Chapter 5

The Orr–Sommerfeld operator

5.1 Introduction

In this chapter, we prove Theorems 1.1.1 and 1.1.2 by obtaining inverse estimates for the Orr–Sommerfeld operator (1.1.7b). As in [3], we use the estimates for the inviscid operator $\mathcal{A}_{\lambda,\alpha}$ from Chapter 2 together with the resolvent estimates for the Schrödinger operators $\mathcal{L}_\beta^{\mathfrak{N},\mathfrak{D}}$ and $\mathcal{L}_\zeta^\beta$ from Chapters 3 and 4. In contrast with [3] we need to consider here many different cases depending on the values of $\Im\lambda$ and α.

Figure 5.1 presents a rough sketch of the various domains where each estimate is valid in the (α, ν) plane ($\nu = \Im\lambda$). The blank domain denotes the domain in the (α, ν) plane where resolvent estimates have not been obtained. We refer the reader to Section 1.2 for a brief explanation of the methods of the proof.

In the following we explain why the division of the (α, ν) plane into 10 subdomains is necessary. Propositions 5.12.1 and 5.12.2 deal with the case where $\nu \notin [0, U(0)]$ making use of the invertibility of $\mathcal{A}_{i\nu,\alpha}$ in these cases. The necessity of Proposition 5.7.1 which deals with the case $\alpha \gtrsim \beta^{1/3}$, and Proposition 5.8.1 (and Proposition 5.8.2) which deals with the case $\alpha \ll \beta^{-1/6}$ is explained in Section 1.2. Proposition 5.6.1 deals with the case $|\nu| < \nu_0 < U(0)$ and $1 \ll \alpha \ll \beta^{1/3}$. In this range of α values we may effectively use the fact that $\|(-d^2/dx^2 + \alpha^2)^{-1}\|$ is small at the conclusion of the proof. Proposition 5.4.1 deals with the case $\nu \geq \beta^{-1/5+\delta}$ for any $0 < \delta < 1/5$ and $\alpha \lesssim 1$. In the proof we use the same methods as in [3], till the value of ν becomes to small due to the non-invertibility of $\mathcal{A}_{0,0}$. For $|\nu| \leq \beta^{-1/5+\delta}$ and $\beta^{-1/10+\delta/2} \ll \alpha \lesssim 1$ we use Proposition 5.5.1. This range of α values allows the application of Proposition 2.5.1 towards the end of the proof. Proposition 5.10.1 deals with the case where $|U(0) - \nu| \lesssim \beta^{-1/2}$. Here, we can approximate U by a quadratic potential near $x = 0$ and use the estimates in Sections 2.9, 3.2, and 4.3. Finally, Proposition 5.11.1 deals with the transition from a linear behavior of $U - U(x_\nu)$ (x_ν is defined in (2.4.5)) to a quadratic behavior near x_ν.

5.2 Preliminaries

We begin by recalling from (4.6.1) the definition of the boundary terms

$$\hat{\psi}_{\lambda,\beta}(x) = \frac{\mathrm{Ai}\big(\beta^{1/3}e^{-i\pi/6}[(1-x) - i\lambda]\big)}{\mathrm{Ai}\big(e^{-i2\pi/3}\beta^{1/3}\lambda\big)}\chi(1-x), \tag{5.2.1}$$

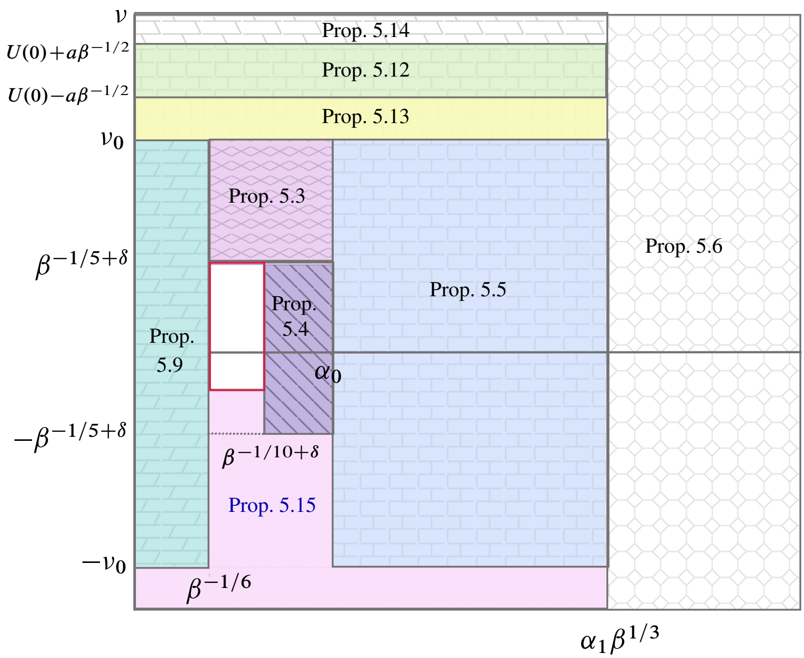

Figure 5.1. Summary of the results in Chapter 5.

where we recall that χ is given by (2.6.20). We also recall from [3, Section 8.3.2, equation (8.91)] that there exists $\Upsilon > 0$ such that, for all $\beta \geq 1$ and $\Re\lambda < \Upsilon\beta^{-1/3}$, it holds that

$$\|(1-x)^s \widehat{\psi}_{\lambda,\beta}\|_1 \leq C\,\lambda_\beta^{-(s+1)/2}\,\beta^{-(s+1)/3}, \quad s \in [0,3]. \tag{5.2.2}$$

Similarly, from [3, Proposition A.8 and equation (A.43c,d)], we can conclude the existence of $C > 0$ such that

$$\|(1-x)^s \widehat{\psi}_{\lambda,\beta}\|_\infty \leq C\,\lambda_\beta^{-s/2}\,\beta^{-s/3}, \quad s \in [0,3]. \tag{5.2.3}$$

We further recall the definition of the inviscid operator in equation (2.1.1), which is the Neumann–Dirichlet realization in $(0,1)$ of

$$\mathcal{A}_{\lambda,\alpha} \overset{\text{def}}{=} (U + i\lambda)\Big(-\frac{d^2}{dx^2} + \alpha^2\Big) + U''$$

for $\lambda \in \mathbb{C}$ and $\alpha \in \mathbb{R}$. We note that $\mathcal{A}_{\lambda,\alpha}$ is invertible when either $\nu \notin [0, U(0)]$ or $|\mu| > 0$, either by Proposition 2.6.1 or by Proposition 4.13 in [3] which holds true since $|U''| > 0$. We introduce in addition

$$\phi_{\lambda,\beta,\alpha} := \mathcal{A}_{\lambda,\alpha}^{-1}(U + i\lambda)\widehat{\psi}_{\lambda,\beta}. \tag{5.2.4}$$

We dedicate this section to two extensions of [3, Lemma 8.1]. These are useful in order to establish the contribution of the boundary terms in (1.2.6). The reader is referred to Section 1.2 for more details on the necessity of these estimates. The first of them is the following lemma which considers the case $\nu > \beta^{-1/5}$. The proof is significantly more complex than the proof of [3, Lemma 8.1] in view of the non-injectivity of $\mathcal{A}_{0,0}$.

Lemma 5.2.1. *Let $U \in C^3([0,1])$ satisfy (2.1.3). There exist positive constants Υ, C, $\widehat{C}$, ν_0, and β_0 such that, for all $\alpha \geq 0$, $\beta \geq \beta_0$, $\lambda \in \mathbb{C}$ for which $\mu < \Upsilon\beta^{-1/3}$, $\mu \neq 0$, and $\beta^{-1/5} < \nu < \nu_0$ it holds that*

$$\|\phi_{\lambda,\beta,\alpha}\|_{1,2} \leq C\nu^{-1}\,[|\lambda|\beta]^{-3/4} \tag{5.2.5}$$

and

$$|\phi_{\lambda,\beta,\alpha}(x_\nu)| \leq \widehat{C}\,[|\lambda|\beta]^{-3/4}. \tag{5.2.6}$$

Proof. Step 1. We prove (5.2.5) and (5.2.6) for the case $\alpha^2 < \nu^{-1}$ and $0 < |\mu| < 1$.

By (2.6.4) applied to the pair (ϕ, v) with $v = (U + i\lambda)\widehat{\psi}$ (see (5.2.4)), it holds that

$$|\phi(x_\nu)|^2 \leq C\,|\langle\phi, \widehat{\psi}\rangle|. \tag{5.2.7}$$

Let $\tilde{x}_\nu = (1 + x_\nu)/2$. To estimate the right-hand side of (5.2.7), we first obtain a bound for $\|\phi'\|_{L^\infty(\tilde{x}_\nu,1)}$. To this end, we integrate the balance $(U + i\lambda)^{-1}\mathcal{A}_{\lambda,\alpha}\phi = \widehat{\psi}$ over $(\tilde{x}_\nu, 1)$ to obtain

$$\|\phi''\|_{L^1(\tilde{x}_\nu,1)} \leq \left(\alpha^2 + \frac{C}{\nu}\right)\|\phi\|_{L^1(\tilde{x}_\nu,1)} + \|\widehat{\psi}\|_{L^1(\tilde{x}_\nu,1)}. \tag{5.2.8}$$

Since $\phi(1) = 0$ it holds that

$$\|\phi\|_{L^1(\tilde{x}_\nu,1)} \leq \|\phi'\|_{L^\infty(\tilde{x}_\nu,1)}\|1 - x\|_{L^1(\tilde{x}_\nu,1)} \leq C\,\|\phi'\|_{L^\infty(\tilde{x}_\nu,1)}\nu^2. \tag{5.2.9}$$

Using (5.2.8), (5.2.9), (5.2.2) for $s = 0$, and the fact that (in this step) $\alpha^2 \leq |\nu|^{-1}$, we obtain that

$$\|\phi''\|_{L^1(\tilde{x}_\nu,1)} \leq C\,(\nu\|\phi'\|_{L^\infty(\tilde{x}_\nu,1)} + [|\lambda|\beta]^{-1/2}). \tag{5.2.10}$$

Clearly, there exists $z_\nu \in [\tilde{x}_\nu, 1]$ such that

$$|\phi'(z_\nu)| \leq |1 - \tilde{x}_\nu|^{-1/2}\|\phi'\|_{L^2(\tilde{x}_\nu,1)} \leq \widehat{C}\,|\nu|^{-1/2}\|\phi'\|_{L^2(\tilde{x}_\nu,1)}.$$

Since for any $x \in (\tilde{x}_\nu, 1)$ it holds that

$$|\phi'(x) - \phi'(z_\nu)| \leq \|\phi''\|_{L^1(\tilde{x}_\nu,1)},$$

we can deduce that

$$\|\phi'\|_{L^\infty(\tilde{x}_\nu,1)} \leq C\nu^{-1/2}\|\phi'\|_{L^2(\tilde{x}_\nu,1)} + \|\phi''\|_{L^1(\tilde{x}_\nu,1)}. \tag{5.2.11}$$

We can then conclude, using (5.2.10), that we can choose $\nu_0 > 0$ and $C > 0$ such that for all $0 < \nu < \nu_0$

$$\|\phi'\|_{L^\infty(\tilde{x}_\nu,1)} \leq C(\nu^{-1/2}\|\phi'\|_{L^2(\tilde{x}_\nu,1)} + [|\lambda|\beta]^{-1/2}). \tag{5.2.12}$$

We now write

$$\begin{aligned}|\langle\phi,\widehat{\psi}\rangle| &\leq |\langle\phi,\widehat{\psi}\rangle_{L^2(\tilde{x}_\nu,1)}| + |\langle\phi,\widehat{\psi}\rangle_{L^2(0,\tilde{x}_\nu)}| \\ &\leq \|\phi'\|_{L^\infty(\tilde{x}_\nu,1)}\|(1-x)\widehat{\psi}\|_{L^1(\tilde{x}_\nu,1)} + \|\phi'\|_2\|(1-x)^{1/2}\widehat{\psi}\|_{L^1(0,\tilde{x}_\nu)},\end{aligned} \tag{5.2.13}$$

and then observe that

$$\begin{aligned}\|(1-x)^{1/2}\widehat{\psi}\|_{L^1(0,\tilde{x}_\nu)} &= \|(1-x)^{-5/2}(1-x)^3\widehat{\psi}\|_{L^1(0,\tilde{x}_\nu)} \\ &\leq (1-\tilde{x}_\nu)^{-5/2}\|(1-x)^3\widehat{\psi}\|_{L^1(0,\tilde{x}_\nu)} \\ &\leq C\,\nu^{-5/2}\,\|(1-x)^3\widehat{\psi}\|_1.\end{aligned} \tag{5.2.14}$$

By (2.6.2) with $v = (U + i\lambda)\widehat{\psi}$ together with (2.6.1) it holds that

$$\|\phi'\|_2 \leq \frac{C}{\nu^{3/2}}\|[(1-x)^{1/2} + \nu^{-1/2}(1-x)]\widehat{\psi}\|_1. \tag{5.2.15}$$

Hence, by (5.2.2) with $s = 1/2, 1, 3$, (5.2.14), and (5.2.15) we obtain that

$$\begin{aligned}&\|\phi'\|_2\|(1-x)^{1/2}\widehat{\psi}\|_{L^1(0,\tilde{x}_\nu)} \\ &\quad\leq \frac{C}{\nu^4}\|[(1-x)^{1/2} + \nu^{-1/2}(1-x)]\widehat{\psi}\|_1\|(1-x)^3\widehat{\psi}\|_1 \\ &\quad\leq \frac{\widehat{C}}{\nu^4}([|\lambda|\beta]^{-11/4} + \nu^{-1/2}[|\lambda|\beta]^{-3}).\end{aligned}$$

Then, since by assumption $\nu > \beta^{-1/5}$, we obtain that

$$\|\phi'\|_2\|(1-x)^{1/2}\widehat{\psi}\|_{L^1(0,\tilde{x}_\nu)} \leq \widetilde{C}\beta^{-1/5}\,[|\lambda|\beta]^{-3/2}. \tag{5.2.16}$$

By (5.2.2) with $s = 1$ and (5.2.12) we have that

$$\|\phi'\|_{L^\infty(\tilde{x}_\nu,1)}\|(1-x)\widehat{\psi}\|_{L^1(\tilde{x}_\nu,1)} \leq C\,(\nu^{-1/2}\|\phi'\|_{L^2(\tilde{x}_\nu,1)} + [|\lambda|\beta]^{-1/2})\,[|\lambda|\beta]^{-1}. \tag{5.2.17}$$

Substituting (5.2.17), together with (5.2.16) into (5.2.13) then yields

$$|\langle\phi,\widehat{\psi}\rangle| \leq C(|\nu|^{-1/2}\|\phi'\|_{L^2(x_\nu,1)} + [|\lambda|\beta]^{-1/2})[|\lambda|\beta]^{-1}. \tag{5.2.18}$$

We now use (2.6.19) and (2.6.1) to obtain that

$$\|\phi'\|_{L^2(x_\nu,1)} \leq C\big[\nu^{-1/2}|\langle\phi,\widehat{\psi}\rangle|^{1/2} + \|[(1-x)^{1/2} + \nu^{-1/2}(1-x)]\widehat{\psi}\|_1\big]. \tag{5.2.19}$$

Substituting (5.2.19) into (5.2.18) yields, with the aid of (5.2.2) and the fact that $\nu > \beta^{-1/5}$

$$|\langle\phi, \widehat{\psi}\rangle| \leq C(\nu^{-1}|\langle\phi, \widehat{\psi}\rangle|^{1/2} + [|\lambda|\beta]^{-1/2})[|\lambda|\beta]^{-1},$$

which immediately implies

$$|\langle\phi, \widehat{\psi}\rangle| \leq C(\nu^{-2}[|\lambda|\beta]^{-1} + [|\lambda|\beta]^{-1/2})[|\lambda|\beta]^{-1} \leq \widetilde{C}\,[|\lambda|\beta]^{-3/2}. \tag{5.2.20}$$

From the above inequality, with the aid of (5.2.7), we conclude (5.2.6). Hence, by (5.2.19), we also get that

$$\|\phi'\|_{L^2(x_\nu,1)} \leq C\,\nu^{-1/2}[|\lambda|\beta]^{-3/4}. \tag{5.2.21}$$

To obtain an effective bound for $\|\phi'\|_2$ we use (2.6.53) and (2.6.1) to obtain, with the aid of (5.2.19), (5.2.21), and (5.2.2)

$$\begin{aligned}\|\phi'\|_2 &\leq C\big(\|[(1-x)^{1/2} + \nu^{-1/2}(1-x)]\widehat{\psi}\|_1 + \nu^{-1/2}\|\phi'\|_{L^2(x_\nu,1)}\big)\\ &\leq \widetilde{C}\,\nu^{-1}[|\lambda|\beta]^{-3/4},\end{aligned} \tag{5.2.22}$$

from which we conclude (5.2.5) by using Poincaré's inequality.

Step 2. We prove (5.2.5) and (5.2.6) for the case $\alpha^2 > \nu^{-1}$ and $|\mu| \leq 1$.

To obtain (5.2.5) for $\alpha^2 > |\nu|^{-1}$ and $\nu < \nu_0$, we observe that for any $A_0 > 0$ we can choose ν_0 such that $\alpha^2 \geq A_0$ for $\nu < \nu_0$, and consequently use (2.6.86) in the form (with $v = (U + i\lambda)\widehat{\psi}$)

$$\begin{aligned}\|\phi\|_{1,2} &\leq C\left\|[(1-x)^{1/2} + \nu^{-1/2}(1-x)]\frac{v}{U+i\lambda}\right\|_1\\ &= C\,\|[(1-x)^{1/2} + \nu^{-1/2}(1-x)]\widehat{\psi}\|_1.\end{aligned}$$

Using (5.2.2) and the fact that $\nu > \beta^{-1/5}$, we obtain,

$$\|\phi\|_{1,2} \leq \widehat{C}([|\lambda|\beta]^{-3/4} + \nu^{-1/2}[|\lambda|\beta]^{-1}) \leq \widetilde{C}\,[|\lambda|\beta]^{-3/4},$$

which implies (5.2.5) for $\alpha^2 > \nu^{-1}$. We now use (5.2.7) to obtain

$$|\phi(x_\nu)|^2 \leq C\|\phi'\|_2\|(1-x)^{1/2}\widehat{\psi}\|_1. \tag{5.2.23}$$

We may then conclude (5.2.6) as well by using (5.2.5) and (5.2.2).

Step 3. We prove (5.2.5) and (5.2.6) for $|\mu| \geq 1$.

The proof of (5.2.5) in this case follows from (2.6.17) which yields

$$\|\phi'\|_2^2 \leq C\,|\langle\phi, \widehat{\psi}\rangle|.$$

Consequently, by (5.2.2) we obtain that

$$\|\phi'\|_2^2 \leq C\|\phi'\|_2\|(1-x)^{1/2}\widehat{\psi}\|_1 \leq \widetilde{C}\,[|\lambda|\beta]^{-3/4}\|\phi'\|_2,$$

yielding, thereby,

$$\|\phi'\|_2 \leq \widehat{C}\,[\|\lambda|\beta]^{-3/4}.$$

We can then complete the proof of (5.2.5) by using Poincaré's inequality. The proof of (5.2.6) follows from (5.2.5) and Sobolev embeddings. ∎

We next consider (as in Proposition 2.5.1) the case $|\lambda| \ll 1$ and α large enough, which will be sufficient to guarantee a satisfactory estimate of $\|\mathcal{A}_{\lambda,\alpha}^{-1}\|$ despite the fact that $\mathcal{A}_{0,0}$ is not injective.

Lemma 5.2.2. *Let $\delta > 0$, and $U \in C^3([0,1])$ satisfy (2.1.3). There exist positive constants Υ_0, λ_0, C, and β_0 such that, for all $\beta \geq \beta_0$, all $\lambda \in \mathbb{C}\setminus\{0\}$ for which $-\lambda_0 < \Re\lambda \leq \beta^{-1/3}\Upsilon_0$, $|\Im\lambda| < \lambda_0$, and $\alpha \geq \alpha_{\lambda,\delta}$, it holds that*

$$|c_\|(\lambda,\beta,\alpha)| \leq \frac{C}{|\alpha^2\|U\|_2^2 + i\lambda|}(\lambda_\beta^{-1}\beta^{-2/3} + \lambda_\beta^{-1/2}|\lambda|\beta^{-1/3}) \tag{5.2.24a}$$

and

$$\begin{aligned}\Big\|\phi_{\lambda,\beta,\alpha} - c_\|(\lambda,\beta,\alpha)U\Big\|_{1,2} \\ \leq C\Big[\lambda_\beta^{-3/4}\beta^{-1/2} + \frac{|\lambda|}{|\alpha^2\|U\|_2^2 + i\lambda|}(\lambda_\beta^{-1}\beta^{-2/3} + \lambda_\beta^{-1/2}|\lambda|\beta^{-1/3})\Big],\end{aligned} \tag{5.2.24b}$$

where, as in (2.3.1),

$$c_\|(\lambda,\beta,\alpha) = \frac{\langle U, \phi_{\lambda,\beta,\alpha}\rangle}{\|U\|_2^2},$$

and as in (2.5.1)

$$\alpha_{\lambda,\delta} = \|U\|_2^{-1}(|\Im\lambda|(1+2\delta))^{1/2}.$$

Proof. We write as in (2.3.1) with $\phi = \phi_{\lambda,\beta,\alpha}$

$$\phi = c_\|\,U + \phi_\perp.$$

Then by (2.5.3a) and (2.5.2b), there exists λ_1 such that for $0 < |\lambda| < \lambda_1$,

$$|c_\|| \leq \frac{1 + C|\lambda|^2\log|\lambda|^{-1}}{|\alpha^2\|U\|_2^2 + i\lambda|}(\|(U+i\lambda)\widehat{\psi}\|_1 + C|\lambda|\|\widehat{\psi}\|_1). \tag{5.2.25}$$

It follows from (5.2.2) (with $s = 0$) that for some positive C

$$\|\widehat{\psi}\|_1 \leq C\,\lambda_\beta^{-1/2}\,\beta^{-1/3}. \tag{5.2.26}$$

Furthermore, by (5.2.26) and (5.2.2) (with $s = 1$)

$$\|(U+i\lambda)\widehat{\psi}\|_1 \leq |\lambda|\,\|\widehat{\psi}\|_1 + C\|(1-x)\widehat{\psi}\|_1 \leq \widehat{C}(|\lambda|\lambda_\beta^{-1/2}\beta^{-1/3} + \lambda_\beta^{-1}\beta^{-2/3}). \tag{5.2.27}$$

Consequently, for β_0 large enough, there exists $\lambda_0 < \lambda_1/\sqrt{2}$ such that for any $|\lambda| \leq \sqrt{2}\lambda_0$ it holds by (5.2.25), that

$$|c_\parallel| \leq \frac{C}{|\alpha^2\,\|U\|_2^2 + i\lambda|}\big(\lambda_\beta^{-1}\beta^{-2/3} + |\lambda|\lambda_\beta^{-1/2}\beta^{-1/3}\big),$$

readily verifying (5.2.24a).

We now apply (2.5.3b) to obtain

$$\|\phi_\perp\|_{1,2} \leq C\Big[\|(1-x)^{1/2}\widehat{\psi}\|_1 + \frac{|\lambda|}{|\alpha^2\|U\|_2^2 + i\lambda|}(\|U\widehat{\psi}\|_1 + C|\lambda|\|\widehat{\psi}\|_1)\Big]$$

which, combined with (5.2.2) (with $s = 1/2$ and $s = 1$), (5.2.26), yields

$$\|\phi_\perp\|_{1,2} \leq C\Big[\lambda_\beta^{-3/4}\beta^{-1/2} + \frac{|\lambda|}{|\alpha^2\|U\|_2^2 + i\lambda|}(\lambda_\beta^{-1}\beta^{-2/3} + \lambda_\beta^{-1/2}|\lambda|\beta^{-1/3})\Big], \tag{5.2.28}$$

establishing, thereby, (5.2.24b). ∎

5.3 Resolvent estimates and Fredholm property

We recall from the introduction that

$$\mathcal{B}^{\mathfrak{N},\mathfrak{D}}_{\lambda,\alpha,\beta} = (\mathcal{L}_\beta - \beta\lambda)\Big(\frac{d^2}{dx^2} - \alpha^2\Big) - i\beta U'' \tag{5.3.1}$$

on $(0, 1)$ with domain (see (1.1.11))

$$D(\mathcal{B}^{\mathfrak{N},\mathfrak{D}}_{\lambda,\alpha,\beta}) = \big\{u \in H^4(0,1), u'(0) = u^{(3)}(0) = 0 \text{ and } u(1) = u'(1) = 0\big\} \tag{5.3.2}$$

and

$$\mathcal{L}_\beta = -\frac{d^2}{dx^2} + i\beta U. \tag{5.3.3}$$

Note that this domain is independent of the parameters (λ, α, β), i.e., $D(\mathcal{B}^{\mathfrak{N},\mathfrak{D}}_{\lambda,\alpha,\beta}) = D(\mathcal{B}^{\mathfrak{N},\mathfrak{D}}_{0,0,0})$.

