Chapter 4

No-slip Schrödinger operators

4.1 Preliminaries

Given the fact that $-\phi'' + \alpha^2\phi$ does not necessarily belong to $D(\mathcal{L}_\beta^{\mathfrak{N},\mathfrak{D}})$, we establish, in this chapter, resolvent estimates for the same differential operator but with one boundary condition replaced by an integral condition which will be satisfied by $-\phi'' + \alpha^2\phi$ (cf. see also [3, Section 6] and the discussion around equation (1.2.8) in the introduction.)

This chapter follows the same path as in [3, Section 6] but this time for symmetric flows in $(-1,+1)$ (so that $U'(0) = 0$), to which end we consider the interval $(0,1)$ and a Neumann condition at $x = 0$.

Let

$$\mathcal{L}_\beta^\zeta = -\frac{d^2}{dx^2} + i\beta U,$$

be defined on

$$D(\mathcal{L}_\beta^\zeta) = \big\{u \in H^2(0,1) \mid \langle \zeta, u\rangle = 0, u'(0) = 0\big\}, \tag{4.1.1}$$

where $\zeta \in H^2(0,1)$.

We will later (see (4.7.1)) confine the discussion to the case where $\zeta_\alpha(x) = \cosh(\alpha x)/\cosh\alpha$.

More precisely, we introduce

$$\mathfrak{U}_0 := \big\{\zeta \in H^2(0,1) \mid \zeta'(0) = 0, \zeta(1) = 1\big\} \tag{4.1.2}$$

and for $\beta \geq 0$, $\gamma > 0$, $\theta > 0$ and $\lambda \in \mathbb{C}$, the subset

$$\mathfrak{U}_1(\beta,\lambda,\gamma,\theta) = \big\{\zeta \in \mathfrak{U}_0, \|\zeta\|_\infty \leq \beta^\gamma, \|\zeta'\|_{L^2(1-\beta^{-\gamma},1)} \leq \theta\beta^{1/6}\lambda_\beta^{1/4}\big\}, \tag{4.1.3}$$

where

$$\lambda_\beta = 1 + |\lambda|\beta^{1/3}. \tag{4.1.4}$$

Let Ai denote Airy's function (See [1, Section 10.4]) and A_0 the generalized Airy function

$$\mathbb{C} \ni z \mapsto A_0(z) = e^{i\pi/6}\int_z^{+\infty} \mathrm{Ai}(e^{i\pi/6}t)\,dt. \tag{4.1.5}$$

(See [1, 10.4] or in [3, Appendix A.2]). We then set

$$\mathcal{S} = \big\{z \mid A_0(iz) = 0\big\}.$$

It is established in [3, Appendix A.2] that $\mathcal{S}$ (which is denoted there by $\mathcal{S}_\lambda$) is non-empty. In [22] (cf. also [3, Appendix A]) it is shown that

$$\vartheta_1^r := \inf_{z\in\mathcal{S}} \Re z > 0. \tag{4.1.6}$$

4.2 Resolvent estimates for $|U(0)-\nu| \gg \beta^{-1/2}$

We can now state the following proposition.

Proposition 4.2.1. *Let $U \in C^2([0,1])$ satisfy (2.1.3). Let further $\gamma < 1/4$. Then, there exist $\Upsilon > 0$, $\beta_0 > 0$, $a > 0$, and $\theta_0 > 0$ such that, for all $\beta \geq \beta_0$, $\theta \in (0,\theta_0]$, and $\lambda \in \mathbb{C}$ satisfying*

$$\max(|U(0)-\nu|^{-1/3},1)\beta^{1/3}\Re\lambda \leq \Upsilon \tag{4.2.1}$$

and

$$|U(0)-\nu| > a\beta^{-1/2}, \tag{4.2.2}$$

there exists a constant $C > 0$ such that, for any $f \in H^1(0,1)$, $\zeta \in \mathfrak{U}_1(\beta,\lambda,\gamma,\theta)$, and $v \in D(\mathcal{L}_\zeta^\beta)$ satisfying

$$(\mathcal{L}_\zeta^\beta - \beta\lambda)v = f, \tag{4.2.3}$$

it holds that

$$\begin{aligned}|v(1)| &\leq C\,\beta^{1/3}\,\lambda_\beta^{1/2}\,\|\zeta\|_\infty \\ &\quad\times \min\big([x_\nu\beta]^{-5/6}\|f\|_2, [|U(0)-\nu|^{1/2}\beta]^{-1}[\|f\|_{1,2} + |f(x_\nu)|\log(1+x_\nu\beta^{1/4})]\big),\end{aligned} \tag{4.2.4}$$

where x_ν is defined by (2.4.5), and that

$$|v(1)| \leq C\,\beta^{1/3}\,\lambda_\beta^{1/2}\,\|\zeta(\mathcal{L}_\beta^{\mathfrak{N}} - \beta\lambda)^{-1}f\|_1. \tag{4.2.5}$$

Furthermore, for any $p > 1$ and $\nu < U(0) - a\beta^{-1/2}$ it holds that

$$|v(1)| \leq C\,\lambda_\beta^{1/2}\beta^{1/3}[x_\nu\beta]^{-1}\big[\|\zeta\|_\infty\|f\|_{1,2} + \|\zeta\|_{1,p}(1+|\log[\nu+i\mathfrak{m}]|)\,|f(x_\nu)|\big], \tag{4.2.6a}$$

where

$$\mathfrak{m} := -\max(-\mu, x_\nu^{2/3}\beta^{-1/3}). \tag{4.2.6b}$$

Finally, if in addition $(x-x_\nu)^{-1}f \in L^2(0,1)$ and $\nu < U(0)$, then we have

$$|v(1)| \leq C\beta^{-2/3}\,\lambda_\beta^{1/2}\,\|\zeta\|_\infty\,|U(0)-\nu|^{-1/4}\Big\|\frac{f}{x-x_\nu}\Big\|_2. \tag{4.2.7}$$

Note that since $\mathfrak{J}_\nu = |U'(x_\nu)| \sim x_\nu \sim [U(0)-\nu]^{1/2}$ for $\nu < U(0)$, (4.2.1) is similar to the condition set on $\Re\lambda$ in (3.1.19). Here, we use a different notation since we need to address the case $\nu > U(0)$ as well.

Proof. As in [3, equation (6.20)] we begin by a decomposition of v into a boundary term associated with $x = 1$ and a solution of the same equation satisfying a Dirichlet condition at $x = 1$. We estimate the boundary term by using a linear approximation of U near $x = 1$ (recall that by (2.1.3) $U(1) = 0$ and $U'(1) = -1$). Let for $\beta > 0$ and $\lambda \in \mathbb{C}$,

$$\widetilde{\psi}_{\lambda,\beta}(x) = e^{-i\pi/6}\frac{\mathrm{Ai}\big(\beta^{1/3}e^{-i\pi/6}[(1-x)-i\lambda]\big)}{\overline{A_0(i\beta^{1/3}\bar{\lambda})}}, \tag{4.2.8}$$

which is (note that $\widetilde{\psi} = \psi_+$, with $J_+ = 1$, in [3, equation (6.8b)]) the decaying solution as $x \to -\infty$ of

$$\begin{cases} \Big(-\dfrac{d^2}{dx^2} + i\beta[(x-1)+i\lambda]\Big)\widetilde{\psi} = 0 & \text{in } (-\infty,1), \\ \displaystyle\int_{-\infty}^{1} \widetilde{\psi}(x)\,dx = \beta^{-1/3}. \end{cases} \tag{4.2.9}$$

Note further that by (4.1.6) $\widetilde{\psi}_{\lambda,\beta}$ is well defined whenever $\Re\lambda < \vartheta_1^r$. For later reference we recall from [3, equation (8.87)] that for any $\hat{\delta}_1$, there exists $C > 0$ and β_0 such that, for $\Re\lambda \le (\vartheta_1^r - \hat{\delta}_1)\beta^{-1/3}$ and $\beta \ge \beta_0$,

$$\frac{1}{C}\lambda_\beta^{1/2} \le |\widetilde{\psi}_{\lambda,\beta}(1)| \le C\,\lambda_\beta^{1/2}. \tag{4.2.10}$$

To guarantee that the Neumann condition at $x = 0$ is satisfied we further set

$$\psi_{\lambda,\beta}(x) = \widetilde{\psi}_{\lambda,\beta}(x)\chi(1-x), \tag{4.2.11}$$

where χ is given by (2.6.20), which we recall here for the convenience of the reader

$$\chi(t) = \begin{cases} 1 & t < 1/2, \\ 0 & t > 3/4. \end{cases}$$

Consequently, $\psi_{\lambda,\beta}$ is supported on $[1/4,+1]$ and $\psi_{\lambda,\beta} = \widetilde{\psi}_{\lambda,\beta}$ on $[1/2,1]$. We omit the subscript (λ,β) when no ambiguity is expected. Consider a pair $(\tilde{v},h) \in L^2(0,1) \times D(\mathcal{L}_\beta^{\mathfrak{N}})$ such that

$$h = \Big(-\frac{d^2}{dx^2} + i\beta(U+i\lambda)\Big)\psi \tag{4.2.12}$$

and

$$(\mathcal{L}_\beta^{\mathfrak{N}} - \beta\lambda)\,\tilde{v} = h. \tag{4.2.13}$$

We note that the assumptions of the proposition, and in particular (4.2.1) and (4.2.2) allow us to apply Propositions 3.1.1, 3.3.1, 3.3.2, and 3.3.3 throughout the proof.

Step 1. We prove that

$$\beta^{-1/6}\|\tilde{v}\|_2 + \|\tilde{v}\|_1 \le C \max(\beta^{-2/3}\lambda_\beta^{-3/4}, \beta^{-5/3}). \tag{4.2.14}$$

By (4.2.9) it holds that

$$h = i\beta\,[U - (1-x)]\,\psi + \chi''(1-x)\tilde{\psi} - 2\chi'(1-x)\tilde{\psi}' \quad \text{in } (0,1). \tag{4.2.15}$$

We note that by [3, equation (6.17)] there exists $\Upsilon > 0$ (in the statement of the proposition) such that whenever $\beta^{1/3}\Re\lambda \le \Upsilon$

$$\|(1-x)^k\,\psi_{\lambda,\beta}\|_2 \le \|(1-x)^k\,\tilde{\psi}_{\lambda,\beta}\|_2 \le C\,\lambda_\beta^{\frac{1-2k}{4}}\,\beta^{-(1+2k)/6} \quad \text{for } k \in [0,4]. \tag{4.2.16}$$

Furthermore, since by [3, Proposition A.1] (or more precisely by [3, equations (A.4), (A.6), (A.19), (A.20)] and the display below [3, equation (A.29)]) it holds that

$$\Psi(x,\lambda) := \frac{\mathrm{Ai}(e^{i\pi/6}[x + i\lambda])}{\mathrm{Ai}(e^{i2\pi/3}\lambda)} = \Big[1 + \frac{i}{4}((-\lambda)^{-1/2}x^2 - \lambda^{-1}x)\Big]e^{-(-\lambda)^{1/2}x} + w_1,$$

where

- for $\mu_0 > 0$
$$\lambda \in \mathcal{V}(\mu_0) := \{\Re\lambda \le \mu_0\} \cap \{|\lambda| > 3\mu_0\},$$
- the square root of $-\lambda$ is chosen such that
$$\Re(-\lambda)^{1/2} > 0,$$
- and the remainder $w_1 \in H^1(\mathbb{R}_+)$ satisfies
$$\|x^4 w_1\|_{L^2(\mathbb{R}_+)} + \|x^4 w_1'\|_{L^2(\mathbb{R}_+)} \le C\,|\lambda|^{-9/4}.$$

