Chapter 3

Neumann–Dirichlet Schrödinger operators

The first part of this chapter is devoted to resolvent estimates for the operator

$$\mathcal{L}_\beta^{\mathfrak{N},\mathfrak{D}} = -\frac{d^2}{dx^2} + i\beta\, U, \qquad (3.0.1a)$$

which is defined on

$$D(\mathcal{L}_\beta^{\mathfrak{N},\mathfrak{D}}) = \{u \in H^2(0,1) \mid u(1) = u'(0) = 0\}. \qquad (3.0.1b)$$

We note that to estimate the inverse norm of the Orr–Sommerfeld operator (1.1.7b) we need a bound of both $\mathcal{A}_{\lambda,\alpha}^{-1}$ and $(\mathcal{L}_\beta^{\mathfrak{N},\mathfrak{D}} - \lambda)^{-1}$, and in addition, to obtain resolvent estimates for the special Schrödinger operators of the next chapter.

For convenience of notation we omit in the sequel the reference to the Dirichlet condition at $x = 1$ and use $\mathcal{L}_\beta^{\mathfrak{N}}$ instead of $\mathcal{L}_\beta^{\mathfrak{N},\mathfrak{D}}$.

Let $\lambda = \mu + i\nu$. Recall the definition of x_ν from (2.4.5). From [2], for instance, we know that the main contribution to the resolvent norm comes from a small region near $x = x_\nu$. We begin this section by estimating the resolvent norm of $\mathcal{L}_\beta^{\mathfrak{N}}$ in the case where $U(0) - \nu \gg \beta^{-1/2}$. In this case one may approximate $U - \nu$ by a linear potential of the form $U'(x_\nu)(x - x_\nu)$. In Section 3.2, we consider the case $|U(0) - \nu| \lesssim \beta^{-1/2}$ where $U - \nu$ will be approximated by the quadratic potential $U''(0)x^2/2 + U(0) - \nu$. Finally, Section 3.3 is devoted to some L^1 estimates that are necessary in Chapter 4.

3.1 Resolvent estimates for $U(0) - \Im\lambda \gg \beta^{-1/2}$

With x_ν defined in (2.4.5), we introduce

$$\mathfrak{I}_\nu = |U'(x_\nu)|. \qquad (3.1.1)$$

We further define $\hat\kappa_1 \in \mathbb{C}$ to be the leftmost eigenvalue of

$$\mathcal{L}_+ = -\frac{d^2}{dx^2} + ix \qquad (3.1.2)$$

in $H^2(\mathbb{R}_+) \cap H_0^1(\mathbb{R}_+)$. The first proposition is similar to [3, Proposition 5.2]. Unlike [3] which defines the problem on $(-1, +1)$ with Dirichlet conditions at $x = \pm 1$, we consider the operator on $(0, 1)$ with a Neumann condition at $x = 0$, which corresponds to a restriction to the space of even functions on $(-1, 1)$, and a Dirichlet condition

at $x = 1$. Furthermore, the velocity field is not strictly monotone as in [3]. Neverthe-less, for $x_\nu \gg \beta^{-1/4}$ (or equivalently for $U(0) - \nu \gg \beta^{-1/2}$) we can still make good use of the estimates in [3].

Proposition 3.1.1. *Let $U \in C^2([0,1])$ satisfy (2.1.3) and $p \in (1,2]$. Then there exist positive Υ, a, C, C_p, and β_0 such that, for all $\beta \geq \beta_0$, $U(0) - \nu > a\beta^{-1/2}$, and $f \in L^\infty(0,1)$,*

$$\sup_{\mu \leq \Upsilon \mathfrak{J}_\nu^{2/3}\beta^{-1/3}} \|(\mathcal{L}_\beta^\mathfrak{N} - \beta\lambda)^{-1} f\|_2 \leq C \min([\mathfrak{J}_\nu\beta]^{-2/3}\|f\|_2, [\mathfrak{J}_\nu\beta]^{-5/6}\|f\|_\infty).$$
(3.1.3a)

Furthermore, for $f \in L^2(0,1)$,

$$\sup_{\mu \leq \Upsilon \mathfrak{J}_\nu^{2/3}\beta^{-1/3}} \left\| \frac{d}{dx}(\mathcal{L}_\beta^\mathfrak{N} - \beta\lambda)^{-1} f \right\|_p \leq \frac{C_p}{[\mathfrak{J}_\nu\beta]^{\frac{2+p}{6p}}} \|f\|_2.$$
(3.1.3b)

Proof. Step 1. For $\Upsilon > 0$, we prove that there exist positive β_0, a_0, and C such that for all $\beta \geq \beta_0$, $U(0) - \nu \geq a\beta^{-\frac{1}{2}}$, $a \geq a_0$ and $\mu \leq \Upsilon \mathfrak{J}_\nu^{2/3}\beta^{-1/3}$ we have

$$\|(\mathcal{L}_\beta^\mathfrak{N} - \beta\lambda)^{-1} f\|_2 + [\mathfrak{J}_\nu\beta]^{-1/3}\left\| \frac{d}{dx}(\mathcal{L}_\beta^\mathfrak{N} - \beta\lambda)^{-1} f \right\|_2 \leq C [\mathfrak{J}_\nu\beta]^{-2/3}\|f\|_2.$$
(3.1.4)

As $U(0) - \nu > a\beta^{-1/2}$, it holds that

$$x_\nu \geq \frac{1}{C} a^{1/2}\beta^{-\frac{1}{4}}.$$
(3.1.5)

For future reference we note, in addition, that

$$\frac{1}{C}x_\nu \leq \mathfrak{J}_\nu \leq Cx_\nu.$$
(3.1.6)

We split the proof of (3.1.4) into two parts according to the sign of $\nu - U(\frac{1}{2})$.

Step 1.1. The case when $U(1/2) < \nu < U(0) - a\beta^{-1/2}$. We note that in this case $x_\nu \in (0, \frac{1}{2})$. Let $\hat{\chi}$ be given as in (2.4.16) by

$$\hat{\chi}(x) = \begin{cases} 0 & |x| < \frac{1}{4}, \\ 1 & |x| > \frac{1}{2}. \end{cases}$$
(3.1.7)

Set further for $x \in [0,1]$

$$\chi_\nu^\pm(x) = \hat{\chi}(x/x_\nu - 1)\mathbf{1}_{\mathbb{R}_+}(\pm(x - x_\nu)).$$
(3.1.8)

and $\hat{\chi}$ can be chosen such that

$$\tilde{\chi}_\nu := \sqrt{1 - (\chi_\nu^+)^2 - (\chi_\nu^-)^2} \in C^\infty([0,1]).$$
(3.1.9)

Note that χ_ν^- is supported on $(0, 3x_\nu/4)$ whereas χ_ν^+ is supported on $(5x_\nu/4, 1)$. The complementary cutoff function $\tilde{\chi}_\nu$ is supported on $(x_\nu/2, 3x_\nu/2)$. The cutoff functions defined in (3.1.8) allow us to obtain estimates for v separately on the intervals $(0, x_\nu/2)$, $(3x_\nu/2, 1)$ (via integration by parts), and $(x_\nu/2, 3x_\nu/2)$.

More precisely, let $(v, f) \in D(\mathcal{L}_\beta^{\mathfrak{N}}) \times L^2(0, 1)$ satisfy $(\mathcal{L}_\beta^{\mathfrak{N}} - \lambda)v = f$. Set

$$\widehat{U}_\nu(x) = \begin{cases} U(x) & x \in (x_\nu/2, 3x_\nu/2), \\ U(x_\nu/2) + U'(x_\nu/2)(x - x_\nu) & x \leq \frac{x_\nu}{2}, \\ U(3x_\nu/2) + U'(3x_\nu/2)(x - x_\nu) & x \geq \frac{3x_\nu}{2}. \end{cases} \tag{3.1.10}$$

Let then

$$\widehat{\mathcal{L}}_{\beta,\nu,\mathbb{R}} = -\frac{d^2}{dx^2} + i\beta\widehat{U}_\nu, \tag{3.1.11a}$$

be defined on the domain

$$D(\widehat{\mathcal{L}}_{\beta,\nu,\mathbb{R}}) = \{u \in H^2(\mathbb{R}) \mid xu \in L^2(\mathbb{R})\}. \tag{3.1.11b}$$

We now apply the unitary dilation operator (corresponding to the change of variable $y = x/x_\nu$)

$$\mathcal{T}_\nu u(x) = x_\nu^{-1/2} u(x/x_\nu), \tag{3.1.12}$$

to obtain

$$\mathcal{T}_\nu^{-1} \widehat{\mathcal{L}}_{\beta,\nu,\mathbb{R}} \mathcal{T}_\nu = x_\nu^{-2}\Big(-\frac{d^2}{dy^2} + i\tilde{\beta}_\nu \tilde{U}_\nu(y)\Big), \tag{3.1.13}$$

where

$$\tilde{U}_\nu(y) = \frac{\widehat{U}_\nu(x_\nu y)}{x_\nu^2} \quad \text{and} \quad \tilde{\beta}_\nu = \beta x_\nu^4. \tag{3.1.14}$$

As there exist positive m, M such that for ν satisfying the assumptions of our proposition

$$0 < m \leq |\tilde{U}_\nu'(y)| \leq M \quad \forall y \in \mathbb{R}, \tag{3.1.15}$$

and in view of the uniform bound

$$\|\tilde{U}_\nu''\|_{L^\infty(\mathbb{R})} \leq C, \tag{3.1.16}$$

we may apply [3, Proposition 5.1] to the family of operators

$$\tilde{\mathcal{L}}_{\tilde{\beta},\mathbb{R}} := -\frac{d^2}{dy^2} + i\tilde{\beta}\,\tilde{U}_\nu(y) \tag{3.1.17}$$

to obtain for $\hat{\beta} \geq \hat{\beta}_0$

$$\sup_{\Re\tilde{\lambda} \leq \Upsilon\tilde{\beta}^{-1/3}} \|(\tilde{\mathcal{L}}_{\tilde{\beta},\mathbb{R}} - \tilde{\beta}\tilde{\lambda})^{-1}\| + \tilde{\beta}^{-1/3}\left\|\frac{d}{dy}(\tilde{\mathcal{L}}_{\tilde{\beta},\mathbb{R}} - \tilde{\beta}\tilde{\lambda})^{-1}\right\| \leq \frac{C}{\tilde{\beta}^{2/3}},$$

where

$$\widetilde{\beta} = \widetilde{\beta}_\nu = \beta x_\nu^4 \quad \text{and} \quad \widetilde{\lambda} = x_\nu^{-2}\lambda. \tag{3.1.18}$$

We observe that, by (3.1.5), for any given $\widehat{\beta}_0 > 0$ there exists $a_0 > 0$ such that $\widetilde{\beta} > \widehat{\beta}_0$ is satisfied for $a \geq a_0$. Taking the inverse dilation transformation, we obtain

$$\sup_{\Re\lambda \leq \Upsilon x_\nu^{2/3}\beta^{-1/3}} \|(\widehat{\mathcal{L}}_{\beta,\nu,\mathbb{R}}-\beta\lambda)^{-1}\| + (\beta x_\nu)^{-1/3}\left\|\frac{d}{dx}(\widehat{\mathcal{L}}_{\beta,\nu,\mathbb{R}}-\beta\lambda)^{-1}\right\| \leq \frac{C}{(x_\nu\beta)^{2/3}}.$$

Given that [3, Proposition 5.1] allows for an arbitrary Υ, we can replace x_ν by $\mathfrak{J}_\nu$ to obtain

$$\sup_{\Re\lambda \leq \Upsilon\mathfrak{J}_\nu^{2/3}\beta^{-1/3}} \|(\widehat{\mathcal{L}}_{\beta,\nu,\mathbb{R}}-\beta\lambda)^{-1}\| + (\beta\mathfrak{J}_\nu)^{-1/3}\left\|\frac{d}{dx}(\widehat{\mathcal{L}}_{\beta,\nu,\mathbb{R}}-\beta\lambda)^{-1}\right\| \leq \frac{C}{(\mathfrak{J}_\nu\beta)^{2/3}}. \tag{3.1.19}$$

We now write

$$(\widehat{\mathcal{L}}_{\beta,\nu,\mathbb{R}} - \beta\lambda)(\widetilde{\chi}_\nu v) = \widetilde{\chi}_\nu f - 2\widetilde{\chi}_\nu' v' - \widetilde{\chi}_\nu'' v. \tag{3.1.20}$$

We can then conclude from (3.1.19) and (3.1.20) that

$$\|\widetilde{\chi}_\nu v\|_2 + [\beta\mathfrak{J}_\nu]^{-1/3}\|(\widetilde{\chi}_\nu v)'\|_2 \leq \frac{C}{[\beta\mathfrak{J}_\nu]^{2/3}}(\|\widetilde{\chi}_\nu f\|_2 + 2\|\widetilde{\chi}_\nu' v'\|_2 + \|\widetilde{\chi}_\nu'' v\|_2). \tag{3.1.21}$$

Note that (3.1.21) implies, by (3.1.6), that

$$\|\widetilde{\chi}_\nu v\|_2 + [\beta x_\nu]^{-1/3}\|(\widetilde{\chi}_\nu v)'\|_2 \leq \frac{C}{[\beta x_\nu]^{2/3}}(\|f\|_2 + x_\nu^{-1}\|v'\|_2 + x_\nu^{-2}\|v\|_2). \tag{3.1.22}$$

To estimate v on $(0, 3x_\nu/4)$ and $(5x_\nu/4, 1)$ we write

$$(\mathcal{L}_\beta^{\mathfrak{N}} - \beta\lambda)(\chi_\nu^\pm v) = \chi_\nu^\pm f - 2(\chi_\nu^\pm)'v' - (\chi_\nu^\pm)''v. \tag{3.1.23}$$