It can be easily verified that $\mathcal{B}^{\mathfrak{N},\mathfrak{D}}_{0,0,0}$ is invertible. Next, we observe that

$$\mathcal{B}^{\mathfrak{N},\mathfrak{D}}_{\lambda,\alpha,\beta}\big(\mathcal{B}^{\mathfrak{N},\mathfrak{D}}_{0,0,0}\big)^{-1} = I + K_{\lambda,\alpha,\beta},$$

where $K_{\lambda,\alpha,\beta}$ is a compact operator from $L^2(0, 1)$ to $L^2(0, 1)$. Hence, $I + K_{\lambda,\alpha,\beta}$ is a Fredholm operator. Considering again the family $\mathcal{B}^{\mathfrak{N},\mathfrak{D}}_{\lambda,\alpha,\beta} = (I + K_{\lambda,\alpha,\beta})\,\mathcal{B}^{\mathfrak{N},\mathfrak{D}}_{0,0,0}$, we can conclude that it is a Fredholm family from $D(\mathcal{B}^{\mathfrak{N},\mathfrak{D}}_{0,0,0})$ into $L^2(0, 1)$.

Since its index depends continuously on (α, β, λ) and vanishes for $(\alpha, \beta, \lambda) = (0, 0, 0)$, it must be zero for all $(\lambda, \alpha, \beta) \in \mathbb{C} \times \mathbb{R}^2$.

The rest of Chapter 5 is dedicated to the estimation of $(\mathcal{B}^{\mathfrak{D},\mathfrak{N}}_{\lambda,\alpha,\beta})^{-1}$. In practice, we first show in each section that for some subset of parameters (λ, α, β),

$$\phi \in D(\mathcal{B}^{\mathfrak{D},\mathfrak{N}}_{\lambda,\alpha,\beta}) \Rightarrow \|\phi\|_{1,2} \le C(\lambda, \alpha, \beta)\|f\|_2, \tag{5.3.4}$$

where $f = \mathcal{B}^{\mathfrak{D},\mathfrak{N}}_{\lambda,\alpha,\beta}\phi$. It follows that $\mathcal{B}^{\mathfrak{D},\mathfrak{N}}_{\lambda,\alpha,\beta}$ is injective and since its index vanishes, we can conclude the existence of $(\mathcal{B}^{\mathfrak{D},\mathfrak{N}}_{\lambda,\alpha,\beta})^{-1} : L^2(0,1) \to D(\mathcal{B}^{\mathfrak{N},\mathfrak{D}}_{0,0,0})$ together with the estimate

$$\|(\mathcal{B}^{\mathfrak{D},\mathfrak{N}}_{\lambda,\alpha,\beta})^{-1}\| + \left\|\frac{d}{dx}(\mathcal{B}^{\mathfrak{D},\mathfrak{N}}_{\lambda,\alpha,\beta})^{-1}\right\| \le C(\lambda, \alpha, \beta). \tag{5.3.5}$$

5.4 Resolvent estimates for $\beta^{-1/5} \ll |\Im\lambda| < U(0)$

The next proposition is somewhat similar to [3, Lemma 8.8] albeit with a significant difference: the fact that $\mathcal{A}_{0,0}$ is not invertible, which makes the estimates become significantly more complex.

Proposition 5.4.1. *Let $U \in C^4([0,1])$ satisfy (2.1.3) and the assumption $U'''(0) = 0$. Let $0 < \delta < 1/5$, $\nu_0 < U(0)$, and α_0 denote positive constants. There exist $C > 0$, and $\beta_0 > 0$ such that for all $\beta \ge \beta_0$, it holds that*

$$\sup_{\substack{0\le\alpha\le\alpha_0\\ \Re\lambda\le\beta^{-2/5-\delta}\\ \beta^{-1/5+\delta}\le\Im\lambda<\nu_0}} \Im\lambda\Big(\|(\mathcal{B}^{\mathfrak{N},\mathfrak{D}}_{\lambda,\alpha,\beta})^{-1}\| + \left\|\frac{d}{dx}(\mathcal{B}^{\mathfrak{N},\mathfrak{D}}_{\lambda,\alpha,\beta})^{-1}\right\|\Big) \le C\beta^{-1/2+\delta}. \tag{5.4.1}$$

Proof. We assume throughout the proof, without any loss of generality, that $0 < \delta \le 1/30$.

Step 1. Preliminaries. Let $\phi \in D(\mathcal{B}^{\mathfrak{N},\mathfrak{D}}_{\lambda,\alpha,\beta})$ and $f = \mathcal{B}^{\mathfrak{N},\mathfrak{D}}_{\lambda,\alpha,\beta}\phi$. Let further $v_{\mathfrak{D}} \in H^2(0,1)$ be defined by

$$v_{\mathfrak{D}} = \mathcal{A}_{\lambda,\alpha}\phi + (U + i\lambda)\phi''(1)\hat{\psi}, \tag{5.4.2}$$

where $\hat{\psi} = \hat{\psi}_{\lambda,\beta}$ is given by (5.2.1). Note that by (1.1.11), (2.1.3) and the fact that $U'''(0) = 0$, we have

$$v_{\mathfrak{D}}(1) = v'_{\mathfrak{D}}(0) = 0, \tag{5.4.3}$$

and hence $v_{\mathfrak{D}} \in D(\mathcal{L}^{\mathfrak{N},\mathfrak{D}}_\beta)$ and we may introduce, as in [3, Lemma 8.8],

$$g_{\mathfrak{D}} := (\mathcal{L}^{\mathfrak{N},\mathfrak{D}}_\beta - \beta\lambda)v_{\mathfrak{D}}, \tag{5.4.4a}$$

which is expressible in the alternative form (using the fact that $f = \mathcal{B}^{\mathfrak{N},\mathfrak{D}}_{\lambda,\alpha,\beta}\phi$)

$$g_{\mathfrak{D}} = (U + i\lambda)(-f + \phi''(1)\hat{g}) - (U''\phi)'' - 2U'\tilde{v}'_{\mathfrak{D}} - U''\tilde{v}_{\mathfrak{D}}, \tag{5.4.4b}$$

wherein

$$\hat{g} := (\mathcal{L}_\beta - \beta\lambda)\widehat{\psi} \tag{5.4.5}$$

and

$$\tilde{v}_{\mathfrak{D}} := \frac{v_{\mathfrak{D}} - U''\phi}{U + i\lambda} = -\phi'' + \alpha^2\phi + \phi''(1)\widehat{\psi}. \tag{5.4.6}$$

We note that

$$(\mathcal{L}_\beta^{\mathfrak{N}} - \beta\lambda)\tilde{v}_{\mathfrak{D}} = i\beta U''(\phi - \phi(x_\nu)) + i\beta U''\phi(x_\nu) - f + \phi''(1)\hat{g}, \tag{5.4.7}$$

where x_ν is given by (2.4.5). Recall that $\mathcal{L}_\beta^{\mathfrak{N}}$ stands for $\mathcal{L}_\beta^{\mathfrak{N},\mathfrak{D}}$, which is defined in (3.0.1).

As in the proof of [3, Lemma 8.8] (see equation (8.90) there) we can integrate by parts to obtain

$$\begin{aligned}\Re\langle (U'')^{-1}\tilde{v}_{\mathfrak{D}}, (\mathcal{L}_\beta^{\mathfrak{N}} - \beta\lambda)\tilde{v}_{\mathfrak{D}} - i\beta U''\phi\rangle \\ = \|(U'')^{-1/2}\tilde{v}_{\mathfrak{D}}'\|_2^2 + \Re\langle \big((U'')^{-1}\big)'\tilde{v}_{\mathfrak{D}}, \tilde{v}_{\mathfrak{D}}'\rangle - \beta\mu\,\|(U'')^{-1/2}\tilde{v}_{\mathfrak{D}}\|_2^2 \\ + \beta\,\Re\langle \phi''(1)\widehat{\psi}, i\phi\rangle.\end{aligned} \tag{5.4.8}$$

We begin the estimation of $\tilde{v}_{\mathfrak{D}}'$ by obtaining a bound for the last term on the right-hand side of (5.4.8).

Step 2: Estimate of $\beta\Re\langle\phi''(1)\widehat{\psi}, i\phi\rangle$. We begin by writing ϕ as the sum

$$\phi = \hat{w} + \phi''(1)w$$

with

$$w(x) = \int_x^1 (\xi - x)\widehat{\psi}(\xi)\,d\xi,$$

and the remainder

$$\hat{w}(x) := \int_x^1 (\xi - x)[\phi''(\xi) - \phi''(1)\widehat{\psi}(\xi)]\,d\xi.$$

Then, we separately estimate the contribution of the terms $\beta\Re\langle\phi''(1)\widehat{\psi}, i\phi''(1)w\rangle$ and $\beta\Re\langle\phi''(1)\widehat{\psi}, i\hat{w}\rangle$. By (5.2.3) it holds that

$$|w(x)| \leq C\,|1-x|^2.$$

Consequently,

$$|\Re\langle\phi''(1)\widehat{\psi}, i\phi''(1)w\rangle| \leq C\,|\phi''(1)|^2\|(1-x)^2\widehat{\psi}\|_1,$$

and hence, by (5.2.2) with $s = 2$, we then obtain that

$$\beta|\Re\langle\phi''(1)\widehat{\psi}, i\phi''(1)w\rangle| \leq C\,[1 + |\lambda|\beta^{1/3}]^{-3/2}\,|\phi''(1)|^2 \tag{5.4.9}$$

(see [3, equation (8.90)]).

To estimate $\beta\Re\langle\phi''(1)\widehat{\psi}, i\hat{w}\rangle$, we first obtain by (5.4.6), for $x \in (0,1)$,

$$\left|\overline{\widehat{\psi}(x)}\hat{w}(x)\right| = \left|\overline{\widehat{\psi}(x)}\int_x^1 (\xi - x)[-\tilde{v}_{\mathfrak{D}}(\xi) + \alpha^2\phi(\xi)]\,d\xi\right|$$
$$\leq C(1-x)^{5/2}\,|\widehat{\psi}(x)|(\|\tilde{v}_{\mathfrak{D}}'\|_2 + \alpha^2\|\phi'\|_2), \tag{5.4.10}$$

where to obtain the last inequality we used the fact that

$$|\phi(x)| \leq (1-x)^{1/2}\|\phi'\|_2 \quad \text{and} \quad |\tilde{v}_{\mathfrak{D}}(x)| \leq (1-x)^{1/2}\|\tilde{v}_{\mathfrak{D}}'\|_2.$$

Using (5.2.2) with $s = 5/2$, we obtain

$$\beta|\Re\langle\phi''(1)\widehat{\psi}, i\hat{w}\rangle| \leq C|\phi''(1)|\lambda_\beta^{-7/4}\,\beta^{-1/6}(\|\tilde{v}_{\mathfrak{D}}'\|_2 + \alpha^2\|\phi'\|_2). \tag{5.4.11}$$

Consequently, from (5.4.9) and (5.4.11), we thus get, as $\alpha \leq \alpha_0$,

$$\beta|\,\Re\langle\phi''(1)\widehat{\psi}, i\phi\rangle| \leq C\big([1 + |\lambda|\beta^{1/3}]^{-3/2}|\phi''(1)|^2$$
$$+ \lambda_\beta^{-7/4}\,\beta^{-1/6}|\phi''(1)|(\|\tilde{v}_{\mathfrak{D}}'\|_2 + \|\phi'\|_2)\big). \tag{5.4.12}$$

(See [3, equation (8.93)].)

Step 3: Estimate $|\phi''(1)|$. Let $\mathfrak{z} = \mathfrak{z}_\alpha$ be given by (4.7.2) and recall that $\phi \in D(\mathcal{B}_{\lambda,\alpha,\beta}^{\mathfrak{N},\mathfrak{D}})$. As $\langle\mathfrak{z}, \phi'' - \alpha^2\phi\rangle = 0$, $\phi'' - \alpha^2\phi$ belongs to the domain of $\mathcal{L}_\beta^{\mathfrak{z}} - \beta\lambda$. Hence, we may write

$$(\mathcal{L}_\beta^{\mathfrak{z}} - \beta\lambda)(\phi'' - \alpha^2\phi) = i\beta U''\phi + f. \tag{5.4.13}$$

We separately solve in $D(\mathcal{L}_\beta^{\mathfrak{z}})$ the equations $(\mathcal{L}_\beta^{\mathfrak{z}} - \beta\lambda)v = f$ and $(\mathcal{L}_\beta^{\mathfrak{z}} - \beta\lambda)v_1 = i\beta U''\phi := f_1$, so that $(\phi'' - \alpha^2\phi) = v_1 + v_2$, and then apply (4.2.6) to the pair (v_1, f_1) (note that the assumptions of Proposition 4.2.1 are satisfied) and (4.2.4) to the pair (v, f) to obtain

$$|\phi''(1)| \leq C\,\lambda_\beta^{1/2}\,\big(\beta^{1/3}[\|\phi\|_{1,2} + |\phi(x_\nu)|\,\big|\log|\nu + i\mathfrak{m}|^{-1}\big|] + \beta^{-1/2}\|f\|_2\big), \tag{5.4.14}$$

where $\mathfrak{m}$ is defined in 4.2.6b. Note that since $\alpha \leq \alpha_0$ it holds that $\|\mathfrak{z}\|_{1,p} \leq C(\alpha_0)$. Note further that

$$|\phi(x_\nu)|\,\big|\log|\nu + i\mathfrak{m}|^{-1}\big| \leq C\,\nu^{1/2}\big|\log|\nu + i\mathfrak{m}|^{-1}\big|\,\|\phi'\|_2 \leq \widehat{C}\,\|\phi'\|_2.$$

Then, for any $\hat{\mathfrak{v}}_0 > 0$, there exists a constant $C > 0$ such that for $|\lambda| \geq \hat{\mathfrak{v}}_0\beta^{-1/3}$, (in particular it holds for $\nu > \beta^{-1/5}$ for sufficiently large β_0).

$$|\phi''(1)| \leq C\,|\lambda|^{1/2}\big(\beta^{1/2}[\|\phi\|_{1,2} + \beta^{-1/3}]\|f\|_2\big). \tag{5.4.15}$$

Substituting (5.4.14) into (5.4.12) yields

$$\beta|\Re\langle\phi''(1)\widehat{\psi}, i\phi\rangle| \leq C\big[\lambda_\beta^{-1/2}\,(\beta^{2/3}\|\phi\|_{1,2}^2 + \beta^{-1}\|f\|_2^2)$$
$$+ \lambda_\beta^{-5/4}(\beta^{1/3}\|\phi\|_{1,2} + \beta^{-1/2}\|f\|_2)\big(\|\tilde{v}_{\mathfrak{D}}'\|_2 + \|\phi'\|_2\big)\big],$$

from which, we conclude that for any $\hat{\delta} > 0$ there exists $C_{\hat{\delta}} > 0$ such that

$$\beta |\Re\langle \phi''(1)\widehat{\psi}, i\phi\rangle| \leq C_{\hat{\delta}}\big(\lambda_\beta^{-1/2}(\beta^{2/3}\|\phi\|_{1,2}^2 + \beta^{-1}\|f\|_2^2) + \hat{\delta}\,\|\tilde{v}_{\mathfrak{D}}'\|_2^2\big). \tag{5.4.16}$$

Since, as in the proof of (4.2.18) (see also [3, equation (6.18)]), we have for $\hat{g}$, introduced in (5.4.5),

$$|\hat{g}(x)| \leq C\,\big(\beta\,(1-x)^2|\widehat{\psi}(x)| + (1-x)^3|\widehat{\psi}'(x)|\big), \tag{5.4.17}$$

we obtain from (4.2.10), (4.2.16), (4.2.17), and (4.6.1) that

$$\beta^{1/3}\lambda_\beta^{-1}\|(U+i\lambda)\hat{g}\|_2 + \|\hat{g}\|_2 \leq C\beta^{1/6}\lambda_\beta^{-5/4}. \tag{5.4.18}$$

Using the fact that $|\lambda| > \beta^{-1/5}$ we can then conclude

$$|\lambda|^{-1}\|(U+i\lambda)\hat{g}\|_2 + \|\hat{g}\|_2 \leq C\,\beta^{-1/4}|\lambda|^{-5/4}. \tag{5.4.19}$$

Step 4: Estimate of $\|\tilde{v}_{\mathfrak{D}}'\|$. From (5.4.8), we obtain, using (5.4.7) and the fact that $U'' \neq 0$

$$\begin{aligned}\frac{1}{C}\|\tilde{v}_{\mathfrak{D}}'\|_2^2 &\leq \Re\langle (U'')^{-1}\tilde{v}_{\mathfrak{D}}, -f + \phi''(1)\hat{g}\rangle \\ &\quad - \Re\langle ((U'')^{-1})'\tilde{v}_{\mathfrak{D}}, \tilde{v}_{\mathfrak{D}}'\rangle + \beta\mu\,\|\tilde{v}_{\mathfrak{D}}\|_2^2 - \beta\,\Re\langle \phi''(1)\widehat{\psi}, i\phi\rangle.\end{aligned} \tag{5.4.20}$$

Next, we obtain from (5.4.20) and (5.4.16) (for sufficiently small $\hat{\delta}$) that

$$\begin{aligned}\|\tilde{v}_{\mathfrak{D}}'\|_2^2 &\leq C\big[\|\tilde{v}_{\mathfrak{D}}\|_2\big(\|f\|_2 + |\phi''(1)|\,\|\hat{g}\|_2\big) \\ &\quad + (\mu_{\beta,+}+1)\|\tilde{v}_{\mathfrak{D}}\|_2^2 + \lambda_\beta^{-1/2}\big(\beta^{2/3}\|\phi\|_{1,2}^2 + \beta^{-1}\|f\|_2^2\big)\big],\end{aligned} \tag{5.4.21}$$

where

$$\mu_{\beta,+} := \max(\mu\beta, 0).$$

We now substitute (5.4.14) and (5.4.18) into (5.4.21) to obtain

$$\begin{aligned}\|\tilde{v}_{\mathfrak{D}}'\|_2^2 &\leq C\big(\|\tilde{v}_{\mathfrak{D}}\|_2\big[\|f\|_2 + \lambda_\beta^{-3/4}\beta^{1/2}\|\phi\|_{1,2}\big] \\ &\quad + \lambda_\beta^{-1/2}(\beta^{2/3}\|\phi\|_{1,2}^2 + \beta^{-1}\|f\|_2^2) + (\mu_{\beta,+}+1)\,\|\tilde{v}_{\mathfrak{D}}\|_2^2\big).\end{aligned} \tag{5.4.22}$$

Hence, for $|\lambda| > \beta^{-1/5}$ we can conclude that

$$\begin{aligned}\|\tilde{v}_{\mathfrak{D}}'\|_2^2 &\leq C\,\big[\|\tilde{v}_{\mathfrak{D}}\|_2(\|f\|_2 + \beta^{1/4}|\lambda|^{-3/4}\|\phi\|_{1,2}) \\ &\quad + (\mu_{\beta,+}+1)\|\tilde{v}_{\mathfrak{D}}\|_2^2 + |\lambda|^{-1/2}\beta^{1/2}\|\phi\|_{1,2}^2 + |\lambda|^{-1/2}\beta^{-7/6}\|f\|_2^2\big].\end{aligned} \tag{5.4.23}$$

Step 5: Estimate of $\|\tilde{v}_{\mathfrak{D}}\|$. Given that $\nu < \nu_0$, we may apply (3.3.10), (3.1.3a), and Hardy's inequality (2.6.13), to (5.4.7) to obtain

$$\|\tilde{v}_{\mathfrak{D}}\|_2 \leq C\big(\|\phi'\|_2 + \beta^{1/6}|\phi(x_\nu)| + \beta^{-2/3}[\|f\|_2 + |\phi''(1)|\,\|\hat{g}\|_2]\big). \tag{5.4.24}$$

Using again (5.4.18) and (5.4.14) we obtain

$$\|\tilde{v}_{\mathfrak{D}}\|_2 \leq$$
$$C\big(\|\phi'\|_2 + \beta^{1/6}|\phi(x_\nu)| + \beta^{-2/3}\big[\|f\|_2 + \lambda_\beta^{-3/4}\big(\beta^{1/2}\|\phi\|_{1,2} + \beta^{-1/3}\|f\|_2\big)\big]\big),$$

which implies for any $\nu < \nu_0$

$$\|\tilde{v}_{\mathfrak{D}}\|_2 \leq C\big(\|\phi\|_{1,2} + \beta^{1/6}|\phi(x_\nu)| + \beta^{-2/3}\|f\|_2\big). \tag{5.4.25}$$

Step 6: Estimate of $\|g_{\mathfrak{D}}\|_2$. Substituting (5.4.25) into (5.4.22) yields that

$$\begin{aligned}\|\tilde{v}_{\mathfrak{D}}'\|_2 \leq C\,\big[&(\beta^{1/3}\lambda_\beta^{-1/4} + 1)\|\phi\|_{1,2} + \beta^{1/6}|\phi(x_\nu)| \\ &+ (1 + \mu_{\beta,+}^{1/2}\beta^{-2/3})\|f\|_2 + \mu_{\beta,+}^{1/2}(\beta^{1/6}|\phi(x_\nu| + \|\phi'\|_2))\big].\end{aligned} \tag{5.4.26}$$

For $|\lambda| \geq \beta^{-1/5}$ and $\mu \leq \beta^{-2/5}$ we conclude from (5.4.26) that

$$\begin{aligned}\|\tilde{v}_{\mathfrak{D}}'\|_2 \leq C\,\big[&(\beta^{1/4}|\lambda|^{-1/4} + 1)\|\phi\|_{1,2} + \beta^{1/6}|\phi(x_\nu)| + \|f\|_2 \\ &+ \mu_{\beta,+}^{1/2}(\beta^{1/6}|\phi(x_\nu|) + \|\phi'\|_2)\big].\end{aligned}$$

Furthermore, as

$$(U''\phi)'' = -U''\big(\tilde{v}_{\mathfrak{D}} + \alpha^2\phi + |\phi''(1)|\,\hat{\psi}\big) + 2U^{(3)}\phi' + U^{(4)}\phi,$$

we obtain from (5.4.25), the boundedness of α, (5.4.14), and (4.6.5), that

$$\|(U''\phi)''\|_2 \leq C\lambda_\beta^{\frac{1}{4}}(\beta^{1/6}\|\phi\|_{1,2} + \beta^{-2/3}\|f\|_2). \tag{5.4.27}$$

For $|\lambda| \geq \beta^{-1/5}$ the above inequality implies

$$\|(U''\phi)''\|_2 \leq C\,|\lambda|^{1/4}\beta^{1/4}(\|\phi\|_{1,2} + \beta^{-5/6}\|f\|_2). \tag{5.4.28}$$

By (5.4.4b) it holds that

$$\|g_{\mathfrak{D}}\| \leq C\big(\|(U + i\lambda)f\|_2 + |\phi''(1)|\,\|(U + i\lambda)\hat{g}\| + \|((U''\phi)'')\| + \|\tilde{v}_{\mathfrak{D}}'\| + \|\tilde{v}_{\mathfrak{D}}\|\big). \tag{5.4.29}$$

We now substitute into (5.4.29) the estimates (5.4.15) and (5.4.19), (5.4.28), (5.4.26), and (5.4.23), to obtain that

$$\begin{aligned}\|g_{\mathfrak{D}}\|_2 \leq C\big[&(1 + |\lambda|)\,\|f\|_2 + \mu_{\beta,+}^{1/2}(\beta^{1/6}|\phi(x_\nu)| + \|\phi\|_{1,2}) \\ &+ \beta^{1/4}(|\lambda|^{-1/4} + |\lambda|^{1/4})\,\|\phi\|_{1,2}\big].\end{aligned} \tag{5.4.30}$$

Step 7: Estimate the contribution of the boundary term $\mathcal{A}_{\lambda,\alpha}^{-1}([U+i\lambda]\phi''(1)\widehat{\psi})$. We continue as in the proof of [3, Lemma 8.8]. We first write, in view of (5.4.2)

$$\phi = \phi_{\mathfrak{D}} + \check{\phi}, \tag{5.4.31a}$$

where

$$\phi_{\mathfrak{D}} = \mathcal{A}_{\lambda,\alpha}^{-1} v_{\mathfrak{D}}; \quad \check{\phi} = -\mathcal{A}_{\lambda,\alpha}^{-1}\big([U+i\lambda]\phi''(1)\widehat{\psi}\big). \tag{5.4.31b}$$

Note here that

$$\check{\phi} = -\phi''(1)\phi_{\lambda,\beta,\alpha}. \tag{5.4.31c}$$

By (5.2.5) and (5.4.15) we have

$$\|\check{\phi}\|_{1,2} \leq C\,|\nu|^{-1}\beta^{-1/4}|\lambda|^{-1/4}\big(\|\phi\|_{1,2} + \beta^{-5/6}\|f\|_2\big). \tag{5.4.32}$$

Consequently, as $|\nu| > \beta^{-1/5+\delta}$, we obtain for sufficiently large β_0

$$\|\check{\phi}\|_{1,2} \leq C\,|\nu|^{-1}\beta^{-1/4}|\lambda|^{-1/4}\big(\beta^{-5/6}\|f\|_2 + \|\phi_{\mathfrak{D}}\|_{1,2}\big). \tag{5.4.33}$$