Consequently, for all $\lambda \in \mathcal{V}(\mu_0)$ it holds by [3, equation (A.20)] that

$$\|x^4\Psi\|_2 + |\lambda|^{-1/2}\|x^4\Psi'\|_2 \le C\,|\lambda|^{-9/4}.$$

Let $0 < \mu_0 < \hat{\kappa}_1$ (given by (3.1.2)). Then, there exists $C(\mu_0) > 0$ such that

$$\sup_{|\lambda| \le 3\mu_0} \|x^4\,\mathrm{Ai}(e^{i\pi/6}[x + i\lambda])\|_{1,2} \le C,$$

and, since all the zeroes of $\mathrm{Ai}(e^{i2\pi/3}\lambda)$ are located in the half-plane $\Re\lambda \ge \hat{\kappa}_1$, we have, since $\mu_0 < \hat{\kappa}_1$, that

$$\sup_{\substack{|\lambda| \le 3\mu_0 \\ \Re\lambda < \mu_0}} \left|\frac{1}{\mathrm{Ai}(e^{i2\pi/3}\lambda)}\right| \le C.$$

Consequently, by the above inequalities, relying on the sole condition that $\Re\lambda \leq \mu_0$, there exists $C>0$ such that

$$\|x^4\Psi\|_2 + [1+|\lambda|^2]^{-1/4}\|x^4\Psi'\|_2 \leq C\,[1+|\lambda|^2]^{-9/8}.$$

Note that

$$\frac{\widetilde{\psi}_{\lambda,\beta}(x)}{\widetilde{\psi}_{\lambda,\beta}(1)} = \Psi(\beta^{1/3}(1-x), \beta^{1/3}\lambda).$$

Using dilation, translation, and (4.2.10) the above yields

$$\|(1-x)^k\,\widetilde{\psi}'_{\lambda,\beta}\|_{L^2(-\infty,1)} \leq C\,\lambda_\beta^{\frac{3-2k}{4}}\,\beta^{(1-2k)/6}. \tag{4.2.17}$$

Hence, rewriting (4.2.15) in the form

$$\begin{aligned} h = i\beta[U-(1-x)]\psi &+ ((1-x)^{-2}\chi''(1-x))(1-x)^2\widetilde{\psi} \\ &- 2((1-x)^{-3}\chi'(1-x))(1-x)^3\widetilde{\psi}', \end{aligned}$$

we obtain

$$|h(x)| \leq C[\beta(1-x)^2|\psi(x)| + (1-x)^3|\widetilde{\psi}'(x)|], \tag{4.2.18}$$

and hence by (4.2.16) with $k=2$ and (4.2.17) with $k=3$,

$$\|h\|_2 \leq C\,\beta^{1/6}\lambda_\beta^{-3/4}. \tag{4.2.19}$$

Recall from (2.4.5) the definition of x_ν:

$$U(x_\nu) = \nu \text{ for } 0<\nu<U(0),\ x_\nu = 1 \text{ if } \nu \leq 0 \text{ and } x_\nu = 0 \text{ if } \nu > U(0).$$

We split the proof of (4.2.14) into two steps, depending on the value of x_ν.

Step 1.1. $x_\nu > 1/4$. In this case we have by (4.2.13), (3.1.3a), (3.1.75), and (3.3.1) (note that $\mathfrak{J}_\nu \geq \mathfrak{J}_{U^{-1}(1/4)} > 0$ in this case) that

$$\beta^{-1/3}\|\tilde{v}\|_2 + \|(U-\nu)\tilde{v}\|_2 + \beta^{-1/6}\|\tilde{v}\|_1 \leq C\,\beta^{-1}\,\|h\|_2.$$

By (4.2.19) we then obtain

$$\beta^{-1/3}\|\tilde{v}\|_2 + \|(U-\nu)\tilde{v}\|_2 + \beta^{-1/6}\|\tilde{v}\|_1 \leq C\beta^{-5/6}\lambda_\beta^{-3/4}, \tag{4.2.20}$$

readily yielding (4.2.14).

Step 1.2. $0 < x_\nu \leq 1/4$. We recall that $x_\nu \geq C a^{1/2}\beta^{-1/4}$. We write

$$(\mathcal{L}_\beta^{\mathfrak{N}} - \beta\lambda)(\chi\tilde{v}) = \chi h - 2\chi'\tilde{v}' - \tilde{\chi}''\tilde{v},$$

to obtain by (3.1.3a) and (3.3.1)

$$\|\chi\tilde{v}\|_2 + [\beta x_\nu]^{1/6}\|\chi\tilde{v}\|_1 \leq \frac{C}{[\beta x_\nu]^{2/3}}(\|\chi h\|_2 + \|\tilde{v}'\|_2 + \|\tilde{v}\|_2). \tag{4.2.21}$$

Let $\tilde{\chi} = \sqrt{1-\chi^2}$ where χ is chosen such that $\tilde{\chi} \in C^\infty(\mathbb{R})$. We note that the support of $\tilde{\chi}$ belongs to $[1/2, +\infty)$. Then, an integration by parts yields, as in (3.1.24)–(3.1.25)

$$\|(\tilde{\chi}\tilde{v})'\|_2^2 = \|\tilde{\chi}'\tilde{v}\|_2^2 + \mu\beta\|\tilde{\chi}\tilde{v}\|_2^2 + \Re\langle\tilde{\chi}\tilde{v}, \tilde{\chi}h\rangle, \tag{4.2.22}$$

and, observing that the sign of $(U-\nu)$ is constant on the support of $\tilde{\chi}$,

$$-\beta\||U-\nu|^{1/2}\tilde{\chi}\tilde{v}\|_2^2 + 2\Im\langle\tilde{\chi}'\tilde{v}, (\tilde{\chi}\tilde{v})'\rangle = \Im\langle\tilde{\chi}\tilde{v}, \tilde{\chi}h\rangle. \tag{4.2.23}$$

Combining (4.2.22), (4.2.23), given the support of $\tilde{\chi}$ and the fact that $x_\nu \leq 1/4$, yields

$$\begin{aligned}\beta\|\tilde{\chi}\tilde{v}\|_2^2 &\leq C(\|\tilde{v}\|_2\,\|(\tilde{\chi}\tilde{v})'\|_2 + \|\tilde{\chi}\tilde{v}\|_2\|h\|_2) \leq C(\|\tilde{v}\|_2^2 + \|(\tilde{\chi}\tilde{v})'\|_2^2 + \|\tilde{\chi}\tilde{v}\|_2\|h\|_2)\\ &\leq 2C(\|\tilde{v}\|_2^2 + \mu\beta\|\tilde{\chi}\tilde{v}\|_2^2 + \|\tilde{\chi}\tilde{v}\|_2\|h\|_2).\end{aligned}$$

Observing that $\mu \leq \Upsilon\mathfrak{J}_\nu\beta^{-\frac{1}{3}} \leq C\beta^{-\frac{1}{3}}$, we may conclude the existence of $\beta_0 > 0$ and $\widehat{C} > 0$ such that for all $\beta > \beta_0$

$$\|\tilde{\chi}\tilde{v}\|_2 \leq C\beta^{-1}\,(\|h\|_2 + \beta^{1/2}\|\tilde{v}\|_2). \tag{4.2.24}$$

Combining (4.2.24) with the control of $\|\chi\tilde{v}\|_2$ given in (4.2.21) yields for sufficiently large β_0

$$\|\tilde{v}\|_2 \leq C(\beta^{-1}\|h\|_2 + [\beta x_\nu]^{-2/3}(\|\chi h\|_2 + \|\tilde{v}'\|_2)). \tag{4.2.25}$$

By (3.2.34) (with $v = \tilde{v}$ and $f = h$) and the fact that $\mu \leq C\beta^{-1/3}x_\nu^{2/3}$ we can conclude that

$$\|\tilde{v}'\|_2 \leq C([\beta x_\nu]^{1/3}\|\tilde{v}\|_2 + \|\tilde{v}\|_2^{1/2}\|h\|_2^{1/2}). \tag{4.2.26}$$

Since by (4.2.2) βx_ν is large for sufficiently large β_0, substituting (4.2.26) into (4.2.25) yields

$$\|\tilde{v}\|_2 \leq C[(\beta^{-1} + [\beta x_\nu]^{-4/3})\|h\|_2 + [\beta x_\nu]^{-2/3}\|\chi h\|_2].$$

Hence, using the fact that $x_\nu \geq \beta^{-1/4}$,

$$\|\tilde{v}\|_2 \leq C(\beta^{-1}\|h\|_2 + \beta^{-1/2}\|\chi h\|_2). \tag{4.2.27}$$

By (4.2.15), (4.2.16), and (4.2.17), we obtain, since χ is supported on $[0, 3/4]$,

$$\|\chi h\|_2 \leq C\left[\beta\|(1-x)^4\psi\|_2 + \|(1-x)^4\psi'\|_2\right] \leq \widehat{C}\beta^{-1/2}\lambda_\beta^{-\frac{5}{4}}(\lambda_\beta^{-\frac{1}{2}} + \beta^{-2/3}). \tag{4.2.28}$$

Substituting (4.2.28) together with (4.2.19) into (4.2.27) gives

$$\|\tilde{v}\|_1 \leq \|\tilde{v}\|_2 \leq C\beta^{-5/6}\lambda_\beta^{-3/4}, \tag{4.2.29}$$

completing the proof of (4.2.14), for $0 < x_\nu \leq 1/4$ as well.

