C\,|\nu|^{-1/2}\|\phi'\|_2\Big\|(1-x)^{1/2}\frac{v}{U+i\lambda}\Big\|_1\\ &\le \hat{C}\,|\nu|^{-3/2}\,\Big\|(1-x)^{1/2}\frac{v}{U+i\lambda}\Big\|_1^2,\end{aligned}$$

readily verifying (2.7.1b). ∎

2.8 The case $\nu_1 \le \Im\lambda < U(0) - \kappa_0|\Re\lambda|$, $|\Re\lambda|$ small

In the following, we establish estimates, similar to (2.5.3), for $\nu_1 \le \nu < U(0) - \kappa_0|\mu|$, for sufficiently small $U(0) - \nu_1$ and $|\mu|$.

Proposition 2.8.1. *Let $U \in C^4([0,1])$ satisfy (2.1.3) and $U'''(0) = 0$. Let further $p \geq 2$. There exist $0 < \nu_1 < U(0)$, $\mu_0 > 0$, $\kappa_0 > 0$, and $C > 0$ such that for λ s.t. $0 < |\mu| \leq \mu_0$ and $\nu_1 < \nu < U(0) - \kappa_0|\mu|$, for all $\alpha \geq 0$ we have, for all pair $(\phi, v) \in D(\mathcal{A}_{\lambda,\alpha}) \times L^\infty(0,1)$ satisfying (2.4.22),*

$$\|\phi\|_{1,2} \leq \frac{C}{|\mu|^{\frac{1}{p}}\, x_\nu^{1/2-1/p}} \|v\|_p \tag{2.8.1a}$$

and

$$\|\phi\|_{1,2} \leq C \frac{\log \frac{x_\nu}{|\mu|^{1/2}}}{x_\nu^{1/2}} \|v\|_\infty. \tag{2.8.1b}$$

Proof. Since $0 < \nu < U(0)$ and since by (2.4.5) $U(x_\nu) = \nu$, we must have that $x_\nu \in (0,1)$.

Step 1. We prove that there exist $C > 0$, $\mu_0 > 0$, and $\nu_1 < U(0)$, such that, for all $\lambda = \mu + i\nu$ such that $\nu_1 < \nu < U(0) - |\mu|$ and $0 < |\mu| \leq \mu_0$ it holds that

$$|\phi(x_\nu)|^2 \leq C x_\nu \Big[\frac{\mu}{x_\nu^2} \|\phi'\|_2^2 + \Big| \Big\langle \phi, \frac{v}{U + i\lambda} \Big\rangle \Big| \Big] \tag{2.8.2}$$

for all pairs $(\phi, v) \in D(\mathcal{A}_{\lambda,\alpha}) \times L^\infty(0,1)$ satisfying (2.4.22).

As in (2.6.6) we write

$$\Big| \Im \Big\langle \phi, \frac{v}{U - \nu + i\mu} \Big\rangle \Big| \geq |\mu| \Big\langle \frac{|U''|}{(U-\nu)^2 + \mu^2}, \frac{1}{2}|\phi(x_\nu)|^2 - |\phi(x) - \phi(x_\nu)|^2 \Big\rangle. \tag{2.8.3}$$

We begin by observing that for some $C > 0$

$$\Big| U(x) - \nu - \frac{1}{2} U''(0)[2x_\nu(x - x_\nu) + (x - x_\nu)^2] \Big| \leq C[(x - x_\nu)^4 + x_\nu^3 |x - x_\nu|]. \tag{2.8.4}$$

By (2.8.4) we have

$$\frac{1}{2}U''(0)|x^2 - x_\nu^2| - C[(x - x_\nu)^4 + x_\nu^3|x - x_\nu|] \\ \leq |U - \nu| \leq \frac{1}{2}U''(0)|x^2 - x_\nu^2| + C[(x - x_\nu)^4 + x_\nu^3|x - x_\nu|]. \tag{2.8.5}$$

From the right inequality in (2.8.5), as $(x - x_\nu)^3 + x_\nu^3 \leq 2(x + x_\nu)$, we conclude that there exists $C_1 > 0$ such that

$$|U - \nu| \leq C_1 |x^2 - x_\nu^2|. \tag{2.8.6}$$

From the left inequality in (2.8.5), there exist $a_0 > 0$ and $\nu_1 < U(0)$ such that for all $\nu_1 < \nu < U(0)$ and $x + x_\nu \leq a_0$ it holds that

$$\frac{1}{2}U''(0)|x^2 - x_\nu^2| - C[(x - x_\nu)^4 + x_\nu^3|x - x_\nu|] \geq \frac{1}{4}U''(0)|x^2 - x_\nu^2|, \tag{2.8.7}$$

from which we can conclude that whenever $x + x_\nu \leq a_0$, we have

$$|U - \nu| \geq \frac{1}{C}|x^2 - x_\nu^2|. \tag{2.8.8}$$

On the other hand we have that, for $\nu_1 < \nu < U(0)$ such that $x_{\nu_1} < \frac{a_0}{2}$,

$$\inf_{x \geq a_0 - x_\nu} \frac{|U - \nu|}{x^2 - x_\nu^2} \geq \frac{1}{a_0} \inf_{x \geq a_0 - x_\nu} \frac{|U - \nu|}{x - x_\nu} \geq \frac{|U'(a_0 - x_{\nu_1})|}{a_0} > 0.$$

Combining the above with (2.8.8) and (2.8.6) yields the existence of $0 < \nu_1 < U(0)$ and $C > 0$ for which

$$\frac{1}{C}(x^2 - x_\nu^2)^2 \leq (U(x) - \nu)^2 \leq C(x^2 - x_\nu^2)^2 \tag{2.8.9}$$

for all $x \in [0, 1]$ and $\nu_1 < \nu < U(0)$.

From (2.8.9) we can get that

$$\int_0^1 \frac{|U''|}{(U - \nu)^2 + \mu^2}\, dx \geq \frac{1}{C} \int_0^1 \frac{1}{(x - x_\nu)^2(x + x_\nu)^2 + \mu^2}\, dx.$$

As

$$\sup_{x \in [1,\infty)} \frac{x^4}{(x - x_\nu)^2(x + x_\nu)^2 + \mu^2} \leq \frac{1}{(1 - x_{\nu_1}^2)^2},$$

we obtain that

$$\begin{aligned}
&\int_0^1 \frac{dx}{(x - x_\nu)^2(x + x_\nu)^2 + \mu^2} \\
&\quad \geq \int_0^\infty \frac{dx}{(x - x_\nu)^2(x + x_\nu)^2 + \mu^2} - \frac{1}{(1 - x_{\nu_1}^2)^2} \int_1^\infty \frac{dx}{x^4} \\
&\quad = \int_0^\infty \frac{dx}{(x - x_\nu)^2(x + x_\nu)^2 + \mu^2} - C_2,
\end{aligned}$$

where $C_2 := \frac{1}{3(1 - x_{\nu_1}^2)^2}$.

Using the substitution $\xi = x/x_\nu$ it can be easily verified that

$$\int_0^\infty \frac{dx}{(x - x_\nu)^2(x + x_\nu)^2 + \mu^2} = \frac{1}{2x_\nu^3} \int_{-\infty}^\infty \frac{d\xi}{(\xi^2 - 1)^2 + \hat{a}^2},$$

where $\hat{a} = |\mu|/x_\nu^2$.

Hence, there exists $C > 0$ such that

$$\int_0^1 \frac{|U''|}{(U - \nu)^2 + \mu^2}\, dx \geq \frac{1}{C}\Big[\frac{1}{x_\nu^3} \int_{-\infty}^\infty \frac{dx}{(x^2 - 1)^2 + \hat{a}^2} - 2C_2\Big]. \tag{2.8.10}$$

Making use of the residue theorem yields

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2-1)^2+\hat{a}^2} = 2\pi i\Big[\operatorname{Res}\Big(\frac{1}{(z^2-1)^2+\hat{a}^2}, \sqrt{1+i\hat{a}}\Big) + \operatorname{Res}\Big(\frac{1}{(z^2-1)^2+\hat{a}^2}, -\sqrt{1-i\hat{a}}\Big)\Big] = \frac{\pi\Re(\sqrt{1+i\hat{a}})}{\hat{a}\sqrt{\hat{a}^2+1}}. \tag{2.8.11}$$

Consequently, given that, by (2.8.6) applied with $x = 0$, it holds, for sufficiently small $U(0) - \nu$, that

$$\mu < U(0) - \nu \le C_1 x_\nu^2,$$

hence $x_\nu^2 \ge c_0\mu$ with $c_0 = 1/C_1$ and finally that $\hat{a} \le \frac{1}{c_0}$.

Consequently, there exists $0 < \nu_1 < U(0)$ and $\mu_0 > 0$ such that for all $\nu_1 < \nu < U(0) - |\mu|$ and $0 < |\mu| \le \mu_0$ we have for some $\widehat{C} > 0$

$$\int_0^1 \frac{|U''|}{(U-\nu)^2+\mu^2}\,dx \ge \frac{1}{C}\Big[\frac{1}{x_\nu^3}\frac{\pi\Re(\sqrt{1+i\hat{a}})}{\hat{a}\sqrt{\hat{a}^2+1}} - 2C_2\Big] \ge \frac{\widehat{C}}{|\mu|x_\nu}. \tag{2.8.12}$$

In a similar manner we can also show the existence of a positive C such that

$$\int_0^1 \frac{|U''|}{(U-\nu)^2+\mu^2}\,dx \le \widetilde{C}\int_0^\infty \frac{dx}{(x-x_\nu)^2(x+x_\nu)^2+\mu^2}\,dx \le \frac{C}{|\mu|x_\nu}. \tag{2.8.13}$$

We proceed with the estimation of the right-hand side of (2.8.3) by observing that, in view of (2.8.9)

$$\Big\langle \frac{|U''|}{(U-\nu)^2+\mu^2}, |\phi(x)-\phi(x_\nu)|^2\Big\rangle \le C\Big\|\frac{\phi(x)-\phi(x_\nu)}{x^2-x_\nu^2}\Big\|_2^2. \tag{2.8.14}$$

Applying Hardy's inequality yields

$$\Big\|\frac{\phi(x)-\phi(x_\nu)}{x^2-x_\nu^2}\Big\|_2^2 \le C\Big\|\Big(\frac{\phi(x)-\phi(x_\nu)}{x+x_\nu}\Big)'\Big\|_2^2. \tag{2.8.15}$$

As

$$\Big\|\Big(\frac{\phi(x)-\phi(x_\nu)}{x+x_\nu}\Big)'\Big\|_2^2 \le 2\Big\|\frac{\phi'}{x+x_\nu}\Big\|_2^2 + 2\Big\|\frac{\phi-\phi(x_\nu)}{(x+x_\nu)^2}\Big\|_2^2,$$

and since by Hardy's inequality (2.2.8)

$$\Big\|\frac{\phi-\phi(x_\nu)}{(x+x_\nu)^2}\Big\|_2^2 \le \frac{1}{x_\nu^2}\Big\|\frac{\phi-\phi(x_\nu)}{x-x_\nu}\Big\|_2^2 \le \frac{C}{x_\nu^2}\|\phi'\|_2^2,$$

we obtain that

$$\Big\|\Big(\frac{\phi(x)-\phi(x_\nu)}{x+x_\nu}\Big)'\Big\|_2^2 \le \frac{C}{x_\nu^2}\|\phi'\|_2^2.$$

Substituting the above into (2.8.15) and then into (2.8.14) yields

$$\Big\langle \frac{|U''|}{(U-\nu)^2+\mu^2}, |\phi(x)-\phi(x_\nu)|^2\Big\rangle \le \frac{C}{x_\nu^2}\|\phi'\|_2^2 \tag{2.8.16}$$

which, when substituted into (2.8.3) together with (2.8.12) leads to (2.8.2).