The real part of the inner product with $\chi_\nu^\pm v$, after integration by parts, is given by

$$\|(\chi_\nu^\pm v)'\|_2^2 = \|(\chi_\nu^\pm)'v\|_2^2 + \mu\beta\|\chi_\nu^\pm v\|_2^2 + \Re\langle\chi_\nu^\pm v, \chi_\nu^\pm f\rangle, \tag{3.1.24}$$

whereas the imaginary part assumes the form

$$\mp\beta\||U - \nu|^{1/2}\chi_\nu^\pm v\|_2^2 + 2\Im\langle(\chi_\nu^\pm)'v, (\chi_\nu^\pm v)'\rangle = +\Im\langle\chi_\nu^\pm v, \chi_\nu^\pm f\rangle. \tag{3.1.25}$$

As, by (2.8.8), $|U - \nu|^{1/2}\chi_\nu^\pm \geq \frac{1}{C} x_\nu \chi_\nu^\pm$, (3.1.25) yields first

$$\frac{1}{C}\beta x_\nu^2\|\chi_\nu^\pm v\|_2^2 \leq \|\chi_\nu^\pm v\|_2 \|\chi_\nu^\pm f\|_2 + \frac{C}{x_\nu}\|v\|_2 \|(\chi_\nu^\pm v)'\|_2.$$

Combining the above with (3.1.24) we obtain

$$\frac{1}{C}\beta x_\nu^2 \|\chi_\nu^\pm v\|_2^2 \leq \|\chi_\nu^\pm v\|_2 \, \|\chi_\nu^\pm f\|_2$$
$$+ \frac{C}{x_\nu}\|v\|_2 \left(\frac{1}{x_\nu}\|v\|_2 + \mu_{\beta,+}^{\frac{1}{2}}\|\chi_\nu^\pm v\|_2 + \sqrt{\|\chi_\nu^\pm v\|_2 \, \|\chi_\nu^\pm f\|_2}\right),$$

where

$$\mu_{\beta,+} = \max(\mu\beta, 0). \tag{3.1.26}$$

For $\mu \leq \Upsilon \Im_\nu^{2/3}\beta^{-1/3}$, we may conclude, using (3.1.6), that

$$\frac{1}{C}\beta x_\nu^2 \|\chi_\nu^\pm v\|_2^2 \leq$$
$$\|\chi_\nu^\pm v\|_2 \, \|\chi_\nu^\pm f\|_2 + \frac{C}{x_\nu}\|v\|_2\left(\frac{1}{x_\nu}\|v\|_2 + (x_\nu\beta)^{\frac{1}{3}}\|\chi_\nu^\pm v\|_2 + \sqrt{\|\chi_\nu^\pm v\|_2 \, \|\chi_\nu^\pm f\|_2}\right),$$

which implies

$$\|\chi_\nu^\pm v\|_2^2 \leq \frac{C}{(\beta x_\nu^2)^2}\|\chi_\nu^\pm f\|_2^2 + C\frac{1}{\beta x_\nu^2}\left(\frac{1}{x_\nu^2} + \frac{1}{\beta^{\frac{1}{3}}x_\nu^{\frac{10}{3}}}\right)\|v\|_2^2.$$

By (3.1.5), there exists $C > 0$ such that

$$\frac{1}{\beta^{\frac{1}{3}}x_\nu^{\frac{10}{3}}} \leq Ca^{-2/3}x_\nu^{-2}.$$

Hence, we may conclude that there exists $C > 0$ such that, if $a \geq 1$ and $U(0) - \nu > a\beta^{-1/2}$,

$$\|\chi_\nu^\pm v\|_2 \leq \frac{C}{\beta x_\nu^2}[\|\chi_\nu^\pm f\|_2 + \beta^{1/2}\|v\|_2]. \tag{3.1.27}$$

Combining (3.1.27) and (3.1.22) leads to, with the aid of (3.1.5),

$$\|v\|_2 \leq C\big(([\beta x_\nu^2]^{-1} + [\beta x_\nu]^{-2/3})\|f\|_2$$
$$+ ([\beta x_\nu^4]^{-1/3} + [\beta x_\nu^4]^{-2/3})\|v\|_2 + [\beta^2 x_\nu^5]^{-1/3}\|v'\|_2\big)$$
$$\leq \check{C}\big([\beta x_\nu]^{-2/3}\|f\|_2 + [\beta^2 x_\nu^5]^{-1/3}\|v'\|_2\big) + \hat{C}a^{-\frac{2}{3}}\|v\|_2.$$

Thus, there exists $a_0 \geq 1$ and $C > 0$ such that for $a \geq a_0$ we obtain

$$\|v\|_2 \leq C\big([\beta x_\nu]^{-2/3}\|f\|_2 + [\beta^2 x_\nu^5]^{-1/3}\|v'\|_2\big). \tag{3.1.28}$$

We now use (3.1.22) together with (3.1.5) to establish that

$$\|(\tilde{\chi}_\nu v)'\|_2 \leq C([\beta \, x_\nu]^{-1/3}\|f\|_2 + [\beta x_\nu^7]^{-1/3}\|v\|_2) + \hat{C}a^{-\frac{2}{3}}\|v'\|_2. \tag{3.1.29}$$

By (3.1.24), as $\mu\beta \leq C(x_\nu\beta)^{\frac{2}{3}}$, it holds that

$$\|(\chi_\nu^\pm v)'\|_2 \leq C\Big(\frac{1}{x_\nu}\|v\|_2 + x_\nu^{\frac{1}{3}}\beta^{\frac{1}{3}}\|\chi_\nu^\pm v\|_2 + |\Re\langle\chi_\nu^\pm v, \chi_\nu^\pm f\rangle|^{\frac{1}{2}}\Big),$$

which leads to

$$\|(\chi_\nu^\pm v)'\|_2 \leq \hat{C}\Big(\frac{1}{x_\nu}\|v\|_2 + (\beta x_\nu)^{\frac{1}{3}}\|\chi_\nu^\pm v\|_2 + (\beta x_\nu)^{-\frac{1}{3}}\|f\|_2\Big). \tag{3.1.30}$$

Then we use (3.1.27), to get first that

$$\|(\chi_\nu^\pm v)'\|_2 \leq C\Big(\frac{1}{x_\nu} + \frac{\beta^{\frac{1}{2}}(\beta x_\nu)^{\frac{1}{3}}}{\beta x_\nu^2}\Big)\|v\|_2 + C\Big((\beta x_\nu)^{-\frac{1}{3}} + \frac{(\beta x_\nu)^{\frac{1}{3}}}{\beta x_\nu^2}\Big)\|f\|_2,$$

and then conclude from (3.1.5) that

$$\|(\chi_\nu^\pm v)'\|_2 \leq C(x_\nu^{-1}\|v\|_2 + (\beta x_\nu)^{-\frac{1}{3}}\|f\|_2). \tag{3.1.31}$$

Combining (3.1.29) and (3.1.31) yields

$$\|v'\|_2 \leq C\big((x_\nu^{-1} + [\beta x_\nu^7]^{-1/3})\|v\|_2 + [\beta x_\nu]^{-1/3}\|f\|_2\big) + \hat{C}a^{-\frac{2}{3}}\|v'\|_2.$$

Using again (3.1.5) we obtain the existence of a_0 that for $a \geq a_0$

$$\|v'\|_2 \leq C(x_\nu^{-1}\|v\|_2 + [\beta x_\nu]^{-1/3}\|f\|_2).$$

Substituting (3.1.28) into the above yields the existence of $a_0 > 0$ that for all $a \geq a_0$

$$\|v'\|_2 \leq C[\beta x_\nu]^{-1/3}\|f\|_2. \tag{3.1.32}$$

By (3.1.28) and (3.1.32) we then obtain

$$\|v\|_2 \leq C\big([\beta x_\nu]^{-2/3} + [\beta^2 x_\nu^5]^{-1/3}[\beta x_\nu]^{-1/3}\big)\|f\|_2,$$

which implies, using (3.1.5),

$$\|v\|_2 \leq C\,[\beta x_\nu]^{-2/3}\|f\|_2. \tag{3.1.33}$$

Having in mind (3.1.6) we finally obtain from (3.1.32) and (3.1.33)

$$\|v\|_2 + [\mathfrak{J}_\nu\beta]^{-1/3}\|v'\|_2 \leq C[\mathfrak{J}_\nu\beta]^{-2/3}\|f\|_2, \tag{3.1.34}$$

which is precisely (3.1.4).

Step 1.2. The case $\nu \leq U(1/2)$. We recall that $x_\nu = 1$ for $\nu < 0$ and observe that $x_\nu \geq \frac{1}{2}$ in this step. Hence, we need to define only a pair of cutoff functions. We thus set

$$\chi_2(x) = \hat{\chi}(2x), \tag{3.1.35}$$

and $\tilde{\chi}_2 = \sqrt{1 - \chi_2^2}$, which is supported on $[0, 1/4]$.

Then, we may write as in (3.1.27)

$$\|\tilde\chi_2 v\|_2 \leq C\beta^{-1}[\|f\|_2 + \beta^{1/2}\|v\|_2]. \tag{3.1.36}$$

Similarly, we obtain as in (3.1.31)

$$\|(\tilde\chi_2 v)'\|_2 \leq \frac{C}{\beta^{1/2}}[\|f\|_2 + \beta\|v\|_2]. $$

Suppose that

$$\Upsilon < [U'(1/4)/|U'(1)|]\Re\hat\kappa_1. \tag{3.1.37}$$

For later reference we note that (3.1.37) implies that $\Upsilon < [U'(1/4)/\Im_\nu]^{2/3}\Re\hat\kappa_1$. As in (3.1.21) we can also conclude, from [3, Proposition 5.2], that

$$\|\chi_2 v\|_2 + \beta^{-1/3}\|(\chi_2 v)'\|_2 \leq \frac{C}{\beta^{2/3}}(\|f\|_2 + \|v'\|_2 + \|v\|_2).$$

Combining the above we may proceed as in the Step 1.1 to conclude (3.1.4) and the L^2-bound on the right-hand side in (3.1.3a).

To complete the proof of (3.1.3a) we need to establish an $\mathcal{L}(L^2, L^\infty)$ bound for $(\mathcal{L}_\beta^\Re - \beta\lambda)^{-1}$.

Step 2. For

$$\Upsilon < [|U'(1/16)|/|U'(1)|]^{2/3}\Re\hat\kappa_1, \tag{3.1.38}$$

we prove that there exist positive β_0, a_0, and C such that for all $\beta \geq \beta_0$, $U(0) - \nu \geq a\beta^{-\frac{1}{2}}$, $a \geq a_0$ and $\mu \leq \Upsilon\Im_\nu^{2/3}\beta^{-1/3}$ we have

$$\|(\mathcal{L}_\beta^\Re - \beta\lambda)^{-1}f\|_2 + [\Im_\nu\beta]^{-1/3}\left\|\frac{d}{dx}(\mathcal{L}_\beta^\Re - \beta\lambda)^{-1}f\right\|_2 \leq C[\Im_\nu\beta]^{-5/6}\|f\|_\infty. \tag{3.1.39}$$

Step 2.1. We consider the case $0 < x_\nu \leq 1/2$. Considering a pair $(v, f) \in D(\mathcal{L}_\beta^\Re) \times L^\infty(0, 1)$ satisfying $(\mathcal{L}_\beta^\Re - \beta\lambda)v = f$, we then write as in (3.1.20)

$$(\hat{\mathcal{L}}_{\beta,\nu,\mathbb{R}} - \beta\lambda)(\tilde\chi_\nu v) = \tilde\chi_\nu f - 2\tilde\chi_\nu' v' - \tilde\chi_\nu'' v. \tag{3.1.40}$$

Let $w_1 \in D(\hat{\mathcal{L}}_{\beta,\nu,\mathbb{R}})$ satisfy

$$(\hat{\mathcal{L}}_{\beta,\nu,\mathbb{R}} - \beta\lambda)w_1 = \tilde\chi_\nu f. \tag{3.1.41}$$

Let $\tilde{\mathcal{L}}_{\tilde\beta,\mathbb{R}}$ be defined by (3.1.17). We now apply [3, Lemma 5.5] to the operator $\tilde{\mathcal{L}}_{\tilde\beta,\mathbb{R}} - i\tilde\beta\tilde{U}_\nu(1)$. Note that, due to (3.1.15) and (3.1.16), there exists $r > 1$ such that the potential $y \mapsto \tilde{U}_\nu(y) - \tilde{U}_\nu(1)$ belongs to S_r^2 (see [3, equation (2.32)] for the

definition of this class). Note further that [3, Lemma 5.5] holds under the assumption $\Upsilon > 0$. Hence, for any $\tilde{a} > 0$ there exists of $\widetilde{C}$ such that

$$\sup_{\Re\tilde{\lambda}\leq\Upsilon\tilde{\beta}^{-1/3}} \|(\widetilde{\mathcal{L}}_{\tilde{\beta},\mathbb{R}} - \tilde{\beta}\tilde{\lambda})^{-1}\tilde{g}\|_{L^2(-\tilde{a},\tilde{a})} \leq \frac{\widetilde{C}}{\tilde{\beta}^{5/6}}\|\tilde{g}\|_\infty. \tag{3.1.42}$$

We apply (3.1.42) with $\tilde{a} = 2$, $\tilde{g}(y) = x_\nu^{1/2}(\chi_\nu f)(x_\nu y)$, $\tilde{\lambda} = x_\nu^{-2}\lambda$ and $\tilde{\beta} = \beta x_\nu^4$ to establish after a change of variable that

$$\|w_1\|_{L^2(0,2x_\nu)} \leq \frac{C}{[\beta\mathfrak{J}_\nu]^{5/6}}\|f\|_\infty. \tag{3.1.43}$$