Step 1.3: $U(0)-\nu \le -a\beta^{-1/2}$. Since

$$\Im\langle \chi^2\tilde{v}, (\mathcal{L}_\beta^{\mathfrak{N}} - \beta\lambda)\tilde{v}\rangle = -\beta\|(\nu-U)^{1/2}\chi\tilde{v}\|_2^2 + 2\Im\langle \chi'\tilde{v}, (\chi\tilde{v})'\rangle, \tag{4.2.30}$$

we may conclude that

$$\beta\|(\nu-U)^{1/2}\chi\tilde{v}\|_2^2 \le \|\chi\tilde{v}\|_2\|\chi h\|_2 + C\|\tilde{v}\|_2\|(\chi\tilde{v})'\|_2. \tag{4.2.31}$$

As

$$\Re\langle \chi^2\tilde{v}, (\mathcal{L}_\beta^{\mathfrak{N}} - \beta\lambda)\tilde{v}\rangle = \|(\chi\tilde{v})'\|_2^2 - \mu\beta\|\chi\tilde{v}\|_2^2 - \|\chi'\tilde{v}\|_2^2,$$

we can conclude that

$$\|(\chi\tilde{v})'\|_2 \le C\big(\mu_{\beta,+}^{1/2}\|\chi\tilde{v}\|_2 + \|\tilde{v}\|_2 + \|\chi\tilde{v}\|_2^{1/2}\|\chi h\|_2^{1/2}\big),$$

where $\mu_{\beta,+}$ is defined in (3.1.26). Substituting the above into (4.2.31) yields

$$\beta\|(\nu-U)^{1/2}\chi\tilde{v}\|_2^2 \le \|\chi\tilde{v}\|_2\|\chi h\|_2 + C\big(\mu_{\beta,+}^{1/2}\|\chi\tilde{v}\|_2\|\tilde{v}\|_2 + \|\tilde{v}\|_2^2 + \|\chi\tilde{v}\|_2\|\chi h\|_2\big).$$

Since $\nu - U \ge \nu - U(0)$ in $[0,1]$ we may now conclude that

$$\|\chi\tilde{v}\|_2 \le \frac{C}{\beta(\nu-U(0))}(\|\chi h\|_2 + (\mu_{\beta,+}^{1/2} + [\beta(\nu-U(0))]^{1/2})\|v\|_2). \tag{4.2.32}$$

Given that (4.2.24) remains valid for $\nu > U(0)$ it holds that

$$\|\tilde{v}\|_2 \le \|\chi\tilde{v}\|_2 + \|\tilde{\chi}\tilde{v}\|_2 \le \|\chi\tilde{v}\|_2 + C\beta^{-1}(\|h\|_2 + \beta^{1/2}\|\tilde{v}\|_2).$$

Consequently, there exists β_0 such that for $\beta \ge \beta_0$ we may write

$$\|\tilde{v}\|_2 \le \|\chi\tilde{v}\|_2 + C\beta^{-1}\|h\|_2. \tag{4.2.33}$$

Substituting the above into (4.2.32) yields, for some sufficiently large β_0 and $\beta \ge \beta_0$

$$\|\chi\tilde{v}\|_2 \le C(\beta^{-1}(\nu-U(0))^{-1}(\|\chi h\|_2 + [\nu-U(0)]^{1/2})\beta^{-1/2}\|h\|_2). \tag{4.2.34}$$

By (4.2.28) and (4.2.19) we then obtain (note that $|\lambda| > U(0)$ as $\nu - U(0) \ge a\beta^{-1/2}$)

$$\|\chi\tilde{v}\|_2 \le C\,\beta^{-4/3}. \tag{4.2.35}$$

Substituting the above into (4.2.33) yields

$$\|\tilde{v}\|_1 \le \|\tilde{v}\|_2 \le C\beta^{-5/6}\lambda_\beta^{-3/4}.$$

Step 2. We prove (4.2.4) and (4.2.6).

Step 2.1. We prove (4.2.4) in the case $\nu < U(0)$.

Consider the pairs $(v, f) \in D(\mathcal{L}_\beta^\zeta) \times L^2(0,1)$ satisfying (4.2.3) and $(\psi, \tilde{v})$ satisfying (4.2.8)–(4.2.13). As in [3, equation (6.20)] there exists $(A, u) \in \mathbb{C} \times D(\mathcal{L}_\beta^{\mathfrak{N}})$ such that

$$v = A(\psi - \tilde{v}) + u, \tag{4.2.36}$$

where

$$u = (\mathcal{L}_\beta^{\mathfrak{N}} - \beta\lambda)^{-1} f. \tag{4.2.37}$$

Taking the inner product with ζ yields in view of (4.1.1)

$$A\langle \zeta, (\psi - \tilde{v})\rangle = -\langle \zeta, u\rangle. \tag{4.2.38}$$

By (4.1.3) and (4.2.20) it holds that for any $0 < \gamma < 1/4$ there exist positive C and β_0 such that

$$|\langle \zeta, \tilde{v}\rangle| \leq C\beta^\gamma \, \|\tilde{v}\|_1 \leq \hat{C}\beta^{-(2/3-\gamma)} \tag{4.2.39}$$

for all $\beta > \beta_0$.

Then, we write

$$\langle \zeta, \psi\rangle = \langle 1, \psi\rangle + \langle \zeta - 1, \psi\rangle. \tag{4.2.40}$$

For the first term on the right-hand side of (4.2.40), we may rely on [3, equation (6.28)] to obtain

$$\langle 1, \tilde{\psi}\rangle = -\beta^{-1/3} + \mathcal{O}(\beta^{-4/3}). \tag{4.2.41}$$

Observing that

$$|\langle 1, \psi - \tilde{\psi}\rangle| \leq C\|(1-x)^4\tilde{\psi}\|_2 \leq C\lambda_\beta^{-7/4}\beta^{-3/2},$$

we obtain

$$\langle 1, \psi\rangle = -\beta^{-1/3} + \mathcal{O}(\beta^{-4/3}). \tag{4.2.42}$$

For the second term on the right-hand side of (4.2.40) we have

$$|\langle \zeta - 1, \psi\rangle| \leq \|\zeta'\|_{L^2(1-\beta^{-\gamma},1)} \| \|(1-x)^{1/2}\psi\|_1 + (1 + \|\zeta\|_\infty)\|\psi\|_{L^1(0,1-\beta^{-\gamma})}. \tag{4.2.43}$$

By [3, equation (6.27)] it holds that

$$\|\psi\|_{L^1(0,1-\beta^{-\gamma})} \leq \beta^{3\gamma}\|(1-x)^3\psi\|_1 \leq C\beta^{3\gamma-4/3},$$

and that

$$\|(1-x)^{1/2}\psi\|_1 \leq C\beta^{-1/2}\lambda_\beta^{-1/4}. \tag{4.2.44}$$

Consequently, we obtain from (4.2.43) that (recall that $\zeta \in \mathfrak{U}_1(\beta, \lambda, \gamma, \theta)$)

$$|\langle \zeta - 1, \psi\rangle| \leq C_0(\theta\beta^{-1/3} + \beta^{4\gamma-4/3}). \tag{4.2.45}$$

As

$$\langle \zeta, (\psi - \tilde{v}) \rangle = \langle 1, \psi \rangle + \langle (\zeta - 1), \psi \rangle - \langle \zeta, \tilde{v} \rangle,$$

we can conclude from (4.2.45), (4.2.42), and (4.2.39) that

$$|\langle \zeta, (\psi - \tilde{v}) \rangle| \geq \beta^{-1/3}(1 - C_0\theta - \widehat{C}\beta^{4\gamma-1} - \widehat{C}\beta^{-1} - \widehat{C}\beta^{\gamma-1/3}). \tag{4.2.46}$$

We choose $\theta_0 = 1/2C_0$, and since $\gamma < 1/4$, there exists β_0 such that under the assumptions of the proposition we can conclude from (4.2.38) and (4.2.46) that

$$|A| \leq C\beta^{1/3}|\langle \zeta, u \rangle| \leq C\beta^{1/3}\|\zeta\|_\infty \|u\|_1. \tag{4.2.47}$$

For $\nu < U(0)$ we may use (3.3.34), the fact that $x_\nu \geq C\beta^{-1/4}$, and the $\mathcal{L}(L^1, L^2)$ estimate in (3.3.1) to obtain that

$$|A| \leq C\|\zeta\|_\infty \min\big(x_\nu^{-5/6}\beta^{-1/2}\|f\|_2, x_\nu^{-1}\beta^{-2/3}[\|f\|_{1,2} + |f(x_\nu)|\log(x_\nu\beta^{1/4})]\big). \tag{4.2.48}$$

We can now conclude (4.2.4) from (4.2.48), (4.2.10), (4.2.11), and the fact (see (4.2.36)) that $v(1) = A\psi(1)$.

Step 2.2. We prove (4.2.6)

To prove (4.2.6) we now write

$$\Big\langle \zeta, \frac{1}{U - \nu + i\mathfrak{m}} \Big\rangle = \zeta(x_\nu) \int_0^1 \frac{dx}{U - \nu + i\mathfrak{m}} + \int_0^1 \frac{[\zeta - \zeta(x_\nu)]\,dx}{U - \nu + i\mathfrak{m}},$$

where $\mathfrak{m}$ is defined in (4.2.6b). For the coefficient of $\zeta(x_\nu)$ in the first term on the right-hand side we write

$$\int_0^1 \frac{dx}{U - \nu + i\mathfrak{m}} = \frac{1}{2\tilde{x}_\nu} \int_0^1 \Big[\frac{1}{[U(0) - U]^{1/2} - \tilde{x}_\nu} - \frac{1}{[U(0) - U]^{1/2} + \tilde{x}_\nu}\Big]\,dx, \tag{4.2.49}$$

where

$$\tilde{x}_\nu = [U(0) - \nu + i\mathfrak{m}]^{1/2}.$$

An integration by parts yields

$$\begin{aligned} &\int_0^1 \frac{dx}{[U(0) - U]^{1/2} \pm \tilde{x}_\nu} \\ &\quad = \frac{2[U(0) - U]^{1/2}}{U'} \log\big([U(0) - U]^{1/2} \pm \tilde{x}_\nu\big)\Big|_0^1 \\ &\qquad - \int_0^1 \Big(\frac{2[U(0) - U]^{1/2}}{U'}\Big)' \log\big([U(0) - U]^{1/2} \pm \tilde{x}_\nu\big)\,dx. \end{aligned} \tag{4.2.50}$$

Given that

$$\Big\|\Big(\frac{2[U(0) - U]^{1/2}}{U'}\Big)'\Big\|_\infty \leq C,$$

it holds, for $\mu \geq -1$ that

$$\Big|\int_0^1 \Big(\frac{2[U(0)-U]^{1/2}}{U'}\Big)' \log\big([U(0)-U]^{1/2} \pm \tilde{x}_\nu\big)\,dx\Big| \leq \hat{C}. \tag{4.2.51}$$

We now observe that

$$\begin{aligned}\frac{2[U(0)-U]^{1/2}}{U'} & \log\big([U(0)-U]^{1/2} \pm \tilde{x}_\nu\big)\Big|_0^1 \\ &= -\Big[\frac{2}{U''(0)}\Big]\log\big(\pm\tilde{x}_\nu\big) + 2\sqrt{U(0)}\log\big([U(0)]^{1/2} \pm \tilde{x}_\nu\big).\end{aligned} \tag{4.2.52}$$

Given that

$$\log\big(-\tilde{x}_\nu\big) - \log\big(+\tilde{x}_\nu\big) = i\pi$$

and

$$\big|[U(0)]^{1/2} - \tilde{x}_\nu\big| \geq \frac{1}{C}\,|\nu + i\mathfrak{m}|,$$

we obtain, by substituting (4.2.52) together with (4.2.51) into (4.2.50), given that $\frac{1}{C}x_\nu \leq \tilde{x}_\nu$,

$$\Big|\int_0^1 \frac{\zeta(x_\nu)dx}{U-\nu+i\mathfrak{m}}\Big| \leq C\,x_\nu^{-1}[1 + |\log|\nu + i\mathfrak{m}|^{-1}|]\|\zeta\|_\infty. \tag{4.2.53}$$

For $\mu \leq -1$ it holds that

$$\Big|\int_0^1 \frac{\zeta(x_\nu)dx}{U-\nu+i\mathfrak{m}}\Big| \leq \frac{C}{|\mu|}\,\|\zeta\|_\infty$$

in accordance with (4.2.53).