Step 2. We prove that there exist $0<\nu_1<U(0)$, positive μ_0, C, and C_0, and $\kappa_0\ge 1$ such that, for all $\alpha \ge C_0/x_\nu$, $\nu_1<\nu<U(0)-\kappa_0|\mu|$, and $0<|\mu|\le\mu_0$, the inequality

$$\|\phi\|_{1,2} \le C\Big|\Big\langle \phi, \frac{v}{U+i\lambda}\Big\rangle\Big|^{1/2}, \tag{2.8.17}$$

holds for any pair $(\phi, v)\in D(\mathcal{A}_{\lambda,\alpha})\times L^\infty(0,1)$ satisfying (2.4.22).

Let

$$\tilde U = U(0)-U \quad\text{and}\quad \kappa=\sqrt{(i\lambda+U(0))}.$$

Note that $|\kappa|>|\mu|^{1/2}>0$. Clearly,

$$\frac{1}{U+i\lambda} = -\frac{1}{(\tilde U^{1/2}-\kappa)(\tilde U^{1/2}+\kappa)} = -\frac{1}{2\kappa}\Big[\frac{1}{\tilde U^{1/2}-\kappa}-\frac{1}{\tilde U^{1/2}+\kappa}\Big].$$

We now write

$$\int_0^1 \frac{U''}{U+i\lambda}\,dx = -\frac{1}{4\kappa}\int_0^1 U''\Big[\frac{1}{\tilde U^{1/2}-\kappa}-\frac{1}{\tilde U^{1/2}+\kappa}\Big]\,dx.$$

An integration by parts now yields

$$\int_0^1 U''\Big[\frac{1}{\tilde U^{1/2}-\kappa}-\frac{1}{\tilde U^{1/2}+\kappa}\Big]\,dx$$
$$= \frac{1}{2}\frac{U''}{\tilde U^{-1/2}U'}\log\frac{\tilde U^{1/2}-\kappa}{\tilde U^{1/2}+\kappa}\Big|_{x=0}^{x=1} + \frac{1}{2}\int_0^1 \Big(\frac{U''}{\tilde U^{-1/2}U'}\Big)'\log\frac{\tilde U^{1/2}-\kappa}{\tilde U^{1/2}+\kappa}\,dx$$

Since $\max_{x\in[0,1]}(\tilde U^{-1/2}U')(x)<0$ and since $\tilde U^{-1/2}U'\in C^2([0,1])$ we can conclude that there exist positive C_1, C_2, C such that

$$\Big|\int_0^1 \frac{U''}{U+i\lambda}\,dx\Big| \le C_1+\frac{C_2}{\kappa}\int_0^1 [|\log\tilde U^{1/2}-\kappa|+\log\tilde U^{1/2}+\kappa|]\,dx \le \frac{C}{\kappa}.$$

Since by (2.8.8) it holds that $\kappa\ge\sqrt{U(0)-\nu}\ge x_\nu/C$, we conclude from the above that there exists $\hat C>0$ such that

$$\Big|\int_0^1 \frac{U''}{U+i\lambda}\,dx\Big| \le \frac{\hat C}{x_\nu}. \tag{2.8.18}$$

Taking the inner product of (2.4.22) with $\phi/(U+i\lambda)$ yields for the real part

$$\Re\Big\langle \phi, \frac{v}{U+i\lambda}\Big\rangle = \|\phi'\|_2^2+\alpha^2\|\phi\|_2^2+\Re\Big\langle U''\phi, \frac{\phi}{U+i\lambda}\Big\rangle. \tag{2.8.19}$$

We then write

$$\|\phi'\|_2^2 + \alpha^2\|\phi\|_2^2 \le \Re\Big\langle \phi, \frac{v}{U+i\lambda}\Big\rangle + \Big[\int_0^1 \frac{|U''(x)|\,\big|\,|\phi(x)|^2 - |\phi(x_\nu)|^2\,\big|}{|U+i\lambda|}\,dx + |\phi(x_\nu)|^2\Big|\int_0^1 \frac{U''(x)}{U+i\lambda}\,dx\Big|\Big]. \tag{2.8.20}$$

By (2.8.18) it holds that

$$|\phi(x_\nu)|^2\Big|\int_0^1 \frac{U''(x)}{U+i\lambda}\,dx\Big| \le \frac{\widehat{C}}{x_\nu}|\phi(x_\nu)|^2. \tag{2.8.21}$$

To estimate the first term on the right-hand side of (2.8.21) we use the inequality

$$\big|\,|\phi(x)|^2 - |\phi(x_\nu)|^2\,\big| \le [|\phi(x)| + |\phi(x_\nu)|]\,|\phi(x) - \phi(x_\nu)|$$

together with Hardy's inequality and (2.8.9) to obtain that for some $C_0 > 0$

$$\begin{aligned}\int_0^1 &\frac{|U''(x)|\big|\,|\phi(x)|^2 - |\phi(x_\nu)|^2\,\big|}{|U+i\lambda|}\,dx \\ &\le \Big\|\frac{x^2 - x_\nu^2}{U+i\lambda}\Big\|_\infty \Big\|\frac{|\phi| + |\phi(x_\nu)|}{x+x_\nu}\Big\|_2 \Big\|\frac{\phi - \phi(x_\nu)}{x - x_\nu}\Big\|_2 \\ &\le C_0[x_\nu^{-1}\|\phi\|_2 + x_\nu^{-1/2}|\phi(x_\nu)|]\,\|\phi'\|_2.\end{aligned} \tag{2.8.22}$$

Note that to obtain the last inequality we need the estimate

$$\Big\|\frac{\phi(x_\nu)}{x+x_\nu}\Big\|_2 \le C\,\frac{|\phi(x_\nu)|}{x_\nu^{1/2}}.$$

Substituting (2.8.22) together with (2.8.21) into (2.8.20) yields the existence of $\widehat{C}_0 > 0$ and $C > 0$ such that, for $\alpha \ge \widehat{C}_0/x_\nu$,

$$\|\phi'\|_2^2 \le \frac{C}{x_\nu}|\phi(x_\nu)|^2 + 2\Big|\Big\langle \phi, \frac{v}{U+i\lambda}\Big\rangle\Big|.$$

Substituting (2.8.2) into the above yields

$$\|\phi'\|_2^2 \le \frac{|\mu|}{x_\nu^2}|\phi(x_\nu)|^2 + (2 + Cx_\nu)\Big|\Big\langle \phi, \frac{v}{U+i\lambda}\Big\rangle\Big|.$$

For sufficiently large κ_0 (or equivalently for sufficiently small $|\mu|/x_\nu^2$) we obtain (2.8.17).

Step 3. With $N_2(v,\lambda)$ given by

$$N_2(v,\lambda) = \Big\|(1-x)^{1/2}\frac{v}{U+i\lambda}\Big\|_1, \tag{2.8.23}$$

we prove that, for any $\widehat{C}_0 > 0$, there exist $C > 0$, $\mu_0 > 0$, $\kappa_0 > 0$, and $\nu_1 > 0$ such that, for $\nu_1 < \nu < U(0) - \kappa_0|\mu|$, $|\alpha| \leq \widehat{C}_0/x_\nu$, and $|\mu| \leq \mu_0$, we have

$$\|\phi'\|_{L^2(0,x_\nu)} \leq C\Big(\Big|\Big\langle\phi, \frac{v}{U+i\lambda}\Big\rangle\Big|^{1/2} + x_\nu^{1/2}N_2(v,\lambda)\Big), \tag{2.8.24}$$

holds for any pair $(\phi, v) \in D(\mathcal{A}_{\lambda,\alpha}) \times W^{1,p}(0,1)$ satisfying (2.4.22).

We seek an estimate for $\|\phi'\|_{L^2(0,x_\nu)}$, depending on $\|\phi\|_{L^2(0,x_\nu)}$. We begin, to this end, by obtaining an L^∞ estimate of ϕ'.

Estimate of $\|\phi'\|_{L^\infty}$. We separately consider the subintervals $(0, x_\nu/2)$ and $(x_\nu/2, x_\nu)$.

Estimate on $(0, x_\nu/2)$. To obtain an estimate for $\|\phi'\|_{L^\infty(0,x_\nu/2)}$ we integrate the relation $\mathcal{A}_{\lambda,\alpha}\phi = v$, to obtain for all $x \in (0, x_\nu)$,

$$|\phi'(x)| = \Big|\int_0^x \phi''(t)\,dt\Big| \leq \Big|\int_0^x \Big(\frac{U''\phi - v}{U+i\lambda} + \alpha^2\phi\Big)\,dt\Big|,$$

which leads to

$$|\phi'(x) \leq \Big|\int_0^x \Big(\frac{U''\phi}{U+i\lambda}\Big)\,dt\Big| + \Big\|\frac{v}{U+i\lambda}\Big\|_{L^1(0,x)} + \alpha^2\|\phi\|_{L^1(0,x)}. \tag{2.8.25}$$

We then use the following decomposition

$$\int_0^x \frac{U''\phi}{U+i\lambda}\,dt = \int_0^x \frac{U''[\phi - \phi(x_\nu)]}{U+i\lambda}\,dt + \phi(x_\nu)\int_0^x \frac{U''}{U+i\lambda}\,dt. \tag{2.8.26}$$

To estimate the first integral on the right-hand side of (2.8.26) we need the following bound which follows from (2.8.4):

$$\Big|\frac{U''}{U-\nu} - \frac{2}{x^2 - x_\nu^2}\Big| \leq \Big|\frac{U''(0)}{U-\nu} - \frac{2}{x^2 - x_\nu^2}\Big| + \Big|\frac{U''(x) - U''(0)}{U-\nu}\Big| \leq C\frac{x_\nu}{x_\nu^2 - x^2}.$$