Furthermore, by (5.2.6) and (5.4.31c), we have

$$|\check{\phi}(x_\nu)| \leq C[|\lambda|\beta]^{-3/4}|\phi''(1)|.$$

Hence, by (5.4.15) we obtain

$$|\check{\phi}(x_\nu)| \leq C[|\lambda|\beta]^{-1/4}\big(\|\phi\|_{1,2} + \beta^{-5/6}\|f\|_2\big),$$

and, therefore, by equation (5.4.33) and Poincaré's inequality we may conclude for $\nu \geq \beta^{-1/5+\delta}$ and β_0 large enough

$$|\check{\phi}(x_\nu)| \leq C[|\lambda|\beta]^{-1/4}\big(\|\phi_{\mathfrak{D}}'\|_2 + \beta^{-5/6}\|f\|_2\big). \tag{5.4.34}$$

Step 8: Estimate $\phi_{\mathfrak{D}}$. Substituting the above into (5.4.30) yields, with the aid of (5.4.31), (5.4.33), and (5.4.34)

$$\begin{aligned}\|g_{\mathfrak{D}}\|_2 \leq C\big[&(1+|\lambda|)\|f\|_2 + \mu_{\beta,+}^{1/2}(\beta^{1/6}|\phi_{\mathfrak{D}}(x_\nu)| + \|\phi_{\mathfrak{D}}\|_{1,2}) \\ &+ \beta^{1/4}(|\lambda|^{-1/4} + |\lambda|^{1/4})\|\phi_{\mathfrak{D}}\|_{1,2}\big].\end{aligned} \tag{5.4.35}$$

By either (2.6.81) or (2.11.1) (for $|\mu|$ large) applied to the pair $(v_{\mathfrak{D}}, \phi_{\mathfrak{D}})$ together with (5.4.3), it holds for any $1 < q < 2$ that

$$|\phi_{\mathfrak{D}}(x_\nu)| \leq C\,\big[\|v_{\mathfrak{D}}\|_{1,q} + \nu^{-1/2}\|v_{\mathfrak{D}}\|_2 + \nu^{-1}\|v_{\mathfrak{D}}\|_1\big]. \tag{5.4.36}$$

We now estimate $\|\phi_{\mathfrak{D}}\|_{1,2}$ by applying (2.6.1) and (2.6.2) to the pair $(v_{\mathfrak{D}}, \phi_{\mathfrak{D}})$. We then conclude that there exists $\mu_0 > 0$ such that for all $|\mu| \leq \mu_0$ and $0 < \nu \leq \nu_0$ it holds that

$$\|\phi_{\mathfrak{D}}\|_{1,2} \leq C\,\nu^{-1} n_{\mathfrak{D}}, \tag{5.4.37}$$

where

$$n_D := (\|v'_{\mathfrak{D}}\|_q + \nu^{-1/2}\|v_{\mathfrak{D}}\|_2 + \nu^{-1}\|v_{\mathfrak{D}}\|_1).$$

For $|\mu| > \mu_0$ we use (2.11.1) to obtain for $0 < \nu \leq \nu_0$

$$\|\phi_{\mathfrak{D}}\|_{1,2} \leq C\|(1-x)^{1/2}v_{\mathfrak{D}}\|_1 \leq C\,\nu^{-1}n_{\mathfrak{D}}. \tag{5.4.38}$$

We can then substitute (5.4.36) and (5.4.37) or (5.4.38) into (5.4.35) to obtain for any $1 < q < 2$

$$\|g_{\mathfrak{D}}\|_2 \leq C\,\big([1+|\lambda|]\|f\|_2 + [\mu_{\beta,+}^{1/2}[\beta^{1/6}+\nu^{-1}] + \beta^{1/4}\nu^{-1}(|\lambda|^{-1/4}+|\lambda|^{1/4})]\,n_{\mathfrak{D}}\big). \tag{5.4.39}$$

Step 9: Estimate $n_{\mathfrak{D}}$. By either (3.1.3a) for $-\beta^{-1/3} \leq \mu < \beta^{-2/5-\delta}$ or (3.1.84) for $\mu < -\beta^{-1/3}$ we have,

$$\|v_{\mathfrak{D}}\|_2 \leq \frac{C}{\beta^{\frac{2}{3}} + |\mu|\beta}\|g_{\mathfrak{D}}\|_2, \tag{5.4.40}$$

whereas by (3.1.3b), which holds for $\mu \leq C\beta^{-1/3}$ and β_0 large, we have for all $\beta > \beta_0$

$$\|v'_{\mathfrak{D}}\|_q \leq C_q\,\beta^{-\frac{2+q}{6q}}\|g_{\mathfrak{D}}\|_2. \tag{5.4.41}$$

Furthermore, by (3.3.1) it holds that

$$\|v_{\mathfrak{D}}\|_1 \leq C_q\,\beta^{-5/6}\|g_{\mathfrak{D}}\|_2. \tag{5.4.42}$$

Substituting (5.4.40), (5.4.41), and (5.4.42) into (5.4.39) yields, for $\delta \leq 1/30$, β_0 large enough, and q satisfying

$$1 < q < \frac{4}{4-15\delta}, \tag{5.4.43}$$

we obtain the existence of $C > 0$ such that for all $\beta \geq \beta_0$

$$\|g_{\mathfrak{D}}\|_2 \leq C(1+|\lambda|)\|f\|_2. \tag{5.4.44}$$

We now use (5.4.40) to obtain

$$\|v_{\mathfrak{D}}\|_2 \leq C\,\frac{1+|\lambda|}{\beta^{\frac{2}{3}}+|\mu|\beta}\|f\|_2 \leq \hat{C}\,\beta^{-2/3}\|f\|_2, \tag{5.4.45}$$

which is valid for all $\mu < \beta^{-\frac{2}{5}-\delta}$.

We next use (5.4.41) and (5.4.39) to obtain

$$\|v'_{\mathfrak{D}}\|_q \leq C\beta^{-\frac{2+q}{6q}}(1+|\lambda|)\|f\|_2. \tag{5.4.46}$$

Finally, use of (5.4.42) and (5.4.39) yields

$$\|v_{\mathfrak{D}}\|_1 \leq C\beta^{-5/6}(1+|\lambda|)\|f\|_2. \tag{5.4.47}$$

For $-\mu_0 \leq \mu < \beta^{-\frac{2}{5}-\delta}$ we have by (5.4.45), (5.4.46), and (5.4.47) that the dominant term is the one involving $\|v'_{\mathfrak{D}}\|_q$ and hence

$$n_{\mathfrak{D}} \leq C\,\beta^{-\frac{2+q}{6q}}\,\|f\|_2. \tag{5.4.48}$$

Step 10: Prove (5.4.1). By (5.4.37) we obtain, for $-\mu_0 \leq \mu \leq \beta^{-\frac{2}{5}-\delta}$,

$$\|\phi_{\mathfrak{D}}\|_{1,2} \leq C\nu^{-1}\,\beta^{-\frac{2+q}{6q}}\,\|f\|_2. \tag{5.4.49}$$

For $\mu < -\mu_0$ we use (2.11.1) and the first inequality in (5.4.45) to obtain

$$\|\phi_{\mathfrak{D}}\|_{1,2} \leq C\,\|v_{\mathfrak{D}}\|_2 \leq \hat{C}\,\beta^{-1}\|f\|_2. \tag{5.4.50}$$

Combining (5.4.33) with (5.4.49) and (5.4.50) yields

$$\|\phi\|_{1,2} \leq C\nu^{-1}\,\beta^{-\frac{2+q}{6q}}\,\|f\|_2.$$

■

5.5 Resolvent estimates for $|\Im\lambda| = \mathcal{O}(\beta^{-1/5})$ and large $\beta^{1/10}\alpha$

In the previous section, we considered the case where $\Im\lambda \gg \beta^{-1/5}$, where the inverse estimates derived in Section 2.6 for the Rayleigh operator $\mathcal{A}_{\lambda,\alpha}$ become effective for all $\alpha \geq 0$. Here, we use the estimates obtained in Section 2.5 assuming $\alpha^2 \gg |\Im\lambda|$. Since we consider here $|\Im\lambda| = \mathcal{O}(\beta^{-1/5})$ we need to consider $\alpha^2 \gg \beta^{-1/5}$.

Proposition 5.5.1. *Let $U \in C^4([0,1])$ satisfy* (2.1.3) *and $U'''(0) = 0$. Let further $0 < \delta < 1/15$ and α_0 denote positive constants. There exist $C > 0$, $\beta_0 > 0$, and $a_0 > 0$ such that for all $\beta \geq \beta_0$, it holds that*

$$\sup_{\substack{a_0\beta^{-1/10+\delta/2}\leq\alpha\leq\alpha_0\\ \Re\lambda\leq\beta^{-1/3-\delta}\\ |\Im\lambda|\leq\beta^{-1/5+\delta}}} \Big[\big\|(\mathcal{B}^{\mathfrak{N},\mathfrak{D}}_{\lambda,\alpha,\beta})^{-1}\big\| + \Big\|\frac{d}{dx}(\mathcal{B}^{\mathfrak{N},\mathfrak{D}}_{\lambda,\alpha,\beta})^{-1}\Big\|\Big] \leq C\,\beta^{-1/2+\delta}. \tag{5.5.1}$$

Proof. Step 1. With $g_{\mathfrak{D}}$ given by (5.4.4b), we prove that

$$\|g_{\mathfrak{D}}\|_2 \leq C\big[(1+|\lambda|)\,\|f\|_2 + (\lambda_\beta^{-1/4}\beta^{1/3} + \beta^{1/3-\delta/2} + \lambda_\beta^{1/4}\beta^{1/6})\|\phi\|_{1,2} + \beta^{1/2-\delta/2}|\phi(x_\nu)|\big]. \tag{5.5.2}$$

Let $\phi \in D(\mathcal{B}^{\mathfrak{N},\mathfrak{D}}_{\lambda,\alpha,\beta})$, $f = \mathcal{B}^{\mathfrak{N},\mathfrak{D}}_{\lambda,\alpha,\beta}\,\phi$ and $v_{\mathfrak{D}} \in H^2(0,1)$ defined by (5.4.2). As before we write

$$(\mathcal{L}^{\mathfrak{N}}_\beta - \beta\lambda)v_{\mathfrak{D}} = g_{\mathfrak{D}},$$

Let $\tilde{v}_{\mathfrak{D}}$ be given by (5.4.6). For the convenience of the reader we repeat here (5.4.25)

$$\|\tilde{v}_{\mathfrak{D}}\|_2 \leq C\big(\|\phi\|_{1,2} + \beta^{1/6}|\phi(x_\nu)| + \beta^{-2/3}\|f\|_2\big),$$

and (5.4.26), which reads

$$\|\tilde{v}_{\mathfrak{D}}'\|_2 \leq C\,\big[(\beta^{1/3}\lambda_\beta^{-1/4}+1)\|\phi\|_{1,2}+\beta^{1/6}|\phi(x_\nu)| \\ +(1+\mu_{\beta,+}^{1/2}\beta^{-2/3})\|f\|_2+\mu_{\beta,+}^{1/2}(\beta^{1/6}|\phi(x_\nu|+\|\phi\|_{1,2}))\big].$$

For $\mu<\beta^{-1/3-\delta}$ with $\delta<1/10$, we then obtain using Poincaré's inequality

$$\|\tilde{v}_{\mathfrak{D}}'\|_2 \leq C\,\big[(\beta^{1/3}\lambda_\beta^{-1/4}+\beta^{1/3-\delta/2})\|\phi\|_{1,2}+\|f\|_2+\beta^{1/2-\delta/2}|\phi(x_\nu)|\big]. \quad (5.5.3)$$

We next estimate $\|g_{\mathfrak{D}}\|_2$, beginning by repeating (5.4.29), which states

$$\|g_{\mathfrak{D}}\|\leq C(\|(U+i\lambda)f\|_2+|\phi''(1)|\,\|(U+i\lambda)\hat{g}\|+\|(U''\phi)''\|+\|\tilde{v}_{\mathfrak{D}}'\|+\|\tilde{v}_{\mathfrak{D}}\|).$$

By (5.4.14), (5.4.18), (5.4.27), (5.5.3), and (5.4.25) it holds, that

$$\|g_{\mathfrak{D}}\|_2\leq C\big[(1+|\lambda|)\,\|f\|_2 \\ +(\lambda_\beta^{-1/4}\beta^{1/3}+\beta^{1/3-\delta/2}+\lambda_\beta^{1/4}\beta^{1/6})\|\phi\|_{1,2}+(\beta^{1/2-\delta/2}|\phi(x_\nu)|)\big],$$

which is precisely (5.5.2).

Step 2: We estimate $\|v_{\mathfrak{D}}\|_{1,q}$ and $\|v_{\mathfrak{D}}\|_1$. By (5.5.2) it holds for $\mu\geq -1$ that

$$\|g_{\mathfrak{D}}\|_2\leq C\,\big[\|f\|_2+\beta^{1/3}\|\phi\|_{1,2}+(\beta^{1/2-\delta/2}|\phi(x_\nu)|)\big]. \quad (5.5.4)$$

By (3.1.3b) we have, for any $1<q<2$ and given that $|\nu|<\beta^{-1/5+\delta}$ and $\delta<1/5$,

$$\|v_{\mathfrak{D}}\|_{1,q}\leq C\beta^{-\frac{2+q}{6q}}\|g_{\mathfrak{D}}\|_2.$$

Hence, by (5.5.4) we obtain for $\mu\geq -1$,

$$\|v_{\mathfrak{D}}\|_{1,q}\leq C([\beta^{-\frac{2+q}{6q}}\|f\|_2+\beta^{-\frac{2-q}{6q}}\|\phi\|_{1,2}+\beta^{\delta_1(q)}|\phi(x_\nu)|]), \quad (5.5.5)$$

where $\delta_1(q)=(q-1)/3q-\delta/2$.

Furthermore, we have by (3.3.1) and (5.5.4), for $\mu\geq -1$,

$$\|v_{\mathfrak{D}}\|_1\leq C\beta^{-5/6}\|g_{\mathfrak{D}}\|_2\leq \hat{C}\big([\beta^{-5/6}\|f\|_2+\beta^{-1/2}\|\phi\|_{1,2}+\beta^{-1/3-\delta/2}|\phi(x_\nu)|]\big). \quad (5.5.6)$$

For $\mu\leq -1$ we use (3.1.84) to obtain

$$\|v_{\mathfrak{D}}\|_1\leq\|v_{\mathfrak{D}}\|_2\leq\frac{C}{|\lambda|\beta}\|g_{\mathfrak{D}}\|_2.$$

We then employ (5.5.2) which implies, for $\delta\leq 1/10$,

$$\|g_{\mathfrak{D}}\|_2\leq C\big[(1+|\lambda|)\,\|f\|_2+(\beta^{1/3-\delta/2}+|\lambda|^{1/4}\beta^{1/4})\|\phi\|_{1,2}+(\beta^{1/2-\delta/2}|\phi(x_\nu)|)\big]$$

and hence,

$$\|v_{\mathfrak{D}}\|_1 \leq C\big[\beta^{-1}\|f\|_2 + \beta^{-2/3-\delta/2}\|\phi\|_{1,2} + \beta^{-1/2-\delta/2}|\phi(x_\nu)|\big]. \tag{5.5.7}$$

Combining (5.5.6) and (5.5.7) yields

$$\|v_{\mathfrak{D}}\|_1 \leq C\,\big([\beta^{-5/6}\|f\|_2 + \beta^{-1/2}\|\phi\|_{1,2} + \beta^{-1/3-\delta/2}|\phi(x_\nu)|]\big). \tag{5.5.8}$$

Step 3. We prove (5.5.1). We continue as in the proof of Proposition 5.4.1. Recall from (5.4.2) that

$$v_{\mathfrak{D}} = \mathcal{A}_{\lambda,\alpha}\phi + (U + i\lambda)\phi''(1)\widehat{\psi}.$$

Set

$$\phi = \phi_{\mathfrak{D}} + \breve{\phi}, \tag{5.5.9}$$

where

$$\phi_{\mathfrak{D}} = \mathcal{A}_{\lambda,\alpha}^{-1}v_{\mathfrak{D}} \quad \text{and} \quad \breve{\phi} = -\mathcal{A}_{\lambda,\alpha}^{-1}\big([U + i\lambda]\phi''(1)\widehat{\psi}\big).$$

Step 3a: We estimate $\|\breve{\phi}\|_{1,2}$. Note that by (5.2.4), we have

$$\breve{\phi} = -\phi''(1)\phi_{\lambda,\beta,\alpha}. \tag{5.5.10}$$

By (5.2.24) there exist $C > 0$ and $\lambda_0 > 0$, and for any $a_0 > \|U\|_2^{-1}(1+\delta)^{1/2}$, $\beta(a_0) > 0$ such that, for $-\lambda_0 < \mu$, $|\nu| \leq \beta^{-1/5+\delta}$, $\alpha \geq a_0\beta^{-1/10+\delta/2}$, and $\beta \geq \beta(a_0)$, we have

$$\|\breve{\phi}\|_{1,2} \leq Ca_0^{-2}\beta^{-1/3}\lambda_\beta^{-1/2}|\phi''(1)|. \tag{5.5.11}$$

For $\mu \leq -\lambda_0 < 0$ we obtain from (2.11.3) applied to the pair $(\breve{\phi}, (U + i\lambda)\phi''(1)\widehat{\psi})$

$$\|\breve{\phi}'\|_2^2 \leq C\,|\langle\breve{\phi}, \widehat{\psi}\rangle\,\phi''(1)| \leq C\,\|\breve{\phi}'\|_2\|(1-x)^{1/2}\widehat{\psi}\|_1\,|\phi''(1)|,$$

and hence by (5.2.2) (with $s = 1/2$) for the first inequality, we conclude for the second inequality that there exists $\beta(a_0) > 0$ such that for $\beta \geq \beta(a_0)$

$$\|\breve{\phi}'\|_2 \leq C\,\lambda_\beta^{-3/4}\beta^{-1/2}|\phi''(1)| \leq \widehat{C}\,a_0^{-2}\beta^{-1/3}\lambda_\beta^{-1/2}|\phi''(1)|.$$

Hence, (5.5.11) holds true for any $\mu < \beta^{-1/3+\delta}$ and $|\nu| \leq \beta^{-1/5+\delta}$. From (5.4.14) we then deduce

$$\|\breve{\phi}\|_{1,2} \leq Ca_0^{-2}\beta^{-1/3}(\beta^{1/3}\|\phi\|_{1,2} + \beta^{-1/2}\|f\|_2) \leq \widehat{C}(\beta^{-5/6}\|f\|_2 + a_0^{-2}\|\phi\|_{1,2}).$$

Hence, for sufficiently large a_0 we conclude with the aid of (5.5.9)

$$\|\breve{\phi}\|_{1,2} \leq C\,\big(\beta^{-5/6}\|f\|_2 + a_0^{-2}\|\phi_{\mathfrak{D}}\|_{1,2}\big). \tag{5.5.12}$$

Step 3b: We estimate $|\check{\phi}(x_\nu)|$. Using (5.5.10), the decomposition

$$\phi_{\lambda,\beta,\alpha} = (\phi_{\lambda,\beta,\alpha} - c_{\|}(\lambda,\beta,\alpha)U) + c_{\|}(\lambda,\beta,\alpha)U,$$

where

$$c_{\|}(\lambda,\beta,\alpha) = \frac{\langle U, \phi_{\lambda,\beta,\alpha}\rangle}{\|U\|_2^2},$$

and Hölder's inequality, we may write

$$|\check{\phi}(x_\nu)| \le C\big(|\nu|\,|c_{\|}(\lambda,\beta,\alpha)| + |\nu|^{1/2}\big\|\phi_{\lambda,\beta,\alpha} - c_{\|}(\lambda,\beta,\alpha)U\big\|_{1,2}\big)\,|\phi''(1)|. \quad (5.5.13)$$

Then, we obtain by (5.2.24) and the fact that $\alpha \ge a_0\beta^{-1/10+\delta/2}$

$$|\check{\phi}(x_\nu)| \le$$
$$C\Big[|\nu|^{1/2}\lambda_\beta^{-3/4}\beta^{-1/2} + \frac{|\mu|\nu^{1/2} + |\nu|}{a_0^2\beta^{-1/5+\delta} + |\mu|}\big(\lambda_\beta^{-1}\beta^{-2/3} + \lambda_\beta^{-1/2}|\lambda|\beta^{-1/3}\big)\Big]|\phi''(1)|. \quad (5.5.14)$$

Using (5.4.14) and the fact that $|\lambda| \le \beta^{-1/3}\lambda_\beta$ we obtain

$$|\check{\phi}(x_\nu)| \le C\big(a_0^{-2}\beta^{-1/3}\lambda_\beta + |\nu|^{1/2}\lambda_\beta^{-1/4}\beta^{-1/6}\big)\big(\|\phi\|_{1,2} + \beta^{-5/6}\|f\|_2\big). \quad (5.5.15)$$

We now consider three different cases.

- For $-\beta^{-1/5+\delta} < \mu < \beta^{-1/3+\delta}$, we have $\lambda_\beta \lesssim \beta^\delta$ and since $|\nu| \le |\lambda|$, we deduce from (5.5.15)

$$|\check{\phi}(x_\nu)| \le C\beta^{-1/5+\delta}\big(\|\phi\|_{1,2} + \beta^{-5/6}\|f\|_2\big). \quad (5.5.16)$$

- For $-\beta^{-1/10+\delta/2} < \mu < -\beta^{-1/5+\delta}$ we have, since $|\nu| \le |\mu|$,

$$\frac{|\mu|\,|\nu|^{1/2} + |\nu|}{a_0^2\beta^{-1/5+\delta} + |\mu|}|\lambda| \le C(|\mu|\,|\nu|^{1/2} + |\nu|) \le \hat{C}\beta^{-1/5+\delta},$$

and hence, as $\lambda_\beta \sim |\lambda|\beta^{1/3}$, we can conclude (5.5.16) in this case as well.

- Finally, for $\mu \le -\beta^{-1/10+\delta/2}$, we use (2.11.2) with $v = (U + i\lambda)\hat{\psi}$, (5.4.31c), and (5.4.14) to obtain that

$$\|\check{\phi}'\|_2 \le C|\mu|^{-1}\|(1-x)^{1/2}\hat{\psi}\|_1|\phi''(1)| \le \hat{C}|\lambda|^{-5/4}\beta^{-1/4}\big(\|\phi\|_{1,2} + \beta^{-5/6}\|f\|_2\big),$$

which implies

$$|\check{\phi}(x_\nu)| \le C|\nu|^{1/2}\|\check{\phi}'\|_2 \le \hat{C}\beta^{-9/40-\delta/8}\big(\|\phi\|_{1,2} + \beta^{-5/6}\|f\|_2\big). \quad (5.5.17)$$

Combining (5.5.17) with (5.5.16) which holds in the two first cases yields for all $\mu < \beta^{-1/3-\delta}$

$$|\breve{\phi}(x_\nu)| \leq C\beta^{-1/5+\delta}\big(\|\phi\|_{1,2} + \beta^{-5/6}\|f\|_2\big). \tag{5.5.18}$$

Then, by (5.5.9) and (5.5.18) it holds that

$$|\breve{\phi}(x_\nu)| \leq C\beta^{-1/5+\delta}\big(\beta^{-5/6}\|f\|_2 + \|\phi_{\mathfrak{D}}\|_{1,2}\big). \tag{5.5.19}$$

Step 3c: We prove (5.5.1) *for* $\mu > -\beta^{-\delta}$. From (5.5.5) and (5.5.19) we get, for $\delta \leq 1/15$ and $\mu > -1$, that

$$\|v_{\mathfrak{D}}\|_{1,q} \leq C\,\big(\beta^{-\frac{2+q}{6q}}\|f\|_2 + \beta^{-\frac{2-q}{6q}}\|\phi\|_{1,2} + \beta^{\delta_1(q)}|\phi_{\mathfrak{D}}(x_\nu)|\big).$$

Using (5.5.12) we then obtain that

$$\|v_{\mathfrak{D}}\|_{1,q} \leq C\big(\beta^{-\frac{2+q}{6q}}\|f\|_2 + \beta^{-\frac{2-q}{6q}}\|\phi_{\mathfrak{D}}\|_{1,2} + \beta^{\delta_1(q)}|\phi_{\mathfrak{D}}(x_\nu)|\big).$$

As

$$-\frac{2-q}{6q} + \frac{1}{6} = \frac{1}{3} - \frac{1}{3q} = \delta_1(q) + \delta/2,$$

we can finally conclude, for $\mu > -1$, that

$$\|v_{\mathfrak{D}}\|_{1,q} \leq C\,\big(\beta^{-\frac{2+q}{6q}}\|f\|_2 + \beta^{-\frac{2-q}{6q}}[\|\phi_{\mathfrak{D}}\|_{1,2} + \beta^{1/6}|\phi_{\mathfrak{D}}(x_\nu)|]\big). \tag{5.5.20}$$

From (5.5.8), (5.5.12), and (5.5.19) we obtain

$$\|v_{\mathfrak{D}}\|_1 \leq C\,\big(\beta^{-5/6}\|f\|_2 + \beta^{-1/2}\|\phi_{\mathfrak{D}}\|_{1,2} + \beta^{-1/3-\delta/2}|\phi_{\mathfrak{D}}(x_\nu)|\big). \tag{5.5.21}$$

As in the proof of (5.5.13) we then write

$$|\phi_{\mathfrak{D}}(x_\nu)| \leq C\,\big(|\nu|\,|c_\parallel^{\mathfrak{D}}| + |\nu|^{1/2}\|\phi_{\mathfrak{D}} - c_\parallel^{\mathfrak{D}}U\|_{1,2}\big), \tag{5.5.22}$$

where

$$c_\parallel^{\mathfrak{D}} = \langle U, \phi_{\mathfrak{D}}\rangle/\|U\|_2^2.$$

We then conclude from (2.5.3a) applied to the pair $(\phi_{\mathfrak{D}}, v_{\mathfrak{D}})$ that for all $-\beta^{-\delta} < \mu$ it holds that

$$|\phi_{\mathfrak{D}}(x_\nu)| \leq C\,\big(\|v_{\mathfrak{D}}\|_1 + |\lambda|N_1(v_{\mathfrak{D}}, \lambda) + |\nu|^{1/2}\|v_{\mathfrak{D}}\|_{1,q}\big), \tag{5.5.23}$$

where $N_1(v_{\mathfrak{D}}, \lambda)$ is defined in (2.5.2b).