Let further $w_2 \in D(\hat{\mathcal{L}}_{\beta,\nu,\mathbb{R}})$ satisfy

$$(\hat{\mathcal{L}}_{\beta,\nu,\mathbb{R}} - \beta\lambda)w_2 = -2\tilde{\chi}_\nu' v' - \tilde{\chi}_\nu'' v.$$

From (3.1.13) we get that

$$\|w_2\|_2 \leq \frac{C}{[\beta\mathfrak{J}_\nu]^{2/3}}(\|\tilde{\chi}_\nu' v'\|_2 + \|\tilde{\chi}_\nu'' v\|_2). \tag{3.1.44}$$

Combining (3.1.44) with (3.1.43) yields as $\tilde{\chi}_\nu v = w_1 + w_2$ and Supp $\tilde{\chi}_\nu \subset (0, 2x_\nu)$

$$\begin{aligned}\|\tilde{\chi}_\nu v\|_2 &= \|\tilde{\chi}_\nu v\|_{L^2(0,2x_\nu)}\\ &\leq C\left([\beta x_\nu]^{-5/6}\|f\|_\infty + [\beta^2 x_\nu^5]^{-1/3}[\|\mathbf{1}_{(\frac{x_\nu}{2},\frac{3x_\nu}{4})}v'\|_2\right.\\ &\quad\left. + \|\mathbf{1}_{(\frac{5x_\nu}{4},\frac{3x_\nu}{2})}v'\|_2 + x_\nu^{-1}\|v\|_2]\right).\end{aligned} \tag{3.1.45}$$

Given the support of χ_ν^- it holds that

$$\|\chi_\nu^- f\|_2 \leq C x_\nu^{1/2}\|f\|_\infty,$$

and hence we can conclude from (3.1.27) that

$$\|\chi_\nu^- v\|_2 \leq \frac{C}{\beta x_\nu^2}\left[x_\nu^{1/2}\|f\|_\infty + \beta^{1/2}\|v\|_2\right]. \tag{3.1.46}$$

To obtain a similar bound for $\chi_\nu^+ v$ we obtain with the aid (2.8.9)

$$\|\chi_\nu^+|U - \nu|^{-1/2}\|_2^2 \leq C\int_{\frac{3x_\nu}{2}}^1 \frac{dx}{x^2 - x_\nu^2} \leq \frac{C}{x_\nu}.$$

Consequently, we can conclude that

$$\|(\chi_\nu^+)^2 v\|_1 \leq \|\chi_\nu^+|U - \nu|^{-1/2}\|_2\|\chi_\nu^+|U - \nu|^{1/2}v\|_2 \leq \frac{C}{x_\nu^{1/2}}\|\chi_\nu^+|U - \nu|^{1/2}v\|_2. \tag{3.1.47}$$

We now use (3.1.25) to obtain that

$$\|\chi_\nu^+ |U - \nu|^{1/2} v\|_2^2 \le C\beta^{-1}(x_\nu^{-1}\|v\|_2\|(\chi_\nu^+ v)'\|_2 + \|(\chi_\nu^+)^2 v\|_1\|f\|_\infty),$$

which implies that for any $\eta > 0$, we have

$$\|\chi_\nu^+ |U - \nu|^{1/2} v\|_2^2 \le C\beta^{-1}\Big(\eta x_\nu^{-2}\|v\|_2^2 + \frac{1}{\eta}\|(\chi_\nu^+ v)'\|_2^2 + \|(\chi_\nu^+)^2 v\|_1\|f\|_\infty\Big).$$

$$(3.1.48)$$

By (3.1.24) and (3.1.5) we have that

$$\|(\chi_\nu^+ v)'\|_2^2 \le C[\beta x_\nu]^{2/3}\|v\|_2^2 + \|(\chi_\nu^+)^2 v\|_1\|f\|_\infty. \qquad (3.1.49)$$

Note that by (3.1.5) we can conclude that $x_\nu^{-2} \le [\beta x_\nu]^{2/3}$. Substituting (3.1.49) into (3.1.48) yields for any $\eta > 0$

$$\|\chi_\nu^+ |U - \nu|^{1/2} v\|_2^2$$
$$\le C\beta^{-1}\Big((\eta x_\nu^{-2} + \frac{1}{\eta}(\beta x_\nu)^{2/3})\|v\|_2^2 + \Big(\frac{1}{\eta} + 1\Big)\|(\chi_\nu^+)^2 v\|_1\|f\|_\infty\Big).$$

Setting $\eta_\nu = x_\nu(\beta x_\nu)^{\frac{1}{3}}$ we observe that $\eta_\nu \ge \frac{1}{C}$ by (3.1.5), and hence

$$\|\chi_\nu^+ |U - \nu|^{1/2} v\|_2^2 \le C\beta^{-1}\big(\beta^{1/3} x_\nu^{-2/3}\|v\|_2^2 + \|(\chi_\nu^+)^2 v\|_1\|f\|_\infty\big). \qquad (3.1.50)$$

We then obtain, for any $\rho > 0$,

$$\|\chi_\nu^+ |U - \nu|^{1/2} v\|_2^2 \le C\Big([\beta x_\nu]^{-2/3}\|v\|_2^2 + \rho\beta^{-2}\|(\chi_\nu^+)^2 v\|_1 + \frac{1}{\rho}\|f\|_\infty^2\Big)$$

which, with the aid of (3.1.47), leads to

$$\|\chi_\nu^+ |U - \nu|^{1/2} v\|_2^2$$
$$\le \widehat{C}\Big([\beta x_\nu]^{-2/3}\|v\|_2^2 + \rho\beta^{-2} x_\nu^{-1}\|\chi_\nu^+ |U - \nu|^{1/2} v\|_2^2 + \frac{1}{\rho}\|f\|_\infty^2\Big).$$

Setting $\rho = [2\widehat{C}]^{-1}\beta^2 x_\nu$ finally leads to

$$\|\chi_\nu^+ |U - \nu|^{1/2} v\|_2^2 \le C\big([\beta x_\nu]^{-2/3}\|v\|_2^2 + [\beta^2 x_\nu]^{-1}\|f\|_\infty^2\big) \qquad (3.1.51)$$

Since for some positive C it holds by (2.8.9) that $\chi_\nu^+ |U - \nu|^{1/2} \ge C^{-1} x_\nu \chi_\nu^+$ we can conclude that

$$\|\chi_\nu^+ v\|_2 \le C(\beta^{-1/3} x_\nu^{-4/3}\|v\|_2 + \beta^{-1} x_\nu^{-3/2}\|f\|_\infty). \qquad (3.1.52)$$

Combining (3.1.46) with (3.1.45) and (3.1.52) then yields with the aid of (3.1.5)

$$
\begin{aligned}
\|v\|_2 &\leq C \left(([\beta x_\nu^{3/2}]^{-1} + [\beta x_\nu]^{-5/6})\|f\|_\infty + [\beta x_\nu^4]^{-1/3}\|v\|_2\right. \\
&\quad \left. + [\beta^2 x_\nu^5]^{-1/3}[\|\mathbf{1}_{(\frac{x_\nu}{2},\frac{3x_\nu}{4})}v'\|_2 + \|\mathbf{1}_{(\frac{5x_\nu}{4},\frac{3x_\nu}{2})}v'\|_2]\right) \\
&\leq \hat{C} \left([\beta x_\nu]^{-5/6}\|f\|_\infty + a^{-1}\|v\|_2 + [\beta^2 x_\nu^5]^{-1/3}[\|\mathbf{1}_{(\frac{x_\nu}{2},\frac{3x_\nu}{4})}v'\|_2\right. \\
&\quad \left. + \|\mathbf{1}_{(\frac{5x_\nu}{4},\frac{3x_\nu}{2})}v'\|_2]\right).
\end{aligned}
$$

Hence, there exist $a_0 > 0$ and $C > 0$ such that for all $a \geq a_0$

$$
\|v\|_2 \leq C \left([\beta x_\nu]^{-5/6}\|f\|_\infty + [\beta^2 x_\nu^5]^{-1/3}[\|\mathbf{1}_{(\frac{x_\nu}{2},\frac{3x_\nu}{4})}v'\|_2 + \|\mathbf{1}_{(\frac{5x_\nu}{4},\frac{3x_\nu}{2})}v'\|_2]\right).
\tag{3.1.53}
$$

Set

$$
\check{\chi}_\nu^\pm(x) = \hat{\chi}(2(x/x_\nu - 1))\mathbf{1}_{\mathbb{R}_+}(\pm(x - x_\nu)),
$$

where $\hat{\chi}$ is defined by (2.4.16). We note that by its definition $\check{\chi}_\nu^- = 1$ on $[0, \frac{3x_\nu}{4})$ and supp $\check{\chi}_\nu^- \subset (-\infty, \frac{7x_\nu}{8})$. Similarly, $\check{\chi}_\nu^+ = 1$ on $[\frac{5x_\nu}{4}, 1]$ and supp $\check{\chi}_\nu^+ \subset (\frac{9x_\nu}{8}, +\infty)$.

Proceeding as in the proof of (3.1.24) integration by parts yields, since v satisfies a Neumann condition at $x = 0$ and Dirichlet condition at $x = 1$, and since we have $(\check{\chi}_\nu^-)'(0) = 0$,

$$
\|(\check{\chi}_\nu^\pm v)'\|_2^2 = \|(\check{\chi}_\nu^\pm)'v\|_2^2 + \mu\beta\|\check{\chi}_\nu^\pm v\|_2^2 + \Re\langle\check{\chi}_\nu^\pm v, \check{\chi}_\nu^\pm f\rangle.
$$

The above identity implies, as $\mu < \Upsilon\mathfrak{J}_\nu\beta^{-\frac{1}{3}}$,

$$
\|(\check{\chi}_\nu^\pm v)'\|_2^2 \leq C([x_\nu\beta]^{2/3} + x_\nu^{-2})\|v\|_2^2 + \||U - \nu|^{1/2}\check{\chi}_\nu^\pm v\|_2\||U-\nu|^{-1/2}\check{\chi}_\nu^\pm\|_2\|f\|_\infty.
$$

By (2.8.9) there exists $0 < \nu_1 < U(0)$ such that for all $\nu_1 < \nu < U(0) - a\beta^{-1/2}$ it holds that

$$
\||U - \nu|^{-1/2}\check{\chi}_\nu^-\|_2^2 \leq C \int_0^{7x_\nu/8} \frac{dx}{x_\nu^2 - x^2} \leq \frac{C}{x_\nu}.
\tag{3.1.54}
$$

Similarly,

$$
\||U - \nu|^{-1/2}\check{\chi}_\nu^+\|_2^2 \leq C \int_{9x_\nu/8}^1 \frac{dx}{x_\nu^2 - x^2} \leq \frac{C}{x_\nu}.
\tag{3.1.55}
$$

For $0 < \nu < \nu_1$ (3.1.54) and (3.1.55) still hold given the support of $\check{\chi}_\nu^\pm$. Consequently, we may conclude that

$$
\|(\check{\chi}_\nu^\pm v)'\|_2^2 \leq C([x_\nu\beta]^{2/3} + x_\nu^{-2})\|v\|_2^2 + x_\nu^{-1/2}\||U - \nu|^{1/2}\check{\chi}_\nu^\pm v\|_2\|f\|_\infty.
\tag{3.1.56}
$$

As in (3.1.25) it holds that

$$
\mp\beta\||U - \nu|^{1/2}\check{\chi}_\nu^\pm v\|_2^2 + 2\Im\langle(\check{\chi}_\nu^\pm)'v, (\check{\chi}_\nu^\pm v)'\rangle = \Im\langle\check{\chi}_\nu^\pm v, \check{\chi}_\nu^\pm f\rangle,
$$

which implies

$$\beta \||U - \nu|^{1/2} \check{\chi}_\nu^\pm v\|_2^2$$
$$\leq \frac{C}{x_\nu} \|v\|_2 \|(\check{\chi}_\nu^\pm v)'\|_2 + \||U - \nu|^{1/2} \check{\chi}_\nu^\pm v\|_2 \||U - \nu|^{-1/2} \check{\chi}_\nu^\pm\|_2 \|f\|_\infty.$$

Consequently, by (3.1.54),

$$\||U - \nu|^{1/2} \check{\chi}_\nu^\pm v\|_2^2 \leq C([\beta x_\nu]^{-1} \|v\|_2 \|(\check{\chi}_\nu^\pm v)'\|_2 + [\beta^2 x_\nu]^{-1} \|f\|_\infty^2). \quad (3.1.57)$$

Substituting (3.1.57) into (3.1.56) then yields, with the aid of (3.1.5),

$$\|(\check{\chi}_\nu^\pm v)'\|_2^2$$
$$\leq C\left([\beta x_\nu]^{2/3} \|v\|_2^2 + [\beta x_\nu]^{-1} \|f\|_\infty^2 + [\beta^{1/2} x_\nu]^{-1} \|v\|_2^{1/2} \|(\check{\chi}_\nu^\pm v)'\|_2^{1/2} \|f\|_\infty\right).$$

By the above inequality we may conclude, first, that

$$\|(\check{\chi}_\nu^\pm v)'\|_2^2 \leq C\left([\beta x_\nu]^{2/3} \|v\|_2^2 + 2[\beta x_\nu]^{-1} \|f\|_\infty^2 + x_\nu^{-1}(\|(\check{\chi}_\nu^\pm v)'\|_2 \|v\|_2)\right),$$

and then, for any $\eta > 0$,

$$\|(\check{\chi}_\nu^\pm v)'\|_2^2 \leq C\left([\beta x_\nu]^{2/3} \|v\|_2^2 + 2[\beta x_\nu]^{-1} \|f\|_\infty^2 + \eta \|(\check{\chi}_\nu^\pm v)'\|_2^2 + \frac{1}{\eta} x_\nu^{-2} \|v\|_2^2\right).$$