Consequently,

$$\begin{aligned}\Big|\int_0^x \frac{U''[\phi - \phi(x_\nu)]}{U+i\lambda}\,dt\Big| &\leq \int_0^x \frac{|U''|\,|\phi - \phi(x_\nu)|}{|U-\nu|}\,dt \\ &\leq (2 + Cx_\nu)\int_0^x \frac{|\phi(t) - \phi(x_\nu)|}{x_\nu^2 - t^2}\,dt,\end{aligned} \tag{2.8.27}$$

and hence, for sufficiently small $U(0) - \nu_1$,

$$\begin{aligned}\Big|\int_0^x \frac{U''[\phi - \phi(x_\nu)]}{U+i\lambda}\,dt\Big| &\leq (2 + Cx_\nu)\log\frac{x_\nu + x}{x_\nu}\|\phi'\|_{L^\infty(0,x)} \\ &\leq (1 + Cx_\nu)\log(9/4)\|\phi'\|_{L^\infty(0,x)}.\end{aligned} \tag{2.8.28}$$

To obtain the last inequality we have used the fact that $x \in (0, x_\nu/2)$. Note that $\log(9/4) < 1$. Hence, the coefficient of $|\phi'\|_{L^\infty(0,x)}$ is smaller than one for sufficiently small $U(0) - \nu_1$. Next, we write for the second integral in the right-hand side of equation (2.8.26)

$$\left|\phi(x_\nu)\int_0^x \frac{U''}{U+i\lambda}\,dt\right| \le |\phi(x_\nu)|\left|\int_0^x \frac{|U''|}{|U-\nu|}\,dt\right| \le (2+Cx_\nu)|\phi(x_\nu)|\left|\int_0^x \frac{dt}{x_\nu^2-t^2}\right.,$$

which leads to

$$\left|\phi(x_\nu)\int_0^x \frac{U''}{U+i\lambda}\,dt\right| \le \hat{C}\,\frac{|\phi(x_\nu)|}{x_\nu}\,\log\frac{x_\nu+x}{x_\nu-x}. \tag{2.8.29}$$

Substituting the above, together with (2.8.29) and (2.8.28) into (2.8.25) yields for all $t \in (0,x] \subset (0, x_\nu/2]$

$$|\phi'(t)| \le \log\frac{9}{4}(1+Cx_\nu)\|\phi'\|_{L^\infty(0,t)} + C\,\frac{|\phi(x_\nu)|}{x_\nu}\log\frac{x_\nu+t}{x_\nu-t} + \left\|\frac{v}{U+i\lambda}\right\|_{L^1(0,x)} + \alpha^2\|\phi\|_{L^1(0,x)}.$$

Taking the supremum over $t \in (0,x]$ yields

$$\|\phi'(t)\|_{L^\infty(0,x)} \le \log\frac{9}{4}(1+Cx_\nu)\|\phi'\|_{L^\infty(0,x)} + C\,\frac{|\phi(x_\nu)|}{x_\nu}\log\frac{x_\nu+x}{x_\nu-x} + \left\|\frac{v}{U+i\lambda}\right\|_{L^1(0,x)} + \alpha^2\|\phi\|_{L^1(0,x)}.$$

Hence, for sufficiently small $U(0) - \nu_1$, we obtain for all $\nu_1 < \nu < U(0)$ and all $x \in [0, x_\nu/2]$,

$$\|\phi'\|_{L^\infty(0,x)} \le C\Big(\frac{|\phi(x_\nu)|}{x_\nu}\log\frac{x+x_\nu}{x-x_\nu} + \alpha^2\|\phi\|_{L^1(0,x)} + \left\|\frac{v}{U+i\lambda}\right\|_{L^1(0,x)}\Big). \tag{2.8.30}$$

Estimate on $(x_\nu/2, x_\nu)$. To obtain a bound for $\|\phi'\|_{L^\infty(x_\nu/2,x)}$ for $x \in (x_\nu/2, x_\nu)$ we write

$$|\phi'(x)| \le |\phi'(x_\nu/2)| + \left|\int_{x_\nu/2}^x \phi''(t)\,dt\right|.$$

The first term can be estimated by using (2.8.32). To obtain a bound for the second term, we follow the same path as in (2.8.27). We get

$$\left|\int_{x_\nu/2}^x \frac{U''[\phi-\phi(x_\nu)]}{U+i\lambda}\,dt\right| \le (2+Cx_\nu)\int_{x_\nu.}^x \frac{|\phi(t)-\phi(x_\nu)|}{x_\nu^2-t^2}\,dt. \tag{2.8.31}$$

As in (2.8.28) we can conclude that

$$\begin{aligned}(2+Cx_\nu)\int_{x_\nu/2}^x \frac{|\phi(t)-\phi(x_\nu)|}{x_\nu^2-t^2}\,dt &\le (2+Cx_\nu)\|\phi'\|_{L^\infty(x_\nu/2,x)}\log\frac{x+x_\nu}{3x_\nu/2}\\ &\le (2+Cx_\nu)\log(4/3)\|\phi'\|_{L^\infty(x_\nu/2,x)}.\end{aligned}$$

Hence, the coefficient of $\|\phi'\|_{L^\infty(x_\nu/2,x)}$ is again smaller than 1 by a suitable choice of ν_1. Repeating the above other steps then yields, for $x \geq x_\nu/2$,

$$\|\phi'\|_{L^\infty(x_\nu/2,x)} \leq C\Big(\frac{1}{x_\nu}|\phi(x_\nu)|\log\frac{x_\nu+x}{x_\nu-x}+\alpha^2\|\phi\|_{L^1(0,x)}+\Big\|\frac{v}{U+i\lambda}\Big\|_{L^1(0,x)}+|\phi'(x_\nu/2)|\Big),$$

which, combined with (2.8.30) for $x = x_\nu/2$, finally gives

$$\|\phi'\|_{L^\infty(x_\nu/2,x)} \leq \hat{C}\Big(\frac{1}{x_\nu}|\phi(x_\nu)|\log\frac{x_\nu+x}{x_\nu-x}+\alpha^2\|\phi\|_{L^1(0,x)}+\Big\|\frac{v}{U+i\lambda}\Big\|_{L^1(0,x)}\Big).$$

Combining the above and (2.8.30) lead to the existence of $C > 0$ such that

$$\|\phi'\|_{L^\infty(0,x)} \leq C\Big(\frac{1}{x_\nu}|\phi(x_\nu)|\log\frac{x_\nu+x}{x_\nu-x}+\alpha^2\|\phi\|_{L^1(0,x)}+\Big\|\frac{v}{U+i\lambda}\Big\|_{L^1(0,x)}\Big) \quad \forall x \in (0,x_\nu). \tag{2.8.32}$$

Estimate of $\|\phi'\|_{L^2(0,x_\nu)}$. Observing that

$$\int_0^{x_\nu}\log^2\frac{x_\nu+x}{x_\nu-x}\,dx \leq x_\nu\int_0^1\log^2\frac{1+t}{1-t}\,dt \leq Cx_\nu, \tag{2.8.33}$$

we may conclude from (2.8.32) by integrating over $(0, x_\nu)$, that

$$\|\phi'\|_{L^2(0,x_\nu)} \leq C\Big(x_\nu^{-1/2}|\phi(x_\nu)| + \alpha^2 x_\nu\|\phi\|_{L^2(0,x_\nu)} + x_\nu^{1/2}\Big\|\frac{v}{U+i\lambda}\Big\|_{L^1(0,x_\nu)}\Big).$$

Note that

$$\Big\|\frac{v}{U+i\lambda}\Big\|_{L^1(0,x_\nu)} \leq (1-x_\nu)^{-\frac{1}{2}}N_2(v,\lambda), \tag{2.8.34}$$

which leads to

$$\|\phi'\|_{L^2(0,x_\nu)} \leq C\big(x_\nu^{-1/2}|\phi(x_\nu)| + \alpha^2 x_\nu\|\phi\|_{L^2(0,x_\nu)} + x_\nu^{1/2}N_2(v,\lambda)\big). \tag{2.8.35}$$

Estimate of $\|\phi\|_{L^2(0,x_\nu)}$. Set

$$\phi = \varphi + \phi(x_\nu),$$

and recall from (2.6.23) that for all $x \in (0, x_\nu)$

$$-\Big((U-\nu)^2\Big(\frac{\varphi}{U-\nu}\Big)'\Big)' + \alpha^2(U-\nu)\varphi = \frac{(U-\nu)v}{U+i\lambda} - \alpha^2\phi(x_\nu)\big((U-\nu)-U''\big) + i\mu\frac{U''\phi}{U+i\lambda}.$$

Taking the inner product, in $L^2(0, x_\nu)$, with w defined as in equation (2.6.22) by $w := (U - \nu)^{-1}\varphi$, yields as in (2.6.24)

$$\begin{aligned}\|(U-\nu)w'\|^2_{L^2(0,x_\nu)} &+ \alpha^2\|\varphi\|^2_{L^2(0,x_\nu)} \\ &= \Big\langle \varphi, \frac{v}{U+i\lambda}\Big\rangle_{L^2(0,x_\nu)} - \langle w, \phi(x_\nu)U''\rangle_{L^2(0,x_\nu)} \\ &\quad - \alpha^2\phi(x_\nu)\langle\varphi,1\rangle_{L^2(0,x_\nu)} + i\mu\Big\langle w, \frac{U''\phi}{U+i\lambda}\Big\rangle_{L^2(0,x_\nu)}. \end{aligned} \tag{2.8.36}$$

We now turn to estimate the various terms on the right-hand side of (2.8.36).