By (2.5.38) (and the fact that $v_{\mathfrak{D}}(1) = 0$) we then obtain that

$$|\phi_{\mathfrak{D}}(x_\nu)| \leq C\,\big(\|v_{\mathfrak{D}}\|_1 + \beta^{-\delta}\|v_{\mathfrak{D}}\|_{1,q}\big). \tag{5.5.24}$$

Substituting (5.5.20) and (5.5.21) into (5.5.24) then yields, for $\mu > -\beta^{-\delta}$, sufficiently large β_0, and $1 < q < (1-3\delta)^{-1}$

$$|\phi_{\mathfrak{D}}(x_\nu)| \leq C\beta^{-\delta}\left(\beta^{-\frac{2+q}{6q}}\|f\|_2 + \beta^{-\frac{2-q}{6q}}\|\phi_{\mathfrak{D}}\|_{1,2}\right). \tag{5.5.25}$$

Substituting (5.5.25) into (5.5.20) then leads to

$$\|v_{\mathfrak{D}}\|_{1,q} \leq C\left(\beta^{-\frac{2+q}{6q}}\|f\|_2 + \beta^{-\frac{2-q}{6q}}\|\phi_{\mathfrak{D}}\|_{1,2}\right). \tag{5.5.26}$$

Similarly, by substituting (5.5.25) into (5.5.21) we obtain that for $\mu > -\beta^{-\delta}$ and $1 < q < (1-3\delta)^{-1}$

$$\|v_{\mathfrak{D}}\|_1 \leq C(\beta^{-5/6}\|f\|_2 + \beta^{-1/2}\|\phi_{\mathfrak{D}}\|_{1,2}). \tag{5.5.27}$$

For sufficiently large a_0 and $|\nu| \leq \beta^{-1/5+\delta}$ we have

$$\frac{1 + C|\lambda|^2\log|\lambda|^{-1}}{|\alpha^2\|U\|_2^2 + i\lambda|} \leq \widetilde{C}\frac{1}{|\alpha^2\|U\|_2^2 - \nu|} \leq \widehat{C}\beta^{1/5-\delta},$$

and (recall that $|\nu| < |\alpha^2\|U\|_2^2 - \nu|$)

$$\frac{|\lambda|(1 + C|\lambda|^2\log|\lambda|^{-1})}{|\alpha^2\|U\|_2^2 + i\lambda|} \leq \widetilde{C}\frac{|\nu| + |\mu|}{|\alpha^2\|U\|_2^2 - \nu| + |\mu|} \leq \widehat{C}.$$

Hence, we obtain from (2.5.3a) and (2.5.38) that

$$|c_{\|}^{\mathfrak{D}}| \leq C\left[\beta^{1/5-\delta}\|v_{\mathfrak{D}}\|_1 + \|v_{\mathfrak{D}}\|_{1,q}\right],$$

and from (2.5.3b) we obtain that

$$\|\phi_{\mathfrak{D}} - c_{\|}^{\mathfrak{D}}U\|_{1,2} \leq C[\|v_{\mathfrak{D}}\|_1 + \|v_{\mathfrak{D}}\|_{1,q}].$$

Consequently, by (5.5.26) and (5.5.27) it holds that

$$\|\phi_{\mathfrak{D}}\|_{1,2} \leq C(\|v_{\mathfrak{D}}\|_{1,q} + \beta^{1/5}\|v_{\mathfrak{D}}\|_1) \leq \widehat{C}\left(\beta^{-\frac{2+q}{6q}}\|f\|_2 + \beta^{-\frac{2-q}{6q}}\|\phi_{\mathfrak{D}}\|_{1,2}\right).$$

It follows that, for sufficiently large β_0,

$$\|\phi_{\mathfrak{D}}\|_{1,2} \leq C\beta^{-\frac{2+q}{6q}}\|f\|_2. \tag{5.5.28}$$

Combining (5.5.28) with (5.5.12) gives (note that $5/6 > (2+q)/(6q)$)

$$\|\check{\phi}\|_{1,2} \leq C\,\beta^{-\frac{2+q}{6q}}\|f\|_2. \tag{5.5.29}$$

As $\phi = \phi_{\mathfrak{D}} + \check{\phi}$, we can deduce from (5.5.28) and (5.5.29) that for any $\delta \in (0, 1/15)$ and $q \in (1, (1-3\delta)^{-1})$ and $\mu > -\beta^{-\delta}$ we have

$$\|\phi\|_{1,2} \leq C\,\beta^{-\frac{2+q}{6q}}\|f\|_2 \leq C\beta^{1/2-\delta}. \tag{5.5.30}$$

Step 3d. We prove (5.5.1) *for* $\mu \leq -\beta^{-\delta}$. For $\mu \leq -\beta^{-\delta}$ we use (2.11.2), Sobolev embeddings, and Poincaré's inequality to obtain that

$$\|\phi_{\mathfrak{D}}\|_{1,2} \leq \frac{C}{|\mu|}\|\phi_{\mathfrak{D}}\|_\infty^{1/2}\|v_{\mathfrak{D}}\|_1^{1/2} \leq \hat{C}\,\beta^\delta\|\phi'_{\mathfrak{D}}\|_2^{1/2}\|v_{\mathfrak{D}}\|_1^{1/2},$$

which implies

$$\|\phi_{\mathfrak{D}}\|_{1,2} \leq C\beta^{2\delta}\|v_{\mathfrak{D}}\|_1. \tag{5.5.31}$$

Consequently, by (5.5.8), (5.5.31), and Sobolev embeddings we obtain that

$$\|\phi_{\mathfrak{D}}\|_{1,2} \leq C\beta^{2\delta}\left(\beta^{-5/6}\|f\|_2 + \beta^{-1/3-\delta/2}\|\phi\|_{1,2}\right).$$

Making use of (5.5.12) we establish that for sufficiently large a_0 and β_0

$$\|\phi\|_{1,2} \leq C\,\beta^{2\delta-5/6}\|f\|_2,$$

which, together with (5.5.30), yields (5.5.1) for $\delta < 1/15$. ■

5.6 Resolvent estimates for intermediate α

In this section, we provide inverse estimates for $\mathcal{B}_{\lambda,\alpha,\beta}$ for $1 \ll \alpha \ll \beta^{1/3}$. Let $\mathfrak{z}_\alpha$ be given by (4.7.2). Since $\|\mathfrak{z}'_\alpha\|_2 \ll \beta^{1/6}$ in this section we may conclude by (4.1.3) that $\mathfrak{z}_\alpha \in \mathfrak{U}_1$. Consequently, we may still use (4.2.4) in this section to estimate $\phi''(1)$. Furthermore, we can use the fact that $\alpha \gg 1$ to obtain a much simpler proof than in the previous section (which is valid only for bounded values of α).

Proposition 5.6.1. *Let* $U \in C^4([0,1])$ *satisfy* (2.1.3) *and* $\nu_2 < U(0)$ *denote a positive constant. There exist* $C > 0$, $\Upsilon > 0$, $\beta_0 > 0$, $\alpha_0 > 0$, *and* $\alpha_1 > 0$ *such that for all* $\beta \geq \beta_0$, *it holds that*

$$\sup_{\substack{\alpha_0\leq\alpha\leq\alpha_1\beta^{1/3}\\ \Re\lambda\leq\Upsilon\beta^{-1/3}\\ |\Im\lambda|\leq\nu_2}} \left\|(\mathcal{B}^{\mathfrak{N},\mathfrak{D}}_{\lambda,\alpha,\beta})^{-1}\right\| + \left\|\frac{d}{dx}(\mathcal{B}^{\mathfrak{N},\mathfrak{D}}_{\lambda,\alpha,\beta})^{-1}\right\| \leq C\beta^{-5/6}. \tag{5.6.1}$$

Proof. Let $f \in L^2(0,1)$, $\phi \in D(\mathcal{B}^{\mathfrak{N},\mathfrak{D}}_{\lambda,\alpha,\beta})$ satisfy

$$\mathcal{B}^{\mathfrak{N},\mathfrak{D}}_{\lambda,\alpha,\beta}\phi = f.$$

We first recall the definition of $\tilde{v}_{\mathfrak{D}}$ from (5.4.6)

$$\tilde{v}_{\mathfrak{D}} = -\phi'' + \alpha^2\phi + \phi''(1)\hat{\psi}, \tag{5.6.2}$$

and rewrite (5.4.7) in the form

$$(\mathcal{L}^{\mathfrak{N},\mathfrak{D}}_\beta - \beta\lambda)\tilde{v}_{\mathfrak{D}} = f + i\beta U''\phi + \phi''(1)\hat{g}, \tag{5.6.3}$$

where $\hat{g}$ is given by (5.4.5).

By (3.3.1) (which is applicable for $|\nu| \leq \nu_2 < U(0)$) it holds that

$$\|(\mathcal{L}_\beta^{\mathfrak{N},\mathfrak{D}} - \beta\lambda)^{-1}(f + \phi''(1)\hat{g})\|_1 \leq C\beta^{-5/6}\|(f + \phi''(1)\hat{g})\|_2 .$$

Furthermore, by (3.3.23) applied with f replaced by $U''\phi$ it holds that

$$\Big\|(\mathcal{L}_\beta^{\mathfrak{N},\mathfrak{D}} - \beta\lambda)^{-1}(U''\phi)+i\,\frac{U''(x_\nu)\phi(x_\nu)}{\beta(U-\nu+i\mathfrak{m})}\Big\|_1 \leq C\beta^{-1}\|U''\phi\|_{1,2} \leq \widehat{C}\beta^{-1}\|\phi\|_{1,2},$$

where

$$\mathfrak{m} = -\max(-\mu, x_\nu^{2/3}\beta^{-1/3}).$$

Hence,

$$\Big\|\tilde{v}_{\mathfrak{D}} - \frac{U''(x_\nu)\phi(x_\nu)}{(U-\nu+i\mathfrak{m})}\Big\|_1 \leq C\,\big(\beta^{-5/6}\|f\|_2 + \|\phi\|_{1,2} + \beta^{-5/6}|\phi''(1)|\,\|\hat{g}\|_2\big). \tag{5.6.4}$$

We next use (4.2.4) together with (5.4.13) to obtain, as in (5.4.14), that

$$|\phi''(1)| \leq C\,\lambda_\beta^{1/2}\,\big(\beta^{1/3}\big[\|\phi\|_{1,2} + |\phi(x_\nu)|\log\beta\big] + \beta^{-1/2}\|f\|_2\big). \tag{5.6.5}$$

For $\alpha \leq \alpha_1\beta^{1/3}$ we may then use (5.6.5), which, combined with (5.4.18), yields, as

$$|\phi(x_\nu)| \leq |\nu|^{1/2}\|\phi'\|_2, \tag{5.6.6}$$

$$|\phi''(1)|\,\|\hat{g}\|_2 \leq C\lambda_\beta^{-3/4}\big(\beta^{1/2}[1 + |\nu|^{1/2}\log\beta]\|\phi\|_{1,2} + \beta^{-1/3}\|f\|_2\big). \tag{5.6.7}$$

Consequently, by substituting (5.6.7) into (5.6.4), we obtain

$$\Big\|\tilde{v}_{\mathfrak{D}} - \frac{U''(x_\nu)\phi(x_\nu)}{U-\nu+i\mathfrak{m}}\Big\|_1 \leq C\,\big(\beta^{-5/6}\|f\|_2 + \|\phi\|_{1,2}\big). \tag{5.6.8}$$

Taking the scalar product of (5.6.2) with ϕ, and integrating by parts gives

$$\Big\langle\phi, \tilde{v}_{\mathfrak{D}} - \frac{U''(x_\nu)\phi(x_\nu)}{U-\nu+i\mathfrak{m}}\Big\rangle = \|\phi'\|_2^2 + \alpha^2\|\phi\|_2^2 + \phi''(1)\langle\phi,\hat{\psi}\rangle - \Big\langle\phi, \frac{U''(x_\nu)\phi(x_\nu)}{U-\nu+i\mathfrak{m}}\Big\rangle. \tag{5.6.9}$$

Using (5.6.8) we then conclude

$$\Big|\Big\langle\phi, \tilde{v}_{\mathfrak{D}} - \frac{U''(x_\nu)\phi(x_\nu)}{U-\nu+i\mathfrak{m}}\Big\rangle\Big| \leq C\,\big(\beta^{-5/6}\|f\|_2 + \|\phi\|_{1,2}\big)\,\|\phi\|_\infty. \tag{5.6.10}$$

We next use (5.2.2), with $s = 1/2$, together with (5.6.5) and a Sobolev inequality to obtain that

$$|\phi''(1)\langle\phi,\hat{\psi}\rangle| \leq C\,\lambda_\beta^{-1/4}\,\big(\beta^{-1/6}[\|\phi\|_{1,2} + |\phi(x_\nu)|\log\beta] + \beta^{-1}\|f\|_2\big)\|\phi'\|_2 .$$

Using (5.6.6), as $\lambda_\beta^{-1/4}\nu^{1/2}\log\beta \leq 1$ for $\beta \geq \beta_0$ with sufficiently large β_0, we can conclude that

$$|\phi''(1)\langle\phi,\hat{\psi}\rangle| \leq C\,\big(\beta^{-1/6}\|\phi\|_{1,2}^2 + \beta^{-11/6}\|f\|_2^2\big). \tag{5.6.11}$$

For the last term on the right-hand side of (5.6.9) we write

$$\Big\langle\phi,\frac{U''(x_\nu)\phi(x_\nu)}{U-\nu+i\mathfrak{m}}\Big\rangle=\Big\langle\phi,\frac{U''(x_\nu)\phi(x_\nu)}{U-\nu+i\mathfrak{m}}\Big\rangle_{L^2(0,x_\nu/2)}+\Big\langle\phi,\frac{U''(x_\nu)\phi(x_\nu)}{U-\nu+i\mathfrak{m}}\Big\rangle_{L^2(x_\nu/2,1)}.$$

For the first term on the right-hand side we have, since $|\nu|\leq\nu_2$,

$$\Big|\Big\langle\phi,\frac{U''(x_\nu)\phi(x_\nu)}{U-\nu+i\mathfrak{m}}\Big\rangle_{L^2(0,x_\nu/2)}\Big|\leq C\|\phi\|_2\|\phi\|_\infty. \tag{5.6.12}$$

For the second term we use integration by parts to obtain

$$\begin{aligned}\Big\langle\phi,\frac{U''(x_\nu)\phi(x_\nu)}{U-\nu+i\mathfrak{m}}\Big\rangle_{L^2(x_\nu/2,1)}&=U''(x_\nu)\phi(x_\nu)\frac{\phi}{U'}\log(U-\nu+i\mathfrak{m})\Big|_{x=x_\nu/2}\\&\quad-U''(x_\nu)\phi(x_\nu)\Big\langle\Big(\frac{\phi}{U'}\Big)',\log(U-\nu+i\mathfrak{m})\Big\rangle_{L^2(x_\nu/2,1)},\end{aligned}$$

from which we readily obtain

$$\Big|\Big\langle\phi,\frac{U''(x_\nu)\phi(x_\nu)}{U-\nu+i\mathfrak{m}}\Big\rangle_{L^2(x_\nu/2,1)}\Big|\leq C\|\phi\|_{1,2}|\phi(x_\nu)|\leq C\|\phi\|_{1,2}\|\phi\|_\infty.$$

In conjunction with (5.6.12) the above inequality yields

$$\Big|\Big\langle\phi,\frac{U''(x_\nu)\phi(x_\nu)}{U-\nu+i\mathfrak{m}}\Big\rangle\Big|\leq C\|\phi\|_{1,2}\|\phi\|_\infty.$$

Substituting the above, together with (5.6.11) and (5.6.10) into (5.6.9) yields

$$\|\phi'\|_2^2+\alpha^2\|\phi\|_2^2\leq C\left(\|\phi\|_{1,2}+\beta^{-5/6}\|f\|_2\right)\|\phi\|_\infty+\beta^{-1/6}\|\phi\|_{1,2}^2.$$

Since $\phi(1)=0$ we have $\|\phi\|_\infty^2\leq 2\|\phi'\|_2\|\phi\|_2$ and hence, for any $\epsilon>0$ there exists $C_\epsilon>0$ such that

$$\|\phi\|_\infty^2\leq\epsilon\|\phi'\|_2^2+C_\epsilon\|\phi\|^2.$$

By choosing sufficiently small ϵ and sufficiently large β_0 we can then conclude, with the aid of Poincaré's inequality,

$$\|\phi'\|_2^2+\alpha^2\|\phi\|_2^2\leq C\left(\beta^{-5/3}\|f\|_2^2+\|\phi\|_2^2\right).$$

We then obtain for sufficiently large α_0 and β_0 the existence of $C>0$ such that, under the conditions of the proposition,

$$\|\phi'\|_2^2+\alpha^2\|\phi\|_2^2\leq C\beta^{-5/3}\,\|f\|_2^2,$$

from which (5.6.1) readily follows. ■

5.7 Resolvent estimates for large α

For $\alpha \gtrsim \beta^{1/3}$, we can no longer use the estimates of the previous section, relying on (4.2.4). In the following we thus establish estimates for the inverse of the Orr–Sommerfeld operator, relying on (4.7.3), which is valid for $\alpha \gtrsim \beta^{1/3}$.

Proposition 5.7.1. *Let $U \in C^4([0,1])$ satisfying (2.1.3), and α_2 denote a positive constant. For any $\Upsilon < \sqrt{-U''(0)}/2$ and any $\widehat{\Upsilon} > 0$ there exist $C > 0$ and $\beta_0 > 0$ such that for all $\beta \geq \beta_0$ it holds that*

$$\sup_{\substack{\alpha_2\beta^{1/3}\leq\alpha \\ \Re\lambda\leq\beta^{-1/3}[\widehat{\mu}_m-\widehat{\Upsilon}-\alpha^2\beta^{-2/3}/2]}} \Big(\big\|(\mathcal{B}^{\mathfrak{N},\mathfrak{D}}_{\lambda,\alpha,\beta})^{-1}\big\| + \Big\|\frac{d}{dx}(\mathcal{B}^{\mathfrak{N},\mathfrak{D}}_{\lambda,\alpha,\beta})^{-1}\Big\|\Big) \leq C\beta^{-5/6}. \tag{5.7.1}$$

Proof. Let $\mathfrak{z} = \mathfrak{z}_\alpha$ be given by (4.7.1)–(4.7.2). Let $f \in L^2(0,1)$, $\phi \in D(\mathcal{B}^{\mathfrak{N},\mathfrak{D}}_{\lambda,\alpha})$ satisfy

$$\mathcal{B}^{\mathfrak{N},\mathfrak{D}}_{\lambda,\alpha}\phi = f.$$

An integration by parts yields $\langle \mathfrak{z}_\alpha, -\phi'' + \alpha^2\phi\rangle = 0$, and hence we may conclude that $-\phi'' + \alpha^2\phi \in D(\mathcal{L}^{\mathfrak{z}_\alpha}_\beta)$. Furthermore, it holds that

$$(\mathcal{L}^{\mathfrak{z}_\alpha}_\beta - \beta\lambda)(-\phi'' + \alpha^2\phi) = f + i\beta U''\phi.$$

By (4.7.3) we then have

$$\| -\phi'' + \theta^2\beta^{2/3}\phi\|_2 \leq C\left(\beta^{1/2}\|\phi\|_2 + \beta^{-1/2}\|f\|_2\right),$$

where $\theta = \alpha\beta^{-1/3}$.

Hence,

$$\|\phi'\|_2^2 + \theta^2\beta^{2/3}\|\phi\|_2^2 = \langle -\phi'' + \theta^2\beta^{2/3}\phi, \phi\rangle \leq C\left(\beta^{1/2}\|\phi\|_2^2 + \beta^{-1/2}\|f\|_2\|\phi\|_2\right). \tag{5.7.2}$$

As $\theta \geq \alpha_2$, we obtain that for sufficiently large β_0,

$$\|\phi'\|_2 \leq C\beta^{-5/6}\|f\|_2.$$

With the aid of Poincaré's inequality we then obtain (5.7.1). ∎

Remark 5.7.2. An improved version of (5.7.1) can be obtained by introducing the effect of $|U(0) - \nu|$ from (4.7.3)

$$\sup_{\substack{\alpha_2\beta^{1/3}\leq\alpha \\ \Re\lambda\leq\Upsilon\beta^{-1/2}}} \big\|(\mathcal{B}^{\mathfrak{N},\mathfrak{D}}_{\lambda,\alpha,\beta})^{-1}\big\| + \Big\|\frac{d}{dx}(\mathcal{B}^{\mathfrak{N},\mathfrak{D}}_{\lambda,\alpha,\beta})^{-1}\Big\| \leq C\frac{\beta^{-5/6}}{1 + |U(0)-\nu|\,\beta^{1/6}}. \tag{5.7.3}$$

5.8 Resolvent estimates for small α

We continue by considering for some positive $\hat{\alpha}_0$ and $0 < \hat{\nu}_0 < U(0)$, the zone

$$0 \le \alpha \le \hat{\alpha}_0 \beta^{-1/6}, \quad |\Im\lambda| \le \hat{\nu}_0, \quad \Re\lambda \le \beta^{-1/2}. \tag{5.8.1}$$

We begin by considering the case $\alpha = 0$ and then obtain estimates of $(\mathcal{B}^{\mathfrak{N},\mathfrak{D}}_{\lambda,\alpha,\beta})^{-1}$ for $0 < \alpha \le \hat{\alpha}_0 \beta^{-1/6}$ by treating it as a perturbation of $(\mathcal{B}^{\mathfrak{N},\mathfrak{D}}_{\lambda,0,\beta})^{-1}$. More precisely, we introduce the set

$$\mathfrak{W}(\beta, \hat{\nu}_0) := \left\{\lambda \in \mathbb{C} : |\Im\lambda| \le \hat{\nu}_0 \text{ and } \mu \le \beta^{-1/2}\right\}$$

and prove the following proposition.