Using again (3.1.5), for η small enough, we finally obtain

$$\|(\check{\chi}_\nu^\pm v)'\|_2^2 \leq C\left([\beta x_\nu]^{2/3} \|v\|_2^2 + 2[\beta x_\nu]^{-1} \|f\|_\infty^2\right).$$

From the above it can be easily verified that

$$\|\mathbf{1}_{(\frac{x_\nu}{2}, \frac{3x_\nu}{4})} v'\|_2 + \|\mathbf{1}_{(\frac{5x_\nu}{4}, \frac{3x_\nu}{2})} v'\|_2$$
$$\leq \|(\check{\chi}_\nu^+ v)'\|_2 + \|(\check{\chi}_\nu^- v)'\|_2 \leq C([\beta x_\nu]^{1/3} \|v\|_2 + [\beta x_\nu]^{-1/2} \|f\|_\infty). \quad (3.1.58)$$

Substituting (3.1.58) into (3.1.53) yields, using (3.1.5)

$$\|v\|_2 \leq C\left([\beta x_\nu]^{-5/6} \|f\|_\infty + [\beta x_\nu^4]^{-1/3} \|v\|_2\right).$$

Hence, there exists $a_0 > 0$ such that we obtain for $a \geq a_0$

$$\|v\|_2 \leq C [\beta x_\nu]^{-5/6} \|f\|_\infty. \quad (3.1.59)$$

Step 2.2. The case $x_\nu \geq 1/8$. Let

$$\Upsilon < [|U'(1/16)|/|U'(1)|]^{2/3} \Re \hat{\kappa}_1. \quad (3.1.60)$$

We set

$$\hat{\eta}_\nu = \chi(-(x - x_\nu)/x_\nu), \quad (3.1.61)$$

which is supported on $(x_\nu/4, 1]$ and satisfies $\widehat{\eta}_\nu \equiv 1$ on $(x_\nu/2, 1]$. Then, as

$$(\mathscr{L}_\beta^{\mathfrak{N}} - \beta\lambda)(\widehat{\eta}_\nu v) = \widehat{\eta}_\nu f - 2\widehat{\eta}_\nu' v' - \widehat{\eta}_\nu'' v, \tag{3.1.62}$$

we may use [3, Propositions 5.2 and 5.4] (both hold for $U \in \mathcal{S}_r^2$ though stated for $U \in \mathcal{S}_r^4$), to obtain that

$$\|\widehat{\eta}_\nu v\|_2 \le \frac{C}{\beta^{2/3}}(\beta^{-1/6}\|f\|_\infty + \|\widehat{\eta}_\nu' v'\|_2 + \|\widehat{\eta}_\nu'' v\|_2). \tag{3.1.63}$$

We note that (3.1.60) implies $\Upsilon < [|U'(x_\nu/4)|/\mathfrak{J}_\nu]^{2/3}\mathfrak{R}\widehat{\kappa}_1$ for all $1/8 \le x_\nu \le 1$. Let $\widetilde{\eta}_\nu = \sqrt{1 - \widehat{\eta}_\nu^2} \in C^\infty(\mathbb{R}, [0, 1])$. Note that $\widetilde{\eta}_\nu$ is supported on $[0, x_\nu/2]$. Consequently, we may obtain, as in (3.1.46) but for $x_\nu \ge \frac{1}{8}$,

$$\|\widetilde{\eta}_\nu v\|_2 \le C\beta^{-1}[\|f\|_2 + \beta^{1/2}\|v\|_2]. $$

Combining the above with (3.1.63) yields

$$\|v\|_2 \le \frac{C}{\beta^{2/3}}(\beta^{-1/6}\|f\|_\infty + \|\mathbf{1}_{(x_\nu/4, x_\nu/2)}v'\|_2). $$

We now use a variant of equation (3.1.58) (which is valid also for $x_\nu > 1/8$) to bound $\|\mathbf{1}_{(x_\nu/4, x_\nu/2)}v'\|_2$ to obtain, with the aid of (3.1.5),

$$\|v\|_2 \le \frac{C}{\beta^{5/6}}\|f\|_\infty. $$

Together with (3.1.4) the above inequality establishes (3.1.3a).

Step 3. We prove (3.1.3b), when

$$\Upsilon < \inf_{x_\nu \in [0,1]}(|U'(x_\nu/2)|/|U'(x_\nu)|)^{2/3}\mathfrak{R}\widehat{\kappa}_1. \tag{3.1.64}$$

Note that for $p = 2$ (3.1.3b) readily follows from (3.1.4). In the following we then assume $p \in (1, 2)$. Suppose first that $x_\nu < 1/2$. As above, we consider a pair (v, f) in $D(\mathscr{L}_\beta^{\mathfrak{N}}) \times L^2(0, 1)$ satisfying $(\mathscr{L}_\beta^{\mathfrak{N}} - \lambda\beta)v = f$. Let

$$\widehat{\mathscr{L}}_\beta^{\mathfrak{D}} : H^2(x_\nu/2, 3x_\nu/2) \cap H_0^1(x_\nu/2, 3x_\nu/2) \to L^2(x_\nu/2, 3x_\nu/2)$$

be associated with the same differential operator as $\mathscr{L}_\beta^{\mathfrak{N}}$. Let

$$\widetilde{\mathscr{L}}_{\widetilde{\beta}}^{\mathfrak{D}} : H^2(1/2, 3/2) \cap H_0^1(1/2, 3/2) \to L^2(1/2, 3/2)$$

be associated with the same differential operator as $\widetilde{\mathscr{L}}_{\widetilde{\beta}, \mathbb{R}}$ in (3.1.17). We recall from [3, Proposition 5.2] that for any $g \in L^2(1/2, 3/2)$ it holds

$$\sup_{\mathfrak{R}\widetilde{\lambda} \le \Upsilon\widetilde{\beta}^{-1/3}} \widetilde{\beta}^{-1/3}\left\|\frac{d}{dy}(\widetilde{\mathscr{L}}_{\widetilde{\beta}}^{\mathfrak{D}} - \widetilde{\beta}\widetilde{\lambda})^{-1}g\right\|_p \le \frac{C}{\widetilde{\beta}^{\frac{2+p}{6p}}}\|g\|_2. \tag{3.1.65}$$

As in (3.1.20) it holds that

$$(\widehat{\mathcal{L}}_\beta^{\mathfrak{D}} - \beta\lambda)(\tilde{\chi}_\nu v) = \tilde{\chi}_\nu f - 2\tilde{\chi}_\nu' v' - \tilde{\chi}_\nu'' v,$$

and hence by applying the inverse of the dilation (3.1.12) to (3.1.65) we can conclude that

$$\|(\tilde{\chi}_\nu v)'\|_p \leq C[\beta x_\nu]^{-\frac{2+p}{6p}}(\|f\|_2 + \|\tilde{\chi}_\nu' v'\|_2 + \|\tilde{\chi}_\nu'' v\|_2). \tag{3.1.66}$$

By (3.1.4) we then obtain

$$\|(\tilde{\chi}_\nu v)'\|_p \leq C[\beta x_\nu]^{-\frac{2+p}{6p}}(1 + \beta^{-1/3}x_\nu^{-4/3} + \beta^{-2/3}x_\nu^{-8/3})\|f\|_2.$$

From (3.1.5) we easily conclude that

$$\|(\tilde{\chi}_\nu v)'\|_p \leq C[\beta x_\nu]^{-\frac{2+p}{6p}}\|f\|_2. \tag{3.1.67}$$

We now seek an estimate for $\chi_\nu^\pm v'$. To this end, we use integration by parts to obtain

$$\Re\langle(\chi_\nu^\pm)^2(U - \nu)v, (\mathcal{L}_\beta^{\mathfrak{N}} - \beta\lambda)v\rangle$$
$$= \mp\|\chi_\nu^\pm|U - \nu|^{1/2}v'\|_2^2 + \Re\langle\chi_\nu^\pm(U'\chi_\nu^\pm + 2(U - \nu)(\chi_\nu^\pm)')v, v'\rangle$$
$$- \mu\beta\|\chi_\nu^\pm|U - \nu|^{1/2}v\|_2^2. \tag{3.1.68}$$

Since

$$|U'(x)| \leq C|U(x) - U(0)|^{1/2} \leq C(|U(x) - \nu|^{1/2} + |U(0) - \nu|^{1/2}), \tag{3.1.69}$$

we can conclude that

$$\|(\chi_\nu^\pm)^2 U'v\|_2 \leq C(x_\nu\|v\|_2 + \|\chi_\nu^\pm|U - \nu|^{1/2}v\|_2).$$

Furthermore, given the support of $(\chi_\nu^\pm)'$ we obtain by (2.8.6)

$$\|\chi_\nu^\pm(U - \nu)(\chi_\nu^\pm)'\|_\infty \leq C\|(x^2 - x_\nu^2)(\chi_\nu^\pm)'\|_\infty \leq Cx_\nu.$$

Combining the above with (3.1.68) yields that

$$\|\chi_\nu^\pm|U - \nu|^{1/2}v'\|_2^2$$
$$\leq \|(U - \nu)v\|_2\|f\|_2 + C\left([\beta x_\nu]^{2/3}\|\chi_\nu^\pm|U - \nu|^{1/2}v\|_2^2 + [x_\nu\|v\|_2\right.$$
$$\left. + \|\chi_\nu^\pm|U - \nu|^{1/2}v\|_2]\|v'\|_2\right). \tag{3.1.70}$$

As

$$\Im\langle(U - \nu)v, (\mathcal{L}_\beta^{\mathfrak{N}} - \beta\lambda)v\rangle = \beta\|(U - \nu)v\|_2^2 + \Im\langle U'v, v'\rangle, \tag{3.1.71}$$

we obtain by (3.1.69) that

$$\beta\|(U - \nu)v\|_2^2 \leq C\left(\beta^{-1}\|f\|_2^2 + [\||U(x) - \nu|^{1/2}v\|_2 + x_\nu\|v\|_2]\|v'\|_2\right).$$

Furthermore, since

$$\||U - v|^{1/2} v\|_2^2 \leq \frac{1}{2} \big[x_v^{-2} \|(U - v)v\|_2^2 + x_v^2 \|v\|_2^2 \big], \tag{3.1.72}$$

we can write

$$\beta \|(U - v)v\|_2^2 \leq C \big(\beta^{-1} \|f\|_2^2 + [x_v \|v\|_2 + x_v^{-1} \|(U - v)v\|_2] \|v'\|_2 \big).$$

Hence,

$$\beta \|(U - v)v\|_2^2 \leq C \big(\beta^{-1} \|f\|_2^2 + x_v \|v\|_2 \|v'\|_2 + \beta^{-1} x_v^{-2} \|v'\|_2^2 \big). \tag{3.1.73}$$

Using (3.1.4) gives the following estimates for $\|v\|_2$ and $\|v'\|_2$

$$\|v'\|_2 \leq C \, [x_v \beta]^{-1/3} \|f\|_2 \quad \text{and} \quad \|v\|_2 \leq C \, [x_v \beta]^{-2/3} \|f\|_2. \tag{3.1.74}$$

Substituting the above into (3.1.73) yields

$$\beta \|(U - v)v\|_2^2 \leq C (\beta^{-1} + \beta^{-1} x_v^{-2} [x_v \beta]^{-2/3}) \|f\|_2^2.$$

From the above, recalling that $\beta^{\frac{1}{4}} x_v$ is bounded from below, we conclude that

$$\|(U - v)v\|_2 \leq C \beta^{-1} \|f\|_2. \tag{3.1.75}$$

Next, we write, using (3.1.74) and (3.1.75),

$$\||U - v|^{1/2} v\|_2^2 \leq \frac{1}{2} [\beta^{1/3} x_v^{-2/3} \|(U - v)v\|_2^2 + \beta^{-1/3} x_v^{2/3} \|v\|_2^2] \leq \frac{C}{\beta^{5/3} x_v^{2/3}} \|f\|_2^2. \tag{3.1.76}$$

Substituting (3.1.76) together with (3.1.75) into (3.1.70) yields with the aid of (3.1.74)

$$\|\chi_v^{\pm} |U - v|^{1/2} v'\|_2^2 \leq C \beta^{-1} \|f\|_2^2. \tag{3.1.77}$$

We now observe that

$$\||U - v|^{-1/2} \chi_v^-\|_q^q \leq C \int_0^{3x_v/4} \frac{dx}{[x_v^2 - x^2]^{q/2}} \leq \frac{C}{x_v^{q-1}}. \tag{3.1.78}$$

Similarly,

$$\||U - v|^{-1/2} \chi_v^+\|_2^2 \leq C \int_{5x_v/4}^1 \frac{dx}{[x_v^2 - x^2]^{q/2}} \leq \frac{C}{x_v^{q-1}}, \tag{3.1.79}$$

which is obtained with the aid of (2.8.9) for $v > v_1$ (for $v \leq v_1$ the above bounds are trivial). Consequently, we obtain that

$$\|(\chi_v^{\pm}) v'\|_p \leq \|\chi_v^{\pm} |U - v|^{1/2} v'\|_2 \|[\mathbf{1}_{[0,3x_v/4]} + \mathbf{1}_{[5x_v/4,1]}] |U - v|^{-1/2}]\|_{\frac{2p}{2-p}}$$

$$\leq \frac{C}{\beta^{1/2} x_v^{\frac{3p-2}{2p}}} \|f\|_2. \tag{3.1.80}$$

Similarly, we write that

$$\|(\chi_\nu^\pm)' v\|_p \leq \|(\chi_\nu^\pm)'|U - \nu|^{1/2} v\|_2 \|[\mathbf{1}_{[0,3x_\nu/4]} + \mathbf{1}_{[5x_\nu/4,1]}|U - \nu|^{-1/2}]\|_{\frac{2p}{2-p}}$$

$$\leq \frac{C}{x_\nu^{\frac{17}{6} - \frac{1}{p}} \beta^{5/6}} \|f\|_2.$$

To obtain the second inequality we have used (3.1.76). Together with (3.1.80) and (3.1.5) the above yields

$$\|(\chi_\nu^\pm v)'\|_p \leq C\beta^{-1/2} x_\nu^{-\frac{3p-2}{2p}} \|f\|_2. \tag{3.1.81}$$

Combining the above with (3.1.67) yields the existence of a_0 such that (3.1.3b) holds for all $a \geq a_0$.