For the second term in (2.8.36) we use the fact that by Hardy's inequality and (2.8.9) it holds that

$$\|w\|_{L^1(0,x_\nu)} \le C\Big\|\frac{\phi-\phi(x_\nu)}{x-x_\nu}\Big\|_{L^2(0,x_\nu)}\Big\|\frac{1}{x+x_\nu}\Big\|_{L^2(0,x_\nu)} \le \frac{\widehat{C}}{x_\nu^{1/2}}\|\phi'\|_{L^2(0,x_\nu)}. \tag{2.8.37}$$

Consequently,

$$|\langle w, \phi(x_\nu)U''\rangle_{L^2(0,x_\nu)}| \le C\frac{|\phi(x_\nu)|}{x_\nu^{1/2}}\|\phi'\|_{L^2(0,x_\nu)}. \tag{2.8.38}$$

For the third term in (2.8.36), it follows from that

$$\alpha^2|\phi(x_\nu)\langle\varphi,1\rangle| \le C\alpha^2 x_\nu^{1/2}(\|\phi\|_{L^2(0,x_\nu)} + x_\nu^{1/2}|\phi(x_\nu)|)|\phi(x_\nu)|. \tag{2.8.39}$$

Finally, for the last term in (2.8.36), proceeding as in the proof of (2.8.22), we obtain by using (2.8.37) and Hardy's inequality

$$\begin{aligned}&\Big|\Big\langle w, \frac{U''\phi}{U+i\lambda}\Big\rangle_{L^2(0,x_\nu)}\Big| \\ &\le |\phi(x_\nu)|\,\|w\|_{L^1(0,x_\nu)}\Big\|\frac{U''}{U+i\lambda}\Big\|_{L^\infty(0,x_\nu)} + \|w\|_{L^2(0,x_\nu)}\Big\|\frac{U''(\phi-\phi(x_\nu))}{U+i\lambda}\Big\|_{L^2(0,x_\nu)} \\ &\le C\Big[\frac{|\phi(x_\nu)|}{|\mu|x_\nu^{1/2}}\|\phi'\|_{L^2(0,x_\nu)} + \frac{\|\phi'\|^2_{L^2(0,x_\nu)}}{x_\nu^2}\Big].\end{aligned}$$

Substituting the above, together with (2.8.39) and (2.8.38) into (2.8.36) yields

$$\begin{aligned}\alpha^2\|\varphi\|^2_{L^2(0,x_\nu)} \le C\Big[&\frac{|\phi(x_\nu)|}{x_\nu^{1/2}}\|\phi'\|_{L^2(0,x_\nu)} + \frac{|\mu|}{x_\nu^2}\|\phi'\|^2_{L^2(0,x_\nu)}\Big] \\ &+ C\alpha^2(x_\nu^{1/2}\|\phi\|_{L^2(0,x_\nu)} + x_\nu|\phi(x_\nu)|)|\phi(x_\nu)| \\ &+ \Big|\Big\langle\varphi, \frac{v}{U+i\lambda}\Big\rangle_{L^2(0,x_\nu)}\Big|,\end{aligned}$$

from which we easily conclude that

$$\alpha^2\|\phi\|^2_{L^2(0,x_\nu)} \leq C\Big[\frac{|\phi(x_\nu)|}{x_\nu^{1/2}}\|\phi'\|_{L^2(0,x_\nu)} + \frac{|\mu|}{x_\nu^2}\|\phi'\|^2_{L^2(0,x_\nu)} + \alpha^2 x_\nu|\phi(x_\nu)|^2\Big] + \Big|\Big\langle\varphi, \frac{v}{U+i\lambda}\Big\rangle_{L^2(0,x_\nu)}\Big|. \tag{2.8.40}$$

Substituting (2.8.40) into (2.8.35) we obtain

$$\|\phi'\|_{L^2(0,x_\nu)} \leq C\Big(\frac{|\phi(x_\nu)|}{x_\nu^{1/2}} + \alpha^2 x_\nu^{3/2}|\phi(x_\nu)| + \alpha|\mu|^{1/2}\|\phi'\|_{L^2(0,x_\nu)} + x_\nu^{1/2}N_2(v,\lambda) + \alpha x_\nu\Big|\Big\langle\varphi, \frac{v}{U+i\lambda}\Big\rangle_{L^2(0,x_\nu)}\Big|^{1/2}\Big).$$

As $|\alpha| \leq C_0/x_\nu$, we obtain for sufficiently large κ_0 (implying that both $|\mu|^{1/2}/x_\nu$ and $\alpha|\mu|^{1/2}$ are sufficiently small)

$$\|\phi'\|_{L^2(0,x_\nu)} \leq \hat{C}\Big(\frac{|\phi(x_\nu)|}{x_\nu^{1/2}} + x_\nu^{1/2}N_2(v,\lambda) + \Big|\Big\langle\varphi, \frac{v}{U+i\lambda}\Big\rangle_{L^2(0,x_\nu)}\Big|^{1/2}\Big). \tag{2.8.41}$$

Note that, by (2.8.34),

$$\begin{aligned}\Big|\Big\langle\varphi, \frac{v}{U+i\lambda}\Big\rangle_{L^2(0,x_\nu)}\Big| &\leq \|\phi-\phi(x_\nu)\|_{L^\infty(0,x_\nu)}\Big\|\frac{v}{U+i\lambda}\Big\|_{L^1(0,x_\nu)} \\ &\leq 2x_\nu^{1/2}\|\phi'\|_{L^2(0,x_\nu)}N_2(v,\lambda).\end{aligned}$$

Combining the above with (2.8.41) and (2.8.2) yields

$$\|\phi'\|_{L^2(0,x_\nu)} \leq C\Big[\frac{\mu^{1/2}}{x_\nu}\|\phi'\|_2 + x_\nu^{1/2}N_2(v,\lambda) + \Big|\Big\langle\phi, \frac{v}{U+i\lambda}\Big\rangle\Big|^{1/2}\Big].$$

For sufficiently large κ_0 we easily obtain (2.8.24).

Step 4. We prove that, for any $C_0 > 0$, there exist $C > 0$, $\mu_0 > 0$, $\kappa_0 > 0$, and $\nu_1 > 0$ such that, for $\nu_1 < \nu < U(0) - \kappa_0|\mu|$, $|\alpha| \leq C_0/x_\nu$, and $|\mu| \leq \mu_0$, we have

$$\|\phi'\|_2 \leq C\Big(\Big|\Big\langle\phi, \frac{v}{U+i\lambda}\Big\rangle\Big|^{1/2} + x_\nu^{1/2}N_2(v,\lambda)\Big), \tag{2.8.42}$$

for any pair (ϕ, v) satisfying (2.4.22).

Observing that $U''(U-\nu) > 0$ on $(x_\nu, 1)$ implies

$$\Re\Big\langle U''\phi, \frac{\phi}{U+i\lambda}\Big\rangle_{L^2(x_\nu,1)} = \int_{x_\nu}^1 \frac{|\phi|^2 U''(U-\nu)}{|U+i\lambda|^2}\,dx \geq 0,$$

and hence we may conclude from (2.8.19), that

$$\|\phi'\|_2^2 + \alpha^2\|\phi\|_2^2 \leq -\Re\Big\langle U''\phi, \frac{\phi}{U+i\lambda}\Big\rangle_{L^2(0,x_\nu)} + \Re\Big\langle\phi, \frac{v}{U+i\lambda}\Big\rangle. \tag{2.8.43}$$

To estimate $\left|\Re\langle U''\phi, \frac{\phi}{U+i\lambda}\rangle_{L^2(0,x_\nu)}\right|$, we now proceed as in the proof of (2.8.21)–(2.8.22) to obtain

$$\left|\Re\Big\langle U''\phi, \frac{\phi}{U+i\lambda}\Big\rangle_{L^2(0,x_\nu)}\right| \\ \le \frac{C}{x_\nu}|\phi(x_\nu)|^2 + C\big(x_\nu^{-1}\|\phi\|_{L^2(0,x_\nu)} + x_\nu^{-1/2}|\phi(x_\nu)|\big)\|\phi'\|_{L^2(0,x_\nu)}. \tag{2.8.44}$$

Using Poincaré's inequality applied in $(0, x_\nu)$ to $(\phi - \phi(x_\nu))$ yields

$$\left|\Re\Big\langle U''\phi, \frac{\phi}{U+i\lambda}\Big\rangle_{L^2(0,x_\nu)}\right| \\ \le \frac{C}{x_\nu}|\phi(x_\nu)|^2 + \hat{C}\big(\|\phi'\|_{L^2(0,x_\nu)} + x_\nu^{-1/2}|\phi(x_\nu)|\big)\|\phi'\|_{L^2(0,x_\nu)}. \tag{2.8.45}$$

Substituting (2.8.45) together with (2.8.24) and (2.8.2) into (2.8.43) yields

$$\|\phi'\|_2^2 + \alpha^2\|\phi\|_2^2 \le C\Big[\frac{|\mu|}{x_\nu^2}\|\phi'\|_2^2 + \Big|\Re\Big\langle\phi, \frac{v}{U+i\lambda}\Big\rangle\Big| + x_\nu N_2(v,\lambda)^2\Big].$$

For sufficiently large κ_0 we readily obtain (2.8.42).

Step 5. We prove (2.8.1). We begin by deriving two conclusions of (2.8.42) and (2.8.17) under the assumptions of the proposition. Since by (2.6.82)

$$\left|\Big\langle\phi, \frac{v}{U+i\lambda}\Big\rangle\right| \le \|\phi'\|_2 N_2(v,\lambda),$$

where N_2 is given by (2.8.23), we obtain by (2.8.42) (for $|\alpha| \ge \frac{C_0}{x_\nu}$) and (2.8.17) (for $|\alpha| \le \frac{C_0}{x_\nu}$) that, under the assumption of the proposition,

$$\|\phi'\|_2 \le C N_2(v,\lambda), \tag{2.8.46}$$

which combined with (2.8.2) yields, for bounded $|\mu|/x_\nu^2$,

$$|\phi(x_\nu)| \le C x_\nu^{1/2} N_2(v,\lambda). \tag{2.8.47}$$

Proof of (2.8.1a). We begin by obtaining a bound on $\|(U+i\lambda)^{-1}\|_q$. By (2.8.9) we have for $q > 1$ and $\nu_1 < \nu < U(0)$

$$\left\|\frac{1}{U+i\lambda}\right\|_{L^q(0,1)}^q \le \frac{C}{|\mu|^{q-1/2}}\int_{\mathbb{R}_+}\frac{ds}{[(s^2-a^2)^2+1]^{q/2}},$$

where $a = x_\nu|\mu|^{-1/2}$.

We estimate the integral on the right-hand side in the following manner:

$$\int_{\mathbb{R}_+}\frac{ds}{[(s^2-a^2)^2+1]^{q/2}} \le \int_{\mathbb{R}_+}\frac{ds}{[a^2(s-a)^2+1]^{q/2}} \\ \le \int_{\mathbb{R}}\frac{d\tau}{[a^2\tau^2+1]^{q/2}} \le \frac{1}{a}\int_{\mathbb{R}}\frac{dt}{[t^2+1]^{q/2}} \le \frac{C}{a}. \tag{2.8.48}$$

It follows that for $q > 1$ and $\nu_1 \leq \nu < U(0)$

$$\left\| \frac{1}{U + i\lambda} \right\|_{L^q(0,1)}^q \leq \frac{C}{x_\nu |\mu|^{q-1}}. \tag{2.8.49}$$

Since for $\nu < \nu_1$ we may establish (2.8.49), using (2.6.10), as in [3] we may conclude that (2.8.49) holds for any $0 \leq \nu < U(0)$.

We continue by estimating $\langle \phi, (U + i\lambda)^{-1} v \rangle$. To thus end we write

$$\left| \left\langle \phi, \frac{v}{U + i\lambda} \right\rangle \right| \leq |\phi(x_\nu)| \left\| \frac{v}{U + i\lambda} \right\|_1 + \left\| \frac{\phi - \phi(x_\nu)}{x - x_\nu} \right\|_2 \left\| \frac{(x - x_\nu)\, v}{U + i\lambda} \right\|_2. \tag{2.8.50}$$

Suppose first that $v \in L^p(0,1)$ for some $p \in [2, \infty)$. Then,

$$\left\| \frac{v}{U + i\lambda} \right\|_1 \leq \|v\|_p \left\| \frac{1}{U + i\lambda} \right\|_q,$$

where $q = p/(p-1)$.