Proposition 5.8.1. *Let $U \in C^4([0,1])$ satisfy (2.1.3). There exist $C > 0$, $\hat{\nu}_0 > 0$ and $\beta_0 > 0$ such that for all $\beta \ge \beta_0$ and $\lambda \in \mathfrak{W}(\beta, \hat{\nu}_0)$ it holds that*

$$\left\|(\mathcal{B}^{\mathfrak{N},\mathfrak{D}}_{\lambda,0,\beta})^{-1}\right\| + \left\|\frac{d}{dx}(\mathcal{B}^{\mathfrak{N},\mathfrak{D}}_{\lambda,0,\beta})^{-1}\right\| + \lambda_\beta^{-1/2}\left\|\frac{d^2}{dx^2}(\mathcal{B}^{\mathfrak{N},\mathfrak{D}}_{\lambda,0,\beta})^{-1}\right\| \le C\,\beta^{-2/3}. \tag{5.8.2}$$

Proof. Step 1. Preliminaries. Let $(\phi, f) \in D(\mathcal{B}^{\mathfrak{N},\mathfrak{D}}_{\lambda,0,\beta}) \times L^2(0,1)$ satisfy $\mathcal{B}^{\mathfrak{N},\mathfrak{D}}_{\lambda,0,\beta}\phi = f$. Setting $\alpha = 0$ in (5.4.2) yields

$$v_{\mathfrak{D}} = -(U + i\lambda)\phi'' + U''\phi + (U + i\lambda)\phi''(1)\hat{\psi}. \tag{5.8.3}$$

Note that $v_{\mathfrak{D}}(1) = 0$ and hence

$$(\mathcal{L}^{\mathfrak{N}}_\beta - \beta\lambda)v_{\mathfrak{D}} = g_{\mathfrak{D}}, \tag{5.8.4}$$

where $g_{\mathfrak{D}}$ is given by (5.4.4b), which we recall here for the benefit of the reader in the equivalent form

$$g_{\mathfrak{D}} + (U + i\lambda)f = (U + i\lambda)\phi''(1)\hat{g} - (U''\phi)'' - 2U'\tilde{v}_{\mathfrak{D}}' - U''\tilde{v}_{\mathfrak{D}}. \tag{5.8.5}$$

In (5.8.5), $\tilde{v}_{\mathfrak{D}}$ is given by setting $\alpha = 0$ in (5.4.6), i.e.,

$$\tilde{v}_{\mathfrak{D}} = -\phi'' + \phi''(1)\hat{\psi}. \tag{5.8.6}$$

Step 2: We estimate $|\phi''(1)|$. Let $(\phi, f) \in D(\mathcal{B}^{\mathfrak{N},\mathfrak{D}}_{\lambda,0,\beta}) \times L^2(0,1)$ satisfy $\mathcal{B}^{\mathfrak{N},\mathfrak{D}}_{\lambda,0,\beta}\phi = f$. An integration by parts yields

$$\begin{aligned}\|(U'')^{-1/2}\phi^{(3)}\|_2^2 = -\Re\langle (U'')^{-1}\phi'', \mathcal{B}_{\lambda,0,\beta}\phi\rangle - \frac{1}{U''(1)}\Re(\bar{\phi}''(1)\phi^{(3)}(1))\\ - \Re\langle [(U'')^{-1}]'\phi'', \phi^{(3)}\rangle + \mu\beta\|(U'')^{-1/2}\phi''\|_2^2.\end{aligned} \tag{5.8.7}$$

To estimate the second term on the right-hand side of (5.8.7) we use the identity (which is obtained via an integration by parts of the balance $\mathcal{B}_{0,0,\beta}\phi = f$)

$$\phi^{(3)}(1) = -\int_0^1 f(x)\,dx. \tag{5.8.8}$$

Hence,

$$\|\phi^{(3)}\|_2^2 \leq C\left(|\phi''(1)|\,\|f\|_2 + \beta^{1/2}\|\phi''\|_2^2 + \|f\|_2\|\phi''\|_2 + \|\phi^{(3)}\|_2\|\phi''\|_2\right),$$

which implies

$$\|\phi^{(3)}\|_2^2 \leq \hat{C}\left(|\phi''(1)|\,\|f\|_2 + \beta^{1/2}\|\phi''\|_2^2 + \|f\|_2\|\phi''\|_2\right). \tag{5.8.9}$$

Sobolev embeddings yield

$$|\phi''(1)|^2 \leq \left(\|\phi^{(3)}\|_2 + \|\phi''\|_2\right)\|\phi''\|_2. \tag{5.8.10}$$

Combining (5.8.9) and (5.8.10) leads to

$$|\phi''(1)| \leq C(\beta^{1/8}\|\phi''\|_2 + \|f\|_2^{1/4}\|\phi''\|_2^{3/4} + \|f\|_2^{1/3}\|\phi''\|_2^{2/3}). \tag{5.8.11}$$

By (5.4.6) and the left inequality of (4.6.5) it holds that

$$\|\phi''\|_2 \leq C\lambda_\beta^{-1/4}\beta^{-1/6}|\phi''(1)| + \|\tilde{v}_{\mathfrak{D}}\|_2. \tag{5.8.12}$$

Using (5.8.11) we then obtain for β_0 large enough

$$\|\phi''\|_2 \leq 2\|\tilde{v}_{\mathfrak{D}}\|_2 + C\lambda_\beta^{-3/4}\beta^{-1/2}\|f\|_2. \tag{5.8.13}$$

By (5.4.18) and (5.4.24) (note that (5.4.24) results from a straightforward application of (3.3.10) and (3.1.3) to (5.4.7)) it holds that

$$\|\tilde{v}_{\mathfrak{D}}\|_2 \leq C\left(\|\phi'\|_2 + \beta^{1/6}|\phi(x_\nu)| + \beta^{-2/3}\|f\|_2 + \beta^{-1/2}|\phi''(1)|\right). \tag{5.8.14}$$

Since

$$|\phi(x_\nu)| = |\phi(x_\nu) - \phi(1)| \leq |\nu|^{1/2}\|\phi'\|_2 \leq |\lambda|^{1/2}\|\phi'\|_2, \tag{5.8.15}$$

we obtain from (5.8.13), (5.8.14), with the aid of (5.8.11) that

$$\|\phi''\|_2 \leq C\left(\lambda_\beta^{1/2}\|\phi'\|_2 + [\lambda_\beta^{-3/4}\beta^{-1/2} + \beta^{-2/3}]\|f\|_2\right). \tag{5.8.16}$$

We now substitute (5.8.16) into (5.8.11) to obtain

$$|\phi''(1)| \leq C\left(\beta^{1/6}\lambda_\beta^{1/2}\|\phi'\|_2 + \beta^{-1/3}\|f\|_2\right). \tag{5.8.17}$$

Step 3: Given some $\mu_0 > 0$, we estimate $\tilde{v}_{\mathfrak{D}}$ and $\tilde{v}'_{\mathfrak{D}}$ under the additional assumption $\mu \geq -\mu_0$. We begin by recalling (5.4.22), which is still valid in the present case and reads

$$\|\tilde{v}'_{\mathfrak{D}}\|_2^2 \leq C\,\Big(\|\tilde{v}_{\mathfrak{D}}\|_2[\|f\|_2 + \lambda_\beta^{-3/4}\beta^{1/2}\|\phi\|_{1,2}] + \lambda_\beta^{-1/2}(\beta^{2/3}\|\phi\|_{1,2}^2 + \beta^{-1}\|f\|_2^2) + (\mu_{\beta,+} + 1)\,\|\tilde{v}_{\mathfrak{D}}\|_2^2\Big).$$

Observing that $\mu_{\beta,+} \leq \beta^{1/2}$, where $\mu_{\beta,+}$ is given by (3.1.26), we conclude that

$$\|\tilde{v}'_{\mathfrak{D}}\|_2^2 \leq C\,\Big(\|\tilde{v}_{\mathfrak{D}}\|_2[\|f\|_2 + \lambda_\beta^{-3/4}\beta^{1/2}\|\phi\|_{1,2}] + \lambda_\beta^{-1/2}(\beta^{2/3}\|\phi\|_{1,2}^2 + \beta^{-1}\|f\|_2^2) + \beta^{1/2}\,\|\tilde{v}_{\mathfrak{D}}\|_2^2\Big). \tag{5.8.18}$$

By (5.8.14), (5.8.15), and (5.8.17) it holds that

$$\|\tilde{v}_{\mathfrak{D}}\|_2 \leq C\,\big(\beta^{-2/3}\|f\|_2 + \lambda_\beta^{1/2}\|\phi'\|_2\big). \tag{5.8.19}$$

Substituting (5.8.19) into (5.8.18) then yields that for given $\mu_0 > 0$ there exists $C > 0$ such that for $-\mu_0 \leq \mu \leq \beta^{-1/2}$ it holds that

$$\|\tilde{v}'_{\mathfrak{D}}\|_2 \leq C\,\big(\beta^{-1/4}\|f\|_2 + \beta^{5/12}\|\phi\|_{1,2}\big). \tag{5.8.20}$$

Step 4: We estimate $\|v_{\mathfrak{D}}\|_\infty$ under the assumptions of Step 3. We begin by estimating the L^2-norm of $g_{\mathfrak{D}} + (U + i\lambda)f$ using (5.8.5). By (5.4.14), (5.4.18), and (5.4.27), it holds that

$$\|(U + i\lambda)\phi''(1)\hat{g}\| + \|(U''\phi)''\| \leq C\,\big(\beta^{1/4}\|\phi\|_{1,2} + \beta^{-7/12}\|f\|_2\big). \tag{5.8.21}$$

Substituting (5.8.21) together with (5.8.16), (5.8.19), and (5.8.20) into (5.8.5) yields

$$\|g_{\mathfrak{D}} + (U + i\lambda)f\|_2 \leq C\,\big(\beta^{-1/4}\|f\|_2 + \beta^{5/12}\|\phi\|_{1,2}\big). \tag{5.8.22}$$

Since, by a Sobolev embedding, we have

$$\|(\mathcal{L}_\beta^{\mathfrak{N}} - \beta\lambda)^{-1}(U - \nu)f\|_\infty \leq \Big\|\frac{d}{dx}(\mathcal{L}_\beta^{\mathfrak{N}} - \beta\lambda)^{-1}(U - \nu)f\Big\|_2^{1/2}\|(\mathcal{L}_\beta^{\mathfrak{N}} - \beta\lambda)^{-1}(U - \nu)f\|_2^{1/2},$$

we can conclude from (3.3.10) and (3.3.11) that

$$\|(\mathcal{L}_\beta^{\mathfrak{N}} - \beta\lambda)^{-1}(U - \nu)f\|_\infty \leq C\beta^{-3/4}\,\|f\|_2. \tag{5.8.23}$$

Furthermore, by (3.1.3) (for $-\Upsilon\beta^{-1/3} \leq \mu \leq \beta^{-1/2}$) or (3.1.84) and (3.1.85) (for $-\mu_0 \leq \mu \leq -\Upsilon\beta^{-1/3}$) we obtain

$$\|(\mathcal{L}_\beta^{\mathfrak{N}} - \beta\lambda)^{-1}i\mu f\|_\infty \leq C\beta^{-3/4}\|f\|_2. \tag{5.8.24}$$

Consequently, by (5.8.23) and (5.8.24) we may infer that for $\mu \geq -\mu_0$

$$\|(\mathcal{L}_\beta^{\mathfrak{N}} - \beta\lambda)^{-1}(U + i\lambda)f\|_\infty \leq C\beta^{-3/4}\,\|f\|_2. \tag{5.8.25}$$

By (5.8.22) and (3.1.3a) it holds that

$$\|(\mathcal{L}_\beta^{\mathfrak{N}} - \beta\lambda)^{-1}(g_{\mathfrak{D}} + [U + i\lambda]f)\|_2 \leq C\left(\beta^{-11/12}\|f\|_2 + \beta^{-1/4}\|\phi\|_{1,2}\right).$$

Furthermore, (5.8.22) and (3.1.3b) with $p = 2$ yield

$$\left\|\frac{d}{dx}(\mathcal{L}_\beta^{\mathfrak{N}} - \beta\lambda)^{-1}(g_{\mathfrak{D}} + [U + i\lambda]f)\right\|_2 \leq C\left(\beta^{-7/12}\|f\|_2 + \beta^{1/12}\|\phi\|_{1,2}\right).$$

Hence, by Sobolev's embeddings

$$\|(\mathcal{L}_\beta^{\mathfrak{N}} - \beta\lambda)^{-1}(g_{\mathfrak{D}} + [U + i\lambda]f)\|_\infty \leq C\left(\beta^{-3/4}\|f\|_2 + \beta^{-1/12}\|\phi\|_{1,2}\right). \tag{5.8.26}$$

In view of (5.8.4) we may combine (5.8.26) with (5.8.25) to obtain, for $\mu > -\mu_0$ that

$$\|v_{\mathfrak{D}}\|_\infty \leq C\left(\beta^{-3/4}\|f\|_2 + \beta^{-1/12}\|\phi\|_{1,2}\right). \tag{5.8.27}$$

Step 5. We prove (5.8.2). Recall from (5.8.1) that $\mu \leq \beta^{-1/2}$ and $|\nu| \leq \hat{\nu}_0 < U(0)$.

Step 5a. With $\mu_0 > 0$, we prove (5.8.2) for $|\nu| \leq \hat{\nu}_0$ and for μ satisfying

$$\mu \in (-\mu_0, -e^{-\beta^{1/24}}) \cup (e^{-\beta^{1/24}}, \beta^{-1/2}). \tag{5.8.28}$$

Set

$$v = \mathcal{A}_{\lambda,0}\phi = -(U + i\lambda)\phi'' + U''\phi,$$

and note from (5.4.2) (with $\alpha = 0$) that

$$v = v_{\mathfrak{D}} - \phi''(1)(U + i\lambda)\hat{\psi}.$$

An integration by parts yields

$$\int_0^1 v\,dx = 0,$$

and hence by (2.4.1) and (2.4.23) it holds that

$$\|\phi'\|_2 \leq C\left(\|v_{\mathfrak{D}}\|_\infty[1 + \log|\mu|^{-1}] + |\phi''(1)|\|\hat{\psi}\|_1\right).$$

By (5.2.2), (5.8.17), and (5.8.27), we obtain for $|\mu| < \mu_0$ that

$$\|\phi'\|_2 \leq C(\beta^{-1/24}\|\phi'\|_2 + \beta^{-2/3}\|f\|_2).$$

For sufficiently large β_0 we obtain for $\beta \geq \beta_0$

$$\|\phi\|_{1,2} \leq C\,\beta^{-2/3}\|f\|_2. \tag{5.8.29}$$

To obtain an estimate for $\|\phi''\|_2$ we use (5.8.16) to obtain

$$\|\phi''\|_2 \leq C\lambda_\beta^{1/2}\beta^{-2/3}\|f\|_2. \tag{5.8.30}$$

Step 5b. With $0 < \hat{\nu}_0 < U(0)$, we prove (5.8.2) for $|\nu| \leq \hat{\nu}_0$ and $|\mu| \leq e^{-\beta^{1/24}}$.

Here, we write for some $0 < \widetilde{\mu} \leq 1/2$

$$\mathcal{B}^{\mathfrak{N},\mathfrak{D}}_{\lambda+\widetilde{\mu}\beta^{-1/2},0,\beta}\phi = -\beta^{1/2}\widetilde{\mu}\phi'' + f. \tag{5.8.31}$$

Note that $\lambda + \widetilde{\mu}\beta^{-1/2}$ meets the assumptions of Step 5a, and hence, we can use (5.8.30) to obtain

$$\|\phi''\|_2 \leq C(\widetilde{\mu}\lambda_\beta^{1/2}\beta^{-1/6}\|\phi''\|_2 + \beta^{-2/3}\|f\|_2).$$

For sufficiently small $\widetilde{\mu}$ and sufficiently large β_0 we obtain (5.8.30) once again. Consequently,

$$\|-\beta^{1/2}\widetilde{\mu}\phi'' + f\|_2 \leq C\|f\|_2.$$

Hence, we can apply (5.8.29) once again to (5.8.31) to establish (5.8.29) for $|\mu| \leq e^{-\beta^{1/24}}$.

Step 5c. With $0 < \hat{\nu}_0 < U(0)$, we prove that there exists $\mu_0 > 0$ such that (5.8.2) holds for $\mu \leq -\mu_0$ and $|\nu| < \hat{\nu}_0$. Since $\mu < 0$, we have, after two integrations by parts,

$$\Re\langle\phi, \mathcal{B}_{\lambda,0,\beta}\phi\rangle = \|\phi''\|_2^2 + |\mu|\beta\,\|\phi'\|_2^2 + \beta\Im\langle U'\phi, \phi'\rangle. \tag{5.8.32}$$

Consequently, using Poincaré's inequality, we obtain

$$\|\phi'\|_2 \leq \frac{C}{|\mu|}(\beta^{-1}\|f\|_2 + \|\phi'\|_2).$$

For sufficiently large μ_0 and β_0 we can then conclude

$$\|\phi'\|_2 \leq \frac{C}{|\lambda|\beta}\|f\|_2.$$

Using (5.8.16) completes the proof of (5.8.2). ■

Using a perturbation argument we now obtain the following proposition.

Proposition 5.8.2. *Let* $0 < \hat{\nu}_0 < U(0)$. *Under the conditions of Proposition* 5.8.1 *there exist* $C > 0$, $\hat{\alpha}_0 > 0$, *and* $\beta_0 > 0$ *such that for all* $\beta \geq \beta_0$ *it holds that*

$$\sup_{\substack{\alpha<\hat{\alpha}_0\beta^{-1/6}\\|\nu|<\hat{\nu}_0\\\mu<\beta^{-1/2}}} \|(\mathcal{B}^{\mathfrak{N},\mathfrak{D}}_{\lambda,\alpha,\beta})^{-1}\| + \Big\|\frac{d}{dx}(\mathcal{B}^{\mathfrak{N},\mathfrak{D}}_{\lambda,\alpha,\beta})^{-1}\Big\| \leq C\beta^{-2/3}. \tag{5.8.33}$$

Proof. Let $(\phi, f) \in D(\mathcal{B}^{\mathfrak{N},\mathfrak{D}}_{\lambda,\alpha,\beta}) \times L^2(0,1)$ satisfy $\mathcal{B}^{\mathfrak{N},\mathfrak{D}}_{\lambda,\alpha,\beta}\phi = f$. We then write

$$\mathcal{B}^{\mathfrak{N},\mathfrak{D}}_{\lambda,0,\beta}\phi = -\alpha^2\Big(-\frac{d^2}{dx^2} + i\beta(U + i\lambda)\Big)\phi + f. \tag{5.8.34}$$

By (5.8.2) we obtain that

$$\|\phi''\|_2 \le C\,[\hat{\alpha}_0^2\lambda_\beta^{1/2}(\beta^{-1}\|\phi''\|_2 + \|\phi\|_2) + \lambda_\beta^{1/2}\beta^{-2/3}\|f\|_2].$$

Hence, for sufficiently large β we obtain that

$$\|\phi''\|_2 \le C\,\big[\hat{\alpha}_0^2\beta^{1/6}\|\phi\|_2 + \beta^{-1/2}\|f\|_2\big].$$

We can thus conclude that

$$\Big\|\Big(-\frac{d^2}{dx^2} + i\beta(U + i\lambda)\Big)\phi\Big\|_2 \le C\,\big(\beta\|\phi\|_2 + \beta^{-1/2}\|f\|_2\big).$$

By (5.8.2) and (5.8.34) we then obtain

$$\|\phi\|_{1,2} \le C\,\big(\hat{\alpha}_0^2\|\phi\|_2 + \beta^{-2/3}\|f\|_2\big).$$

For sufficiently small $\hat{\alpha}_0$ we may now conclude (5.8.33). ■

5.9 Some auxiliary results

This section is devoted to the proof of two auxiliary results which will become useful in the next two sections.

Lemma 5.9.1. *Let $U \in C^4(0,1)$ satisfy (2.1.3). Let further κ_0 and ν_1 denote positive constants. There exist positive β_0, Υ, α_1, and C such that, for $\lambda = \mu + i\nu$, where ν and μ satisfy $\nu_1 < \nu < U(0) + \kappa_0\beta^{-1/2}$ and $\mu < \Upsilon\beta^{-1/2}$, $\beta > \beta_0$, $0 \le \alpha \le \alpha_1\beta^{1/3}$, and any $(\phi, f) \in D(\mathcal{B}^{\mathfrak{N},\mathfrak{D}}_{\lambda,\alpha,\beta}) \times L^2(0,1)$ satisfying $\mathcal{B}^{\mathfrak{N},\mathfrak{D}}_{\lambda,\alpha,\beta}\phi = f$, it holds that*

$$\begin{aligned}|\phi''(1)| \le C\,\Big(&\beta^{-1/3}[\beta^{-1/4} + x_\nu]^{-5/6}\|f\|_2 \\ &+ \beta^{1/2}\Big[\beta^{-1/4} + \frac{x_\nu}{\log(1 + x_\nu\beta^{1/4})}\Big]^{-1}|\phi(x_\nu)| \\ &+ \beta^{1/2}[\beta^{-1/4} + x_\nu]^{-1/2}\|\phi'\|_2\Big),\end{aligned} \tag{5.9.1}$$

where x_ν is defined by (2.4.5).

Proof. Consider first the case $U(0) - \kappa_1\beta^{-1/2} < \nu < U(0) + \kappa_0\beta^{-1/2}$ for some $\kappa_1 > 0$. In this case we have $x_\nu \le C\beta^{-1/4}$. As in the proof of (5.4.14) we use the

$\mathcal{L}(L^2, L^\infty)$ estimate of (4.3.3) applied to $f = i\beta U''\phi(x_\nu)$ and (4.3.4) applied to $i\beta U''(\phi - \phi(x_\nu))$. We obtain

$$|\phi''(1)| \leq C\left(\beta^{-1/8}\|f\|_2 + \beta^{3/4}|\phi(x_\nu)| + \beta^{5/8}\|\phi'\|_2\right).$$

For $\nu_1 < \nu < U(0) - \kappa_1\beta^{-1/2}$ with sufficiently large κ_1 and β_0, we use (4.2.4) and (4.2.7), applied to the pair $(\phi'' - \alpha^2\phi, i\beta U''\phi + f)$ using the decomposition

$$i\beta U''\phi = i\beta U''(\phi - \phi(x_\nu)) + i\beta U''\phi(x_\nu)$$

and Hardy's inequality, to obtain

$$\begin{aligned}&|\phi''(1)|\\ &\leq C\left(\beta^{-1/3}x_\nu^{-5/6}\|f\|_2 + \beta^{1/2}x_\nu^{-1}\log(1 + x_\nu\beta^{1/4})|\phi(x_\nu)| + \beta^{1/2}x_\nu^{-1/2}\|\phi'\|_2\right).\end{aligned}$$

Combining the above pair of inequalities yields (5.9.1). ∎

Lemma 5.9.2. *Let $U \in C^4(0,1)$ satisfy (2.1.3) and κ_0, Υ and ν_1 denote positive constants. Let further*

$$\check{x}_\mu(\Upsilon, \beta) = \begin{cases} \min(\Upsilon\mu_{\beta,+}^{-1/2}, \beta^{-1/8}) & \mu > 0, \\ \beta^{-1/8} & \mu < 0, \end{cases} \tag{5.9.2}$$

where $\mu_{\beta,+}$ is defined by (3.1.26). Suppose that $\lambda = \mu + i\nu$, where $\beta^{-1} < |\mu|$, $\mu < \Upsilon\beta^{-1/2}$, $\nu_1 < \nu < U(0) + \kappa_0\beta^{-1/2}$ and

$$x_\nu < \check{x}_\mu(\Upsilon, \beta), \tag{5.9.3}$$

where x_ν is defined by (2.4.5). Then, there exist positive Υ_0, β_0, and C such that for all $\beta > \beta_0$ and $\Upsilon < \Upsilon_0$ it holds that

$$\begin{aligned}\|\tilde{v}_{\mathfrak{D}}\|_2 \leq C\Big[&\Upsilon^{-5/2}\beta^{-1/4}\|f\|_2 + (\Upsilon^{-3/8}\mu_{+,\beta}^{3/4} + \Upsilon^{3/2}\beta^{3/16})|\phi(x_\nu)| \\ &+ (\mu_{+,\beta}^{1/2} + \Upsilon^2\beta^{1/8})\|\phi'\|_2\Big]\end{aligned} \tag{5.9.4}$$

in which $\tilde{v}_{\mathfrak{D}}$ is given by (5.4.6).

Proof. Let $\delta := \delta(\beta, \Upsilon) \in (0, 1/4)$ be much greater than $\beta^{-1/4}$. More precisely, we introduce for sufficiently small Υ and $\beta \geq \beta_0(\Upsilon)$ with $\beta_0(\Upsilon)$ large enough

$$\delta(\beta, \Upsilon) := \begin{cases} \min(\Upsilon^{1/4}\mu_{\beta,+}^{-1/2}, \Upsilon^{-1}\beta^{-1/8}) & \mu > 0, \\ \Upsilon^{-1}\beta^{-1/8} & \mu < 0. \end{cases} \tag{5.9.5}$$

Recall the definition of $\chi \in C_0^\infty(\mathbb{R}, [0,1])$ from (2.6.20)

$$\chi(x) = \begin{cases} 1 & x < 1/2, \\ 0 & x > 3/4. \end{cases}$$

We further set $\chi_\delta(x) = \chi(x/\delta)$, and $\tilde{\chi}_\delta = 1 - \chi_\delta$. Note that χ_δ is supported in $(0, 3\delta/4)$ and that $\tilde{\chi}_\delta$ is supported in $[\delta/2, +\infty)$.