Consider next the case $x_\nu \geq 1/2$ (in which no dilation transformation is necessary). Let

$$\Upsilon < (|U'(1/4)|/|U'(1)|)^{2/3} \Re \hat{\kappa}_1. \tag{3.1.82}$$

We now set

$$\iota_\nu = \sqrt{\tilde{\chi}_\nu^2 + (\chi_\nu^+)^2}.$$

Then, as

$$(\hat{\mathcal{L}}_\beta^{\mathfrak{D}} - \beta\lambda)(\iota_\nu v) = \iota_\nu f - 2\iota_\nu' v' - \iota_\nu'' v.$$

We obtain using [3, Proposition 5.2] and (3.1.4) that

$$\|(\iota_\nu v)'\|_p \leq \frac{C}{\beta^{\frac{2+p}{6p}}} \|f\|_2.$$

Since (3.1.81) holds true for $\chi_\nu^- v$ in the case $x_\nu \geq 1/2$, we can combine it with the above to extend the validity (3.1.3b) to this case as well.

Given (3.1.37), (3.1.38), (3.1.60), (3.1.64), and (3.1.82) it follows that there exists $\Upsilon > 0$ for which Proposition 3.1.1 holds true. ∎

Remark 3.1.2. As in [3, Proposition 5.1] we can obtain better estimates for the case where $\mu < 0$. Thus, setting $\chi_\nu^\pm \equiv 1$ in (3.1.24) yields for $\mu < 0$

$$\|v'\|_2^2 + |\mu|\beta\|v\|_2^2 = \Re\langle v, f \rangle. \tag{3.1.83}$$

From here we conclude that

$$\|v\|_2 \leq [|\mu|\beta]^{-1}\|f\|_2, \tag{3.1.84}$$

which is stronger that (3.1.3) when $|\mu| \gg \beta^{-1/3}$.

Note that for $\mu < 0$

$$\|v'\|_2 \leq [|\mu|\beta]^{-1/2}\|f\|_2. \tag{3.1.85}$$

3.2 Resolvent estimates for $|U(0) - \Im\lambda| = \mathcal{O}(\beta^{-1/2})$

In this case, we will approximate $U - \nu$ by its quadratic potential

$$x \mapsto U''(0)x^2/2 + U(0) - \nu$$

and then use a proper resolvent estimate established by R. Henry in [16].

More precisely, we prove the following proposition.

Proposition 3.2.1. *Let* $U \in C^3([0,1])$ *satisfy* (2.1.3), $a > 0$ *and* $\Upsilon < \sqrt{-U''(0)}/2$. *Then there exist* $C > 0$ *and* $\beta_0 > 0$ *such that, for all* $\beta \geq \beta_0$,

- *if* $f \in L^\infty(0,1)$,

$$\sup_{\substack{\mu \leq \Upsilon\beta^{-1/2} \\ |\nu-U(0)|<a\beta^{-1/2}}} \Big(\|(\mathscr{L}_\beta^{\mathfrak{N},\mathfrak{D}} - \beta\lambda)^{-1}f\|_2 + \beta^{-1/4}\|\frac{d}{dx}(\mathscr{L}_\beta^{\mathfrak{N},\mathfrak{D}} - \beta\lambda)^{-1}f\|_2$$

$$+ \beta^{1/8}\|(\mathscr{L}_\beta^{\mathfrak{N},\mathfrak{D}} - \beta\lambda)^{-1}f\|_1 \Big)$$

$$\leq C \min(\beta^{-1/2}\|f\|_2, \beta^{-5/8}\|f\|_\infty), \qquad (3.2.1a)$$

- *if* $(x - x_\nu)^{-1}f \in L^2(0,1)$

$$\sup_{\substack{\mu \leq \Upsilon\beta^{-1/2} \\ |\nu-U(0)|<a\beta^{-1/2}}} \Big(\left\|(\mathscr{L}_\beta^{\mathfrak{N},\mathfrak{D}} - \beta\lambda)^{-1}f\right\|_2 + \beta^{-1/4}\left\|\frac{d}{dx}(\mathscr{L}_\beta^{\mathfrak{N},\mathfrak{D}} - \beta\lambda)^{-1}f\right\|_2$$

$$+ \beta^{1/8}\left\|(\mathscr{L}_\beta^{\mathfrak{N},\mathfrak{D}} - \beta\lambda)^{-1}f\right\|_1 \Big)$$

$$\leq C\beta^{-3/4}\left\|\frac{f}{x - x_\nu}\right\|_2. \qquad (3.2.1b)$$

Proof. All the estimates established in this proof assume that

$$\mu \leq \Upsilon\beta^{-1/2} \quad \text{and} \quad U(0) - a\beta^{-1/2} < \nu < U(0) + a\beta^{-1/2}. \qquad (3.2.2)$$

By the second condition it holds that

$$0 \leq x_\nu \leq C_a\beta^{-\frac{1}{4}}. \qquad (3.2.3)$$

Consequently, for any $\nu_1 < U(0)$, there exists β_0 such that, for all $\beta \geq \beta_0$, we have

$$\nu > \nu_1 \quad \text{and} \quad x_\nu < 1/4. \qquad (3.2.4)$$

Step 1: $\mathcal{L}(L^2, L^2)$ estimate. By [16, Theorem 1.3] we immediately obtain that, under (3.2.2),

$$\|(\mathcal{L}_\beta^\mathfrak{N} - \beta\lambda)^{-1} f\|_2 \leq C\beta^{-1/2} \|f\|_2. \tag{3.2.5}$$

To prove the second inequality in (3.2.1a), let $f \in L^2(0,1)$ and $v \in D(\mathcal{L}_\beta^\mathfrak{N})$ satisfy $(\mathcal{L}_\beta^\mathfrak{N} - \beta\lambda)v = f$. Taking the scalar product with v, an integration by parts yields for the real part, with the aid of (3.2.2),

$$\|v'\|_2^2 = \mu\beta\|v\|_2^2 + \Re\langle v, f\rangle \leq C(\beta^{1/2}\|v\|_2^2 + \beta^{-1/2}\|f\|_2^2).$$

By (3.2.5) we can then conclude that

$$\left\|\frac{d}{dx}(\mathcal{L}_\beta^\mathfrak{N} - \beta\lambda)^{-1} f\right\|_2 \leq C\beta^{-1/4}\|f\|_2, \tag{3.2.6}$$

which together with (3.2.5) establishes the $\mathcal{L}(L^2, L^2)$ estimate in (3.2.1a).

Step 2: $\mathcal{L}(L^\infty, L^2)$ estimate. Next, we obtain an $\mathcal{L}(L^\infty, L^2)$ estimate for $(\mathcal{L}_\beta^\mathfrak{N} - \beta\lambda)^{-1}$ under (3.2.2). Let $\hat{\chi}$ be given (2.4.16). As before, we set

$$\chi_\gamma(x) = \hat{\chi}(\gamma\beta^{1/4}x),$$

for some positive

$$\gamma < [8C_a + 1]^{-1} < 1.$$

In particular, γ satisfies

$$0 < \gamma < [8\beta^{\frac{1}{4}}x_\nu + 1]^{-1}.$$

We note that χ_γ satisfies

$$\text{supp}\chi_\gamma \subset \left[2x_\nu + \frac{1}{4}\beta^{-\frac{1}{4}}, 1\right) \quad \text{and} \quad |\chi_\gamma'| \leq C\gamma\beta^{\frac{1}{4}}. \tag{3.2.7}$$

Set further $\tilde{\chi}_\gamma = \sqrt{1 - \chi_\gamma^2} \in C^\infty(\mathbb{R})$ and note that $\tilde{\chi}_\gamma$ satisfies

$$\text{supp}\tilde{\chi}_\gamma \subset \left[0, \frac{1}{2\gamma}\beta^{-\frac{1}{4}}\right) \quad \text{and} \quad |\tilde{\chi}_\gamma'| \leq C\gamma\beta^{\frac{1}{4}}. \tag{3.2.8}$$

Let $f \in L^2(0,1)$ and $v \in D(\mathcal{L}_\beta^\mathfrak{N})$ satisfy $(\mathcal{L}_\beta^\mathfrak{N} - \beta\lambda)v = f$.

We begin by estimating $\chi_\gamma v$. An integration by parts yields

$$\|(\chi_\gamma v)'\|_2^2 - \|\chi_\gamma' v\|_2^2 - \mu\beta\|\chi_\gamma v\|_2^2 = \Re\langle \chi_\gamma v, \chi_\gamma f\rangle, \tag{3.2.9}$$

from which we conclude, given that $\mu\beta^{1/2}$ and γ are bounded from above,

$$\|(\chi_\gamma v)'\|_2^2 \leq \|\chi_\gamma v\|_1 \|f\|_\infty + C\beta^{1/2}\|v\|_2^2. \tag{3.2.10}$$

Furthermore, we have that

$$-\beta\||U-v|^{1/2}\chi_\gamma v\|_2^2 + 2\Im\langle\chi_\gamma' v, (\chi_\gamma v)'\rangle = \Im\langle\chi_\gamma v, \chi_\gamma f\rangle. \tag{3.2.11}$$

Hence, with the aid of (3.2.10), we can conclude that

$$\beta\||U-v|^{1/2}\chi_\gamma v\|_2^2 \le C(\gamma\beta^{1/2}\|v\|_2^2 + \|\chi_\gamma v\|_1\|f\|_\infty). \tag{3.2.12}$$

We now write (note that $\chi_\gamma\chi_{2\gamma} = \chi_\gamma$)

$$\|\chi_\gamma v\|_1 \le \|\chi_{2\gamma}|U-v|^{-1/2}\|_2\||U-v|^{1/2}\chi_\gamma v\|_2. \tag{3.2.13}$$

For $x \in [x_v + \frac{1}{8}\beta^{-\frac{1}{4}}, 1)$, it holds by (2.9.6), (2.9.7), and (3.2.4)

$$|U(x) - v| \ge \frac{1}{C}(x - x_v)^2 \ge \frac{1}{4C}\beta^{-\frac{1}{2}}, \tag{3.2.14}$$

which implies, for $\beta \ge \beta_0$,

$$\int_{x_v+\frac{1}{8}\beta^{-\frac{1}{4}}}^1 |U-v|^{-1}\,dx \le C\int_{\frac{1}{4}\beta^{-\frac{1}{4}}}^1 y^{-2}\,dy \le \hat{C}\beta^{1/4}.$$

We can then conclude, using (3.2.7), that

$$\|\chi_{2\gamma}|U-v|^{-1/2}\|_2 \le C\beta^{1/8}. \tag{3.2.15}$$

Hence, by (3.2.12) and (3.2.13) we obtain that

$$\|\chi_\gamma v\|_1 \le C\beta^{-3/8}(\gamma^{\frac{1}{2}}\beta^{1/4}\|v\|_2 + \|\chi_\gamma v\|_1^{1/2}\|f\|_\infty^{1/2}),$$

from which we conclude that

$$\|\chi_\gamma v\|_1 \le C\beta^{-1/8}(\gamma^{1/2}\|v\|_2 + \beta^{-5/8}\|f\|_\infty).$$

Substituting the above into (3.2.12) then yields

$$\||U-v|^{1/2}\chi_\gamma v\|_2 \le C\beta^{-1/4}(\gamma^{1/2}\|v\|_2 + \beta^{-5/8}\|f\|_\infty). \tag{3.2.16}$$

Since by (3.2.14),

$$|U-v|^{1/2}\chi_\gamma \ge \frac{1}{C}\beta^{-1/4}\chi_\gamma, \tag{3.2.17}$$

we may write

$$\|\chi_\gamma v\|_2 \le C(\gamma^{1/2}\|v\|_2 + \beta^{-5/8}\|f\|_\infty). \tag{3.2.18}$$

We now attempt to estimate $\tilde{\chi}_\gamma v$. As

$$(\mathcal{L}_\beta^\mathfrak{N} - \beta\lambda)(\tilde{\chi}_\gamma v) = \tilde{\chi}_\gamma f - 2\tilde{\chi}_\gamma' v' - \tilde{\chi}_\gamma'' v, \tag{3.2.19}$$

we may conclude from [16, Theorem 1.3] (which can be used since $U \in C^3([0, 1])$ and $\Upsilon < [-U''(0)]^{1/2}/2$), that

$$\|\tilde{\chi}_\gamma v\|_2 \leq C\beta^{-1/2}(\|\tilde{\chi}_\gamma f\|_2 + \|\tilde{\chi}'_\gamma v'\|_2 + \|\tilde{\chi}''_\gamma v\|_2)$$
$$\leq \hat{C}(\beta^{-5/8}\|f\|_\infty + \gamma\beta^{-1/4}\|v'\| + \gamma^2\|v\|_2).$$