Consequently, by (2.8.49)

$$\left\| \frac{v}{U + i\lambda} \right\|_1 \leq \frac{C}{|\mu|^{\frac{1}{p}} x_\nu^{1-\frac{1}{p}}} \|v\|_p. \tag{2.8.51}$$

Next, we estimate the second term on the right-hand side of (2.8.50). Consider first the case $p = 2$. Here, we write with the aid of (2.8.9)

$$\left\| \frac{(x - x_\nu)v}{U + i\lambda} \right\|_2 \leq \|v\|_2 \left\| \frac{x - x_\nu}{U + i\lambda} \right\|_\infty \leq \frac{C}{x_\nu} \|v\|_2. \tag{2.8.52}$$

For $p > 2$ we have

$$\left\| \frac{(x - x_\nu)v}{U + i\lambda} \right\|_2 \leq \|v\|_p \left\| \frac{x - x_\nu}{U + i\lambda} \right\|_{\tilde{q}},$$

where $\tilde{q} = 2p/(p-2)$.

As above we write

$$\left\| \frac{x - x_\nu}{U + i\lambda} \right\|_{\tilde{q}}^{\tilde{q}} \leq C \int_0^1 \frac{dx}{[x + x_\nu]^{\tilde{q}}} \leq \frac{\widehat{C}}{x_\nu^{\tilde{q}-1}}. \tag{2.8.53}$$

Consequently,

$$\left\| \frac{(x - x_\nu)v}{U + i\lambda} \right\|_2 \leq \frac{C}{x_\nu^{\frac{1}{2}+\frac{1}{p}}} \|v\|_p, \tag{2.8.54}$$

which is in accordance with (2.8.52) for $p = 2$.

Using (2.8.50) once again, we deduce from (2.8.51) and (2.8.54) that

$$\left| \left\langle \phi, \frac{v}{U + i\lambda} \right\rangle \right| \leq C \left(\frac{1}{|\mu|^{\frac{1}{p}} x_\nu^{1-\frac{1}{p}}} |\phi(x_\nu)| + \frac{1}{x_\nu^{\frac{1}{2}+\frac{1}{p}}} \left\| \frac{\phi - \phi(x_\nu)}{x - x_\nu} \right\|_2 \right) \|v\|_p.$$

By Hardy's inequality we then have

$$\left|\left\langle \phi, \frac{v}{U+i\lambda}\right\rangle\right| \le C\left(\frac{1}{|\mu|^{\frac{1}{p}} x_\nu^{1-\frac{1}{p}}}|\phi(x_\nu)| + \frac{1}{x_\nu^{\frac{1}{2}+\frac{1}{p}}}\|\phi'\|_2\right)\|v\|_p. \tag{2.8.55}$$

By (2.8.23) and (2.8.51), we have that

$$N_2(v,\lambda) \le \left\|\frac{v}{U+i\lambda}\right\|_1 \le \frac{C}{|\mu|^{\frac{1}{p}} x_\nu^{1-\frac{1}{p}}}\|v\|_p. \tag{2.8.56}$$

The above combined with (2.8.46) yields

$$\|\phi'\|_2 \le \frac{C}{|\mu|^{\frac{1}{p}} x_\nu^{1-\frac{1}{p}}}\|v\|_p, \tag{2.8.57}$$

which is weaker than (2.8.1a).

We obtain a better estimate in the following manner. By (2.8.55), (2.8.56), and (2.8.42) it holds that

$$\|\phi'\|_2 \le$$
$$C\left(\left[\frac{1}{|\mu|^{\frac{1}{2p}} x_\nu^{\frac{1}{2}-\frac{1}{2p}}}|\phi(x_\nu)|^{1/2} + \frac{1}{x_\nu^{\frac{1}{4}+\frac{1}{2p}}}\|\phi'\|_2^{1/2}\right]\|v\|_p^{1/2} + |\mu|^{-\frac{1}{p}} x_\nu^{-\frac{1}{2}+\frac{1}{p}}\|v\|_p\right),$$

from which we get

$$\begin{aligned}\|\phi'\|_2 &\le C\frac{1}{x_\nu^{\frac{1}{4}+\frac{1}{2p}}}\|\phi'\|_2^{\frac{1}{2}}\|v\|_p^{\frac{1}{2}} + C\left(x_\nu^{-1/2}|\phi(x_\nu)| + |\mu|^{-\frac{1}{p}} x_\nu^{-\frac{1}{2}+\frac{1}{p}}\|v\|_p\right)\\ &\le \hat{C}|\mu|^{\frac{1}{p}} x_\nu^{-\frac{2}{p}}\|\phi'\|_2 + \hat{C}\left(x_\nu^{-1/2}|\phi(x_\nu)| + |\mu|^{-\frac{1}{p}} x_\nu^{-\frac{1}{2}+\frac{1}{p}}\|v\|_p\right).\end{aligned}$$

Using the fact that $|\mu|^{1/2}x_\nu^{-1}$ can be assumed to be small for a suitable choice of κ_0, we can then conclude that

$$\|\phi'\|_2 \le C\left(x_\nu^{-1/2}|\phi(x_\nu)| + |\mu|^{-\frac{1}{p}} x_\nu^{-\frac{1}{2}+\frac{1}{p}}\|v\|_p\right). \tag{2.8.58}$$

By (2.8.2) and (2.8.55) it holds that

$$|\phi(x_\nu)|^2 \le Cx_\nu\left[\frac{\mu}{x_\nu^2}\|\phi'\|_2^2 + \left(|\mu|^{-\frac{1}{p}} x_\nu^{-1+\frac{1}{p}}|\phi(x_\nu)| + \frac{1}{x_\nu^{\frac{1}{2}+\frac{1}{p}}}\|\phi'\|_2\right)\|v\|_p\right],$$

and hence we obtain that for any $\delta > 0$ there exist $C_\delta > 0$ and $\kappa_0(\delta)$ such that, under the conditions of the proposition with $\kappa_0 = \kappa_0(\delta)$,

$$|\phi(x_\nu)| \le x_\nu^{1/2}\delta\|\phi'\|_2 + C_\delta\frac{x_\nu^{1/p}}{|\mu|^{1/p}}\|v\|_p. \tag{2.8.59}$$

Substituting (2.8.59) into (2.8.58) yields

$$\|\phi'\|_2 \leq C|\mu|^{-\frac{1}{p}} x_\nu^{-\frac{1}{2}+\frac{1}{p}} \|v\|_p.$$

As $\phi(1) = 0$, we may use Poincaré's inequality to establish (2.8.1a).

Proof of (2.8.1b). Suppose now that $v \in L^\infty(0,1)$. Then,

$$\Big\|\frac{v}{U+i\lambda}\Big\|_1 \leq \|v\|_\infty \Big\|\frac{1}{U+i\lambda}\Big\|_1.$$

Then, we may use (2.8.9) to obtain

$$\Big\|\frac{1}{U+i\lambda}\Big\|_1 \leq C\int_0^1 \frac{dx}{[(x^2-x_\nu^2)^2+\mu^2]^{1/2}} \leq \frac{C}{|\mu|^{1/2}}\int_{\mathbb{R}_+}\frac{ds}{[(s^2-a^2)^2+1]^{1/2}}, \tag{2.8.60}$$

Then, we write, using the fact that for sufficiently large κ_0 we have $a \geq 2$,

$$\int_{\mathbb{R}_+}\frac{ds}{[(s^2-a^2)^2+1]^{1/2}} \leq \int_0^{2a}\frac{ds}{[a^2(s-a)^2+1]^{1/2}} + \int_{2a}^\infty \frac{ds}{[(s-a)^4+1]^{1/2}} \leq C\Big(\frac{\log a}{a}\Big). \tag{2.8.61}$$

Combining the above yields for $\nu_1 < \nu < U(0)$

$$\Big\|\frac{1}{U+i\lambda}\Big\|_1 \leq C\frac{\log\frac{x_\nu}{|\mu|^{1/2}}}{x_\nu}. \tag{2.8.62}$$

Note that by (2.6.10) the above estimate holds for $\nu \leq \nu_1$ as well (see [3]). From (2.8.62), we deduce immediately

$$x_\nu^{1/2}\Big\|\frac{v}{U+i\lambda}\Big\|_1 \leq C\frac{\log\frac{x_\nu}{|\mu|^{1/2}}}{x_\nu^{1/2}}\|v\|_\infty. \tag{2.8.63}$$

Next, we estimate the second term on the right-hand side of (2.8.50) in the case $p = \infty$. Using (2.8.53) we obtain that

$$\Big\|\frac{(x-x_\nu)v}{U+i\lambda}\Big\|_2 \leq \|v\|_\infty\Big\|\frac{x-x_\nu}{U+i\lambda}\Big\|_2 \leq \frac{C}{x_\nu^{1/2}}\|v\|_\infty. \tag{2.8.64}$$

Combining (2.8.64) with (2.8.63), (2.8.17), (2.8.42), and (2.8.24) yields (2.8.1b). Note that by (2.8.2) and (2.8.50) we obtain that

$$|\phi(x_\nu)| \leq C\log\Big(\frac{x_\nu}{|\mu|^{1/2}}\Big)\|v\|_\infty \tag{2.8.65}$$

This completes the proof of the proposition. ∎

2.9 The case $U(0) - \kappa_0|\Re\lambda| \le \Im\lambda \le U(0) + \kappa_0|\Re\lambda|$

In the following, we consider the case where ν is very close to $U(0)$. Here, we need to address the quadratic behavior of $U - U(0)$ near $x = 0$. This case deserves special attention whenever $|\nu - U(0)| \lesssim |\mu|$.