For the second term we have by (3.3.27), for any $p > 1$

$$\Big|\int_0^1 \frac{\zeta-\zeta(x_\nu)dx}{U-\nu+i\mathfrak{m}}\Big| \leq C\|\zeta'\|_p \int_0^1 \frac{|x-x_\nu|^{\frac{p-1}{p}}\,dx}{x_\nu|x-x_\nu|} \leq \frac{\hat{C}}{x_\nu}\|\zeta'\|_p.$$

Combining the above with (4.2.53) yields

$$\Big|\Big\langle\zeta, \frac{1}{U-\nu+i\mathfrak{m}}\Big\rangle\Big| \leq \frac{C}{x_\nu}[1 + |\log|\nu + i\mathfrak{m}|^{-1}|]\|\zeta\|_{1,p}. \tag{4.2.54}$$

By the first inequality of (4.2.47) it holds that

$$|A| \leq C\beta^{1/3}\Big\|\Big[\zeta\Big[u + i\beta^{-1}\frac{f(x_\nu)}{U-\nu+i\mathfrak{m}}\Big]\Big\|_1 + \beta^{-1}|f(x_\nu)|\Big|\Big\langle\zeta, \frac{1}{U-\nu+i\mathfrak{m}}\Big\rangle\Big|\Big]. \tag{4.2.55}$$

Substituting (4.2.54) into (4.2.55) yields with the aid of (3.3.23)

$$|A| \leq C\beta^{-2/3}x_\nu^{-1}\big[\|\zeta\|_\infty\|f\|_{1,2} + (1 + |\log|\nu + i\mathfrak{m}|^{-1}|)\|\zeta\|_{1,p}|f(x_\nu)|\big]. \tag{4.2.56}$$

We can now conclude (4.2.6) from (4.2.56), (4.2.10), (4.2.11), and the fact that $v(1) = A\psi(1)$.

Step 2.3. We prove (4.2.4) in the case $\nu > U(0)$.

In this case we write

$$\|u\|_1 \leq \|(\nu - U)^{-1/2}\|_2 \, \|(\nu - U)^{1/2} u\|_2,$$

and as

$$\|(\nu - U)^{-1/2}\|_2^2 \leq C \int_0^1 \frac{dx}{x^2 + \nu - U(0)} \leq \hat{C} \, |\nu - U(0)|^{-1/2},$$

we may conclude that

$$\|u\|_1 \leq \check{C} \, [\nu - U(0)]^{-1/4} \, \|(\nu - U)^{1/2} u\|_2. \tag{4.2.57}$$

We now use (4.2.30) with $\chi \equiv 1$, $\tilde{v} = u$ and $h = f$ to obtain that

$$\beta \|(\nu - U)^{1/2} u\|_2^2 = -\Im \langle u, f \rangle. \tag{4.2.58}$$

Consequently,

$$\|(\nu - U)^{1/2} u\|_2 \leq \frac{1}{\beta^{1/2}} \|u\|_1^{1/2} \|f\|_\infty^{1/2},$$

and hence, by (4.2.57) we obtain that

$$\|u\|_1 \leq \frac{C}{\beta[\nu - U(0)]^{1/2}} \|f\|_\infty \leq \frac{\hat{C}}{\beta[\nu - U(0)]^{1/2}} \|f\|_{1,2}. \tag{4.2.59}$$

Next, we use the fact that

$$\nu - U(0) \leq \nu - U(x) \quad \text{for } x \in [0, 1], \tag{4.2.60}$$

to obtain from (4.2.58) that

$$\|u\|_2 \leq \frac{1}{\beta[\nu - U(0)]} \|f\|_2. \tag{4.2.61}$$

As

$$\|(\nu - U)^{-1}\|_2^2 \leq C \int_0^1 \frac{dx}{[x^2 + \nu - U(0)]^2} \leq \frac{\hat{C}}{|\nu - U(0)|^{3/2}},$$

we may write

$$\|u\|_1 \leq \|(U - \nu)^{-1}\|_2 \|(U - \nu) u\|_2 \leq C [\nu - U(0)]^{-3/4} \|(U - \nu) u\|_2. \tag{4.2.62}$$

By (3.1.71) (applied with $v = u$) and (3.1.69) combined with (4.2.60), it holds that

$$\begin{aligned} \beta \|(U - \nu) u\|_2^2 &\leq \|(U - \nu) u\|_2 \|f\|_2 + C \|(U - \nu)^{1/2} u\|_2 \|u'\|_2 \\ &\leq \|(U - \nu) u\|_2 \|f\|_2 + C \|(U - \nu) u\|_2^{1/2} \|u\|_2^{1/2} \|u'\|_2. \end{aligned} \tag{4.2.63}$$

Furthermore, by (3.2.34) and (4.2.1)

$$\|u'\|_2 \leq \mu_{\beta,+}^{1/2}\|u\|_2 + \|u\|_2^{1/2}\|f\|_2^{1/2} \leq C(\nu - U(0))^{1/6}\beta^{1/3}\|u\|_2 + \|u\|_2^{1/2}\|f\|_2^{1/2}.$$

We now use (4.2.61) to deduce from above

$$\|u'\|_2 \leq C\left[(\nu - U(0))^{-5/6}\beta^{-2/3} + (\nu - U(0))^{-1/2}\beta^{-1/2}\right]\|f\|_2,$$

which implies using (4.2.2)

$$\|u'\|_2 \leq \widehat{C}(\nu - U(0))^{-1/2}\beta^{-1/2}\|f\|_2. \tag{4.2.64}$$

Substituting (4.2.61) and (4.2.64) into (4.2.63) yields

$$\beta\|(U-\nu)u\|_2^2 \leq \|(U-\nu)u\|_2\|f\|_2 + \frac{C}{\beta(\nu - U(0))}\|(U-\nu)u\|_2^{1/2}\|f\|_2^{3/2},$$

from which we conclude using (4.2.2) that

$$\|(U-\nu)u\|_2 \leq C\beta^{-1}[1 + \beta^{-1/3}[\nu - U(0)]^{2/3}]\|f\|_2 \leq \frac{\widehat{C}}{\beta}\|f\|_2.$$

Substituting the above into (4.2.62) yields together with (4.2.2)

$$\|u\|_1 \leq \frac{C}{\beta[\nu - U(0)]^{3/4}}\|f\|_2 \leq \frac{\widehat{C}}{(\beta[\nu - U(0)]^{1/2})^{5/6}}\|f\|_2,$$

which, together with (4.2.59) proves that

$$\|u\|_1 \leq C\min\big((\beta[\nu - U(0)]^{1/2})^{-5/6}\|f\|_2, (\beta[\nu - U(0)]^{1/2})^{-1}\|f\|_{1,2}\big).$$

As $\psi_{\lambda,\beta}(1) = \widetilde{\psi}_{\lambda,\beta}(1)$ we may infer from (4.2.10)

$$\frac{1}{C}\lambda_\beta^{1/2} \leq |\psi_{\lambda,\beta}(1)| \leq C\,\lambda_\beta^{1/2}, \tag{4.2.65}$$

and hence we can conclude by (4.2.36) and (4.2.47) (which remains valid for $\nu > U(0)$) that

$$|v(1)| = |A\psi(1)| \leq C\,\lambda_\beta^{1/2}\min((\beta[\nu - U(0)]^{1/2})^{-5/6}\|f\|_2, x_\nu^{-1}\beta^{-2/3}\|f\|_{1,2})$$

which verifies (4.2.4) for $\nu > U(0)$.

Step 3. We prove (4.2.5).

The proof of (4.2.5) which reads

$$|v(1)| \leq C\,\lambda_\beta^{1/2}\,\beta^{1/3}\,\|\zeta u\|_1,$$

follows immediately from the first inequality in (4.2.47), from (4.2.10), and again from the fact that $v(1) = A\psi(1)$.

Step 4. We prove (4.2.7). To prove it for $x_\nu < 1/4$ we set

$$f = (x - x_\nu)g,$$

and assume that $g \in L^2(0,1)$. Recall the definition of $\chi_\nu^\pm$ and $\tilde{\chi}_\nu$ from (3.1.8). Since by (3.1.78)–(3.1.79)

$$\||U - \nu|^{-1/2}\chi_\nu^\pm\|_2^2 \le C \int_{\frac{5x_\nu}{4}}^1 \frac{dx}{x^2 - x_\nu^2} + C \int_0^{\frac{3x_\nu}{4}} \frac{dx}{x_\nu^2 - x^2} \le \frac{\hat{C}}{x_\nu},$$

we can conclude that

$$\|(\chi_\nu^\pm)^2 u\|_1 \le \||U - \nu|^{-1/2}\chi_\nu^\pm\|_2 \, \||U - \nu|^{1/2}\chi_\nu^\pm u\|_2 \le C x_\nu^{-1/2} \||U - \nu|^{1/2}\chi_\nu^\pm u\|_2.$$

By (3.3.22) and (3.3.11) we then obtain

$$\|(\chi_\nu^\pm)^2 u\|_1 \le C([\beta x_\nu]^{-1/6}\|u\|_2 + \beta^{-1} x_\nu^{-1/2}\|g\|_2).$$

Hence, by (3.3.10) (which reads $\|u\|_2 \le C\,[\Im_\nu \beta]^{-1}\,\|g\|_2$), we can conclude that

$$\|(\chi_\nu^\pm)^2 u\|_1 \le C([\beta x_\nu]^{-7/6} + [\beta^{-1} x_\nu^{-1/2}])\|g\|_2.$$

Given that $x_\nu \ge \frac{1}{C}\beta^{-1/4}$ we obtain that

$$\|(\chi_\nu^\pm)^2 u\|_1 \le C\beta^{-1} x_\nu^{-1/2}\|g\|_2. \tag{4.2.66}$$

Employing again (3.3.10) we write

$$\|\tilde{\chi}_\nu u\|_2 \le \|u\|_2 \le C[\beta x_\nu]^{-1}\|g\|_2.$$

Consequently, since $\tilde{\chi}_\nu$ is supported on $[x_\nu/2, 3x_\nu/2]$

$$\|\tilde{\chi}_\nu^2 u\|_1 \le x_\nu^{1/2}\|\tilde{\chi}_\nu u\|_2 \le C\beta^{-1} x_\nu^{-1/2}\|g\|_2. \tag{4.2.67}$$

Combining (4.2.67) with (4.2.66), (4.2.5), and (4.2.47) yields (4.2.7) for the case $x_\nu < 1/4$.