To obtain the second inequality we used the fact that supp $\tilde{\chi}_\gamma \subseteq (0, \check{C}\beta^{-\frac{1}{4}})$. Combining the above with (3.2.18) yields the existence of $\gamma_0 > 0$ such that for all $\gamma \in (0, \gamma_0)$,

$$\|v\|_2 \leq C(\beta^{-5/8}\|f\|_\infty + \gamma\beta^{-1/4}\|v'\|_2). \tag{3.2.20}$$

As in (3.2.10) (replacing χ_γ by 1), we obtain that

$$\|v'\|_2^2 \leq \|v\|_1\|f\|_\infty + C\beta^{1/2}\|v\|_2^2. \tag{3.2.21}$$

By (2.9.24), applied with $q = 1$ and $\mu = \beta^{-\frac{1}{2}}$, and (3.2.2) it holds that

$$\|(U - v + i\beta^{-1/2})^{-1/2}\|_2^2 = \|(U - v + i\beta^{-1/2})^{-1}\|_1 \leq C\beta^{1/4}.$$

From the above we conclude that

$$\|v\|_1 \leq \||U - v + i\beta^{-1/2}|^{-1/2}\|_2\||U - v + i\beta^{-1/2}|^{1/2}v\|_2$$
$$\leq C\beta^{1/8}(\||U - v|^{1/2}\chi_\gamma v\|_2 + \||U - v|^{1/2}\tilde{\chi}_\gamma v\|_2 + \beta^{-1/4}\|v\|_2).$$

By (2.8.9) (which is valid by (3.2.4))

$$|U - v|^{1/2}\tilde{\chi}_\gamma \leq C \sup_{x \in (0, \check{C}\beta^{-\frac{1}{4}})} |x^2 - x_v^2|^{1/2} \leq C\beta^{-1/4},$$

we obtain, with the aid of (3.2.16) that

$$\|v\|_1 \leq C(\beta^{-1/8}\|v\|_2 + \beta^{-3/4}\|f\|_\infty). \tag{3.2.22}$$

Substituting (3.2.22) into (3.2.21) yields

$$\|v'\|_2 \leq C(\beta^{1/4}\|v\|_2 + \beta^{-3/8}\|f\|_\infty),$$

which when substituted into (3.2.20) yields for sufficiently small γ_0 and $\gamma \in (0, \gamma_0)$

$$\|v\|_2 \leq C\beta^{-5/8}\|f\|_\infty, \tag{3.2.23}$$

and then

$$\|v'\|_2 \leq C\beta^{-3/8}\|f\|_\infty. \tag{3.2.24}$$

Substituting (3.2.23) into (3.2.22) yields

$$\|v\|_1 \leq C\beta^{-3/4}\|f\|_\infty. \tag{3.2.25}$$

By (3.2.11) it holds that

$$\beta\||U-v|^{1/2}\chi_\gamma v\|_2^2 \le C(\gamma\beta^{1/2}\|v\|_2^2 + \|\chi_\gamma v\|_2\|f\|_2),$$

from which we conclude by combining it with (3.2.5)

$$\||U-v|^{1/2}\chi_\gamma v\|_2 \le C\beta^{-3/4}\|f\|_2.$$

Consequently,

$$\|\chi_\gamma^2 v\|_1 \le \||U-v|^{-1/2}\chi_\gamma\|_2 \||U-v|^{1/2}\chi_\gamma v\|_2 \le C\beta^{-5/8}\|f\|_2.$$

Use of (3.2.15) has been made to obtain the second inequality.

Since by (3.2.5)

$$\|\tilde\chi_\gamma^2 v\|_1 \le C\beta^{-1/8}\|v\|_2 \le C\beta^{-5/8}\|f\|_2, \tag{3.2.26}$$

we may conclude that

$$\|v\|_1 \le C\beta^{-5/8}\|f\|_2, \tag{3.2.27}$$

which together with (3.2.25) completes the proof of (3.2.1a).

Step 3: Proof of (3.2.1b). To prove (3.2.1b) we set

$$f = (x - x_v)g,$$

where $g \in L^2(0,1)$.

Then, as in (3.2.10), we use (3.2.9) to obtain

$$\|(\chi_\gamma v)'\|_2^2 \le \|(x-x_v)\chi_\gamma v\|_2\|g\|_2 + C\beta^{1/2}\|v\|_2^2.$$

By (3.2.7) and (3.2.14) there exists $C > 0$ such that, for all $x \in [0,1]$,

$$0 \le (x-x_v)\chi_\gamma(x) \le C(v-U(x))^{1/2}\chi_\gamma(x). \tag{3.2.28}$$

Hence,

$$\|(\chi_\gamma v)'\|_2^2 \le C\big(\||U-v|^{1/2}\chi_\gamma v\|_2\|g\|_2 + \beta^{1/2}\|v\|_2^2\big).$$

Next, we use (3.2.11) to obtain, as in (3.2.12), with the aid of the above and (3.2.28)

$$\beta\||U-v|^{1/2}\chi_\gamma v\|_2^2 \le C\big(\gamma\beta^{1/2}\|v\|_2^2 + \|\chi_\gamma |U-v|^{1/2}v\|_2\|g\|_2\big),$$

from which we conclude that

$$\||U-v|^{1/2}\chi_\gamma v\|_2 \le C\beta^{-1/4}(\gamma^{1/2}\|v\|_2 + \beta^{-3/4}\|g\|_2). \tag{3.2.29}$$

Combining the above with (3.2.17) yields

$$\|\chi_\gamma v\|_2 \le C(\gamma^{1/2}\|v\|_2 + \beta^{-3/4}\|g\|_2). \tag{3.2.30}$$

Furthermore, with the aid of (3.2.15) we can conclude that

$$
\begin{aligned}
\|\chi_\gamma^2 v\|_1 &\leq \| \, |U - \nu|^{-1/2}\chi_\gamma\|_2 \, \||U - \nu|^{1/2}\chi_\gamma v\|_2 \\
&\leq C(\gamma^{1/2}\beta^{-1/8}\|v\|_2 + \beta^{-7/8}\|g\|_2).
\end{aligned}
\tag{3.2.31}
$$

We now use (3.2.19) to obtain, as in (3.2.20),

$$
\|\tilde{\chi}_\gamma v\|_2 \leq C(\beta^{-3/4}\|g\|_2 + \gamma\beta^{-1/4}\|v'\| + \gamma^2\|v\|_2).
\tag{3.2.32}
$$

Combining (3.2.32) with (3.2.30) yields for sufficiently small γ

$$
\|v\|_2 \leq C \, (\beta^{-3/4}\|g\|_2 + \gamma\beta^{-1/4}\|v'\|).
\tag{3.2.33}
$$

Then we write

$$
\|v'\|_2^2 = \mu\beta\|v\|_2^2 + \Re\langle v, f\rangle = \mu\beta\|v\|_2^2 + \Re\langle (x - x_\nu)v, g\rangle.
\tag{3.2.34}
$$

To estimate the second term on the right-hand side of (3.2.34) we first note that by (3.2.28), (3.2.3) and (3.2.8),

$$
\begin{aligned}
\|(x - x_\nu)v\|_2 &\leq \|(x - x_\nu)\chi_\gamma v\|_2 + \|(x - x_\nu)\tilde{\chi}_\gamma v\|_2 \\
&\leq C(\||U - \nu|^{1/2}\chi_\gamma v\|_2 + \beta^{-1/4}\|v\|_2).
\end{aligned}
$$

With the aid of (3.2.29) we then obtain

$$
\|(x - x_\nu)v\|_2 \leq C(\beta^{-1/4}\|v\|_2 + \beta^{-3/4}\|g\|_2).
\tag{3.2.35}
$$

Hence, by (3.2.34) and since $\mu < C\beta^{-1/2}$ we may conclude that

$$
\|v'\|_2 \leq C(\beta^{1/4}\|v\|_2 + \beta^{-1/2}\|g\|_2).
\tag{3.2.36}
$$

Substituting (3.2.36) into (3.2.33) yields for small enough γ,

$$
\|v\|_2 \leq C\beta^{-3/4}\|g\|_2.
\tag{3.2.37}
$$

By (3.2.37), the first inequality of (3.2.26), and (3.2.31) we obtain

$$
\|v\|_1 \leq C\beta^{-7/8}\|g\|_2.
\tag{3.2.38}
$$

Together with (3.2.37) and (3.2.36), (3.2.38) verifies (3.2.1b). ∎

3.3 L^1 estimates for $U(0) - \Im\lambda \gg \beta^{-1/2}$

It is not difficult to show that the resolvent of the operator $-d^2/dx^2 + ix$ is not bounded in $\mathscr{L}(L^\infty(\mathbb{R}), L^1(\mathbb{R}))$, a fact that can be easily established from the identity

$$
\left(-\frac{d^2}{dx^2} + ix\right)\frac{1}{\sqrt{x^2 + 1}} = \frac{2x^2 - 1}{[x^2 + 1]^{5/2}} + i\frac{x}{x^2 + 1}.
$$

For the resolvent of the operator $\mathcal{L}_\beta^{\mathfrak{N},\mathfrak{D}}$ on $(0, 1)$, this unboundedness manifests itself through a logarithmic dependence on β as we can clearly see in the following proposition.

Proposition 3.3.1. *Let $U \in C^2([0, 1])$ satisfy (2.1.3). There exist $\Upsilon > 0$, $a > 0$, $C > 0$, and $\beta_0 > 1$ such that, for $\beta \geq \beta_0$, $\Im\lambda < U(0) - a\beta^{-1/2}$, $\Re\lambda \leq \mathfrak{J}_\nu^{2/3}\Upsilon\beta^{-1/3}$, and $f \in L^\infty(0, 1)$ we have*

$$\|(\mathcal{L}_\beta^{\mathfrak{N},\mathfrak{D}} - \beta\lambda)^{-1}f\|_1 \leq C \min\left([\mathfrak{J}_\nu\beta]^{-5/6}\|f\|_2, [\mathfrak{J}_\nu\beta]^{-1}\log\beta\|f\|_\infty\right). \quad (3.3.1)$$

Proof. We assume that $\Upsilon > 0$ is sufficiently small so that Proposition 3.1.1 holds true. We begin by recalling that by (3.1.5), for any $N > 0$, there exists $a_0 > 0$ such that for all $a \geq a_0$, we have under the conditions of the proposition,

$$\beta x_\nu \geq \beta x_\nu^4 \geq N,$$

where x_ν is defined by (2.4.5). Let $(v, f) \in D(\mathcal{L}_\beta^{\mathfrak{N}}) \times L^\infty(0, 1)$ satisfy $(\mathcal{L}_\beta^{\mathfrak{N}} - \beta\lambda)v = f$. By (2.8.49) applied with $\mu = \beta^{-1/3}x_\nu^{2/3}$ and $q = 2$, it holds that

$$\|(U - v + i[\beta x_\nu^{-2}]^{-1/3})^{-1}\|_2^2 \leq C x_\nu^{-5/3}\beta^{1/3}.$$

We may then conclude that

$$\begin{aligned}
\|v\|_1 &\leq \|(U - v + i[\beta x_\nu^{-2}]^{-1/3})^{-1}\|_2\|(U - v + i[\beta x_\nu^{-2}]^{-1/3})v\|_2 \\
&\leq C\beta^{1/6}x_\nu^{-5/6}[\|(U - v)v\|_2 + [\beta x_\nu^{-2}]^{-1/3}\|v\|_2]. \quad (3.3.2)
\end{aligned}$$

By (3.1.75) and (3.1.3a)

$$\|v\|_1 \leq C[\beta x_\nu]^{-5/6}\|f\|_2. \quad (3.3.3)$$

By (2.8.62) we can conclude that

$$\|(U - v + i[\beta x_\nu^{-2}]^{-1/3})^{-1/2}\|_2^2 = \|(U - v + i[\beta x_\nu^{-2}]^{-1/3})^{-1}\|_1 \leq \frac{C}{x_\nu}\log(\beta x_\nu^4).$$

Hence, we can complete the proof of (3.3.1) by writing

$$\begin{aligned}
\|v\|_1 &\leq \|(U - v + i[\beta x_\nu^{-2}]^{-1/3})^{-1/2}\|_2\|(U - v + i[\beta x_\nu^{-2}]^{-1/3})^{1/2}v\|_2 \\
&\leq \frac{C}{x_\nu^{1/2}}[\log(\beta x_\nu^4)]^{1/2}[\||U - v|^{1/2}v\|_2 + [\beta x_\nu^{-2}]^{-1/6}\|v\|_2],
\end{aligned}$$

which implies

$$\|v\|_1 \leq \frac{C}{x_\nu^{1/2}}[\log(\beta)]^{1/2}[\||U - v|^{1/2}v\|_2 + [\beta x_\nu^{-2}]^{-1/6}\|v\|_2]. \quad (3.3.4)$$

We note that by (3.1.3a) (which holds for $a \geq a_0$ with a_0 large enough) and (3.1.5) we have

$$[\beta x_\nu^{-2}]^{-1/6} \|v\|_2 \leq C\beta^{-1} x_\nu^{-\frac{1}{2}} \|f\|_\infty.$$

Hence, we obtain from (3.3.4)

$$\|v\|_1 \leq \frac{C}{x_\nu^{1/2}} [\log(\beta)]^{1/2} \big[\||U-v|^{1/2}v\|_2 + \beta^{-1} x_\nu^{-1/2} \|f\|_\infty \big]. \qquad (3.3.5)$$