Proposition 2.9.1. *Let $p \in (2, +\infty]$, $\kappa_0 > 0$ and $U \in C^3([0,1])$ satisfy (2.1.3). There exist $\mu_0 > 0$ and $C > 0$ such that, for any $\alpha \ge 0$ and any λ for which $0 < |\mu| \le \mu_0$ and $U(0) - \kappa_0|\mu| \le \nu \le U(0) + \kappa_0|\mu|$, we have, for every pair $(\phi, v) \in D(\mathcal{A}_{\lambda,\alpha}) \times L^\infty(0,1)$ satisfying (2.4.22)*

$$|\mu|^{\frac{1}{2p}+\frac{1}{4}}\|\phi\|_{1,2} \le C\|v\|_p. \tag{2.9.1}$$

Proof. For $\nu \le U(0)$ we choose $x_\nu \in [0,1)$ so that $U(x_\nu) = \nu$. In the case $\nu > U(0)$ we set $x_\nu = 0$ and proceed in a similar manner. Obviously, the assumptions made on U and λ imply that there exists $C > 0$ such that

$$x_\nu < C|\mu|^{1/2} \quad \text{for all } 0 < |\mu| \le 1. \tag{2.9.2}$$

Step 1. We prove that there exist $C > 0$ and $\mu_0 > 0$ such that, for all λ such that $0 < |\mu| \le \mu_0$ and $U(0) - \kappa_0|\mu| \le \nu \le U(0) + \kappa_0|\mu|$ it holds that

$$|\phi(x_\nu)|^2 \le C|\mu|^{1/2}\Big[\|\phi'\|_2^2 + \Big|\Big\langle \phi, \frac{v}{U + i\lambda}\Big\rangle\Big|\Big] \tag{2.9.3}$$

for all pairs $(\phi, v) \in D(\mathcal{A}_{\lambda,\alpha}) \times L^\infty(0,1)$ satisfying (2.4.22).

We note that (2.8.3) can be rewritten in the form

$$\begin{aligned} &\frac{|\mu|}{2}|\phi(x_\nu)|^2 \int_0^1 \frac{|U''|}{(U-\nu)^2 + \mu^2}\,dx \\ &\quad \le \Big|\Im\Big\langle \phi, \frac{v}{U - \nu + i\mu}\Big\rangle\Big| + |\mu|\Big\langle \frac{|U''|}{(U-\nu)^2+\mu^2}|\phi(x) - \phi(x_\nu)|^2\Big\rangle. \end{aligned} \tag{2.9.4}$$

Since

$$(U-\nu)^2 \le C(x^2 + |\nu - U(0)|)^2 \le C(x^2 + \kappa_0|\mu|)^2 \le \hat{C}(x^4 + |\mu|^2),$$

we obtain that

$$\int_0^1 \frac{|U''|}{(U-\nu)^2 + \mu^2}\,dx \ge \frac{1}{\hat{C}}\int_0^1 \frac{dx}{x^4 + \mu^2}.$$

Using the substitution $x = |\mu|^{1/2}\xi$ yields

$$\int_0^1 \frac{dx}{x^4+\mu^2} = |\mu|^{-3/2}\int_0^{|\mu|^{-1/2}} \frac{d\xi}{\xi^4+1} = |\mu|^{-3/2}\Big[\int_0^\infty \frac{d\xi}{\xi^4+1} - \int_{|\mu|^{-1/2}}^\infty \frac{d\xi}{\xi^4+1}\Big].$$

As

$$\int_{|\mu|^{-1/2}}^{\infty} \frac{d\xi}{\xi^4+1} \leq C|\mu|^{3/2},$$

we obtain the existence of $\mu_0 > 0$ and $\widehat{C}$ such that, under the conditions of this step

$$\int_0^1 \frac{|U''|}{(U-\nu)^2+\mu^2}\,dx \geq \frac{1}{\widehat{C}|\mu|^{3/2}}. \tag{2.9.5}$$

By (2.8.16) we have that

$$\Big\langle \frac{|U''|}{(U-\nu)^2+\mu^2}, |\phi(x)-\phi(x_\nu)|^2 \Big\rangle \leq \frac{C}{|\mu|}\Big\|\frac{\phi-\phi(x_\nu)}{|U-\nu|^{1/2}}\Big\|_2^2.$$

By (2.8.9) we have for all $\nu_1 < \nu < U(0)$ for some positive $\nu_1 > 0$ that (note for sufficiently small μ_0 we clearly have $\nu > U(0) - \kappa_0|\mu| > \nu_1$)

$$|U-\nu| \geq \frac{1}{C}(x^2 - x_\nu^2) \geq \frac{1}{C}(x-x_\nu)^2, \tag{2.9.6}$$

which remains valid also for $\nu \geq U(0)$ given that

$$|U-\nu| \geq |U-U(0)| \geq \frac{1}{C}x^2 = \frac{1}{C}(x-x_\nu)^2. \tag{2.9.7}$$

Hence, by Hardy's inequality (2.6.13),

$$\Big\langle \frac{|U''|}{(U-\nu)^2+\mu^2}, |\phi(x)-\phi(x_\nu)| \Big\rangle \leq \frac{C}{|\mu|}\Big\|\frac{\phi(x)-\phi(x_\nu)}{x-x_\nu}\Big\|_2^2 \leq \frac{C}{|\mu|}\|\phi'\|_2^2.$$

Combining the above with (2.9.5) and (2.9.4) yields (2.9.3).

Step 2. We prove that for any $\kappa_1 > 0$ there exist positive C and μ_0 such that, for all $\nu > U(0) - \kappa_1|\mu|$ and $|\mu| \leq \mu_0$ it holds that

$$\|\phi'\|_2 \leq C\Big[\mu^{1/4}\Big\|\frac{v}{U+i\lambda}\Big\|_1 + \Big\|\frac{(x-x_\nu)v}{U+i\lambda}\Big\|_2\Big] \tag{2.9.8}$$

holds for any pair $(\phi, v) \in D(\mathcal{A}_{\lambda,\alpha}) \times L^\infty(0,1)$ satisfying (2.4.22).

We begin by restating (2.8.43):

$$\|\phi'\|_2^2 + \alpha^2\|\phi\|_2^2 \leq -\Re\Big\langle U''\phi, \frac{\phi}{U+i\lambda}\Big\rangle_{L^2(0,x_\nu)} + \Re\Big\langle \phi, \frac{v}{U+i\lambda}\Big\rangle. \tag{2.9.9}$$

Then, we write

$$\begin{aligned}\Re\Big\langle \phi, \frac{U''\phi}{U+i\lambda}\Big\rangle_{L^2(0,x_\nu)} \\ = \Re\Big\langle \phi(x_\nu), \frac{U''\phi}{U+i\lambda}\Big\rangle_{L^2(0,x_\nu)} + \Re\Big\langle \phi-\phi(x_\nu), \frac{U''\phi}{U+i\lambda}\Big\rangle_{L^2(0,x_\nu)}.\end{aligned}$$

For the first term on the right-hand side we use (2.9.3) to obtain

$$\Big|\Big\langle \phi(x_\nu), \frac{U''\phi}{U+i\lambda}\Big\rangle_{L^2(0,x_\nu)}\Big| \leq C|\mu|^{1/4}x_\nu^{1/2}\Big[\|\phi'\|_2 + \Big|\Big\langle \phi, \frac{v}{U+i\lambda}\Big\rangle\Big|^{1/2}\Big]\Big\|\frac{\phi}{U+i\lambda}\Big\|_2.$$

Using (2.6.5) and the fact that $U''<0$, we obtain that

$$\Big\|\frac{\phi}{U+i\lambda}\Big\|_2^2 = \int_0^1 \frac{|\phi|^2}{(U-\nu)^2+\mu^2}dx \leq C\int_0^1 \frac{-U''|\phi|^2}{(U-\nu)^2+\mu^2}dx \leq \frac{C}{\mu}\Big|\Big\langle \phi, \frac{v}{U+i\lambda}\Big\rangle\Big|.$$

Hence, it holds that

$$\Big\|\frac{\phi}{U+i\lambda}\Big\|_2 \leq C|\mu|^{-1/2}\Big|\Big\langle \phi, \frac{v}{U+i\lambda}\Big\rangle\Big|^{1/2} \tag{2.9.10}$$

and we can conclude that

$$\Big|\Big\langle \phi(x_\nu), \frac{U''\phi}{U+i\lambda}\Big\rangle_{L^2(0,x_\nu)}\Big| \leq C\Big[\|\phi'\|_2 + \Big|\Big\langle \phi, \frac{v}{U+i\lambda}\Big\rangle\Big|^{1/2}\Big]\Big|\Big\langle \phi, \frac{v}{U+i\lambda}\Big\rangle\Big|^{1/2}. \tag{2.9.11}$$

For the second term on the right-hand side we use (2.9.10), (2.6.12), and (2.9.2) to obtain

$$\begin{aligned}\Big|\Big\langle \phi-\phi(x_\nu), \frac{U''\phi}{U+i\lambda}\Big\rangle_{L^2(0,x_\nu)}\Big| &= \Big|\Big\langle \frac{\phi-\phi(x_\nu)}{x-x_\nu}, \frac{(x-x_\nu)U''\phi}{U+i\lambda}\Big\rangle_{L^2(0,x_\nu)}\Big| \\ &\leq C\,x_\nu\|\phi'\|_2\Big\|\frac{\phi}{U+i\lambda}\Big\|_2 \\ &\leq \hat{C}\,\|\phi'\|_2\Big|\Big\langle \phi, \frac{v}{U+i\lambda}\Big\rangle\Big|^{1/2}.\end{aligned}$$

Hence,

$$\Big|\Big\langle \phi-\phi(x_\nu), \frac{U''\phi}{U+i\lambda}\Big\rangle_{L^2(0,x_\nu)}\Big| \leq \hat{C}\,\|\phi'\|_2\Big|\Big\langle \phi, \frac{v}{U+i\lambda}\Big\rangle\Big|^{1/2}. \tag{2.9.12}$$

Substituting (2.9.12) together with (2.9.11) into (2.9.9) yields

$$\|\phi'\|_2^2 \leq C\Big|\Big\langle \phi, \frac{v}{U+i\lambda}\Big\rangle\Big|. \tag{2.9.13}$$

Note that by combining (2.9.13) with (2.9.3) we can also conclude that

$$|\phi(x_\nu)|^2 \leq C|\mu|^{1/2}\Big|\Big\langle \phi, \frac{v}{U+i\lambda}\Big\rangle\Big|. \tag{2.9.14}$$

To prove (2.9.8) we now write, with the aid of Hardy's inequality

$$\begin{aligned}\Big|\Big\langle \phi, \frac{v}{U+i\lambda}\Big\rangle\Big| &\leq \Big|\Big\langle \phi(x_\nu), \frac{v}{U+i\lambda}\Big\rangle + \Big|\Big\langle \phi-\phi(x_\nu), \frac{v}{U+i\lambda}\Big\rangle\Big| \\ &\leq |\phi(x_\nu)|\,\Big\|\frac{v}{U+i\lambda}\Big\|_1 + \|\phi'\|_2\Big\|\frac{(x-x_\nu)v}{U+i\lambda}\Big\|_2.\end{aligned}$$

Combining the above with (2.9.14) gives

$$\left|\left\langle\phi, \frac{v}{U+i\lambda}\right\rangle\right| \leq C\Big(|\mu|^{\frac{1}{2}}\Big\|\frac{v}{U+i\lambda}\Big\|_1^2 + \|\phi'\|_2\Big\|\frac{(x-x_\nu)v}{U+i\lambda}\Big\|_2\Big). \tag{2.9.15}$$

Then, (2.9.15) and (2.9.13) imply (2.9.8).