In the case $x_\nu \ge 1/4$, (4.2.7) immediately follows from (3.3.10) and the fact that $\|u\|_1 \le \|u\|_2$. ∎

4.3 Resolvent estimates for $|U(0) - \Im\lambda| = \mathcal{O}(\beta^{-1/2})$

Here, we introduce, for $\beta > 0$, $\lambda \in \mathbb{C}$, and $\theta > 0$,

$$\mathfrak{U}_2(\beta, \theta, \lambda) = \{\zeta \in \mathfrak{U}_0 \mid \|\zeta'\|_2 \le \theta\beta^{1/6}\lambda_\beta^{1/4}\}, \tag{4.3.1}$$

where $\mathfrak{U}_0$ is introduced in (4.1.2). In the present context λ lies in a bounded set and hence

$$\|\zeta'\|_2 \le C\theta\beta^{1/4}.$$

Proposition 4.3.1. *Let* $U \in C^3([0,1])$ *satisfy* (2.1.3), $\Upsilon < \sqrt{-U''(0)}/2$, $\mu_1 > 0$, *and* $a > 0$. *Then, there exist* $\beta_0 > 0$ *and* $\theta_0 > 0$ *such that for all* $\beta \geq \beta_0$, $\theta \in (0, \theta_0]$, $\lambda \in \mathbb{C}$ *satisfying*

$$U(0) - a\beta^{-1/2} < \nu < U(0) + a\beta^{-1/2} \tag{4.3.2a}$$

and

$$-\mu_1 \leq \mu < \Upsilon\beta^{-1/2} \tag{4.3.2b}$$

for any ζ *in* $\mathfrak{U}_2(\beta,\theta,\lambda)$, *and* $(f,v) \in H^1(0,1) \times D(\mathcal{L}_\zeta^\beta)$ *satisfying* (4.2.3), *it holds that*

$$|v(1)| \leq C\|\zeta\|_\infty \min(\beta^{-1/8}\|f\|_2, \beta^{-1/4}\|f\|_\infty). \tag{4.3.3}$$

Furthermore, for f *satisfying* $(x-x_\nu)^{-1}f \in L^2(0,1)$ *we have*

$$|v(1)| \leq C\beta^{-3/8}\,\|\zeta\|_\infty \Big\|\frac{f}{x-x_\nu}\Big\|_2. \tag{4.3.4}$$

Proof. By (4.2.12), (4.2.13), (4.2.19), and (3.2.1a) in Proposition 3.2.1 it holds that

$$\|\tilde{v}\|_2 + \beta^{1/8}\|\tilde{v}\|_1 \leq C\beta^{-1/2}\|h\|_2 \leq \widehat{C}\,\beta^{-1/3}\,\lambda_\beta^{-3/4}. \tag{4.3.5}$$

Since $|\lambda| > U(0)/2$ we obtain for $\beta \geq \beta_0$ with β_0 large enough

$$\|\tilde{v}\|_2 + \beta^{1/8}\|\tilde{v}\|_1 \leq C\beta^{-1/3}[\beta^{1/3}]^{-3/4} = C\beta^{-7/12}. \tag{4.3.6}$$

Given that for $\zeta \in \mathfrak{U}_2(\beta,\theta,\lambda)$ it holds that

$$\|\zeta\|_\infty \leq (1 + C\|\zeta'\|_2) \leq C(1 + \theta\beta^{1/6}\lambda_\beta^{1/4}), \tag{4.3.7}$$

and hence we can conclude, from (4.3.3) and (4.3.4), that

$$\|\zeta\|_\infty \leq C\beta^{1/4}. \tag{4.3.8}$$

We then obtain, using (4.3.6),

$$|\langle\zeta,\tilde{v}\rangle| \leq \|\zeta\|_\infty\|\tilde{v}\|_1 \leq \widehat{C}\,\beta^{-11/24}. \tag{4.3.9}$$

Furthermore, we have, using (4.2.44) and the fact that $\|\zeta'\|_2 \leq \theta\beta^{1/4}$ that

$$|\langle\zeta - 1,\psi\rangle| \leq \|\zeta'\|_{L^2(0,1)}\|\|(1-x)^{1/2}\psi\|_1 \leq C\theta\beta^{-1/3}. \tag{4.3.10}$$

Since v is still expressible by (4.2.36), we can now conclude, as in (4.2.47), with (4.2.39) and (4.2.45) respectively replaced by (4.3.9) and (4.3.10) that there exist θ_0 and β_0 such that, for $\theta \leq \theta_0$ and $\beta \geq \beta_0$, it holds that

$$|A| \leq C\beta^{1/3}\|\zeta\|_\infty\|u\|_1, \tag{4.3.11}$$

where u is given by (4.2.37).

We now use (3.2.1a) in Proposition 3.2.1 to obtain that

$$|A| \leq C\|\zeta\|_\infty \min(\beta^{-7/24}\|f\|_2, \beta^{-5/12}\|f\|_\infty, \beta^{-13/24}\|(x-x_\nu)^{-1}f\|_2).$$

Consequently, by (4.2.36) and (4.2.10) we obtain that

$$|v(1)| \leq C\,\lambda_\beta^{1/2}\|\zeta\|_\infty \min(\beta^{-7/24}\|f\|_2, \beta^{-5/12}\|f\|_\infty, \beta^{-13/24}\|(x-x_\nu)^{-1}f\|_2).$$

Given that (4.3.2) provides a uniform bound on $|\lambda|$, we have

$$\lambda_\beta^{1/2} \leq C\beta^{1/6},$$

hence we can conclude that

$$|v(1)| \leq C\|\zeta\|_\infty \min(\beta^{-1/8}\|f\|_2, \beta^{-1/4}\|f\|_\infty, \beta^{-3/8}\|(x-x_\nu)^{-1}f\|_2). \qquad \blacksquare$$

4.4 Resolvent estimates for negative $\Re\lambda$

Although Propositions 4.2.1 and 4.3.1 provide estimates when the spectral parameter λ belongs to domains in $\mathbb{C}$ that include $\Re\lambda \leq 0$, one can obtain a better estimate if we assume $\Re\lambda \leq -\mu_0$ for some fixed $\mu_0 > 0$, or at least $\Re\lambda \leq -C\,[U(0)-\nu]$ for $\nu < U(0)$.

Proposition 4.4.1. *Let $U \in C^2([0,1])$ satisfy (2.1.3). Let further a, μ_0, and ν_0 denote positive constants. Then, there exist $C > 0$, $\beta_0 > 0$, and $\theta_0 > 0$ such that for all $\beta \geq \beta_0$, $\theta \in (0, \theta_0]$ and $\lambda = \mu + i\nu \in \mathbb{C}$ satisfying*

$$-\nu_0 < \nu < U(0) + a\beta^{-1/2} \tag{4.4.1}$$

and

$$\mu \leq -\mu_0 \tag{4.4.2}$$

for any $\zeta \in \mathfrak{U}_2(\beta, \theta, \lambda)$, given by (4.3.1), and any pair $(f, v) \in L^2(0,1) \times D(\mathcal{L}_\zeta^\beta)$ satisfying (4.2.3), it holds that

$$|v(1)| \leq C\,\beta^{-1/2}\,\|\zeta\|_\infty\|f\|_2. \tag{4.4.3}$$

Proof. As in (4.2.36) we write

$$v = A(\psi - \tilde{v}) + u,$$

where $A \in \mathbb{C}$, $\psi = \psi_{\lambda,\beta}$ is given by (4.2.11), $\tilde{v}$ by (4.2.13), and u by (4.2.37).

As $\zeta \in \mathfrak{U}_2(\beta, \theta, \lambda)$, we obtain, given that $-\nu_0 < \nu < U(0) - a\beta^{-1/2}$, and in view of (4.4.2), (4.3.7), and (4.2.14)

$$|\langle \zeta, \tilde{v}\rangle| \leq \|\zeta\|_\infty\|\tilde{v}\|_1 \leq C\beta^{-1/2}\lambda_\beta^{-1/2} \leq \hat{C}\,\beta^{-2/3}. \tag{4.4.4}$$

Note that while both Propositions 4.2.1 and 4.3.1 assume $\mu \geq -\mu_0$, both (4.2.14) and (4.3.7) are valid for $\mu < -\mu_0$ as well.

In the case

$$U(0) - a\beta^{-1/2} < \nu < U(0) + a\beta^{-1/2},$$

we proceed as in the proof of (4.4.4) but use (4.3.5) instead of (4.2.14) and (4.2.19) which continues to hold in this case. Hence, we obtain the weaker estimate

$$|\langle \zeta, \tilde{v} \rangle| \leq \hat{C}\, \beta^{-11/24}.$$

Combining the above with (4.4.4) yields the existence of $C > 0$ such that for any λ satisfying (4.4.1) and (4.4.2) it holds that

$$|\langle \zeta, \tilde{v} \rangle| \leq C\beta^{-11/24}. \tag{4.4.5}$$

Furthermore, as in (4.3.10) we write

$$|\langle \zeta - 1, \psi \rangle| \leq \|\zeta'\|_{L^2(0,1)} \|\|(1-x)^{1/2}\psi\|_1,$$

from which we conclude, using the fact that $\zeta \in \mathfrak{U}_2(\beta, \theta, \lambda)$ and (4.2.44)

$$|\langle \zeta - 1, \psi \rangle| \leq C\theta\beta^{1/6}\lambda_\beta^{1/4} \times \beta^{-1/2}\lambda_\beta^{-1/4}.$$

Consequently, it holds that

$$|\langle \zeta - 1, \psi \rangle| \leq C\theta\beta^{-1/3}. \tag{4.4.6}$$

Hence, as in (4.3.11) we obtain that, choosing θ_0 small enough

$$|A| \leq C\beta^{1/3}\|\zeta\|_\infty \|u\|_1. \tag{4.4.7}$$

To estimate $\|u\|_1$ we observe that

$$\Re\langle u, (\mathcal{L}_\beta^{\mathfrak{N}} - \beta\lambda)u \rangle = \|u'\|_2^2 - \mu\beta\|u\|_2^2, \tag{4.4.8}$$

where $\mathcal{L}_\beta^{\mathfrak{N}}$ is defined in (3.0.1), from which we conclude that

$$\|u\|_1 \leq \|u\|_2 \leq \frac{1}{|\mu|\beta}\|f\|_2. \tag{4.4.9}$$

The proof of the proposition can now be completed by using (4.2.10) and the fact that $v(1) = A\psi(1)$. Thus, by (4.2.10) we obtain from (4.4.7) that

$$|v(1)| \leq C\beta^{1/3}\lambda_\beta^{1/2}\|\zeta\|_\infty \|u\|_1. \tag{4.4.10}$$

Since for $\mu \leq -\mu_0$ it holds by (4.4.1) that

$$|\lambda| \leq |\mu| + |\nu| \leq C|\mu|,$$

we may conclude that

$$|v(1)| \leq C\beta^{1/2}|\mu|^{1/2}\|\zeta\|_\infty \|u\|_1 .$$

Hence, by (4.4.9) we can conclude that

$$|v(1)| \leq C\beta^{-1/2}|\mu|^{-1/2}\|\zeta\|_\infty\|f\|_2 \leq \hat{C}\beta^{-1/2}|\mu_0|^{-1/2} \|\zeta\|_\infty \|f\|_2 .$$

The proposition is proved. ■

We next consider the case $-\mu_0 < \mu < -\frac{|U(0)-\nu|}{\kappa_1}$ for some $\kappa_1 > 0$. While (4.2.5) and (4.3.3) hold true under this assumption, it is necessary, in the next section, to obtain better estimates since Proposition 2.8.1 is inapplicable in this case.