To complete the proof we need an estimate for $\||U-v|^{1/2}v\|_2$. In a similar manner to (3.1.8) we let

$$\chi_s^\pm(x) = \hat{\chi}(s(x-x_\nu)) \mathbf{1}_{\mathbb{R}_+}(\pm(x-x_\nu)) \text{ with } s = [\beta x_\nu]^{1/3},$$

where $\hat{\chi}$ is defined by (3.1.7). An integration by parts yields, as in (3.1.24),

$$\|(\chi_s^\pm v)'\|_2^2 - \|(\chi_s^\pm)'v\|_2^2 - \mu\beta\|\chi_s^\pm v\|_2^2 = \Re\langle \chi_s^\pm v, \chi_s^\pm f \rangle,$$

from which we conclude, given that $\mu\beta \leq Cs^2$

$$\|(\chi_s^\pm v)'\|_2^2 \leq \|\chi_s^\pm v\|_1 \|f\|_\infty + \hat{C}s^2 \|v\|_2^2.$$

Furthermore, (see (3.1.25)), we have that

$$\mp\beta\| |U-v|^{1/2}\chi_s^\pm v\|_2^2 + 2\Im\langle(\chi_s^\pm)'v, (\chi_s^\pm v)'\rangle = \Im\langle\chi_s^\pm v, \chi_s^\pm f\rangle, \qquad (3.3.6)$$

and hence, with the aid of above, we obtain that

$$\||U-v|^{1/2}\hat{\chi}_s v\|_2^2 \leq C\beta^{-1}([\beta x_\nu]^{2/3}\|v\|_2^2 + \|\chi_s^\pm v\|_1 \|f\|_\infty).$$

By (3.1.3a), we then have

$$\||U-v|^{1/2}\chi_s^\pm v\|_2 \leq C(x_\nu^{-1/2}\beta^{-1}\|f\|_\infty + \beta^{-1/2}\|v\|_1^{1/2}\|f\|_\infty^{1/2}). \qquad (3.3.7)$$

Let $\tilde{\chi}_s = \sqrt{1 - (\chi_s^+)^2 - (\chi_s^-)^2}$. Since $s = [\beta x_\nu]^{1/3}$ it holds that

$$\operatorname{supp}\tilde{\chi}_s \subset \left[x_\nu - \frac{1}{2}(\beta x_\nu)^{-1/3}, x_\nu + \frac{1}{2}(\beta x_\nu)^{-1/3} \right).$$

As

$$\||U-v|^{1/2}\tilde{\chi}_s v\|_2 \leq C [\beta x_\nu]^{-1/6} x_\nu^{1/2} \|\tilde{\chi}_s v\|_2 \leq C\beta^{-1/6} x_\nu^{1/3} \|v\|_2,$$

we may use (3.1.3a) once again to obtain

$$\||U-v|^{1/2}\tilde{\chi}_s v\|_2 \leq Cx_\nu^{-\frac{1}{2}}\beta^{-1}\|f\|_\infty.$$

Combining the above with (3.3.7) yields

$$\||U - \nu|^{1/2}\nu\|_2 \leq C\big(x_\nu^{-1/2}\beta^{-1}\|f\|_\infty + \beta^{-1/2}\|\nu\|_1^{1/2}\|f\|_\infty^{1/2}\big). \qquad (3.3.8)$$

Substituting (3.3.8) into (3.3.5) yields, for any $\eta > 0$,

$$\|\nu\|_1 \leq \frac{C}{x_\nu^{1/2}}[\log(\beta)]^{1/2}\big(\beta^{-1}x_\nu^{-1/2}\|f\|_\infty + \beta^{-1/2}\|\nu\|_1^{1/2}\|f\|_\infty^{1/2}\big)$$

$$\leq \frac{C}{x_\nu^{1/2}}[\log(\beta)]^{1/2}\Big(\beta^{-1}x_\nu^{-1/2}\|f\|_\infty + \beta^{-1/2}\Big(\frac{1}{\eta}\|f\|_\infty + \eta\|\nu\|_1\Big)\Big).$$

Choosing $\eta > 0$ such that

$$C\,\eta\,\beta^{-1/2}x_\nu^{-1/2}[\log(\beta)]^{1/2} = \frac{1}{2},$$

we obtain

$$\|\nu\|_1 \leq \widehat{C}\,\frac{1}{\beta x_\nu}\log(\beta)\|f\|_\infty. \qquad (3.3.9)$$

Combining the above with (3.3.3), completes the proof of the proposition. ∎

If $U - \nu \neq 0$ in $[0, 1]$ it can be easily verified (see (3.1.75)) that

$$\|(-d^2/dx^2 + i\beta[U - \nu])^{-1}\| \lesssim \beta^{-1}.$$

In contrast, when $U(x) = \nu$ for some $x \in (0, 1)$ the best estimate we can obtain (see (3.1.3)) is

$$\|(-d^2/dx^2 + i\beta[U - \nu])^{-1}\| \lesssim \beta^{-2/3}.$$

The zero of $U - \nu$ at $x = x_\nu$, thus, has a significant effect on the resolvent norm. Nevertheless, if $f(x_\nu) = 0$ and f is small in the neighbourhood of x_ν one may expect that $\|(-d^2/dx^2 + i\beta[U - \nu])^{-1}f\|_2$ would be smaller in that case.

This heuristic argument is manifested, more precisely, in the following proposition.

Proposition 3.3.2. *Let $U \in C^2([0, 1])$ satisfy (2.1.3). There exist $\Upsilon > 0$, $C > 0$, $a > 0$, and $\beta_0 > 0$ such that, for $\beta \geq \beta_0$, $\nu \leq U(0) - a\beta^{-1/2}$, $\mu < \Upsilon\mathfrak{J}_\nu^{2/3}\beta^{-1/3}$, and $f \in L^2(0, 1)$ such that $(x - x_\nu)^{-1}f \in L^2(0, 1)$, we have*

$$\|(\mathscr{L}_\beta^{\mathfrak{N}} - \beta\lambda)^{-1}f\|_2 \leq C(\mathfrak{J}_\nu\beta)^{-1}\left\|\frac{f}{x - x_\nu}\right\|_2 \qquad (3.3.10)$$

and

$$\left\|\frac{d}{dx}(\mathscr{L}_\beta^{\mathfrak{N}} - \beta\lambda)^{-1}f\right\|_2 \leq C\beta^{-1/2}\left\|\frac{f}{x - x_\nu}\right\|_2. \qquad (3.3.11)$$

Proof. Let $(v, f, \lambda) \in D(\mathscr{L}_\beta^\mathfrak{N}) \times L^2(0, 1) \times \mathbb{C}$ satisfy $(\mathscr{L}_\beta^\mathfrak{N} - \beta\lambda)v = f$.

Step 1. We prove (3.3.10).

Set

$$f = (U - v)g.$$

Given that $\mu < \Upsilon \Im_v{}^{2/3}\beta^{-1/3}$, it follows from (3.1.3) that there exists $u \in D(\mathscr{L}_\beta^\mathfrak{N})$ satisfying

$$(\mathscr{L}_\beta^\mathfrak{N} - \beta\lambda)u = g. \tag{3.3.12}$$

Let

$$w = (U - v)u.$$

Then, it holds that

$$(\mathscr{L}_\beta^\mathfrak{N} - \beta\lambda)w = f - 2U'u' - U''u.$$

Consequently,

$$v = w + (\mathscr{L}_\beta^\mathfrak{N} - \beta\lambda)^{-1}(2U'u' + U''u). \tag{3.3.13}$$

We now recall (3.1.69)

$$|U'| \le C|U - U(0)|^{1/2} \le C(|U - v|^{1/2} + x_v).$$

By the above and (3.1.3a) it holds that

$$\|(\mathscr{L}_\beta^\mathfrak{N} - \beta\lambda)^{-1}(2U'u' + U''u)\|_2$$
$$\le C\,[\Im_v\beta]^{-2/3}\big(x_v\|u'\|_2 + \||U - v|^{1/2}u'\|_2 + \|u\|_2\big). \tag{3.3.14}$$

Let $\chi_v^\pm$ be defined by (3.1.8). Clearly, in view of (3.1.9) and the fact that $U'(0) = 0$,

$$\||U - v|^{1/2}u'\|_2 \le \|\chi_v^+|U - v|^{1/2}u'\|_2 + \|\chi_v^-|U - v|^{1/2}u'\|_2 + Cx_v\|u'\|_2. \tag{3.3.15}$$

By (3.1.77), applied to the pair (u, g), (3.3.15), and (3.1.3b) we then have

$$x_v\|u'\|_2 + \||U - v|^{1/2}u'\|_2 \le C(\beta^{-1/2} + x_v^{2/3}\beta^{-1/3})\|g\|_2. \tag{3.3.16}$$

Using (3.1.3a) together with (3.3.14) and (3.3.12) then yields, as $x_v \ge \beta^{-1/4}$,

$$\|(\mathscr{L}_\beta^\mathfrak{N} - \beta\lambda)^{-1}(2U'u' + U''u)\|_2 \le C\beta^{-1}\|g\|_2. \tag{3.3.17}$$

By (3.1.75), applied to the pair (u, g), it holds that

$$\|w\|_2 \le C\beta^{-1}\|g\|_2. \tag{3.3.18}$$

Substituting the above together with (3.3.17) into (3.3.13) yields

$$\|v\|_2 \le C\beta^{-1}\|g\|_2.$$

Since

$$|U(x) - v| \geq \frac{1}{C} x_\nu |x - x_\nu|,$$

it holds that

$$\|g\|_2 \leq C x_\nu^{-1} \|(x - x_\nu)^{-1} f\|_2,$$

and hence, we can now conclude (3.3.10) from the above and (3.3.18).

Step 2. We prove (3.3.11).

Taking the real part of the scalar product with $\langle v, (\mathscr{L}_\beta^\mathfrak{N} - \beta\lambda)v\rangle$ we write

$$\|v'\|_2^2 = \mu\beta \|v\|_2^2 + \mathfrak{R}\langle (x - x_\nu)v, (x - x_\nu)^{-1} f\rangle.$$

From here we deduce that

$$\|v'\|_2^2 \leq \mu_+ \beta \|v\|_2^2 + |\langle (x - x_\nu)v, (x - x_\nu)^{-1} f\rangle|, \tag{3.3.19}$$

where

$$\mu_+ = \max(\mu, 0). \tag{3.3.20}$$

Since $|x - x_\nu| \leq C |U - v|^{1/2}$ we obtain by (3.3.10), as $x_\nu \geq \beta^{-1/4}$ and $\mu_+ \leq C \mathfrak{F}_\nu^{2/3} \beta^{-1/3}$,

$$\|v'\|_2^2 \leq C \big(\beta^{-1} \|(x - x_\nu)^{-1} f\|_2 + \||U - v|^{1/2} v\|_2 \|(x - x_\nu)^{-1} f\|_2 \big). \tag{3.3.21}$$

By (3.1.25) it holds that

$$\beta \||U - v|^{1/2} \chi_\nu^\pm v\|_2^2 \leq C x_\nu^{-1} \|v\|_2 \|v'\|_2 + \|(x - x_\nu)v\|_2 \|(x - x_\nu)^{-1} f\|_2.$$

Hence, in view of (3.1.9),

$$\beta \||U - v|^{1/2} v\|_2^2 \leq C(x_\nu^{-1} \|v\|_2 \|v'\|_2 + \beta x_\nu^2 \|v\|_2^2) + \|(x - x_\nu)v\|_2 \|(x - x_\nu)^{-1} f\|_2.$$

Since by (3.3.10) it holds that

$$\beta x_\nu^2 \|v\|_2^2 \leq C\beta^{-1} \|(x - x_\nu)^{-1} f\|_2^2,$$

we may obtain that

$$\begin{aligned} \beta \||U - v|^{1/2} v\|_2^2 \leq C \big(& x_\nu^{-2} \beta^{-1} \|(x - x_\nu)^{-1} f\|_2 \|v'\|_2 \\ & + \|(U - v)^{1/2} v\|_2 \|(x - x_\nu)^{-1} f\|_2 + \beta^{-1} \|(x - x_\nu)^{-1} f\|_2^2 \big). \end{aligned}$$

From the above we conclude, as $x_\nu \geq \beta^{-1/4}$,

$$\||U - v|^{1/2} v\|_2^2 \leq C(\beta^{-1} \|v'\|_2^2 + \beta^{-2} \|(x - x_\nu)^{-1} f\|_2^2). \tag{3.3.22}$$

Substituting the above into (3.3.21) yields

$$\|v'\|_2 \leq C\beta^{-1/2} \|(x - x_\nu)^{-1} f\|_2,$$

verifying (3.3.11). ∎

We now seek an estimate for $(\mathcal{L}_\beta - \beta\lambda)^{-1}$ in $\mathcal{L}(H^1, L^1)$. To this end we write $f = f - f(x_\nu) + f(x_\nu)$, and estimate first $(\mathcal{L}_\beta - \beta\lambda)^{-1}(f - f(x_\nu))$ using (3.3.10). Then, we estimate $(\mathcal{L}_\beta - \beta\lambda)^{-1} f(x_\nu)$ by observing first that the leading order term is $-if(x_\nu)[\beta(U + i\lambda)]^{-1}$ for $|\mu| > x_\nu^{2/3}\beta^{-1/3}$.

Proposition 3.3.3. *Let $U \in C^2([0,1])$ satisfy* (2.1.3). *Then there exist $\Upsilon > 0$, $C > 0$, $a > 0$, and $\beta_0 > 0$ such that, for $\beta \geq \beta_0$, $U(0) - \nu > a\beta^{-1/2}$, and $\mu \leq \Im_\nu^{2/3}\Upsilon\beta^{-1/3}$ and $f \in H^1(0,1)$ we have*

$$\left\| (\mathcal{L}_\beta^\mathfrak{N} - \beta\lambda)^{-1} f + i \frac{f(x_\nu)}{\beta[U - \nu - i \max(-\mu, x_\nu^{2/3}\beta^{-1/3})]} \right\|_1 \leq C[\Im_\nu\beta]^{-1} \|f\|_{1,2}.$$
(3.3.23)

Proof. Let $u = (\mathcal{L}_\beta^\mathfrak{N} - \beta\lambda)^{-1} f$.