Step 1: We estimate $\tilde{\chi}_\delta \tilde{v}_{\mathfrak{D}}$. Using (5.4.7) we now write

$$(\mathcal{L}_\beta^{\mathfrak{N},\mathfrak{D}} - \beta\lambda)(\tilde{\chi}_\delta \tilde{v}_{\mathfrak{D}}) = f_\delta + i\beta U'' \tilde{\chi}_\delta \,\phi, \tag{5.9.6a}$$

where

$$f_\delta = 2\delta^{-1}\tilde{\chi}'(\cdot/\delta)\tilde{v}_{\mathfrak{D}}' + \delta^{-2}\tilde{\chi}''(\cdot/\delta)\tilde{v}_{\mathfrak{D}} + \tilde{\chi}_\delta\,[-f + \phi''(1)\hat{g}]. \tag{5.9.6b}$$

Setting $\gamma = 2^{-1}\beta^{-1/4}\delta^{-1}$ and $v = \tilde{\chi}_\delta \tilde{v}_{\mathfrak{D}}$ in (3.2.11) yields

$$\begin{aligned}\beta \|\, |U-\nu|^{1/2}\tilde{\chi}_\delta \tilde{v}_{\mathfrak{D}}\|_2^2 &\leq \|\tilde{\chi}_\delta \tilde{v}_{\mathfrak{D}}\|_2\, \|f_\delta\|_2 \\ &+ C\beta\, \|\, |U-\nu|^{1/2}\tilde{\chi}_\delta \tilde{v}_{\mathfrak{D}}\|_2 (\|\phi'\|_2 + \|(U-\nu)^{-1/2}\tilde{\chi}_\delta\|\, |\phi(x_\nu)|),\end{aligned} \tag{5.9.7}$$

where we have used the identities $\tilde{\chi}_\delta \chi_\gamma = \tilde{\chi}_\delta$ and $\tilde{\chi}_\delta \chi_\gamma' = 0$, and the inequality (relying on Hardy's inequality)

$$\begin{aligned}&|\langle \tilde{\chi}_\delta \tilde{v}_{\mathfrak{D}}, \tilde{\chi}_\delta[(\phi - \phi(x_\nu)) + \phi(x_\nu)]\rangle| \\ &\quad \leq C \||U-\nu|^{1/2}\tilde{\chi}_\delta \tilde{v}_{\mathfrak{D}}\|_2 (\|\phi'\|_2 + \|(U-\nu)^{-1/2}\tilde{\chi}_\delta\|\, |\phi(x_\nu)|).\end{aligned}$$

Since by (5.9.3) and (5.9.5) there exist positive C and $\widehat{C}$ such that

$$(U(0) - \nu)_+ \leq C x_\nu^2 \leq \widehat{C}\,\Upsilon^{3/2}\delta^2,$$

we have, for sufficiently small Υ, the existence of $\widetilde{C} > 0$ such that

$$|U-\nu|^{1/2}\tilde{\chi}_\delta \geq \frac{1}{\widetilde{C}}\delta\, \tilde{\chi}_\delta. \tag{5.9.8}$$

Hence, by (5.9.7) and (5.9.8),

$$\|\, |U-\nu|^{1/2}\tilde{\chi}_\delta \tilde{v}_{\mathfrak{D}}\|_2^2 \leq \beta^{-1}\|\tilde{\chi}_\delta \tilde{v}_{\mathfrak{D}}\|_2\, \|f_\delta\|_2 + C(\|\phi'\|_2^2 + \delta^{-1}\, |\phi(x_\nu)|^2), \tag{5.9.9}$$

which implies, using again (5.9.8),

$$\|\tilde{\chi}_\delta \tilde{v}_{\mathfrak{D}}\|_2 \leq C\, \big((\delta^2\beta)^{-1}\, \|f_\delta\|_2 + \delta^{-1}\|\phi'\|_2 + \delta^{-3/2}|\phi(x_\nu)|\big). \tag{5.9.10}$$

Consequently, by (5.9.6b) and (5.4.19)

$$\begin{aligned}\|\tilde{\chi}_\delta \tilde{v}_{\mathfrak{D}}\|_2 \leq C\big([\delta^2\beta]^{-1}\big(\delta^{-1}\|\mathbf{1}_{[\delta/2,\delta]}\tilde{v}_{\mathfrak{D}}'\|_2 + \delta^{-2}\|\mathbf{1}_{[\delta/2,\delta]}\tilde{v}_{\mathfrak{D}}\|_2 + \|f\|_2\big) \\ + \delta^{-3/2}|\phi(x_\nu)| + \delta^{-1}\|\phi'\|_2 + \delta^{-2}\beta^{-5/4}|\phi''(1)|\big).\end{aligned} \tag{5.9.11}$$

Substituting (5.10.13) into (5.9.11) yields in view of (5.9.5) (note that $\delta^2\mu_{\beta,+} \leq \Upsilon^{1/2}$)

$$\begin{aligned}\|\tilde{\chi}_\delta \tilde{v}_{\mathfrak{D}}\|_2^2 \leq C\big([\delta^2\beta]^{-2}\big(\delta^{-2}\|\mathbf{1}_{[\delta/2,\delta]}\tilde{v}_{\mathfrak{D}}'\|_2^2 + \delta^{-4}\|\mathbf{1}_{[\delta/2,\delta]}\tilde{v}_{\mathfrak{D}}\|_2^2\big) \\ + [\delta^2\beta]^{-2}\|f\|_2^2 + \delta^{-3}|\phi(x_\nu)|^2 + \delta^{-2}\|\phi'\|_2^2\big).\end{aligned} \tag{5.9.12}$$

Step 2: We estimate $\chi_{2\delta}\tilde{v}_{\mathfrak{D}}$. Taking the inner product of (5.4.7) with $\chi_{2\delta}^2(U'')^{-1}\tilde{v}_{\mathfrak{D}}$ we obtain (see also (5.4.8)) that

$$\begin{aligned}&\Re\langle\chi_{2\delta}^2(U'')^{-1}\tilde{v}_{\mathfrak{D}},(\mathcal{L}_\beta^{\mathfrak{N}}-\beta\lambda)\tilde{v}_{\mathfrak{D}}-i\beta U''\phi\rangle\\ &\quad=\|[\chi_{2\delta}(U'')^{-1/2}\tilde{v}_{\mathfrak{D}}]'\|_2^2-\|[\chi_{2\delta}(U'')^{-1/2}]'\tilde{v}_{\mathfrak{D}}\|_2^2-\beta\mu\,\|\chi_{2\delta}(U'')^{-1/2}\tilde{v}_{\mathfrak{D}}\|_2^2\\ &\qquad+\beta\Re\langle\chi_{2\delta}^2\phi''(1)\hat{\psi},i\phi\rangle. \end{aligned}\tag{5.9.13}$$

As $|U^{(3)}\chi_{2\delta}|\leq C\delta$ (given that $U^{(3)}(0)=0$) we can conclude that

$$\|[\chi_{2\delta}(U'')^{-1/2}\tilde{v}_{\mathfrak{D}}]'\|_2^2\geq\frac{1}{C}\|[\chi_{2\delta}\tilde{v}_{\mathfrak{D}}]'\|_2^2-C\delta^2\|\chi_{2\delta}\tilde{v}_{\mathfrak{D}}\|_2^2.\tag{5.9.14}$$

Furthermore, as $|(\chi_{2\delta})'|\leq C\delta^{-1}\tilde{\chi}_\delta$, we obtain, using again the fact that $U^{(3)}(0)=0$,

$$\|[\chi_{2\delta}(U'')^{-1/2}]'\tilde{v}_{\mathfrak{D}}\|_2^2\leq C(\delta^2\|\chi_{2\delta}\tilde{v}_{\mathfrak{D}}\|_2^2+\delta^{-2}\|\tilde{\chi}_\delta\tilde{v}_{\mathfrak{D}}\|_2^2).$$

Substituting the above, together with (5.9.14) into (5.9.13), recalling that by (5.2.2)

$$\beta|\langle\chi_{2\delta}^2\phi''(1)\hat{\psi},i\phi\rangle|\leq\beta|\phi''(1)|\,\|\phi'\|_2\|(1-x)^3\hat{\psi}\|_1\leq C\beta^{-1}|\phi''(1)|\,\|\phi'\|_2,$$

and that by (5.4.17), (5.2.2), (5.2.3), (4.2.10), and (4.2.17)

$$|\langle\chi_{2\delta}(U'')^{-1}\tilde{v}_{\mathfrak{D}},\phi''(1)\tilde{\chi}_{2\delta}\hat{g}\rangle|\leq C\beta^{-3/4}\|\chi_{2\delta}\tilde{v}_{\mathfrak{D}}\|_2|\phi''(1)|$$

yields

$$\begin{aligned}\|[\chi_{2\delta}\tilde{v}_{\mathfrak{D}}]'\|_2^2\leq C\big[&\Upsilon^{-1}\delta^2\|f\|_2^2+\beta^{-3/4}|\phi''(1)|\,\|\chi_{2\delta}\tilde{v}_{\mathfrak{D}}\|_2\\ &+(\Upsilon\delta^{-2}+\mu_{\beta,+})\|\chi_{2\delta}\tilde{v}_{\mathfrak{D}}\|_2^2+\delta^{-2}\|\tilde{\chi}_\delta\tilde{v}_{\mathfrak{D}}\|_2^2+\beta^{-1}|\phi''(1)|\|\phi'\|_2\big].\end{aligned}$$

By Poincaré's inequality we have

$$\|\chi_{2\delta}\tilde{v}_{\mathfrak{D}}\|_2^2\leq C\delta^2\|[\chi_{2\delta}\tilde{v}_{\mathfrak{D}}]'\|_2^2.\tag{5.9.15}$$

Hence, in view of (5.9.5) and (5.9.15), we obtain for sufficiently small Υ

$$\begin{aligned}&\|\chi_{2\delta}\tilde{v}_{\mathfrak{D}}\|_2^2+\delta^2\|[\chi_{2\delta}\tilde{v}_{\mathfrak{D}}]'\|_2^2\leq\\ &C\big[\Upsilon^{-1}\delta^4\|f\|_2^2+\|\tilde{\chi}_\delta\tilde{v}_{\mathfrak{D}}\|_2^2+\beta^{-1}\delta^2|\phi''(1)|(\|\phi'\|_2+\delta^2\beta^{-1/2}|\phi''(1)|)\big].\end{aligned}\tag{5.9.16}$$

Substituting (5.9.1) into (5.9.16) yields, in view of (5.9.5)

$$\|\chi_{2\delta}\tilde{v}_{\mathfrak{D}}\|_2^2+\delta^2\|[\chi_{2\delta}\tilde{v}_{\mathfrak{D}}]'\|_2^2\leq C\big(\Upsilon^{-1}\delta^4\|f\|_2^2+(\beta^{-1/4}\delta^2+\delta^4)\|\phi'\|_2^2+\|\tilde{\chi}_\delta\tilde{v}_{\mathfrak{D}}\|_2^2\big).\tag{5.9.17}$$

Combining (5.9.12) with (5.9.17), and (5.9.5) we obtain

$$\begin{aligned}\|\chi_{2\delta}\tilde{v}_{\mathfrak{D}}\|_2^2+\delta^2\|[\chi_{2\delta}\tilde{v}_{\mathfrak{D}}]'\|_2^2\leq C\big[&\Upsilon^{-1}\delta^4\|f\|_2^2+\delta^{-6}\beta^{-2}\|\mathbf{1}_{[\delta/2,\delta]}\tilde{v}_{\mathfrak{D}}'\|_2^2\\ &+\delta^{-8}\beta^{-2}\|\mathbf{1}_{[\delta/2,\delta]}\tilde{v}_{\mathfrak{D}}\|_2^2+\delta^{-3}|\phi(x_\nu)|^2+\delta^{-2}\|\phi'\|_2^2\big].\end{aligned}\tag{5.9.18}$$

As $\mathbf{1}_{[\delta/2,\delta]} \leq \chi_{2\delta}$, and $\delta^{-8}\beta^{-2} \leq \Upsilon^{-2}\beta^2\mu_+^4 + \Upsilon^8\beta^{-1}$, where $\mu_+ = \max(\mu, 0)$, we obtain

$$\begin{aligned}&\|\chi_{2\delta}\tilde{v}_{\mathfrak{D}}\|_2^2 + \delta^2\|[\chi_{2\delta}\tilde{v}_{\mathfrak{D}}]'\|_2^2 \\ &\quad\leq C\big[\Upsilon^{-1}\delta^4\|f\|_2^2 + (\Upsilon^{-2}\beta^2\mu_+^4 + \Upsilon^8\beta^{-1})\delta^2\|[\chi_{2\delta}\tilde{v}_{\mathfrak{D}}]'\|_2^2 \\ &\qquad\quad + (\Upsilon^{-2}\beta^2\mu_+^4 + \Upsilon^8\beta^{-1})\|\chi_{2\delta}\tilde{v}_{\mathfrak{D}}\|_2^2 + \delta^{-3}|\phi(x_\nu)|^2 + \delta^{-2}\|\phi'\|_2^2\big].\end{aligned}$$

For sufficiently small Υ and β_0^{-1} we then conclude (as $\Upsilon^{-2}\beta^2\mu_+^4 \leq \Upsilon^2$) that

$$\|\chi_{2\delta}\tilde{v}_{\mathfrak{D}}\|_2^2 + \delta^2\|[\chi_{2\delta}\tilde{v}_{\mathfrak{D}}]'\|_2^2 \leq C\big[\Upsilon^{-1}\delta^4\|f\|_2^2 + \delta^{-3}|\phi(x_\nu)|^2 + \delta^{-2}\|\phi'\|_2^2\big]. \tag{5.9.19}$$

Similarly, by (5.9.12) and (5.9.5) (note that $\delta^{-4}\beta^{-2} \leq (\Upsilon^{-2}\beta^2\mu_+^4 + \Upsilon^8\beta^{-1})\delta^4$), it holds that

$$\begin{aligned}\|\tilde{\chi}_\delta\tilde{v}_{\mathfrak{D}}\|_2^2 \leq C\big((&\Upsilon^{-2}\beta^2\mu_+^4 + \Upsilon^8\beta^{-1})\delta^2\|[\chi_{2\delta}\tilde{v}_{\mathfrak{D}}]'\|_2^2 \\ &+(\Upsilon^{-2}\beta^{-2}\mu_+^4+\Upsilon^8\beta^{-1})\|\chi_{2\delta}\tilde{v}_{\mathfrak{D}}\|_2^2+(\Upsilon^{-2}\beta^2\mu_+^4+\Upsilon^8\beta^{-1})\delta^4\|f\|_2^2 \\ &+\delta^{-3}|\phi(x_\nu)|^2 + \delta^{-2}\|\phi'\|_2^2\big).\end{aligned} \tag{5.9.20}$$

Since $\chi_{2\delta} + \tilde{\chi}_\delta > \frac{1}{C}$ we obtain from (5.9.19) and (5.9.20) (as $\Upsilon^{-1}\delta^4 \leq C\Upsilon^{-5}\beta^{-1/2}$ by (5.9.5) and since $|\mu| > \beta^{-1}$) that for Υ and β_0^{-1} small enough

$$\begin{aligned}\|\tilde{v}_{\mathfrak{D}}\|_2 \leq C\big[&\Upsilon^{-5/2}\beta^{-1/4}\|f\|_2 + (\Upsilon^{-3/8}\mu_{+,\beta}^{3/4} + \Upsilon^{3/2}\beta^{3/16})|\phi(x_\nu)| \\ &+ (\mu_{+,\beta}^{1/2} + \Upsilon^2\beta^{1/8})\|\phi'\|_2\big],\end{aligned}$$

which is precisely (5.9.4). ∎

5.10 Resolvent estimates for $|\Im\lambda - U(0)| = \mathcal{O}(\beta^{-1/2})$

We consider, for given positive $\kappa_0, \alpha_1, \lambda = \mu + i\nu$, and some positive Υ, the zone

$$\mathcal{E}(\alpha_1, \beta_0, \Upsilon, \kappa_0) := \left\{ \begin{array}{c} (\lambda, \alpha, \beta) \in \mathbb{C} \times \mathbb{R}_+^2,\ 0 \leq \alpha \leq \alpha_1\beta^{1/3}, \beta \geq \beta_0 \\ \mu < \Upsilon\beta^{-1/2},\ U(0) - \kappa_0\beta^{-1/2} \leq \nu \leq U(0) + \kappa_0\beta^{-1/2} \end{array} \right\}. \tag{5.10.1}$$

Proposition 5.10.1. *Let $U \in C^4([0,1])$ satisfy (2.1.3) and $U^{(3)}(0) = 0$. Let further $\alpha_1 > 0$ and $\kappa_0 > 0$. Then, there exist positive Υ, β_0 and C, such that for $(\lambda, \alpha, \beta) \in \mathcal{E}(\alpha_1, \beta_0, \Upsilon, \kappa_0)$, it holds*

$$\max(1, |\mu\beta|^{1/4})\Big(\|(\mathcal{B}_{\lambda,\alpha,\beta}^{\mathfrak{N},\mathfrak{D}})^{-1}\| + \Big\|\frac{d}{dx}(\mathcal{B}_{\lambda,\alpha,\beta}^{\mathfrak{N},\mathfrak{D}})^{-1}\Big\|\Big) \leq C\beta^{-3/8}. \tag{5.10.2}$$

Proof. Step 1: Preliminaries. We follow the same outlines as in the proof of Proposition 5.4.1. Nevertheless, given that $|\nu - U(0)| \sim \mathcal{O}(\beta^{-1/2})$, we need to address here the quadratic behavior of $U(x) - U(0)$ in the vicinity of $x = 0$ (see Section 3.2 for instance).

Let $\tilde{v}_{\mathfrak{D}}$ be given by (5.4.6). For the convenience of the reader, we repeat (5.4.8), which reads

$$\begin{aligned}
&\Re\langle (U'')^{-1}\tilde{v}_{\mathfrak{D}}, (\mathcal{L}_\beta^{\mathfrak{N}} - \beta\lambda)\tilde{v}_{\mathfrak{D}} + i\beta U''\phi\rangle \\
&\quad = \|(U'')^{-1/2}\tilde{v}_{\mathfrak{D}}'\|_2^2 + \Re\langle \big((U'')^{-1}\big)'\tilde{v}_{\mathfrak{D}}, \tilde{v}_{\mathfrak{D}}'\rangle \\
&\qquad - \beta\mu\|(U'')^{-1/2}\tilde{v}_{\mathfrak{D}}\|_2^2 + \beta\Re\langle \phi''(1)\hat{\psi}, i\phi\rangle,
\end{aligned} \tag{5.10.3}$$

where $\hat{\psi} = \hat{\psi}_{\lambda,\beta}$ is introduced in (5.2.1) and $\phi \in D(\mathcal{B}_{\lambda,\alpha,\beta}^{\mathfrak{N},\mathfrak{D}})$ satisfies for $f \in L^2(0,1)$

$$\mathcal{B}_{\lambda,\alpha,\beta}^{\mathfrak{N},\mathfrak{D}}\phi = f.$$

We begin by estimating the last term on the right-hand side of (5.10.3). For technical reasons we distinguish between the case $\mu < -\mu_0$ (for some sufficiently small $\mu_0 > 0$) and $\mu > -\mu_0$.

Step 2: We estimate $\Re\langle \phi''(1)\hat{\psi}, i\phi\rangle$ for $\mu \geq -\mu_0$. As in Step 2 of the proof of Proposition 5.4.1 we write

$$\phi(x) = \int_x^1 (\xi - x)\phi''(\xi)\,d\xi = \phi''(1)w + \int_x^1 (\xi - x)[\phi''(\xi) - \phi''(1)\hat{\psi}(\xi)]\,d\xi,$$

where

$$w(x) = \int_x^1 (\xi - x)\hat{\psi}(\xi)\,d\xi.$$

Then we write

$$\Re\langle \phi''(1)\hat{\psi}, i\phi\rangle = -|\phi''(1)|^2\Im\langle \hat{\psi}, w\rangle + \Re\langle \phi''(1)\hat{\psi}, i(\phi - \phi''(1)w)\rangle. \tag{5.10.4}$$

For the first term on the right-hand side we write, using the fact that $w'' = \hat{\psi}$, and integration by parts,

$$\Im\langle \hat{\psi}, w\rangle = \Im\langle w'', w\rangle = -\Im\{\bar{w}'(0)w(0)\}. \tag{5.10.5}$$

We now use [3, Proposition A.1] to obtain the following improvement of (5.2.2):

$$\hat{\psi}(x) = e^{-\beta^{1/2}(-\lambda)^{1/2}(1-x)} + \hat{\psi}_1(x), \tag{5.10.6}$$

where

$$\|\hat{\psi}_1\|_1 + \beta^{1/2}\|(1-x)\hat{\psi}_1\|_1 \leq C\beta^{-1}. \tag{5.10.7}$$

Next, we write, using (5.10.1), (5.10.6), (5.10.7)

$$\begin{aligned}\bar{w}'(0) &= -\int_0^1 \overline{\widehat{\psi}}(\xi)\,d\xi = -\int_0^1 [e^{-\beta^{1/2}(-\bar{\lambda})^{1/2}(1-\xi)} + \overline{\widehat{\psi}}_1(\xi)]\,d\xi \\ &= -\frac{1}{(-\beta\bar{\lambda})^{1/2}} + \mathcal{O}(\beta^{-1}) \\ &= -\frac{e^{-i\pi/4}\beta^{-1/2}}{[U(0)]^{1/2}}[1 + \mathcal{O}(|\mu|^{1/2})] + \mathcal{O}(\beta^{-1}). \end{aligned} \tag{5.10.8}$$

To obtain (5.10.8) we used the identities

$$\int_0^1 e^{-\beta^{1/2}(-\bar{\lambda})^{1/2}(1-x)}dx = \beta^{-1/2}(-\bar{\lambda})^{-1/2}(1 - e^{-\beta^{1/2}(-\lambda)^{1/2}})$$

and

$$\beta^{-1/2}(-[\mu - i\nu])^{-1/2} = e^{-i\pi/4}\beta^{-1/2}(U(0) + [\nu - U(0) + i\mu])^{-1/2},$$

which shows the exponentially small behavior of $e^{-\beta^{1/2}(-\lambda)^{1/2}}$ if $\mu_0 > 0$ is chosen small enough.

Furthermore, it holds that

$$w(0) = \int_0^1 \xi\widehat{\psi}(\xi)\,d\xi = -w'(0) - \int_0^1 (1-\xi)\widehat{\psi}(\xi)\,d\xi,$$

which implies by (5.2.2), (5.10.6), and (5.10.7)

$$w(0) = -w'(0) - \frac{i\beta^{-1}}{U(0)}[1 + \mathcal{O}(|\mu|^{1/2})] + \mathcal{O}(\beta^{-3/2}). \tag{5.10.9}$$

Combining (5.10.8) and (5.10.9) yields

$$\Im\{\bar{w}'(0)w(0)\} = \frac{\beta^{-3/2}}{[U(0)]^{3/2}\sqrt{2}}[1 + \mathcal{O}(|\mu|^{1/2})] + \mathcal{O}(\beta^{-2}). \tag{5.10.10}$$

Substituting (5.10.10) into (5.10.5) yields

$$\beta\Re\langle\widehat{\psi}, iw\rangle = \frac{\beta^{-1/2}}{[U(0)]^{3/2}\sqrt{2}}[1 + \mathcal{O}(|\mu|^{1/2})] + \mathcal{O}(\beta^{-1}).$$

For sufficiently large β_0 and sufficiently small μ_0, $\beta\Re\langle\widehat{\psi}, iw\rangle$ is positive and hence, by (5.10.4) we can conclude that

$$\Re\langle\phi''(1)\widehat{\psi}, i\phi\rangle \geq \Re\langle\phi''(1)\widehat{\psi}, i(\phi - \phi''(1)w)\rangle. \tag{5.10.11}$$

As in the proof of Proposition 5.4.1 (Step 2) we now apply (5.4.11), recalling that $\alpha \leq \alpha_1 \beta^{1/3}$, to obtain (note that $\lambda_\beta \geq \frac{1}{2}|U(0)|\beta^{1/3}$ by (5.10.1))

$$\begin{aligned}\beta\,\Re\langle\phi''(1)\widehat{\psi}, i\phi\rangle &\geq -C\,\beta^{-1/6}\lambda_\beta^{-7/4}\,|\phi''(1)|(\|\tilde{v}_{\mathfrak{D}}'\|_2 + \beta^{2/3}\|\phi'\|_2)\\ &\geq -\widehat{C}\,\beta^{-3/4}|\phi''(1)|(\|\tilde{v}_{\mathfrak{D}}'\|_2 + \beta^{2/3}\|\phi'\|_2). \qquad (5.10.12)\end{aligned}$$

Let x_ν be defined by (2.4.5). By the assumption on ν it holds that

$$x_\nu \leq C\beta^{-1/4}.$$

Hence, by (5.9.1)

$$|\phi''(1)| \leq C\left(\beta^{-1/8}\|f\|_2 + \beta^{3/4}|\phi(x_\nu)| + \beta^{5/8}\|\phi'\|_2\right), \qquad (5.10.13)$$

from which we conclude

$$\beta\,\Re\langle\phi''(1)\widehat{\psi}, i\phi\rangle \geq -C\,(\|\phi\|_{1,2} + \beta^{-7/8}\|f\|_2)(\|\tilde{v}_{\mathfrak{D}}'\|_2 + \beta^{2/3}\|\phi'\|_2). \qquad (5.10.14)$$

Step 3: We estimate $\tilde{v}_{\mathfrak{D}}$ and $\tilde{v}_{\mathfrak{D}}'$. Here, we follow the Steps 4 and 5 in the proof of Proposition 5.4.1 with (5.4.15) replaced by (5.10.13).

By (3.2.1a), (5.4.7), (5.4.19), and (5.10.13), we obtain (compare with (5.4.24)) that

$$\|\tilde{v}_{\mathfrak{D}}\|_2 \leq C(\beta^{1/4}\|\phi'\|_2 + \beta^{3/8}|\phi(x_\nu)| + \beta^{-1/2}\|f\|_2). \qquad (5.10.15)$$

Substituting (5.10.15), together with (5.10.13), (5.10.14), and (5.4.19), into (5.4.20) yields

$$\begin{aligned}&\frac{1}{C}\|\tilde{v}_{\mathfrak{D}}'\|_2^2\\ &\quad\leq (\beta^{1/4}\|\phi'\|_2 + \beta^{3/8}|\phi(x_\nu)| + \beta^{-1/2}\|f\|_2)(\|f\|_2 + \beta^{3/8}\|\phi'\|_2 + \beta^{1/2}|\phi(x_\nu)|)\\ &\qquad + (1+\mu_{\beta,+})(\beta^{1/4}\|\phi'\|_2 + \beta^{3/8}|\phi(x_\nu)| + \beta^{-1/2}\|f\|_2)^2\\ &\qquad + (\|\phi\|_{1,2} + \beta^{-7/8}\|f\|_2)(\|\tilde{v}_{\mathfrak{D}}'\|_2 + \beta^{2/3}\|\phi'\|_2).\end{aligned}$$

Hence,

$$\|\tilde{v}_{\mathfrak{D}}'\|_2 \leq C\left((\beta^{1/8} + \mu_{\beta,+}^{1/2})(\beta^{1/4}\|\phi'\|_2 + \beta^{3/8}|\phi(x_\nu)|) + \beta^{-1/8}\|f\|_2\right). \qquad (5.10.16)$$

Recall that

$$\mu_{\beta,+} = \beta\mu_+ = \beta\max(\mu, 0).$$

Since (5.10.16) is unsatisfactory, given that the coefficient of $\beta^{1/4}\|\phi'\|_2 + \beta^{3/8}|\phi(x_\nu)|$ is not necessarily small, as will become clear in the sequel, we obtain an improved estimate in the next step.