Proposition 4.4.2. *Let $U \in C^2([0,1])$ satisfy (2.1.3). Let further a, κ_1, ν_0 and μ_0 denote positive constants. Then, there exist $C > 0$, $\beta_0 > 0$, and $\theta_0 > 0$ such that, for all $\beta \geq \beta_0$, $\theta \in (0, \theta_0]$ and $\lambda = \mu + i\nu \in \mathbb{C}$ satisfying (see (4.2.2) and (4.4.1))*

$$-\nu_0 < \nu < U(0) + a\beta^{-1/2} \tag{4.4.11}$$

and

$$-\mu_0 < \mu < -\frac{|U(0)-\nu|}{\kappa_1} \tag{4.4.12}$$

for any $\zeta \in \mathfrak{U}_2(\beta,\theta,\lambda)$, and any pair $(v,f) \in D(\mathcal{L}_\zeta^\beta) \times L^2(0,1)$ satisfying (4.2.3), it holds that

$$|v(1)| \leq C\,\|\zeta\|_\infty \min(|\mu|^{-1/2}\beta^{-1/2}\|f\|_\infty, |\mu|^{-3/4}\beta^{-1/2}\|f\|_2). \tag{4.4.13}$$

Proof. Step 1. We prove that

$$|v(1)| \leq C\,|\mu|^{-3/4}\beta^{-1/2}\|\zeta\|_\infty\|f\|_2 . \tag{4.4.14}$$

As in the proof of Proposition 4.4.1 and since for sufficiently small θ_0, (4.4.10) still holds under the assumptions of this proposition we obtain that

$$|v(1)| \leq C\beta^{1/3}\lambda_\beta^{1/2}\|\zeta\|_\infty \|u\|_1 \leq \hat{C}\beta^{1/2}\|\zeta\|_\infty\|u\|_1 , \tag{4.4.15}$$

where u is given by (4.2.37). Note that under (4.4.11) and (4.4.12), $|\lambda|$ is bounded. To obtain an estimate for $\|u\|_1$ we now write

$$\|u\|_1 \leq \|(U+i\lambda)^{-1}\|_2\,\|(U+i\lambda)u\|_2 . \tag{4.4.16}$$

By (3.1.75) and (4.4.8) we have that

$$\|(U+i\lambda)u\|_2 \leq \|(U-\nu)u\|_2 + |\mu|\,\|u\|_2 \leq \frac{\hat{C}}{\beta}\|f\|_2 . \tag{4.4.17}$$

By (2.9.24) (with $q = 2$) it holds that

$$\|(U + i\lambda)^{-1}\|_2^2 \leq C\,|\mu|^{-3/2}.$$

Consequently, we may conclude from (4.4.16) and (4.4.17) that

$$\|u\|_1 \leq C\,|\mu|^{-3/4}\beta^{-1}\,\|f\|_2.$$

Substituting into (4.4.15) then yields (4.4.14).

Step 2. We prove that

$$|v(1)| \leq C\,|\mu|^{-1/2}\beta^{-1/2}\|\zeta\|_\infty\,\|f\|_\infty. \tag{4.4.18}$$

Suppose that $f \in L^\infty(0,1)$. Then by (4.4.8) it holds for negative values of μ that

$$\|u'\|_2^2 + |\mu|\beta\|u\|_2^2 \leq \|u\|_1\,\|f\|_\infty. \tag{4.4.19}$$

Set

$$\chi_\beta^\pm(x) = \hat{\chi}([|\mu|\beta]^{1/2}(x - x_\nu))\mathbf{1}_{\mathbb{R}_+}(\pm(x - x_\nu)),$$

where $\hat{\chi}$ be given by (2.4.16).

Note that χ_β^+ is supported in $(x_\nu + [|\mu|\beta]^{-1/2}/4, +\infty)$ whereas χ_β^- is supported in $(-\infty, x_\nu - [|\mu|\beta]^{-1/2}/4)$. Let further $\tilde{\chi}_\beta = \sqrt{1 - (\chi_\beta^+)^2 - (\chi_\beta^-)^2}$. Note that by (4.4.12) it follows that we can choose β_0 such that for all $\beta > \beta_0$

$$x_\nu - [|\mu|\beta]^{-1/2}/2 > 0. \tag{4.4.20}$$

Hence, the support of $\tilde{\chi}_\beta$ is contained in

$$\Big(x_\nu - \frac{1}{2}[|\mu|\beta]^{-1/2},\, x_\nu + \frac{1}{2}[|\mu|\beta]^{-1/2}\Big) \subset [0,1].$$

As in (3.1.25) (with $\chi_\nu^\pm$ replaced by $\chi_\beta^\pm$), we now obtain

$$\beta\||U - \nu|^{1/2}\chi_\beta^\pm u\|_2^2 \leq 2\,\|(\chi_\beta^\pm)'u\|_2\|\chi_\beta^\pm u'\|_2 + \|u\|_1\|f\|_\infty. \tag{4.4.21}$$

Consequently, by (4.4.21) and the definition of $\chi_\beta^\pm$ we conclude that

$$\beta\||U - \nu|^{1/2}\chi_\beta^\pm u\|_2^2 \leq C|\mu\beta|^{-1/2}\|u\|_2\|u'\|_2 + \|u\|_1\|f\|_\infty.$$

By (4.4.19) we then have

$$\||U - \nu|^{1/2}\chi_\beta^\pm u\|_2^2 \leq C\,\beta^{-1}\,\|u\|_1\|f\|_\infty. \tag{4.4.22}$$

By (4.4.11) and (4.4.12) we have that

$$\frac{1}{C}\,\beta^{-1/2} \leq x_\nu^2 \leq C|\mu|. \tag{4.4.23}$$

Given the definition of $\tilde{\chi}_\beta$ we obtain that

$$|U(x)-\nu| \leq C\Big(\sup_{x\in \operatorname{supp}\tilde{\chi}_\beta} |U'(x)|\Big)\,|\mu\beta|^{-1/2}.$$

Hence, by (4.4.23) and (4.4.20) we can conclude that

$$\||U-\nu|^{1/2}\tilde{\chi}_\beta u\|_2^2 \leq C x_\nu [|\mu|\beta]^{-1/2}\|u\|_2^2.$$

Combining the above with (4.4.22) now yields

$$\||U-\nu|^{1/2}u\|_2 \leq C\,\beta^{-1/2}\,\|u\|_1^{1/2}\|f\|_\infty^{1/2}.$$

By (2.9.24) (with $q=1$) it holds that

$$\|(U+i\lambda)^{-1/2}\|_2^2 = \|(U+i\lambda)^{-1}\|_1 \leq C\,|\mu|^{-1/2}.$$

Consequently,

$$\begin{aligned}\|u\|_1 &\leq \|(U+i\lambda)^{-1/2}\|_2\,\|(U+i\lambda)^{1/2}u\|_2\\ &\leq C\,|\mu|^{-1/4}\big[\||U-\nu|^{1/2}u\|_2 + |\mu|^{1/2}\|u\|_2\big]\\ &\leq \hat{C}\,|\mu|^{-1/4}\,\beta^{-1/2}\,\|u\|_1^{1/2}\|f\|_\infty^{1/2}.\end{aligned}$$

Hence,

$$\|u\|_1 \leq C\,|\mu|^{-1/2}\beta^{-1}\,\|f\|_\infty,$$

which, when substituted into (4.4.15), establishes (4.4.18).

Together with (4.4.14) the above inequality completes the proof of the proposition. ∎

4.5 Rapidly decaying functions

When considering large values of α in the next section, it is useful to consider, as in [3], the operator $\mathcal{L}_\zeta^\beta$ where ζ decays rapidly away from the boundary at $x=1$. Set then for $\lambda\in\mathbb{C}$ and positive β,θ,α

$$\mathfrak{U}_3(\beta,\theta,\alpha,\lambda) = \big\{\zeta\in\mathfrak{U}_2(\beta,\theta,\lambda)\ |\ |\zeta(x)| \leq e^{-\alpha(1-x)}\|\zeta\|_\infty,\ \forall x\in[0,1]\big\}, \quad (4.5.1)$$

where $\mathfrak{U}_2(\beta,\theta,\lambda)$ is introduced in (4.3.1).

Proposition 4.5.1. *Let $U\in C^3([0,1])$ satisfy (2.1.3). Let further $a>0$, $\mu_0>0$, and $\Upsilon<\sqrt{-U''(0)}/2$. Then, there exist $C>0$, $\beta_0>0$, and $\theta_0>0$ such that for all $\beta\geq\beta_0$, $\theta\in(0,\theta_0]$, all $\alpha\geq 1$ and $\lambda\in\mathbb{C}$ satisfying*

$$-\mu_0 \leq \Re\lambda \leq \Upsilon\beta^{-1/2} \quad (4.5.2)$$

and

$$\frac{1}{2}U(0) < \nu < U(0) + a\beta^{-1/2}, \tag{4.5.3}$$

for all $\zeta \in \mathfrak{U}_3(\beta,\theta,\alpha,\lambda)$*, and for all pair* $(f,v) \in L^2(0,1) \times D(\mathcal{L}_\zeta^\beta)$ *satisfying* (4.2.3)*, it holds that*

$$|v(1)| \leq C\alpha^{-1/2}\,(\beta^{-1/2} + e^{-\alpha/C})\|\zeta\|_\infty\,\|f\|_2. \tag{4.5.4}$$

Proof. Step 1. Control of $v(1)$. Since (4.3.9) and (4.3.10) remain valid under our assumptions (4.5.2) and (4.5.3), we can follow the same procedure as in Proposition 4.3.1 to obtain

$$|v(1)| \leq C\,\beta^{1/3}\lambda_\beta^{1/2}|\langle\zeta,u\rangle| \leq C\beta^{1/2}|\langle\zeta,u\rangle|, \tag{4.5.5}$$

where u is given by (4.2.37).

Step 2. We prove under (4.5.2) and (4.5.3) that

$$\|u\|_2 \leq C\,\beta^{-1/2}\,\|f\|_2. \tag{4.5.6}$$

We first consider the case where

$$U(0)/2 < \nu < U(0) - a_1\beta^{-1/2}, \tag{4.5.7}$$

where $a_1 \geq a$ will be determined in the sequel. In this case, we can use (3.1.3) which reads (for $\mathfrak{J}_\nu = |U'(x_\nu)|$)

$$\|u\|_2 \leq C(\mathfrak{J}_\nu\beta)^{-2/3}\|f\|_2$$

and holds under the condition $\mu \leq \Upsilon_0\,{\mathfrak{J}_\nu}^{2/3}\,\beta^{-1/3}$ for some sufficiently small $\Upsilon_0 > 0$. Note that for $\nu - U(0) > a_1\beta^{-1/2}$ there exists $\widehat{C} > 0$ such that

$$\mathfrak{J}_\nu \geq \frac{1}{\widehat{C}}\,a_1^{1/2}\,\beta^{-1/4}.$$

Consequently, there exists $C > 0$ such that (3.1.3) is applicable for all

$$\mu \leq Ca_1^{1/3}\beta^{-1/2}.$$

For sufficiently large $a_1 \geq a$ the above set of μ values contains (4.5.2) and hence we can conclude (4.5.6) when (4.5.7) holds true.