Step 1. We prove (3.3.23) in the case $-\mu \leq x_\nu^{2/3}\beta^{-1/3}$.

We apply the decomposition

$$v = u + i \frac{f(x_\nu)}{\beta(U - \nu - ix_\nu^{2/3}\beta^{-1/3})}.$$
(3.3.24)

Then,

$$(\mathcal{L}_\beta^\mathfrak{N} - \beta\lambda)v = f + i\beta^{-1} f(x_\nu)((\mathcal{L}_\beta^\mathfrak{N} - \beta\lambda)(U - \nu - ix_\nu^{2/3}\beta^{-1/3})^{-1}).$$

We next observe that

$$(\mathcal{L}_\beta^\mathfrak{N} - \beta\lambda)(U - \nu - ix_\nu^{2/3}\beta^{-1/3})^{-1}$$
$$= \beta \frac{(-\lambda + iU)}{(U - \nu - ix_\nu^{2/3}\beta^{-1/3})} - \frac{2|U'|^2}{(U - \nu - ix_\nu^{2/3}\beta^{-1/3})^3} + \frac{U''}{(U - \nu - ix_\nu^{2/3}\beta^{-1/3})^2},$$

and that

$$-i(\lambda - iU)(U - \nu - ix_\nu^{2/3}\beta^{-1/3})^{-1} + 1$$
$$= (U - \nu - ix_\nu^{2/3}\beta^{-1/3})^{-1}(-i[(\mu + i\nu) - U] + U - \nu - ix_\nu^{2/3}\beta^{-1/3}).$$

Consequently, it holds that

$$(\mathcal{L}_\beta^\mathfrak{N} - \beta\lambda)v = f - f(x_\nu) + f(x_\nu)h,$$
(3.3.25)

where

$$h = i \frac{U''}{\beta(U - \nu - ix_\nu^{2/3}\beta^{-1/3})^2} - 2i \frac{|U'|^2}{\beta(U - \nu - ix_\nu^{2/3}\beta^{-1/3})^3}$$
$$- i \frac{\mu + x_\nu^{2/3}\beta^{-1/3}}{U - \nu - ix_\nu^{2/3}\beta^{-1/3}}.$$
(3.3.26)

Since for $U \in C^2([0, 1])$ satisfying (2.1.3), we have

$$|U(x) - v| \geq \frac{1}{C} x_v |x - x_v|, \tag{3.3.27}$$

we may conclude that for $k > 1$,

$$\int_0^1 \frac{dx}{|U - v - i x_v^{2/3} \beta^{-1/3}|^k} \leq C \int_0^1 \frac{dx}{x_v^k |x - x_v|^k + x_v^{2k/3} \beta^{-k/3}} \leq \hat{C} x_v^{-k} [\beta x_v]^{\frac{k-1}{3}}. \tag{3.3.28}$$

For later reference we mention that for $k = 1$

$$\int_0^1 \frac{dx}{|U - v + i x_v^{2/3} \beta^{-1/3}|}$$
$$\leq C \left[\frac{1}{x_v} \int_0^{2x_v} \frac{dx}{|x - x_v| + x_v^{-1/3} \beta^{-1/3}} + \int_{2x_v}^1 \frac{dx}{|x - x_v|^2} \right]$$
$$\leq \frac{\hat{C}}{x_v} \left[\log(x_v^{4/3} \beta^{1/3}) + 1 \right]. \tag{3.3.29}$$

Using (3.3.28) and the fact that $x_v \geq \frac{1}{C} \beta^{1/4}$, together with (3.1.69), it can be verified that there exist positive C and β_0 such that for $\beta \geq \beta_0$ and $|\mu| \leq x_v^{2/3} \beta^{-1/3}$,

$$\|h\|_2 \leq C [\beta x_v]^{-1/6}. \tag{3.3.30}$$

Consequently, by (3.3.1)

$$\|(\mathcal{L}_\beta^{\mathfrak{N}} - \beta\lambda)^{-1} h\|_1 \leq C(\beta x_v)^{-1}. \tag{3.3.31}$$

By (3.3.10) and Hardy's inequality (2.2.8) it holds that

$$\|(\mathcal{L}_\beta^{\mathfrak{N}} - \beta\lambda)^{-1}(f - f(x_v))\|_1 \leq \|(\mathcal{L}_\beta^{\mathfrak{D}} - \beta\lambda)^{-1}(f - f(x_v))\|_2$$
$$\leq C(\beta x_v)^{-1} \left\| \frac{f - f(x_v)}{x - x_v} \right\|_2 \leq \hat{C}(\beta x_v)^{-1} \|f'\|_2.$$

Substituting the above, together with (3.3.31) into (3.3.25) yields

$$\|v\|_1 \leq C(\beta x_v)^{-1}(\|f'\|_2 + |f(x_v)|) \leq \hat{C}(\beta x_v)^{-1} \|f\|_{1,2}.$$

Step 2. We prove (3.3.23) in the case $\mu \leq -x_v^{2/3} \beta^{-1/3}$.

In this case we consider instead the decomposition

$$v = u + i \frac{f(x_v)}{\beta(U + i\lambda)}.$$

Then we obtain

$$(\mathcal{L}_\beta^{\mathfrak{N}} - \beta\lambda)v = f + i\beta^{-1} f(x_v)((\mathcal{L}_\beta^{\mathfrak{N}} - \beta\lambda)(U + i\lambda)^{-1}).$$

As

$$(\mathcal{L}_\beta^\mathfrak{N} - \beta\lambda)(U + i\lambda)^{-1} = i\beta - \frac{2|U'|^2}{(U + i\lambda)^3} + \frac{U''}{(U + i\lambda)^2},$$

we obtain that

$$(\mathcal{L}_\beta^\mathfrak{N} - \beta\lambda)v = f - f(x_\nu) + f(x_\nu)\tilde{h}, \qquad (3.3.32)$$

where

$$\tilde{h} = i\frac{U''}{\beta(U + i\lambda)^2} - 2i\frac{|U'|^2}{(U + i\lambda)^3} \qquad (3.3.33)$$

and proceed in a similar manner using the lower bound for $|\mu|$.   ∎

An immediate consequence of Proposition 3.3.3 now follows by using (3.3.29).

Corollary 3.3.4. *Under the conditions of Proposition* 3.3.3, *it holds (with sufficiently large a) that*

$$\|(\mathcal{L}_\beta^\mathfrak{N} - \beta\lambda)^{-1}f\|_1 \le C\,[\Im_\nu\beta]^{-1}\big(\|f\|_{1,2} + |f(x_\nu)|\log(x_\nu^{4/3}\beta^{1/3})\big). \qquad (3.3.34)$$

We conclude this section by another auxiliary estimate which will be useful in Sections 5.10 and 5.11.

Proposition 3.3.5. *Let* $U \in C^3([0, 1])$ *satisfying* (2.1.3), $a > 0$ *and* $\Upsilon < \sqrt{-U''(0)}/2$. *Then there exist* $C > 0$, $\beta_0 > 0$ *such that, for all* $\beta \ge \beta_0$,

$$\sup_{\substack{\mu \le \Upsilon\beta^{-1/2} \\ \nu < U(0) + a\beta^{-1/2}}} \Big(\|(\mathcal{L}_\beta^{\mathfrak{N},\mathfrak{D}} - \beta\lambda)^{-1}(U - \nu)f\|_2$$

$$+ \beta^{-1/2}\|\frac{d}{dx}(\mathcal{L}_\beta^{\mathfrak{N},\mathfrak{D}} - \beta\lambda)^{-1}(U - \nu)f\|_2\Big)$$

$$\le C\beta^{-1}\|f\|_2. \qquad (3.3.35)$$

Note that if we apply (3.3.10) (for $\nu < U(0) - a_0\beta^{-1/2}$ with some sufficiently large a_0) or (3.2.1b) (in the case $|\nu - U(0)| \le a_0\beta^{-1/2}$) we obtain that

$$\|(\mathcal{L}_\beta^{\mathfrak{N},\mathfrak{D}} - \beta\lambda)^{-1}(U - \nu)f\|_2 \le C\beta^{-3/4}[1 + x_\nu\beta^{1/4}]^{-1}\|f\|_2,$$

which is weaker than (3.3.35).

Proof. Let $v \in D(\mathcal{L}_\beta^{\mathfrak{N},\mathfrak{D}})$ such that

$$(\mathcal{L}_\beta^{\mathfrak{N},\mathfrak{D}} - \beta\lambda)v = (U - \nu)f. \qquad (3.3.36)$$

Let $w \in D(\mathcal{L}_\beta^{\mathfrak{N},\mathfrak{D}})$ satisfy

$$(\mathcal{L}_\beta^{\mathfrak{N},\mathfrak{D}} - \beta\lambda)w = f.$$

It can be easily verified that

$$(\mathcal{L}_\beta^{\mathfrak{N},\mathfrak{D}} - \beta\lambda)([U - v]w) = (U - v)f - 2U'w' - U''w.$$

Hence,

$$v = (U - v)w + (\mathcal{L}_\beta^{\mathfrak{N},\mathfrak{D}} - \beta\lambda)^{-1}(2U'w' + U''w). \tag{3.3.37}$$

Let $a_0 > 0$. Consider first the case where $U(0) - a_0\beta^{-1/2} < v < U(0) + a_0\beta^{-1/2}$. By (3.1.71) (with v replaced by w) and since by (2.1.3) we have $|U'(x)| \le x$, it holds that

$$\beta\|(U - v)w\|_2^2 \le C(\beta^{-1}\|f\|_2^2 + [\|(x - x_v)w\|_2 + x_v\|w\|_2]\|w'\|_2).$$

We can now use (3.2.35), given that $x_v \le C\beta^{-1/4}$, to obtain

$$\beta\|(U - v)w\|_2^2 \le C(\beta^{-1}\|f\|_2^2 + [\beta^{-3/4}\|f\|_2 + \beta^{-1/4}\|w\|_2]\|w'\|_2).$$

We may now apply (3.2.1a) to the pair (w, f) to conclude that

$$\|(U - v)w\|_2 \le C\beta^{-1}\|f\|_2. \tag{3.3.38}$$

In (3.1.75) we have established that there exists $a > 0$ such that (3.3.38) holds also whenever $v < U(0) - a\beta^{-1/2}$ under the conditions of Proposition 3.1.1.

Next, we use once again either (3.2.1a) (in the case when $|v - U(0)| < a\beta^{-1/2}$) or (3.1.3a) (in the case when $v < U(0) - a\beta^{-1/2}$), with f replaced by $U''w$, to obtain that

$$\begin{aligned}\|(\mathcal{L}_\beta^{\mathfrak{N},\mathfrak{D}} - \beta\lambda)^{-1}(U''w)\|_2 &\le C\beta^{-1/2}[1 + x_v\beta^{1/4}]^{-2/3}\|w\|_2 \\ &\le \tilde{C}\beta^{-1}[1 + x_v\beta^{1/4}]^{-4/3}\|f\|_2 \\ &\le \hat{C}\beta^{-1}\|f\|_2. \end{aligned} \tag{3.3.39}$$

Finally, we write

$$\begin{aligned}\|(\mathcal{L}_\beta^{\mathfrak{N},\mathfrak{D}} - \beta\lambda)^{-1}(U'w')\|_2 &\le \|(\mathcal{L}_\beta^{\mathfrak{N},\mathfrak{D}} - \beta\lambda)^{-1}([U' - U'(x_v)]w')\|_2 \\ &\quad + \|(\mathcal{L}_\beta^{\mathfrak{N},\mathfrak{D}} - \beta\lambda)^{-1}(U'(x_v)w')\|_2.\end{aligned}$$

For the second term on the right-hand side we have by (3.2.1a) and (3.1.3)

$$\|(\mathcal{L}_\beta^{\mathfrak{N},\mathfrak{D}} - \beta\lambda)^{-1}(U'(x_v)w')\|_2 \le C\beta^{-1/2}x_v[1 + x_v\beta^{1/4}]^{-2/3}\|w'\|_2 \le \hat{C}\beta^{-1}\|f\|_2.$$

For the first term we use instead either (3.2.1b) or (3.3.10) with $f = [U' - U'(x_v)]w'$ and then (3.2.1a) and (3.1.3) for the second one to obtain

$$\|(\mathcal{L}_\beta^{\mathfrak{N},\mathfrak{D}} - \beta\lambda)^{-1}([U' - U'(x_v)]w')\|_2 \le C\beta^{-3/4}\left\|\frac{U' - U'(x_v)}{x - x_v}w'\right\|_2 \le \hat{C}\beta^{-1}\|f\|_2.$$

Hence,

$$\|(\mathcal{L}_\beta^{\mathfrak{N},\mathfrak{D}} - \beta\lambda)^{-1}(U'w')\|_2 \le C\beta^{-1}\|f\|_2. \qquad (3.3.40)$$

By (3.3.37), (3.3.38), (3.3.39), (3.3.39), and (3.3.40), we then conclude

$$\|v\|_2 \le C\beta^{-1}\|f\|_2. \qquad (3.3.41)$$

The estimate of v' in (3.3.35) follows immediately from the identity

$$\|v'\|_2^2 = \mu\beta\|v\|_2^2 + \Re\langle v, (U - v)f\rangle,$$

together with (3.3.41) and the fact that $\mu \le \Upsilon\beta^{-1/2}$. ∎