Step 4. For $|\mu| > \beta^{-1}$ we prove under the assumptions of the proposition that

$$\|\phi'' - \phi''(1)\widehat{\psi}\|_2 \leq C\big[\Upsilon^{-5/2}\beta^{-1/4}\|f\|_2 + (\Upsilon^{-3/8}\mu_{+,\beta}^{3/4} + \beta^{1/3})|\phi(x_\nu)| + (\mu_{+,\beta}^{1/2} + \Upsilon^2\beta^{1/8})\|\phi'\|_2\big]. \tag{5.10.17}$$

Using the definition of $\tilde{v}_{\mathfrak{D}}$ given in (5.4.6), an integration by parts yields

$$\langle -\phi'' + \phi''(1)\widehat{\psi}, \tilde{v}_{\mathfrak{D}}\rangle = \|\phi'' - \phi''(1)\widehat{\psi}\|_2^2 + \alpha^2\|\phi'\|_2^2 + \langle\phi''(1)\widehat{\psi}, \alpha^2\phi\rangle. \tag{5.10.18}$$

By (5.2.2) for $s = 1/2$ and (5.10.13) it holds that

$$\begin{aligned}|\langle\phi''(1)\widehat{\psi}, \alpha^2\phi\rangle| &\leq \alpha^2|\phi''(1)|\,\|(1-x)^{1/2}\widehat{\psi}\|_1\,\|\phi'\|_2 \\ &\leq C\alpha^2\left(\beta^{-7/8}\|f\|_2 + |\phi(x_\nu)| + \beta^{-1/8}\|\phi'\|_2\right)\|\phi'\|_2.\end{aligned}$$

Substituting the above into (5.10.18) yields for sufficiently large β_0

$$\begin{aligned}&\|\phi'' - \phi''(1)\widehat{\psi}\|_2^2 + \alpha^2\|\phi'\|_2^2 \\ &\leq \|-\phi''+\phi''(1)\widehat{\psi}\|_2\|\tilde{v}_{\mathfrak{D}}\|_2 + C\alpha^2\big(\beta^{-7/8}\|f\|_2 + |\phi(x_\nu)| + \beta^{-1/8}\|\phi'\|_2\big)\|\phi'\|_2.\end{aligned}$$

For sufficiently large β_0 we then obtain that

$$\|\phi'' - \phi''(1)\widehat{\psi}\|_2^2 \leq C\left[\|\tilde{v}_{\mathfrak{D}}\|_2^2 + \alpha^2(\beta^{-7/4}\|f\|_2^2 + |\phi(x_\nu)|^2)\right]. \tag{5.10.19}$$

We now obtain (5.10.17) from (5.9.4) and the fact that $\alpha \leq \alpha_1\beta^{1/3}$.

Step 5. We estimate $\|v_{\mathfrak{D}}\|_\infty$ under the assumption of the proposition and the additional conditions $|\mu| > \beta^{-1}$ and $\mu \geq -\mu_0$.

Let $v_{\mathfrak{D}}$ be given by (5.4.2). For the convenience of the reader we recall here (5.4.4)

$$(\mathcal{L}_\beta^{\mathfrak{N},\mathfrak{D}} - \beta\lambda)v_{\mathfrak{D}} = g_{\mathfrak{D}}, \tag{5.10.20a}$$

where (after reordering)

$$\begin{aligned}g_{\mathfrak{D}} = (U + i\lambda)(-f + \phi''(1)\hat{g}) - 2U'\tilde{v}_{\mathfrak{D}}' - U''\phi''(1)\widehat{\psi} \\ + U''([\phi''(1)\widehat{\psi} - \phi''] - \tilde{v}_{\mathfrak{D}}) - 2U^{(3)}\phi' - U^{(4)}\phi.\end{aligned} \tag{5.10.20b}$$

Next, we obtain a bound for $\|(\mathcal{L}_\beta^{\mathfrak{N},\mathfrak{D}} - \beta\lambda)^{-1}g_{\mathfrak{D}}\|_\infty$ by separately estimating the contribution of each of the six terms on the right-hand side of (5.10.20b).

To obtain the L^∞ estimates we repeatedly use the following Sobolev embedding inequality

$$\|v_{\mathfrak{D}}\|_\infty \leq \|v_{\mathfrak{D}}\|_2^{1/2}\,\|v_{\mathfrak{D}}'\|_2^{1/2}. \tag{5.10.21}$$

Writing $(U + i\lambda) = (U - \nu) + i\mu$ we obtain by (3.1.84)–(3.1.85) (for $\mu < 0$), (3.2.1a) (for $\mu > 0$), and (3.3.35), that

$$\begin{aligned}&\|(\mathcal{L}_\beta^{\mathfrak{N},\mathfrak{D}} - \beta\lambda)^{-1}(U + i\lambda)(-f + \phi''(1)\hat{g})\|_\infty \\ &\qquad \leq C\big(\beta^{-3/4} + |\mu|^{1/4}\beta^{-3/4} + \mu_+\beta^{-3/8}\big)\big(\|f\|_2 + |\phi''(1)|\,\|\hat{g}\|_2\big).\end{aligned}$$

Since $-\mu_0 < \mu < \Upsilon\beta^{-1/2}$, we obtain

$$\|(\mathcal{L}_\beta^{\mathfrak{N},\mathfrak{D}} - \beta\lambda)^{-1}(U + i\lambda)(-f + \phi''(1)\hat{g})\|_\infty \leq C\beta^{-3/4}(\|f\|_2 + |\phi''(1)|\,\|\hat{g}\|_2). \tag{5.10.22a}$$

Using (3.2.1a) and (3.2.1b) we obtain, recalling that, for $|\nu - U(0)| \leq \kappa_0\beta^{-1/2}$, we have $x_\nu \leq C\beta^{-1/4}$,

$$\begin{aligned}&\|(\mathcal{L}_\beta^{\mathfrak{N},\mathfrak{D}} - \beta\lambda)^{-1}(U'\tilde{v}_{\mathfrak{D}}')\|_\infty \\ &\qquad = \|(\mathcal{L}_\beta^{\mathfrak{N},\mathfrak{D}} - \beta\lambda)^{-1}(U'(x_\nu)) + (U'(x) - U'(x_\nu))\tilde{v}_{\mathfrak{D}}'\|_\infty \\ &\qquad \leq C\,\big(x_\nu\beta^{-3/8} + \beta^{-5/8}\big)\|\tilde{v}_{\mathfrak{D}}'\|_2 \leq \hat{C}\beta^{-5/8}\|\tilde{v}_{\mathfrak{D}}'\|_2.\end{aligned} \tag{5.10.22b}$$

By (4.6.3) and (3.2.1a) we have

$$\begin{aligned}&\|(\mathcal{L}_\beta^{\mathfrak{N},\mathfrak{D}} - \beta\lambda)^{-1}(\phi''(1)U''\hat{\psi})\|_\infty \\ &\qquad = \|(\mathcal{L}_\beta^{\mathfrak{N},\mathfrak{D}} - \beta\lambda)^{-1}\phi''(1)[U''(1) + (U''(x) - U''(1))]\hat{\psi}\|_\infty \\ &\qquad \leq C\,\big(\beta^{-1} + \beta^{-3/8}\|(1-x)\hat{\psi}\|_2\big)\,|\phi''(1)|.\end{aligned}$$

By (4.2.10), (4.2.16), and (4.6.1) it holds that

$$\|(1-x)^k\,\hat{\psi}\|_2 \leq C\,\lambda_\beta^{-(1+2k)/4}\beta^{-(1+2k)/6}. \tag{5.10.22c}$$

Using (5.10.22c) with $k = 1$ yields

$$\|(\mathcal{L}_\beta^{\mathfrak{N},\mathfrak{D}} - \beta\lambda)^{-1}(\phi''(1)U''\hat{\psi})\|_\infty \leq C\beta^{-1}|\phi''(1)|. \tag{5.10.22d}$$

For the next term we use (3.2.1a) and (3.2.1b) to obtain that

$$\|(\mathcal{L}_\beta^{\mathfrak{N},\mathfrak{D}} - \beta\lambda)^{-1}U''([\phi''(1)\hat{\psi} - \phi''] - \tilde{v}_{\mathfrak{D}})\|_\infty \leq C\beta^{-3/8}\,\big(\|\phi''(1)\hat{\psi} - \phi''\|_2 + \|\tilde{v}_{\mathfrak{D}}\|_2\big).$$

We then use (5.9.4) and (5.10.17) to obtain that

$$\begin{aligned}&\|(\mathcal{L}_\beta^{\mathfrak{N},\mathfrak{D}} - \beta\lambda)^{-1}U''([\phi''(1)\hat{\psi} - \phi''] - \tilde{v}_{\mathfrak{D}})\|_\infty \\ &\qquad \leq C\beta^{-3/8}\big[\Upsilon^{-5/2}\beta^{-1/4}\|f\|_2 + (\Upsilon^{-3/8}\mu_{+,\beta}^{3/4} + \beta^{1/3})|\phi(x_\nu)| \\ &\qquad\qquad + (\mu_{+,\beta}^{1/2} + \Upsilon^2\beta^{1/8})\|\phi'\|_2\big].\end{aligned} \tag{5.10.22e}$$

As by (5.4.6)

$$\phi''(1)\hat{\psi} - \phi'' - \tilde{v}_{\mathfrak{D}} = \alpha^2\phi,$$

we may also write

$$\|(\mathcal{L}_\beta^{\mathfrak{N},\mathfrak{D}}-\beta\lambda)^{-1}U''([\phi''(1)\widehat{\psi}-\phi'']-\tilde{v}_{\mathfrak{D}})\|_\infty=\|(\mathcal{L}_\beta^{\mathfrak{N},\mathfrak{D}}-\beta\lambda)^{-1}U''\alpha^2\phi\|_\infty$$
$$\leq\|(\mathcal{L}_\beta^{\mathfrak{N},\mathfrak{D}}-\beta\lambda)^{-1}U''\alpha^2[\phi(x_\nu)]\|_\infty+\|(\mathcal{L}_\beta^{\mathfrak{N},\mathfrak{D}}-\beta\lambda)^{-1}U''\alpha^2[\phi-\phi(x_\nu)]\|_\infty.$$

Then we use (3.2.1a) and (3.2.1b) together with Hardy's inequality for the second term to obtain,

$$\|(\mathcal{L}_\beta^{\mathfrak{N},\mathfrak{D}}-\beta\lambda)^{-1}U''\alpha^2\phi\|_\infty\leq C\alpha^2\beta^{-1/2}\big(|\phi(x_\nu)|+\beta^{-1/8}\|\phi'\|_2\big).\qquad(5.10.22f)$$

In the sequel we use (5.10.22e) for $\alpha\geq\beta^{1/8}$ and (5.10.22f) for $\alpha<\beta^{1/8}$.

We estimate the next term as in (5.10.22a) (as $|U^{(3)}(x)|\leq Cx$):

$$\|(\mathcal{L}_\beta^{\mathfrak{N},\mathfrak{D}}-\beta\lambda)^{-1}(2U^{(3)}\phi')\|_\infty\leq C\,\beta^{-5/8}\|\phi'\|_2.\qquad(5.10.22g)$$

Finally, we estimate the last term as in (5.10.22e):

$$\|(\mathcal{L}_\beta^{\mathfrak{N},\mathfrak{D}}-\beta\lambda)^{-1}(2U^{(4)}\phi)\|_\infty\leq C\beta^{-1/2}(|\phi(x_\nu)|+\beta^{-1/8}\|\phi'\|_2).\qquad(5.10.22h)$$

Combining (5.10.22a)–(5.10.22h) then yields

$$\|v_{\mathfrak{D}}\|_\infty\leq C\big[\beta^{-3/4}\gamma(\alpha,\beta)\|f\|_2+\beta^{-3/4}|\phi''(1)|\{\|\hat{g}\|_2+\beta^{-1/4}\}+\beta^{-5/8}\|\tilde{v}_{\mathfrak{D}}'\|_2$$
$$+([\beta\Upsilon]^{-3/8}\mu_{+,\beta}^{3/4}+\beta^{-1/24})|\phi(x_\nu)|+(\beta^{-3/8}\mu_{+,\beta}^{1/2}+\beta^{-1/4})\|\phi'\|_2\big],\qquad(5.10.23)$$

where

$$\gamma(\alpha,\beta)=\begin{cases}1 & \alpha<\beta^{1/8},\\ \Upsilon^{-5/2}\beta^{1/8} & \alpha\geq\beta^{1/8}.\end{cases}$$

Substituting (5.10.13), (5.10.16), (5.4.19), and (5.9.4) into (5.10.23), yields, with the aid of (4.6.5)

$$\|v_{\mathfrak{D}}\|_\infty\lesssim\beta^{-3/4}\gamma(\alpha,\beta)\|f\|_2$$
$$+([\beta\Upsilon]^{-3/8}\mu_{+,\beta}^{3/4}+\beta^{-1/24})|\phi(x_\nu)|+(\beta^{-3/8}\mu_{+,\beta}^{1/2}+\beta^{-1/4})\|\phi'\|_2.$$

By the assumption on μ_+, we may finally conclude

$$\|v_{\mathfrak{D}}\|_\infty\leq C\big(\gamma(\alpha,\beta)\beta^{-3/4}\|f\|_2+[\beta^{-1/4}+\mu_+^{1/2}\beta^{1/8}]\|\phi\|_{1,2}+[\beta^{-1/24}$$
$$+\mu_+^{3/8}\beta^{3/16}]|\phi(x_\nu)|\big).\qquad(5.10.24)$$

Step 6. We prove (5.10.2) in the case $\mu>-\mu_0$ and $\alpha\leq\alpha_0\beta^{1/8}$.

Step 6a. Preliminaries. We continue as in the proof of [3, Lemma 8.8]. We first write, as in (5.4.31)

$$\phi = \phi_{\mathfrak{D}} + \breve{\phi}, \tag{5.10.25a}$$

where

$$\phi_{\mathfrak{D}} = \mathcal{A}_{\lambda,\alpha}^{-1} v_{\mathfrak{D}}, \quad \breve{\phi} = -\mathcal{A}_{\lambda,\alpha}^{-1}\big([U + i\lambda]\phi''(1)\widehat{\psi}\big) = -\phi''(1)\phi_{\lambda,\alpha,\beta}. \tag{5.10.25b}$$

By Propositions 2.8.1 and 2.9.1, there exists (sufficiently small) $C > 0$ so that we can use (2.8.47), for $|\mu| \leq C\, x_\nu^2$, and (2.9.14) for $|\mu| \geq C x_\nu^2$, both holding for sufficiently small μ_0. Hence, we can conclude that for any pair $(\tilde{v}, \tilde{\phi}) \in W^{1,p}(0,1) \times D(\mathcal{A}_{\lambda,\alpha})$ satisfying $\tilde{v} = \mathcal{A}_{\lambda,\alpha}\tilde{\phi}$,

$$|\tilde{\phi}(x_\nu)|^2 \leq C\Big(|\mu|^{1/2}\Big|\Big\langle\tilde{\phi}, \frac{\tilde{v}}{U + i\lambda}\Big\rangle\Big| + x_\nu\Big\|(1-x)^{1/2}\frac{\tilde{v}}{U + i\lambda}\Big\|_1^2\Big). \tag{5.10.26}$$

We apply the above inequality to the pair $(-(U + i\lambda)\phi''(1)\widehat{\psi}, \breve{\phi})$ to obtain

$$|\breve{\phi}(x_\nu)|^2 \leq C|\phi''(1)|\,(|\mu|^{1/2}|\langle\breve{\phi}, \widehat{\psi}\rangle| + x_\nu|\phi''(1)|\|(1-x)^{1/2}\widehat{\psi}|_1^2).$$

Given that $\breve{\phi}(1) = 0$ we write

$$|\langle\phi, \widehat{\psi}\rangle| \leq \|\phi'\|_2\|(1-x)^{1/2}\widehat{\psi}\|_1 \leq C\,[|\lambda|\beta]^{-3/4}\|\phi'\|_2, \tag{5.10.27}$$

to obtain, with the aid of (5.2.2) and (5.4.31),

$$|\langle\breve{\phi}, \widehat{\psi}\rangle| \leq \|\breve{\phi}'\|_2\|(1-x)^{1/2}\widehat{\psi}\|_1 \leq C\,[|\lambda|\beta]^{-3/4}\|\breve{\phi}'\|_2, \tag{5.10.28}$$

and consequently it holds that

$$|\breve{\phi}(x_\nu)|^2 \leq \widehat{C}\,(|\mu|^{1/2}\beta^{-3/4}\|\breve{\phi}'\|_2|\phi''(1)| + x_\nu\beta^{-3/2}|\phi''(1)|^2). \tag{5.10.29}$$

Combining (5.10.29) with (5.10.13) yields

$$\begin{aligned}|\breve{\phi}(x_\nu)|^2 \leq \widehat{C}\,\big(|\mu|^{1/2}\|\breve{\phi}'\|_2 + x_\nu\big(\beta^{-7/8}\|f\|_2 + |\phi(x_\nu)| + \beta^{-1/8}\|\phi'\|_2\big)\big)\\ \times\big(\beta^{-7/8}\|f\|_2 + |\phi(x_\nu)| + \beta^{-1/8}\|\phi'\|_2\big).\end{aligned}$$

Since

$$|\mu|^{1/2}\|\breve{\phi}'\|_2|\phi(x_\nu)| \leq C(\delta^{-2}|\mu|\|\breve{\phi}'\|_2^2 + \delta^2|\phi(x_\nu)|^2)$$

for any $\delta > 0$ and $x_\nu < C\beta^{-1/4}$, we can conclude that there exists $C > 0$ such that for any $\delta > 0$

$$\begin{aligned}|\breve{\phi}(x_\nu)| \leq C\,\big(\delta^{-1}(|\mu|^{1/2} + \beta^{-1/8})[\|\breve{\phi}'\|_2 + \|\phi_{\mathfrak{D}}'\|_2]\\ + (\delta + \beta^{-1/8})[|\breve{\phi}(x_\nu)| + |\phi_{\mathfrak{D}}(x_\nu)|] + \delta^{-1}\beta^{-7/8}\|f\|_2\big).\end{aligned} \tag{5.10.30}$$

By applying to the pair $(\phi_{\mathfrak{D}}, v_{\mathfrak{D}})$ (2.8.65) for $U(0) - \kappa_0\beta^{-1/2} \leq \nu \leq U(0) - \kappa_0|\mu|$, (2.9.29) for $U(0) - \kappa_0 \min(|\mu|, \beta^{-1/2}) \leq \nu \leq U(0) + \kappa_0 \min(|\mu|, \beta^{-1/2})$, and (2.10.12) for $U(0) + \kappa_0|\mu| \leq \nu \leq U(0) + \kappa_0\beta^{-1/2}$, we obtain that

$$|\phi_{\mathfrak{D}}(x_\nu)| \leq C\Big(1 + \log \frac{\max(x_\nu, |\mu|^{1/2})}{|\mu|^{1/2}}\Big)\|v_{\mathfrak{D}}\|_\infty.$$

With the aid of (5.10.24) we then obtain

$$\begin{aligned}&|\phi_{\mathfrak{D}}(x_\nu)| \leq C\Big(1 + \log \frac{\max(x_\nu, |\mu|^{1/2})}{|\mu|^{1/2}}\Big)\\ &\times\big(\gamma(\alpha,\beta)\beta^{-3/4}\|f\|_2 + [\beta^{-1/4} + \mu_+^{1/2}\beta^{1/8}]\|\phi\|_{1,2} + [\beta^{-1/24} + \mu_+^{3/8}\beta^{3/16}]|\phi(x_\nu)|\big),\end{aligned}$$

(where μ_+ is given by (3.3.20)), from which we conclude by (5.10.25) that

$$\begin{aligned}\mu_+|\phi_{\mathfrak{D}}(x_\nu)| \leq C\Big(1 + \log \frac{\max(x_\nu, |\mu|^{1/2})}{|\mu|^{1/2}}\Big)&\\ \times\big(\gamma(\alpha,\beta)\beta^{-3/4}\|f\|_2 + (\beta^{-1/4} + \mu_+^{1/2}\beta^{1/8})[\|\check{\phi}'\|_2 + \|\phi_{\mathfrak{D}}'\|_2]&\\ + (\beta^{-1/24} + \mu_+^{3/8}\beta^{3/16})[|\check{\phi}(x_\nu)| + |\phi_{\mathfrak{D}}(x_\nu)|]\big).&\end{aligned} \tag{5.10.31}$$

Combining (5.10.30) and (5.10.31) yields,

$$\begin{aligned}|\check{\phi}(x_\nu)| + |\phi_{\mathfrak{D}}(x_\nu)| \leq C\Big(1 + \log \frac{\max(x_\nu, |\mu|^{1/2})}{|\mu|^{1/2}}\Big)&\\ \times\big([\beta^{-1/24} + \mu_+^{3/8}\beta^{3/16} + \delta][|\check{\phi}(x_\nu)| + |\phi_{\mathfrak{D}}(x_\nu)|]&\\ + [\beta^{-1/4} + \mu_+^{1/2}\beta^{1/8} + \delta^{-1}|\mu|^{1/2}][\|\check{\phi}'\|_2 + \|\phi_{\mathfrak{D}}'\|_2]&\\ + \gamma(\alpha,\beta)\beta^{-3/4}\|f\|_2\big).&\end{aligned} \tag{5.10.32}$$

Step 6b. We prove the existence of $\Upsilon > 0$, $\mu_0 > 0$, and $\beta_0 > 0$ such that (5.10.2) holds for $\Upsilon^{-1}\beta^{-1} < \mu < \Upsilon\beta^{-1/2}$ or $-\mu_0 < \mu < -\Upsilon^{-1}\beta^{-1}$, with $\beta \geq \beta_0$.

Let

$$\delta = \hat{\delta}\Big/\Big(1 + \log \frac{\max(x_\nu, |\mu|^{1/2})}{|\mu|^{1/2}}\Big),$$

where $\hat{\delta} > 0$ is independent of β.