We now look at the case

$$U(0) - a_1\beta^{-1/2} < \nu < U(0) + a\beta^{-1/2},$$

Here, we can apply (3.2.1a) (with a replaced by a_1) to obtain (4.5.6) which, combined with (4.5.5), leads to

$$|v(1)| \leq C\,\|\zeta\|_\infty\|f\|_2.$$

Note that at this stage it is sufficient to assume that $\zeta \in \mathfrak{U}_2(\beta,\theta,\alpha,\lambda)$.

Step 3. With $\hat{x}_\nu \in (0,1)$ satisfying

$$U(\hat{x}_\nu) = \frac{\nu}{2}, \tag{4.5.8}$$

we prove that

$$|\langle \zeta, u\rangle| \leq C\,\alpha^{-1/2}\,\beta^{-1/2}(\beta^{-1/2} + e^{-\alpha(1-\hat{x}_\nu)})\,\|\zeta\|_\infty\,\|f\|_2. \tag{4.5.9}$$

Consider the decomposition

$$\langle \zeta, u\rangle = \langle \mathbf{1}_{L^2(0,\hat{x}_\nu)}\zeta, u\rangle + \langle \mathbf{1}_{(\hat{x}_\nu,1)}\zeta, u\rangle.$$

We first obtain using (4.5.6) that

$$|\langle \mathbf{1}_{L^2(0,\hat{x}_\nu)}\zeta, u\rangle| \leq \|\mathbf{1}_{L^2(0,\hat{x}_\nu)}\zeta\|_2\,\|u\|_2 \leq C\,\alpha^{-1/2}\,\beta^{-1/2}\,e^{-\alpha(1-\hat{x}_\nu)}\,\|\zeta\|_\infty\|f\|_2. \tag{4.5.10}$$

Moreover, it holds that

$$|\langle \mathbf{1}_{(\hat{x}_\nu,1)}\zeta, u\rangle| \leq C\,\|\zeta\|_2\,\|\mathbf{1}_{(\hat{x}_\nu,1)}u\|_2 \leq \frac{C}{\alpha^{1/2}}\|\zeta\|_\infty\,\|\mathbf{1}_{(\hat{x}_\nu,1)}u\|_2. \tag{4.5.11}$$

Let

$$\check{\chi}_\nu(x) = \chi\Big(\frac{x-x_\nu}{\hat{x}_\nu - x_\nu}\Big)\mathbf{1}_{\mathbb{R}_+}(x-x_\nu), \tag{4.5.12}$$

where χ is given by (2.4.16). Note that $\check{\chi}_\nu$ is supported in the interval $(x_\nu + (\hat{x}_\nu - x_\nu)/4, +\infty)$ and equals 1 on $[(x_\nu + \hat{x}_\nu)/2, 1]$. Integration by part yields

$$\|(\check{\chi}_\nu u)'\|_2^2 = \|\check{\chi}_\nu' u\|_2^2 + \mu\beta\|\check{\chi}_\nu u\|_2^2 + \Re\langle \check{\chi}_\nu u, \check{\chi}_\nu f\rangle, \tag{4.5.13a}$$

and, given that $(U-\nu)$ has constant sign on the support of $\check{\chi}_\nu$,

$$-\beta\||U-\nu|^{1/2}\check{\chi}_\nu u\|_2^2 + 2\Im\langle \check{\chi}_\nu' u, (\check{\chi}_\nu u)'\rangle = \Im\langle \check{\chi}_\nu u, \check{\chi}_\nu f\rangle. \tag{4.5.13b}$$

Combining the above we obtain, given the support of $\check{\chi}_\nu$

$$\begin{aligned}\|\check{\chi}_\nu u\|_2^2 &\leq C\||U-\nu|^{1/2}\check{\chi}_\nu u\|_2^2 \leq C\beta^{-1}\big[\|u\|_2\|(\check{\chi}_\nu u)'\|_2 + \|\check{\chi}_\nu u\|_2\|\check{\chi}_\nu f\|_2\big]\\ &\leq C\beta^{-1}\big[\|u\|_2(\|u\|_2 + \mu_{\beta,+}^{1/2}\|\check{\chi}_\nu u\|_2 + \|\check{\chi}_\nu u\|_2^{1/2}\|\check{\chi}_\nu f\|_2^{1/2})\\ &\qquad + \|\check{\chi}_\nu u\|_2\|\check{\chi}_\nu f\|_2\big].\end{aligned} \tag{4.5.14}$$

From here we deduce that

$$\begin{aligned}&\|\check{\chi}_\nu u\|_2^2\\ &\leq C\beta^{-1}\big[\|u\|_2(\|u\|_2 + \beta^{1/4}\|\check{\chi}_\nu u\|_2 + \|\check{\chi}_\nu u\|_2^{1/2}\|\check{\chi}_\nu f\|_2^{1/2}) + \|\check{\chi}_\nu u\|_2\|\check{\chi}_\nu f\|_2\big],\end{aligned}$$

which implies

$$\|\check{\chi}_\nu u\|_2^2 \leq C\Big(\frac{1}{\beta^2}\|\check{\chi}_\nu f\|_2^2 + \frac{1}{\beta}\|u\|_2^2\Big). \tag{4.5.15}$$

Combining (4.5.15) and (4.5.6) leads to

$$\|\mathbf{1}_{(\hat{x}_\nu,1)}u\|_2 \leq \|\check{\chi}_\nu u\|_2 \leq C\,\beta^{-1}\,\|f\|_2. \tag{4.5.16}$$

For later reference we note that by (4.5.13a) and (4.5.16) it holds that

$$\|\mathbf{1}_{(\hat{x}_\nu,1)}u'\|_2 \leq \|(\check{\chi}_\nu u)'\|_2 \leq C\,\beta^{-1/2}\,\|f\|_2. \tag{4.5.17}$$

Combining (4.5.16) with (4.5.10) and (4.5.11) yields (4.5.9), which, together with (4.5.5) yields (4.5.4). ∎

4.6 Auxiliary estimates

We recall that for $(\lambda,\beta) \in \mathbb{C}\times\mathbb{R}_+$, $\psi = \psi_{\lambda,\beta}$ is given by (4.2.11). We now set, for $x \in (0,1)$

$$\widehat{\psi}_{\lambda,\beta}(x) = \frac{\psi_{\lambda,\beta}(x)}{\psi_{\lambda,\beta}(1)}. \tag{4.6.1}$$

The following auxiliary estimate will become useful in the next section.

Lemma 4.6.1. *Let $\widetilde{\nu}_1 \in (0,U(0))$, $a>0$, and $\Upsilon < \sqrt{-U''(0)}/2$. Then there exist C and β_0 such that, for $\beta \geq \beta_0$,*

$$\nu \in (U(0)-\widetilde{\nu}_1, U(0)+a\beta^{-1/2}) \quad \textit{and} \quad \mu < \Upsilon\beta^{-1/2} \tag{4.6.2}$$

such that

$$\|(\mathcal{L}_\beta^{\mathfrak{N}}-\beta\lambda)^{-1}\widehat{\psi}_{\lambda,\beta}\|_2 + \beta^{-1/2}\Big\|\frac{d}{dx}(\mathcal{L}_\beta^{\mathfrak{N}}-\beta\lambda)^{-1}\widehat{\psi}_{\lambda,\beta}\Big\|_2 \leq C\,\beta^{-5/4}. \tag{4.6.3}$$

For convenience of omit the subscript (λ,β) from $\widehat{\psi}_{\lambda,\beta}$ in the sequel.

Proof. Let $v \in D(\mathcal{L}_\beta^{\mathfrak{N}})$ satisfy

$$(\mathcal{L}_\beta^{\mathfrak{N}}-\beta\lambda)v = \widehat{\psi}. \tag{4.6.4}$$

Let $\mu_0>0$. We begin by considering the case $\mu \geq -\mu_0$. Let further $\check{\chi}_\nu$ be given by (4.5.12). As in (4.5.15) we obtain

$$\|\check{\chi}_\nu v\|_2 \leq C\,[\beta^{-1/2}\|v\|_2 + \beta^{-1}\|\widehat{\psi}\|_2].$$

By (4.5.6) it holds that

$$\|v\|_2 \leq C\beta^{-1/2}\|\widehat{\psi}\|_2.$$

Furthermore, using (4.2.16) (with $k = 0$) and (4.2.10) we have for $|\lambda| \geq |\nu| \geq U(0)/2$

$$\|\widehat{\psi}\|_2 \leq C\,\lambda_\beta^{-1/4}\beta^{-1/6} \leq \widehat{C}\,\beta^{-1/4}. \tag{4.6.5}$$

Hence, we obtain that

$$\|\check{\chi}_\nu v\|_2 \leq C\beta^{-5/4}. \tag{4.6.6}$$

Furthermore, since by (4.5.13a)

$$\|(\check{\chi}_\nu v)'\|_2 \leq [\mu_{\beta,+}]^{1/2}\|\check{\chi}_\nu v\|_2 + C(\|v\|_2 + \|\check{\chi}_\nu v\|_2^{1/2}\|\widehat{\psi}\|_2^{1/2}), \tag{4.6.7}$$

where $\mu_{\beta,+}$ is given by (3.1.26), we can conclude from (4.6.6) that

$$\|(\check{\chi}_\nu v)'\|_2 \leq C(\beta^{-3/4} + \|v\|_2). \tag{4.6.8}$$

Let

$$\hat{\chi}_\nu = \sqrt{1 - \check{\chi}_\nu^2}. \tag{4.6.9}$$

Clearly,

$$(\mathcal{L}_\beta^{\mathfrak{N}} - \beta\lambda)(\hat{\chi}_\nu v) = -\hat{\chi}_\nu'' v - 2\hat{\chi}_\nu' v' + \hat{\chi}_\nu\widehat{\psi}. \tag{4.6.10}$$

By (4.6.6) it holds that

$$\|v\|_2 \leq C(\|\hat{\chi}_\nu v\|_2 + \beta^{-5/4}).$$

Furthermore, we have, by (4.6.8) together with (4.6.6) for the last line, that

$$\begin{aligned}\|v'\|_2 &\leq \|(\check{\chi}_\nu v)'\|_2 + \|(\hat{\chi}_\nu v)'\| \\ &\leq C(\beta^{-3/4} + \|v\|_2 + \|(\hat{\chi}_\nu v)'\|) \\ &\leq \widehat{C}(\beta^{-3/4} + \|\hat{\chi}_\nu v\|_2 + \|(\hat{\chi}_\nu v)'\|).\end{aligned}$$

We now apply either (3.2.1a) or (3.1.3) (see the proof of Proposition 4.5.1, Step 2) to (4.6.10) to obtain

$$\begin{aligned}&\|\hat{\chi}_\nu v\|_2 + \beta^{-1/4}\|(\hat{\chi}_\nu v)'\|_2 \\ &\quad \leq C\beta^{-1/2}(\|v\|_2 + \|v'\|_2 + \|\hat{\chi}_\nu\widehat{\psi}\|_2) \\ &\quad \leq \widehat{C}\beta^{-1/2}(\beta^{-3/4} + \|(\hat{\chi}_\nu v)'\|_2 + \|(\hat{\chi}_\nu v)'\|_2 + \|\hat{\chi}_\nu\widehat{\psi}\|_2).\end{aligned}$$

Hence,

$$\|\hat{\chi}_\nu v\|_2 + \beta^{-1/4}\|(\hat{\chi}_\nu v)'\|_2 \leq C\beta^{-1/2}(\beta^{-3/4} + \|\hat{\chi}_\nu\widehat{\psi}\|_2).$$

By (4.2.16) and (4.2.10) we obtain that

$$\|\hat{\chi}_\nu\widehat{\psi}\|_2 \leq C\beta^{-3/4},$$

and hence

$$\|\hat{\chi}_\nu v\|_2 + \beta^{-1/4}\|(\hat{\chi}_\nu v)'\|_2 \leq C\beta^{-5/4}.$$

Combining the above with (4.6.6) and (4.6.10) yields (4.6.3).