Step 3. We estimate $\|\frac{1}{U+i\lambda}\|_q$ for $q\geq 1$ and $\|\frac{x-x_\nu}{U+i\lambda}\|_q$ for $q\geq 2$ for $\nu\in(U(0)-\kappa_0|\mu|, U(0)+\kappa_0|\mu|)$.

We consider two separate cases by splitting $(U(0)-\kappa_0|\mu|, U(0)+\kappa_0|\mu|)$ into two subintervals.

The case $\nu\in(U(0)-\kappa_0|\mu|, U(0)-\delta|\mu|)$, $\delta\in(0,\kappa_0)$. In this case, observing that $x_\nu\geq C^{-1}|\delta\mu|^{\frac{1}{2}}$ we use (2.8.49) to establish that for any $\delta>0$ there exists C such that for all $\delta|\mu|<U(0)-\nu<\kappa_0|\mu|$ it holds for all $q>1$

$$\Big\|\frac{1}{U+i\lambda}\Big\|_q^q \leq \frac{C}{|\delta|^{\frac{q}{2}}\,|\mu|^{q-1/2}}. \tag{2.9.16}$$

In a similar manner we obtain from (2.8.60) and (2.8.61) (note that $a=x_\nu|\mu|^{-1/2}\geq C^{-1}\delta^{\frac{1}{2}}$ in the present regime of ν values) that there exists $C_\delta>0$ such that

$$\Big\|\frac{1}{U+i\lambda}\Big\|_1 \leq \frac{C_\delta}{|\mu|^{1/2}}. \tag{2.9.17}$$

Finally, we use (2.8.53) and the fact that $x_\nu\geq C^{-1}|\delta\mu|^{\frac{1}{2}}$ to obtain for all $q\geq 2$

$$\Big\|\frac{x-x_\nu}{U+i\lambda}\Big\|_q^q \leq \frac{C_\delta}{|\mu|^{(q-1)/2}}. \tag{2.9.18}$$

The case $\nu\in(U(0)-\delta|\mu|, U(0)+\kappa_0|\mu|)$, $\delta\in(0,\kappa_0)$. In this case we use (2.8.9) to obtain that

$$(U-\nu)^2\geq\frac{1}{C}(x^2-x_\nu^2)^2\geq\frac{1}{C}\Big(\frac{x^4}{2}-x_\nu^4\Big)\geq\frac{1}{C_1}x^4-C_2\delta^2|\mu|^2.$$

The above inequality implies that there exist C and $\delta_0\leq\kappa_0$ such that, for $\delta\in(0,\delta_0)$ and $\nu\in(U(0)-\delta|\mu|, U(0)+\kappa_0|\mu|)$,

$$\frac{1}{(U(x)-\nu)^2+|\mu|^2}\leq\frac{C}{x^4+|\mu|^2} \tag{2.9.19}$$

for all $x\in[0,1]$.

Consequently, for all $q\geq 1$, using the substitution $x=|\mu|^{1/2}\xi$, we obtain

$$\Big\|\frac{1}{U+i\lambda}\Big\|_q^q\leq\int_0^1\frac{C}{[x^2+|\mu|]^q}\,dx\leq\frac{C}{|\mu|^{q-1/2}}\int_0^\infty\frac{d\xi}{[\xi^2+1]^q}\leq\frac{C}{|\mu|^{q-1/2}}. \tag{2.9.20}$$

Finally, we use (2.9.19) to obtain that

$$\Big\|\frac{x-x_\nu}{U+i\lambda}\Big\|_q^q \leq C\Big(|\mu|^{q/2}\Big\|\frac{1}{U+i\lambda}\Big\|_q^q + \Big\|\frac{x}{x^2+|\mu|}\Big\|_q^q\Big). \tag{2.9.21}$$

We now observe that for all $q > 1$

$$\Big\|\frac{x}{x^2+|\mu|}\Big\|_q^q \leq \frac{1}{|\mu|^{(q-1)/2}}\int_0^\infty \frac{\xi^q}{[\xi^2+1]^q}\,d\xi \leq \frac{C}{|\mu|^{(q-1)/2}}. \tag{2.9.22}$$

Together with (2.9.22) and (2.9.20), (2.9.21) yields the existence of $C > 0$

$$\Big\|\frac{x-x_\nu}{U+i\lambda}\Big\|_q^q \leq \frac{C}{|\mu|^{(q-1)/2}}. \tag{2.9.23}$$

The general case. Combining (2.9.16) and (2.9.17), (2.9.20) yields, for $q \geq 1$, the existence of $C > 0$, such that for $|U(0) - \nu| < \kappa_0|\mu|$

$$\Big\|\frac{1}{U+i\lambda}\Big\|_q^q \leq \frac{C}{|\mu|^{q-1/2}}. \tag{2.9.24}$$

By (2.9.18) and (2.9.23) we may conclude, for all $q \geq 1$, that there exists $C > 0$ such that, for $|U(0) - \nu| < \kappa_0|\mu|$

$$\Big\|\frac{x-x_\nu}{U+i\lambda}\Big\|_q^q \leq \frac{C}{|\mu|^{(q-1)/2}}. \tag{2.9.25a}$$

Note that

$$\begin{aligned}\Big\|\frac{x-x_\nu}{U+i\lambda}\Big\|_\infty &\leq C\Big\|\frac{x-x_\nu}{|x^2-x_\nu^2|+|\mu|}\Big\|_\infty \\ &\leq C\Big(\Big\|\mathbf{1}_{|x-x_\nu|<|\mu|^{1/2}}\frac{x-x_\nu}{|\mu|}\Big\|_\infty + \Big\|\mathbf{1}_{|x-x_\nu|\geq|\mu|^{1/2}}\frac{x-x_\nu}{|x^2-x_\nu^2|}\Big\|_\infty\Big) \leq \frac{\widehat{C}}{|\mu|^{1/2}},\end{aligned}$$

hence

$$\Big\|\frac{x-x_\nu}{U+i\lambda}\Big\|_\infty \leq \frac{\widehat{C}}{|\mu|^{1/2}}. \tag{2.9.25b}$$

Step 4. We prove (2.9.1). The proof is similar to Step 4 of the proof of Proposition 2.8.1. We estimate the right-hand side of (2.9.8) separately for $p \in [2, +\infty)$ and for $p = \infty$. Suppose first that $v \in L^p(0,1)$ for some $p \in [2, +\infty)$.

For the first term on the right-hand side, we deduce from (2.9.24),

$$|\mu|^{1/4}\Big\|\frac{v}{U+i\lambda}\Big\|_1 \leq C\frac{\|v\|_p}{|\mu|^{\frac{1}{2p}+\frac{1}{4}}}. \tag{2.9.26}$$

To estimate the second term we use (2.9.25) to obtain for all $p \geq 2$

$$\Big\|\frac{(x-x_\nu)v}{U+i\lambda}\Big\|_2 \leq \frac{C}{|\mu|^{\frac{1}{2p}+\frac{1}{4}}}\|v\|_p. \tag{2.9.27}$$

Suppose now that $v \in L^\infty(0,1)$. Then, by (2.9.24) we may conclude for the first term on the right-hand side of (2.9.8) that

$$|\mu|^{1/4}\Big\|\frac{v}{U+i\lambda}\Big\|_1 \le \frac{C}{|\mu|^{1/4}}\|v\|_\infty. \tag{2.9.28}$$

Next, we estimate the second term on the right-hand side of (2.9.8). Using (2.9.25) we obtain that

$$\Big\|\frac{(x-x_\nu)v}{U+i\lambda}\Big\|_2 \le \frac{C}{|\mu|^{1/4}}\|v\|_\infty.$$

Together with (2.9.28) the above yields (2.9.1) for $p=+\infty$. ■

Remark 2.9.2. Note that, under the assumptions of Proposition 2.9.1, by (2.9.14), (2.9.15), (2.9.25a) and (2.9.24) it holds that

$$|\phi(x_\nu)|^2 \le C\left(|\phi(x_\nu)|\,\|v\|_\infty + |\mu|^{1/4}\|\phi'\|_2\|v\|_\infty\right).$$

Using (2.9.1) for $p=+\infty$ then yields the existence of $C>0$ such that

$$|\phi(x_\nu)| \le C\|v\|_\infty. \tag{2.9.29}$$

2.10 The case $\Im\lambda > U(0)$

In the case where $\Im\lambda = \nu > U(0)$, we get a better estimate of $\mathcal{A}_{\lambda,\alpha}^{-1}$, measured by a negative power of $|\mu| + (\nu - U(0))$. More precisely, we have the following proposition.

Proposition 2.10.1. *Let $p \in [2,+\infty]$. There exist $\mu_0 > 0$ and $C>0$ such that for all $U \in C^3([0,1])$ satisfying (2.1.3), $\nu > U(0)$, $|\mu| < \mu_0$, $\alpha \ge 0$, and $(\phi, v) \in D(\mathcal{A}_{\lambda,\alpha}) \times L^\infty(0,1)$ satisfying (2.4.22) it holds that*

$$\|\phi\|_{1,2} \le \frac{C}{[|\mu| + |\nu - U(0)|]^{\frac{1}{2p}+\frac{1}{4}}}\|v\|_p, \tag{2.10.1}$$

Proof. We begin by restating (2.8.19)

$$\|\phi'\|_2^2 + \alpha^2\|\phi\|_2^2 + \Big\langle \frac{U''(U-\nu)\phi}{(U-\nu)^2+\mu^2}, \phi\Big\rangle = \Re\Big\langle \phi, \frac{v}{U+i\lambda}\Big\rangle. \tag{2.10.2}$$

Since $\nu > U(0)$, it holds by (2.1.3) that the third term on the left-hand side is positive, and hence we can conclude that

$$\|\phi'\|_2^2 \le \Re\Big\langle \phi, \frac{v}{U+i\lambda}\Big\rangle. \tag{2.10.3}$$

We split the proof into two separate parts in accordance with the magnitude of $(U(0)-\nu)^2 + \mu^2$.

Step 1. The case $(U(0)-\nu)^2+\mu^2$ *small.* We consider here the case where

$$(U(0)-\nu)^2+\mu^2<\varepsilon$$

for some sufficiently small $\varepsilon>0$.