Consider now the case where $\mu < -\mu_0$. Here, we use (3.1.84) and (3.1.85), applied to the pair $(v, \widehat{\psi})$, and then (4.6.5) to obtain that

$$\|v\|_2 + \beta^{-1/2}\|v'\|_2 \leq C\beta^{-1}\|\widehat{\psi}\|_2 \leq \widehat{C}\,\beta^{-5/4},$$

establishing, thereby, (4.6.3) for the case $\mu < -\mu_0$. ■

Remark 4.6.2. Let for $(\lambda, \beta) \in \mathbb{C} \times \mathbb{R}_+$, $\hat{g}(x) = h(x)\chi(1-x)/\psi_{\lambda,\beta}(1)$ where h is given by (4.2.15). By the same arguments used to establish (4.6.3) we may conclude under the assumptions of Lemma 4.6.1 that there exists C and β_0 such that for $\beta \geq \beta_0$,

$$\nu \in (U(0) - \widetilde{\nu}_1, U(0) + a\beta^{-1/2}) \quad \text{and} \quad -\mu_0 \leq \mu < \Upsilon\beta^{-1/2},$$

we have

$$\|(\mathcal{L}_\beta^{\mathfrak{N}} - \beta\lambda)^{-1}\hat{g}\|_2 + \beta^{-1/2}\Big\|\frac{d}{dx}(\mathcal{L}_\beta^{\mathfrak{N}} - \beta\lambda)^{-1}\hat{g}\Big\|_2 \leq C\,\beta^{-5/4}. \tag{4.6.11}$$

4.7 Resolvent estimates for large α

In this section, we will adapt the results of Section 6.3 in [3] to the present setting, involving a Neumann condition at $x = 0$. For $\alpha > 0$, we consider $\mathfrak{z}_\alpha$ to be the solution of

$$\begin{cases} -\mathfrak{z}'' + \alpha^2\mathfrak{z} = 0 & \text{for } x \in (0,1), \\ \mathfrak{z}(1) = 1 \text{ and } \mathfrak{z}'(0) = 0. \end{cases} \tag{4.7.1}$$

The solution of (4.7.1) is given by

$$\mathfrak{z}_\alpha(x) = \cosh(\alpha x)/\cosh(\alpha), \tag{4.7.2}$$

and hence, for large α decays exponentially fast away from $x = 1$.

Proposition 4.7.1. *Let $\theta_1 > 0$, $U \in C^2([0,1])$ satisfy (2.1.3), and $\Upsilon < \sqrt{-U''(0)}/2$. Let further $\widehat{\mu}_m > 0$ be given by* [3, equation (6.57)]*, Then, for any $\widehat{\Upsilon} > 0$, there exist $\beta_0 > 0$ and $C > 0$ such that, for $\beta \geq \beta_0$ and $\alpha \geq \theta_1\beta^{1/3}$,*

$$\sup_{\substack{\Re\lambda \leq \min(\Upsilon\beta^{-1/2}, \\ \beta^{-1/3}[\widehat{\mu}_m - \widehat{\Upsilon} - \alpha^2\beta^{-2/3}/2])}} \|(\mathcal{L}_\beta^{\mathfrak{z}_\alpha} - \beta\lambda)^{-1}\| \leq \frac{C}{\beta^{1/2}[1 + \beta^{1/6}|U(0) - \nu|^{1/3}]}. \tag{4.7.3}$$

Proof. The proof follows the same lines of the proof of [3, Proposition 6.11], and hence we bring only its main ingredients.

Let $\theta = \alpha\beta^{-1/3}$ and

$$F(\lambda,\theta) = \int_{\mathbb{R}_+} e^{-\theta x}\ \mathrm{Ai}(e^{i\pi/6}(x+i\lambda))\,dx.$$

Let further

$$\omega(\beta,\lambda,\theta) := \frac{F(\beta^{1/3}\lambda,0)}{F(\beta^{1/3}\lambda,\theta)}.$$

We then define

$$\psi_\theta = \omega(\beta,\lambda,\theta)\psi,$$

where $\psi = \psi_{\lambda,\beta}$ is defined in (4.2.11), and

$$h_\theta = \omega(\beta,\lambda,\theta)h,$$

where h is defined by (4.2.12). Set

$$\tilde{v}_\theta = (\mathcal{L}_\beta^{\mathfrak{N}} - \beta\lambda)^{-1}h_\theta.$$

By [3, equation (6.79)] for any $\widehat{\Upsilon} > 0$ there exists $C > 0$ such that

$$\sup_{\Re\lambda\leq\beta^{-1/3}[\hat{\mu}_m-\widehat{\Upsilon}-\alpha^2\beta^{-2/3}/2]} |\omega(\beta,\lambda,\theta)| \leq C\,\theta \quad \forall\theta\geq\theta_1, \tag{4.7.4}$$

and hence by (4.2.14) and (4.3.6) we obtain that

$$\|\tilde{v}_\theta\|_1 \leq C\theta\beta^{-2/3}. \tag{4.7.5}$$

Note that for any $\Upsilon > 0$, there exist β_0 and a such that for $\beta \geq \beta_0$ and $\nu < U(0) - a\beta^{-1/2}$, we have

$$\frac{\sqrt{-U''(0)}}{2}\beta^{-1/2} \leq \Upsilon \min(1, |U(0)-\nu|^{1/3})\beta^{-1/3},$$

which allows for the application of (4.2.14).

Suppose now that $(v,g) \in D(\mathcal{L}_\beta^{\mathfrak{z}\alpha}) \times L^2(0,1)$ satisfy

$$(\mathcal{L}_\beta^{\mathfrak{z}\alpha} - \beta\lambda)v = g.$$

Then as in [3, equation (6.20)] or (4.2.36)–(4.2.37) we may write v in the form

$$v = A(\psi_\theta - \tilde{v}_\theta) + u, \tag{4.7.6}$$

where

$$u = (\mathcal{L}_\beta^{\mathfrak{N}} - \beta\lambda)^{-1}g.$$

Taking the inner product of (4.7.6) with $\mathfrak{z}_\alpha$ yields

$$A\langle \mathfrak{z}_\alpha, (\psi_\theta - \tilde{v}_\theta)\rangle = \langle \mathfrak{z}_\alpha, u\rangle.$$

As in [3], using the approximation $\mathfrak{z}_\alpha \approx e^{-\alpha(1-x)}$ (see the display below [3, equation (6.95)] and its proof with minor changes), we then show that

$$\langle \mathfrak{z}_\alpha, (\psi_\theta - \tilde{v}_\theta)\rangle = \beta^{-1/3}[1 + \mathcal{O}(\beta^{-1/3})].$$

Consequently,

$$|A| \leq C\beta^{1/3}|\langle \mathfrak{z}_\alpha, u\rangle|. \tag{4.7.7}$$

To complete the proof we need an estimate for $|\langle \mathfrak{z}_\alpha, u\rangle|$. The proof in the case $\nu \leq U(1/2)$ is identical with the derivation of [3, equation (6.93)], given that (3.1.3) together with (3.1.84) give [3, equation (5.4)] in this case. Hence, we consider here only the case where $\nu > U(1/2)$.

We thus write, with the aid of either (3.1.3) or (3.2.5)

$$|\langle \mathfrak{z}_\alpha, \mathbf{1}_{[0,\hat{x}_\nu]}u\rangle| \leq Ce^{-\alpha/2}\|\mathbf{1}_{[0,\hat{x}_\nu]}u\|_2 \leq Ce^{-\theta\beta^{1/3}/2}\|g\|_2, \tag{4.7.8}$$

where $\hat{x}_\nu$ is given by (4.5.8) so that $U(\hat{x}_\nu) = \nu/2$. Furthermore,

$$|\langle \mathfrak{z}_\alpha, \mathbf{1}_{[\hat{x}_\nu,1]}u\rangle| \leq \|\mathfrak{z}_\alpha\|_1\|\mathbf{1}_{[\hat{x}_\nu,1]}u\|_\infty \leq \frac{C}{\theta\beta^{1/3}}\|\mathbf{1}_{[\hat{x}_\nu,1]}u\|_\infty. \tag{4.7.9}$$

Since $u(1) = 0$ it holds that

$$\|\mathbf{1}_{[\hat{x}_\nu,1]}u\|_\infty \leq \|\mathbf{1}_{[\hat{x}_\nu,1]}u\|_2\|\mathbf{1}_{[\hat{x}_\nu,1]}u'\|_2.$$

By (4.5.16) and (4.5.17), and (4.7.9) we then have

$$|\langle \mathfrak{z}_\alpha, \mathbf{1}_{[\hat{x}_\nu,1]}u\rangle| \leq \frac{C}{\theta\beta^{13/12}}\|g\|_2.$$

Substituting the above, together with (4.7.8) into (4.7.7) yields

$$|A| \leq \frac{C}{\theta\beta^{3/4}}\|g\|_2.$$

By (4.7.6), (4.2.16), (4.2.14), and (4.3.6) we then have

$$\|v\|_2 \leq \frac{C}{\theta\beta^{3/4}}(\|\psi_\theta\|_2 + \|\tilde{v}_\theta\|_2)\|g\|_2 + \|u\|_2. \tag{4.7.10}$$

By (4.2.16) with $k = 0$ and (4.7.4) it holds that

$$\frac{1}{\theta\beta^{3/4}}\|\psi_\theta\|_2 \leq C\,\lambda_\beta^{\frac{1}{4}}\beta^{-11/12}.$$

By (4.3.6) and (4.7.4) it holds that

$$\frac{1}{\theta\beta^{3/4}}\|\tilde{v}_\theta\|_2 \leq C\,\beta^{-4/3}.$$

Finally, (3.1.3) and (3.2.1a) establish the existence of $\Upsilon > 0$ such that

$$\|u\|_2 \leq \frac{C}{\beta^{1/2}[1+\beta^{1/6}|U(0)-\nu|^{1/3}]}\|g\|_2 \tag{4.7.11}$$

for all $U(0)/2 < \nu < U(0) + a\beta^{-1/2}$.

For $\nu \geq U(0) + a\beta^{-1/2}$ we can use (4.5.13b) with $\check{\chi}_\nu \equiv 1$ and $f = g$ to obtain

$$\|u\|_2 \leq \frac{C}{\beta|U(0)-\nu|}\|g\|_2,$$

and hence (4.7.11) holds true for all $\nu > U(0)/2$.

Combining the above yields

$$\|v\|_2 \leq \frac{C}{\beta^{1/2}[1+\beta^{1/6}|U(0)-\nu|^{1/3}]}\|g\|_2,$$

verifying thereby (4.7.3). ∎