To properly bound the right-hand side of (2.10.3) in that case we need an estimate for $|\phi(0)|$. To this end we use (2.10.2) once again to obtain

$$\Big\langle\frac{U''(U-\nu)\phi}{(U-\nu)^2+\mu^2},\phi\Big\rangle\leq\Re\Big\langle\phi,\frac{v}{U+i\lambda}\Big\rangle,$$

from which we can conclude that

$$\begin{aligned}&\Big\langle\frac{U''(U-\nu)}{(U-\nu)^2+\mu^2},1\Big\rangle|\phi(0)|^2\\&\quad\leq 2\Big\langle\frac{U''(U-\nu)(\phi-\phi(0))}{(U-\nu)^2+\mu^2},\phi-\phi(0)\Big\rangle+2\Re\Big\langle\phi,\frac{v}{U+i\lambda}\Big\rangle.\end{aligned}\tag{2.10.4}$$

We continue by bounding from below the left-hand side of (2.10.4). To this end we observe that since

$$\min_{x\in[0,1]}|U''(x)|\geq\frac{1}{C_0}>0\quad\text{and}\quad U(0)-U(x)\geq\frac{1}{2C_0}x^2,$$

we can conclude that

$$\begin{aligned}\Big\langle\frac{U''(U-\nu)}{(U-\nu)^2+\mu^2},1\Big\rangle&\geq\frac{1}{C}\int_0^1\frac{x^2+\nu-U(0)}{[x^2+\nu-U(0)]^2+\mu^2}\,dx\\&\geq\frac{1}{C}\int_0^1\frac{x^2}{[x^2+\nu-U(0)]^2+\mu^2}\,dx\\&\geq\frac{1}{2C}\int_0^1\frac{x^2}{x^4+(\nu-U(0)^2+\mu^2)}\,dx\\&\geq\frac{1}{2C}\Big[\int_0^\infty\frac{x^2}{x^4+[\nu-U(0)]^2+\mu^2}\,dx-1\Big].\end{aligned}$$

Setting

$$x=([\nu-U(0)]^2+\mu^2)^{1/4}s$$

yields

$$\begin{aligned}\Big\langle\frac{U''(U-\nu)}{(U-\nu)^2+\mu^2},1\Big\rangle&\geq\frac{1}{2C}\Big(\frac{1}{([\nu-U(0)]^2+\mu^2)^{1/4}}\int_0^\infty\frac{s^2}{s^4+1}\,ds-1\Big)\\&\geq\frac{1}{\widehat{C}\{[\nu-U(0)]^2+\mu^2\}^{1/4}}-\widehat{C}.\end{aligned}$$

For sufficiently small ε we obtain that

$$\Big\langle\frac{U''(U-\nu)}{(U-\nu)^2+\mu^2},1\Big\rangle \geq \frac{1}{2\hat{C}\{[\nu-U(0)]^2+\mu^2\}^{1/4}}. \tag{2.10.5}$$

To estimate the first term on the right-hand side of (2.10.4) we first write

$$\Big\langle\frac{U''(U-\nu)(\phi-\phi(0))}{(U-\nu)^2+\mu^2},\phi-\phi(0)\Big\rangle \leq \Big\|\frac{U''(U-\nu)x^2}{(U-\nu)^2+\mu^2}\Big\|_\infty\Big\|\frac{\phi-\phi(0)}{x}\Big\|_2^2.$$

As

$$\Big\|\frac{U''(U-\nu)x^2}{(U-\nu)^2+\mu^2}\Big\|_\infty \leq C\Big\|\frac{x^4+[\nu-U(0)]x^2}{[x^2+\nu-U(0)]^2+\mu^2}\Big\|_\infty \leq \hat{C},$$

we may use Hardy's inequality to obtain that

$$\Big\langle\frac{U''(U-\nu)(\phi-\phi(0))}{(U-\nu)^2+\mu^2},\phi-\phi(0)\Big\rangle \leq C\,\|\phi'\|_2^2. \tag{2.10.6}$$

Equation (2.10.6) together with (2.10.3) and (2.10.5) yields, when substituted into (2.10.4),

$$|\phi(0)|^2 \leq C(|\mu|^{1/2}+|\nu-U(0)|^{1/2})\Re\Big\langle\phi,\frac{v}{U+i\lambda}\Big\rangle. \tag{2.10.7}$$

To complete the proof we now estimate the right-hand side of (2.10.3). To this end we write

$$\Re\Big\langle\phi,\frac{v}{U+i\lambda}\Big\rangle \leq |\phi(0)|\Big\|\frac{v}{U+i\lambda}\Big\|_1+\Big\|\frac{\phi-\phi(0)}{x}\Big\|_2\Big\|\frac{xv}{U+i\lambda}\Big\|_2.$$

Using Hardy's inequality together with (2.10.7) yields

$$\Re\Big\langle\phi,\frac{v}{U+i\lambda}\Big\rangle \leq C\Big[(|\mu|^{1/2}+|\nu-U(0)|^{1/2})\Big\|\frac{v}{U+i\lambda}\Big\|_1^2+\|\phi'\|_2\Big\|\frac{xv}{U+i\lambda}\Big\|_2\Big].$$

Using (2.10.3) once again yields

$$\|\phi'\|_2 \leq C\Big[(|\mu|^{1/4}+|\nu-U(0)|^{1/4})\Big\|\frac{v}{U+i\lambda}\Big\|_1+\Big\|\frac{xv}{U+i\lambda}\Big\|_2\Big]. \tag{2.10.8}$$

The proof of (2.10.1) is now verified by following the same path as in the proof of Step 4 of Propositions 2.8.1 and 2.9.1. Thus, since

$$(U-\nu) \geq C(x^2+\nu-U(0)),$$

we obtain as in (2.9.20) that for all $q \geq 1$

$$\begin{aligned}\Big\|\frac{1}{U+i\lambda}\Big\|_q^q &\leq \int_0^1 \frac{C}{[x^2+\nu-U(0)+|\mu|]^q}\,dx\\ &\leq \frac{C}{[|\mu+\nu-U(0)]|^{q-1/2}}\int_0^\infty \frac{d\xi}{[\xi^2+1]^q}\\ &\leq \frac{\hat{C}}{[|\mu|+\nu-U(0)]^{q-1/2}}.\end{aligned} \tag{2.10.9}$$

Hence, for all $p \in [2, +\infty]$,

$$\left\| \frac{v}{U + i\lambda} \right\|_1 \le C[|\mu| + \nu - U(0)]^{-\frac{1}{2p} - \frac{1}{2}} \|v\|_p. \tag{2.10.10}$$

As in (2.9.21) and (2.9.22) we then write for all $q > 1$

$$\begin{aligned} \left\| \frac{x}{U + i\lambda} \right\|_q^q &\le C \left\| \frac{x}{x^2 + \nu - U(0) + |\mu|} \right\|_q^q \\ &\le \frac{\widehat{C}}{[|\mu| + \nu - U(0)]^{(q-1)/2}} \int_0^\infty \frac{\xi^q}{[\xi^2 + 1]^q} \, d\xi \\ &\le \frac{\widetilde{C}}{[|\mu| + \nu - U(0)]^{(q-1)/2}}. \end{aligned}$$

Note also that for $q = +\infty$ we have

$$\left\| \frac{x}{U + i\lambda} \right\|_\infty \le \frac{C}{[|\mu| + \nu - U(0)]^{1/2}}.$$

Consequently, for all $p \in [2, +\infty]$

$$\left\| \frac{xv}{U + i\lambda} \right\|_2 \le \frac{C}{[|\mu| + \nu - U(0)]^{\frac{1}{2p} + \frac{1}{4}}} \|v\|_p. \tag{2.10.11}$$

Substituting the above, together with (2.10.10), into (2.10.8) yields (2.10.1) for sufficiently small $\varepsilon > 0$.

Step 2. The case where, for some $\varepsilon > 0$,

$$(U(0) - \nu)^2 + \mu^2 \ge \varepsilon.$$

By Poincaré's inequality and (2.10.10), we obtain

$$\begin{aligned} \left| \Re\left\langle \phi, \frac{v}{U + i\lambda} \right\rangle \right| &\le \|\phi\|_\infty \left\| \frac{v}{U + i\lambda} \right\|_1 \\ &\le \frac{C}{[(U(0) - \nu)^2 + \mu^2]^{1/2}} \|\phi'\|_2 \|v\|_p \\ &\le \frac{C}{\varepsilon^{\frac{2-p}{4}} [(U(0) - \nu)^2 + \mu^2]^{\frac{1}{2p} + \frac{1}{4}}} \|\phi'\|_2 \|v\|_p, \end{aligned}$$

which together with (2.10.3) readily gives (2.10.1) in this case. ■

Remark 2.10.2. As in Remark 2.9.2 we may use (2.10.7), (2.10), and (2.10.8), under the assumptions of Proposition 2.10.1, to obtain

$$|\phi(0)|^2 \le C(|\mu| + |\nu - U(0)|) \left\| \frac{v}{U + i\lambda} \right\|_1^2 + (|\mu| + |\nu - U(0)|)^{1/2} \left\| \frac{xv}{U + i\lambda} \right\|_2^2.$$

From (2.10.10) we then obtain the existence of $C > 0$ such that

$$|\phi(0)| \le C \|v\|_\infty. \tag{2.10.12}$$

2.11 The case $|\Re\lambda| \geq \mu_1 > 0$

The results in the preceding subsections were all obtained under the assumption that $|\mu| \leq \mu_0$ for some sufficiently small μ_0. Hence, it remains to treat the case when $|\mu| \geq \mu_1$ where $\mu_1 > 0$ is arbitrary. We complement Proposition 2.6.1 by addressing the large $|\mu|$ case.

Proposition 2.11.1. *For any $\mu_1 > 0$ and $p > 1$ there exists $C > 0$ such that for all $(\phi, v) \in D(\mathcal{A}_{\lambda,\alpha}) \times W^{1,p}(0,1)$ satisfying $\mathcal{A}_{\lambda,\alpha}\phi = v$, and for all $|\mu| \geq \mu_1$, $\nu \in \mathbb{R}$, and $\alpha \geq 0$,*

$$\|\phi\|_{H^1(0,1)} \leq C\|(1-x)^{1/2}v\|_1. \tag{2.11.1}$$

Proof. Since (2.6.17) holds true for any $\mu \neq 0$ we can conclude that

$$\|\phi'\|_2 \leq \Big(1 + \frac{C}{|\mu|^{1/2}}\Big)\Big|\Big\langle\phi, \frac{v}{U + i\lambda}\Big\rangle\Big|^{1/2}. \tag{2.11.2}$$

Hence, there exists C such that for $|\mu| \geq \mu_1$

$$\|\phi'\|_2^2 \leq C\,\Big|\Big\langle\phi, \frac{v}{U + i\lambda}\Big\rangle\Big|. \tag{2.11.3}$$

Consequently, as $|\phi(x)| \leq \|\phi'\|_2(1-x)^{1/2}$ and $|\mu| > \mu_1$,

$$\|\phi'\|_2^2 \leq C\|\phi'\|_2\|(1-x)^{1/2}v\|_1\Big\|\frac{1}{U + i\lambda}\Big\|_\infty \leq \frac{C}{\mu_1}\|\phi'\|_2\|(1-x)^{1/2}v\|_1,$$

from which (2.11.1) follows with the aid of Poincaré's inequality. ∎