Note that for $\beta^{-1} < |\mu| < x_\nu^2$, we have, for any $s > 0$,

$$\Big(\frac{|\mu|^{1/2}}{x_\nu}\Big)^s\Big(1 + \log \frac{\max(x_\nu, |\mu|^{1/2})}{|\mu|^{1/2}}\Big) \leq C_s. \tag{5.10.33}$$

The above inequality implies (with $s = 1$), since $x_\nu \leq C\beta^{-1/4}$, the existence of $\beta_0 > 0$ such that for all $\beta > \beta_0$,

$$\delta^{-1}|\mu|^{1/2} \leq C\beta^{-1/4}. \tag{5.10.34a}$$

Furthermore, (5.10.34) with $s = 1/2$ leads to

$$\Big(1 + \log \frac{\max(x_\nu, |\mu|^{1/2})}{|\mu|^{1/2}}\Big)\, \mu_+^{3/8}\beta^{3/16} \leq C\mu_+^{1/8} x_\nu^{1/2}\beta^{3/16} \leq \widehat{C}\,\Upsilon^{1/8}, \tag{5.10.34b}$$

and with $s = 1/12$ to

$$\Big(1 + \log \frac{\max(x_\nu, |\mu|^{1/2})}{|\mu|^{1/2}}\Big)\, \beta^{-1/24} \leq C_s \beta^{-1/24}(x_\nu/|\mu|^{1/2})^s \leq \widehat{C}\,\beta^{-1/48}. \tag{5.10.34c}$$

Hence, we obtain from (5.10.32) and (5.10.34) that, for sufficiently small Υ, $\hat{\delta}$, and μ_0 and sufficiently large β_0,

$$\begin{aligned} |\check{\phi}(x_\nu)| + |\phi_{\mathfrak{D}}(x_\nu)| &\leq C\Big(1 + \log \frac{\max(x_\nu, |\mu|^{1/2})}{|\mu|^{1/2}}\Big) \\ &\times \big(\gamma(\alpha,\beta)\beta^{-3/4}\|f\|_2 + [\beta^{-1/4} + \mu_+^{1/2}\beta^{1/8} + |\mu|^{1/2}][\|\check{\phi}'\|_2 + \|\phi_{\mathfrak{D}}'\|_2]\big). \end{aligned} \tag{5.10.35}$$

Next, we apply (2.8.1b), for $U(0) - \kappa_0\beta^{-1/2} \leq \nu \leq U(0) - \kappa_0|\mu|$, (2.9.1) for $U(0) - \kappa_0 \min(|\mu|, \beta^{-1/2}) \leq \nu \leq U(0) + \kappa_0 \min(|\mu|, \beta^{-1/2})$, and (2.10.1) for $U(0) + \kappa_0|\mu| \leq \nu \leq U(0) + \kappa_0\beta^{-1/2}$, for $p = +\infty$, to the pair $(\phi_{\mathfrak{D}}, v_{\mathfrak{D}})$ to obtain that, for $|\mu| > \Upsilon^{-1}\beta^{-1}$,

$$\Big(1 + \log \frac{\max(x_\nu, |\mu|^{1/2})}{|\mu|^{1/2}}\Big)\|\phi_{\mathfrak{D}}'\|_2 \leq C\,|\mu|^{-1/4}\|v_{\mathfrak{D}}\|_\infty. \tag{5.10.36}$$

Note that while applying (2.8.1b) we have, since $x_\nu^2 > \frac{1}{C}|\mu|$ in this case, that

$$\frac{[\log \frac{x_\nu}{|\mu|^{1/2}}]^2}{x_\nu^{1/2}} \leq \widehat{C}\,|\mu|^{-1/4}.$$

Then (5.10.24) and (5.10.36) yield with the aid of (5.10.25a),

$$\begin{aligned} \|\phi_{\mathfrak{D}}'\|_2 \leq C\Big(1 + \log \frac{\max(x_\nu, |\mu|^{1/2})}{|\mu|^{1/2}}\Big)^{-1} \Big(&|\mu|^{-1/4}\gamma(\alpha,\beta)\beta^{-3/4}\|f\|_2 \\ &+ (|\mu|^{-1/4}\beta^{-1/4} + \mu_+^{1/4}\beta^{1/8})[\|\check{\phi}'\|_2 + \|\phi_{\mathfrak{D}}'\|_2] \\ &+ (\beta^{-1/24}|\mu|^{-1/4} + \mu_+^{1/8}\beta^{3/16})[|\check{\phi}(x_\nu)| + |\phi_{\mathfrak{D}}(x_\nu)|]\Big). \end{aligned} \tag{5.10.37}$$

Substituting (5.10.35) into (5.10.37) yields

$$\begin{aligned} \|\phi_{\mathfrak{D}}'\|_2 \leq C\big(&|\mu|^{-1/4}\gamma(\alpha,\beta)\beta^{-3/4}\|f\|_2 \\ &+ [|\mu|^{-1/4}\beta^{-1/4} + \mu_+^{1/4}\beta^{1/8} + |\mu|^{1/4}][\|\check{\phi}'\|_2 + \|\phi_{\mathfrak{D}}'\|_2]\big). \end{aligned} \tag{5.10.38}$$

We now use (2.9.13) for the pair $(\check{\phi}, \phi''(1)(U + i\lambda)\widehat{\psi})$, together with (5.10.28) to obtain that

$$\|\check{\phi}'\|_2 \leq C\,[\lambda|\beta|]^{-3/4}|\phi''(1)|. \tag{5.10.39}$$

Using (5.10.13), we deduce from (5.10.39) for sufficiently large β_0

$$\|\check{\phi}'\|_2 \leq C([|\check{\phi}(x_\nu)| + |\phi_{\mathfrak{D}}(x_\nu)|] + \beta^{-1/8}[\|\check{\phi}'\|_2 + \|\phi_{\mathfrak{D}}'\|_2] + \beta^{-7/8}\|f\|_2). \tag{5.10.40}$$

We then obtain from (5.10.35)

$$\|\check{\phi}'\|_2 \leq C\Big(1 + \log\frac{\max(x_\nu, |\mu|^{1/2})}{|\mu|^{1/2}}\Big) \times \big(\gamma(\alpha,\beta)\beta^{-3/4}\|f\|_2 + [\beta^{-1/4} + \mu_+^{1/2}\beta^{1/8} + |\mu|^{1/2}][\|\check{\phi}'\|_2 + \|\phi_{\mathfrak{D}}'\|_2]\big). \tag{5.10.41}$$

To obtain the coefficient of $\|f\|_2$, we set $s = 1/2$ in (5.10.33) to conclude for $|\mu| < x_\nu^2$

$$\Big(1 + \log\frac{\max(x_\nu, |\mu|^{1/2})}{|\mu|^{1/2}}\Big) \leq C x_\nu^{1/2}|\mu|^{-1/4} \leq \widetilde{C}\beta^{-1/4}|\mu|^{-1/4} \leq \widehat{C}|\mu|^{-1/4}.$$

Hence, combining (5.10.41) with (5.10.38) yields

$$\|\check{\phi}'\|_2 + \|\phi_{\mathfrak{D}}'\|_2 \leq C\Bigg(|\mu|^{-1/4}\gamma(\alpha,\beta)\beta^{-3/4}\|f\|_2 + [\beta^{-1/4} + \mu_+^{1/2}\beta^{1/8} + |\mu|^{1/2}]\Big(1 + \log\frac{\max(x_\nu, |\mu|^{1/2})}{|\mu|^{1/2}}\Big)[\|\check{\phi}'\|_2 + \|\phi_{\mathfrak{D}}'\|_2]\Bigg).$$

Hence, with the aid of (5.10.34), we obtain that there exist $\Upsilon > 0$ and $\beta_0 > 0$ (so that $\Upsilon + \beta_0^{-1}$ is small enough) such that for either $\Upsilon^{-1}\beta^{-1} \leq \mu \leq \Upsilon\beta^{-1/2}$ or $-\mu_0 \leq \mu \leq -\Upsilon^{-1}\beta^{-1}$ we have

$$\|\phi'\|_2 \leq C\,|\mu|^{-1/4}\gamma(\alpha,\beta)\beta^{-3/4}\|f\|_2 \leq \widehat{C}\gamma(\alpha,\beta)\beta^{-1/2}\|f\|_2. \tag{5.10.42}$$

Combined with Poincaré's inequality (5.10.42) yields (5.10.2).

In the next step we use a shifting argument and hence it is necessary to obtain first an estimate for $\|\phi'' - \alpha^2\phi\|_2$. By (5.10.35) and the first inequality of (5.10.42), we obtain that

$$|\phi(x_\nu)| \leq |\check{\phi}(x_\nu)| + |\phi_{\mathfrak{D}}(x_\nu)| \leq C\gamma(\alpha,\beta)\beta^{-3/4}\log\beta\,\|f\|_2. \tag{5.10.43}$$

Substituting the first inequality of (5.10.42) and (5.10.43) into (5.10.15) yields

$$\|\tilde{v}_{\mathfrak{D}}\|_2 \leq C\,\gamma(\alpha,\beta)\big(\beta^{-3/8}\log\beta + |\mu|^{-1/4}\beta^{-1/2}\big)\,\|f\|_2. \tag{5.10.44}$$

Consequently, by (5.4.6), (5.10.13), and (4.6.5) we obtain, from (5.10.42), (5.10.43) and (5.10.44),

$$\|\phi'' - \alpha^2\phi\|_2 \leq C\gamma(\alpha,\beta)\big(\beta^{-1/4}\log\beta + |\mu|^{-1/4}\beta^{-3/8}\big)\,\|f\|_2. \tag{5.10.45}$$

Hence, we have proven, under the additional condition that $\alpha \leq \beta^{1/8}$, that there exist $\Upsilon > 0$ and $\beta_0 > 0$ such that for all $\beta > \beta_0$ and either $\Upsilon^{-1}\beta^{-1} \leq \mu \leq \Upsilon\beta^{-1/2}$ or $-\mu_0 \leq \mu \leq -\Upsilon^{-1}\beta^{-1}$,

$$\|\phi'' - \alpha^2\phi\|_2 \leq C[\beta^{-1/4}\log\beta + |\mu|^{-1/4}\beta^{-3/8}]\|f\|_2. \tag{5.10.46}$$

For $\alpha \geq \beta^{1/8}$ (5.10.45) is deficient (in this case $\gamma(\alpha,\beta) = \Upsilon^{-5/2}\beta^{1/8}$), hence we use (4.5.4) (with $v = \phi'' - \alpha^2\phi$ and f replaced by $f + i\beta U''\phi$) instead of (5.10.13), to obtain that

$$|\phi''(1)| \leq C\beta^{7/16}(\|\phi\|_2 + \beta^{-1}\|f\|_2).$$

Then, with the aid of (5.10.44) and (4.6.5) we establish (5.10.46) for $\alpha \geq \beta^{1/8}$ as well.

Step 6c. We prove (5.10.2) for $|\mu| < \Upsilon^{-1}\beta^{-1}$, where $\Upsilon > 0$ has been determined in the previous step.

Here, we use a shifting argument. We begin by writing

$$\mathcal{B}_{\lambda+2\Upsilon^{-1}\beta^{-1},\alpha}\phi = f + 2\Upsilon^{-1}(\phi'' - \alpha^2\phi), \tag{5.10.47}$$

and observe that $\hat{\lambda} := \lambda + 2\Upsilon^{-1}\beta^{-1}$ satisfies the assumptions of Step 6b. We then have by (5.10.46), with λ replaced by $\hat{\lambda}$,

$$\|\phi'' - \alpha^2\phi\|_2 \leq C(\beta^{-1/8} + \hat{\mu}^{-1/4}\beta^{-3/8})[\|\phi'' - \alpha^2\phi\|_2 + \|f\|_2].$$

Consequently,

$$\|\phi'' - \alpha^2\phi\|_2 \leq C\beta^{-1/8}\|f\|_2.$$

We now apply (5.10.42) to (5.10.47) to obtain, with the aid of the above inequality,

$$\|\phi'\|_2 \leq C\gamma(\alpha,\beta)\beta^{-1/2}(\|f\|_2 + \|\phi'' - \alpha^2\phi\|_2) \leq \hat{C}\gamma(\alpha,\beta)\beta^{-1/2}\|f\|_2.$$

Combining the above with Poincaré's inequality yields (5.10.2).

Step 7: The case $\mu \leq -\mu_0$. Here, we use (4.4.3), applied to the pair $(\phi'' - \alpha^2\phi, f + i\beta U''\phi)$, and (1.1.7b) to obtain that

$$|\phi''(1)| \leq C(\beta^{1/2}\|\phi\|_2 + \beta^{-1/2}\|f\|_2). \tag{5.10.48}$$

We then use (2.11.1) for the pair $(\check{\phi}, \phi''(1)(U + i\lambda)\hat{\psi})$, together with (5.2.2) for $s = 1/2$ to conclude

$$\|\check{\phi}'\|_2 \leq C\,|\lambda\beta|^{-3/4}|\phi''(1)|. \tag{5.10.49}$$

From (5.10.48) and (5.10.49), we obtain, with the aid of Poincaré's inequality, for sufficiently large β_0

$$\|\check{\phi}'\|_2 \leq C\left(\beta^{-1/4}\|\phi'_{\mathfrak{D}}\|_2 + \beta^{-5/4}\|f\|_2\right). \tag{5.10.50}$$

To estimate $\|\phi'_{\mathfrak{D}}\|_2$ we apply (2.11.1) to the pair $(\phi_{\mathfrak{D}}, v_{\mathfrak{D}})$ to obtain

$$\|\phi'_{\mathfrak{D}}\|_2 \leq C\,\|v_{\mathfrak{D}}\|_2. \tag{5.10.51}$$

We use (5.4.7), (5.10.48), (5.4.19), and (3.1.85), applied to the pair $(\tilde{v}_{\mathfrak{D}}, i\beta U''\phi - f + \phi''(1)\hat{g})$, to obtain, that

$$\|\tilde{v}'_{\mathfrak{D}}\|_2 \leq C\,|\mu|^{-1/2}\big(\beta^{1/2}\|\phi'\|_2 + \beta^{-1/2}\|f\|_2\big). \tag{5.10.52}$$

By (3.1.84) applied to the pair $(v_{\mathfrak{D}}, g_{\mathfrak{D}})$ and (5.4.4) it holds that

$$\|v_{\mathfrak{D}}\|_2 \leq \frac{C}{\beta|\mu|}\big((1+|\mu|)(\|f\|_2 + |\phi''(1)|\,\|\hat{g}\|_2) + \|\tilde{v}'_{\mathfrak{D}}\|_2 + \|\tilde{v}_{\mathfrak{D}}\|_2 + \|\phi''\|_2 + \|\phi\|_{1,2}\big). \tag{5.10.53}$$

To bound $\|\phi''\|_2$ we use the identity

$$\|\phi''\|_2^2 + \alpha^2\|\phi'\|_2^2 = \langle \phi'', \phi'' - \alpha^2\phi\rangle, \tag{5.10.54}$$

to obtain, with the aid of (5.4.6),

$$\|\phi''\|_2 \leq \|\tilde{v}_{\mathfrak{D}}\|_2 + |\phi''(1)|\,\|\widehat{\psi}\|_2. \tag{5.10.55}$$

Consequently, we obtain from the substitution of (5.4.19), (5.10.52), (5.10.55), and (4.6.5), into (5.10.53)

$$\|v_{\mathfrak{D}}\|_2 \leq C\big(\beta^{-1/2}\|\phi'\|_2 + \beta^{-1}\|f\|_2\big). \tag{5.10.56}$$

Next, we apply (5.10.51) and (5.10.56) to obtain

$$\|\phi'_{\mathfrak{D}}\|_2 \leq C\,\|v_{\mathfrak{D}}\|_2 \leq \hat{C}\big(\beta^{-1/2}\|\phi'\|_2 + \beta^{-1}\|f\|_2\big).$$

Combining the above inequality with (5.10.50) yields

$$\|\phi'\|_2 \leq C\,\beta^{-1}\|f\|_2. \tag{5.10.57}$$

The above inequality, combined with Poincaré's inequality, completes the proof of (5.11.2). ∎

5.11 Resolvent estimates for $\beta^{-1/2} \ll U(0) - \nu < U(0) - U(1/2)$

We now consider the case where $\beta^{-1/4} \ll x_\nu < 1/2$. More precisely, given some positive Υ and $\nu_1 < U(1/2)$, we consider for suitable $\nu_2 > 0$ and β_0 the zone

$$\mathcal{C}_1(\nu_1, \nu_2, \Upsilon, \alpha_1, \beta_0) = \left\{ \begin{array}{l} (\lambda, \alpha, \beta) \in \mathbb{C} \times \mathbb{R}_+^2, \beta \geq \beta_0, \mu < \Upsilon\beta^{-1/2} \\ \nu_1 \leq \nu \leq U(0) - \nu_2\beta^{-1/2}, 0 \leq \alpha \leq \alpha_1\beta^{1/3} \end{array} \right\}, \tag{5.11.1}$$

for some sufficiently small $\alpha_1 > 0$.

Proposition 5.11.1. *Let $U \in C^4([0,1])$ satisfy (2.1.3). Let further $\nu_1 < U(1/2)$ denote a positive constant. Then, there exist $\Upsilon > 0$, $\alpha_1 > 0$, $\beta_0 > 0$, $\nu_2 > 0$, and $C > 0$, such that for $(\lambda, \alpha, \beta) \in \mathcal{C}_1(\nu_1, \nu_2, \Upsilon, \alpha_1, \beta_0)$ it holds that*

$$\|(\mathcal{B}_{\lambda,\alpha,\beta}^{\mathfrak{N},\mathfrak{D}})^{-1}\| + \Big\|\frac{d}{dx}(\mathcal{B}_{\lambda,\alpha,\beta}^{\mathfrak{N},\mathfrak{D}})^{-1}\Big\| \leq C\,\beta^{-1/2}\log\beta. \tag{5.11.2}$$

Proof. We refer to the notation introduced in (5.4.2)–(5.4.7) for $v_{\mathfrak{D}}$, $\tilde{v}_{\mathfrak{D}}$ and $g_{\mathfrak{D}}$.

Step 1. We estimate $\tilde{v}_{\mathfrak{D}}$ and $\tilde{v}'_{\mathfrak{D}}$ in L^2 for $\mu > -\mu_0$, for some, sufficiently small, $\mu_0 > 0$.

For the convenience of the reader we repeat here once again (5.4.8)

$$\begin{aligned}
&\Re\langle (U'')^{-1}\tilde{v}_{\mathfrak{D}}, (\mathcal{L}_\beta^{\mathfrak{N}} - \beta\lambda)\tilde{v}_{\mathfrak{D}} + i\beta U''\phi\rangle \\
&\quad = \|(U'')^{-1/2}\tilde{v}'_{\mathfrak{D}}\|_2^2 + \Re\langle ((U'')^{-1})'\tilde{v}_{\mathfrak{D}}, \tilde{v}'_{\mathfrak{D}}\rangle - \beta\mu\,\|\tilde{v}_{\mathfrak{D}}\|_2^2 + \beta\Re\langle \phi''(1)\widehat{\psi}, i\phi\rangle.
\end{aligned}$$

As in (5.10.12) we obtain that

$$\beta\,\Re\langle \phi''(1)\widehat{\psi}, i\phi\rangle \geq -C\beta^{-3/4}|\phi''(1)|(\|\tilde{v}'_{\mathfrak{D}}\|_2 + \beta^{2/3}\|\phi'\|_2). \tag{5.11.3}$$

Let x_ν be defined by (2.4.5). Since by assumption we have $x_\nu \geq C\nu_2^{1/2}\beta^{-1/4}$ it holds by (5.9.1) that, for sufficiently large ν_2

$$\begin{aligned}
|\phi''(1)| \leq C\big(&\beta^{-1/3}x_\nu^{-5/6}\|f\|_2 \\
&+ \beta^{1/2}x_\nu^{-1}\log(\beta^{1/4}x_\nu)|\phi(x_\nu)| + \beta^{1/2}x_\nu^{-1/2}\|\phi'\|_2\big).
\end{aligned} \tag{5.11.4}$$

For $\alpha > \beta^{1/6}$ we use (4.5.4) for the pair $(\phi'' - \alpha^2\phi, f + i\beta U''\phi)$ to obtain, with the aid of Poincaré's inequality, that

$$|\phi''(1)| \leq C\alpha^{-1/2}(\beta^{1/2}\|\phi'\|_2 + \beta^{-1/2}\|f\|_2) \tag{5.11.5}$$

By substituting (5.11.4) into (5.11.3), we get

$$\begin{aligned}
\beta\,\Re\langle \phi''(1)\widehat{\psi}, i\phi\rangle \geq -C\,\big(&\beta^{-1/4}[x_\nu^{-1/2}\|\phi\|_{1,2} + x_\nu^{-1}\log(\beta^{1/4}x_\nu)|\phi(x_\nu)|] \\
&+ \beta^{-13/12}x_\nu^{-5/6}\|f\|_2\big)\big(\|\tilde{v}'_{\mathfrak{D}}\|_2 + \beta^{2/3}\|\phi'\|_2\big).
\end{aligned} \tag{5.11.6}$$

Next, by (5.4.7), (5.4.19), (5.11.4), (3.1.3a), and (3.3.10), we obtain, for the parameter range set in (5.11.1), that (compare with (5.10.15))

$$\|\tilde{v}_{\mathfrak{D}}\|_2 \leq C\,\big(x_\nu^{-1}\|\phi'\|_2 + \beta^{1/6}x_\nu^{-5/6}|\phi(x_\nu)| + [\beta x_\nu]^{-2/3}\|f\|_2\big). \tag{5.11.7}$$

Substituting (5.11.7) together with (5.11.6), (5.4.19), and (5.11.4) into (5.4.20) yields, with the aid of (5.4.7)

$$\begin{aligned}
&\frac{1}{C}\|\tilde{v}'_{\mathfrak{D}}\|_2^2 \\
&\quad\le \big(x_\nu^{-1}\|\phi'\|_2 + \beta^{1/6}x_\nu^{-5/6}|\phi(x_\nu)| + [\beta x_\nu]^{-2/3}\|f\|_2\big) \\
&\qquad\times\big(\|f\|_2 + \beta^{1/4}x_\nu^{-1/2}\|\phi'\|_2 + \beta^{1/4}x_\nu^{-1}\log(\beta^{1/4}x_\nu)|\phi(x_\nu)|\big) \\
&\qquad+\max(1,\mu\beta)\big(x_\nu^{-1}\|\phi'\|_2 + \beta^{1/6}x_\nu^{-5/6}|\phi(x_\nu)| + [\beta x_\nu]^{-2/3}\|f\|_2\big)^2 \\
&\qquad+\big(\beta^{-1/4}[x_\nu^{-1/2}\|\phi\|_{1,2} + x_\nu^{-1}\log(\beta^{1/4}x_\nu)|\phi(x_\nu)|] + \beta^{-13/12}x_\nu^{-5/6}\|f\|_2\big) \\
&\qquad\times\big(\|\tilde{v}'_{\mathfrak{D}}\|_2 + \beta^{2/3}\|\phi'\|_2\big).
\end{aligned}$$

Hence,

$$\begin{aligned}
\|\tilde{v}'_{\mathfrak{D}}\|_2 \le C\,\big[&(\beta^{1/8} + \mu_+^{1/2}\beta^{1/2})(x_\nu^{-1}\|\phi'\|_2 + \beta^{1/6}x_\nu^{-5/6}|\phi(x_\nu)|) \\
&+ \beta^{5/24}x_\nu^{-1/4}\|\phi'\|_2 + \beta^{-1/8}\|f\|_2\big]. \qquad (5.11.8)
\end{aligned}$$

Step 2. We prove that

$$\begin{aligned}
&\|\phi'' - \phi''(1)\widehat{\psi}\|_2 \\
&\le C\,\big(\|\tilde{v}_{\mathfrak{D}}\|_2 + \beta^{-11/12}x_\nu^{-5/6}\|f\|_2 + \beta^{-1/12}x_\nu^{-1}\log(\beta^{1/4}x_\nu)|\phi(x_\nu)|\big). \qquad (5.11.9)
\end{aligned}$$

Step 2.1: Prove (5.11.9) *in the case* $0 \le \alpha \le \beta^{1/6}$. We write, as in Step 5 of the proof of Proposition 5.10.1 and with the aid of (5.11.4) and (5.2.2) (with $s = 1/2$)

$$\begin{aligned}
|\langle\phi''(1)\widehat{\psi},\alpha^2\phi\rangle| &\le \alpha^2|\phi''(1)|\,\|(1-x)^{1/2}\widehat{\psi}\|_1\,\|\phi'\|_2 \\
&\le C\alpha^2\,\big(\beta^{-13/12}x_\nu^{-5/6}\|f\|_2 + \beta^{-1/4}[x_\nu^{-1}\log(\beta^{1/4}x_\nu)|\phi(x_\nu)| \\
&\quad + x_\nu^{-1/2}\|\phi'\|_2]\big)\|\phi'\|_2. \qquad (5.11.10)
\end{aligned}$$

Substituting (5.11.10) into (5.10.18) yields

$$\begin{aligned}
&\|\phi'' - \phi''(1)\widehat{\psi}\|_2^2 + \alpha^2\|\phi'\|_2^2 \\
&\le \|-\phi'' + \phi''(1)\widehat{\psi}\|_2\,\|\tilde{v}_{\mathfrak{D}}\|_2 + C\alpha^2\big(\beta^{-13/12}x_\nu^{-5/6}\|f\|_2 \\
&\quad + \beta^{-1/4}[x_\nu^{-1}\log(\beta^{1/4}x_\nu)|\phi(x_\nu)| + x_\nu^{-1/2}\|\phi'\|_2]\big)\|\phi'\|_2.
\end{aligned}$$

For sufficiently large β_0 we then obtain, as $x_\nu > \beta^{-1/4}$,

$$\begin{aligned}
&\|\phi'' - \phi''(1)\widehat{\psi}\|_2 \\
&\quad\le C\,\big(\|\tilde{v}_{\mathfrak{D}}\|_2 + \alpha(\beta^{-13/12}x_\nu^{-5/6}\|f\|_2 + \beta^{-1/4}x_\nu^{-1}\log(\beta^{1/4}x_\nu)|\phi(x_\nu)|)\big).
\end{aligned}$$

For $\alpha < \beta^{1/6}$ (5.11.9) is readily verified.

Step 2.2. We prove (5.11.9) in the case $\beta^{1/6} \le \alpha \le \alpha_1 \beta^{1/3}$. In this case, we obtain, instead of (5.11.10), with the aid of (5.11.5) and again (5.2.2) (with $s = 1/2$) that

$$|\langle \phi''(1)\widehat{\psi}, \alpha^2 \phi\rangle| \le C\alpha^{3/2}\left(\beta^{-5/4}\|f\|_2 + \beta^{-1/4}\|\phi'\|_2\right)\|\phi'\|_2 .$$

Substituting the above into (5.10.18) yields, as above

$$\|\phi'' - \phi''(1)\widehat{\psi}\|_2 \le C(\|\tilde{v}_{\mathfrak{D}}\|_2 + \beta^{-7/6}\|f\|_2).$$

Consequently, (5.11.9) is valid also for $\beta^{1/6} \le \alpha < \alpha_1\beta^{1/3}$.

Step 3. We estimate $v_{\mathfrak{D}}$ in L^∞ for $\mu \ge -\mu_0$. Let $v_{\mathfrak{D}}$ be given by (5.4.2). Recall that by (5.10.20) $(\mathcal{L}_\beta - \beta\lambda)v_{\mathfrak{D}} = g_{\mathfrak{D}}$. To obtain an estimate for $\|v_{\mathfrak{D}}\|_\infty$ we begin, as in Step 6 in Proposition 5.10.1, by rewriting (5.10.20b) as the sum of five terms:

$$\begin{aligned} g_{\mathfrak{D}} = {} & [(U-\nu)(-f + \phi''(1)\hat{g})] + [i\mu(-f + \phi''(1)\hat{g})] \\ & - [2(U' - U'(x_\nu))\tilde{v}'_{\mathfrak{D}}] + [-2U'(x_\nu)\tilde{v}'_{\mathfrak{D}} - U''([\phi'' - \phi''(1)\widehat{\psi}] + \tilde{v}_{\mathfrak{D}}) \\ & - 2U^{(3)}\phi' - U^{(4)}\phi] - [U''\phi''(1)\widehat{\psi}]. \end{aligned} \tag{5.11.11}$$

We separately estimate the contribution of each term on the right-hand side of equation (5.11.11), using the interpolation inequality (5.10.21). For the first term on the right-hand side of (5.11.11), we apply (3.3.35) with f replaced by $-f + \phi''(1)\hat{g}$. For the second term, we use (3.1.3a), and (3.1.3b) with $p = 2$ (both valid for sufficiently large ν_2) for $-\beta^{-1/2} \le \mu < \Upsilon\beta^{-1/2}$ and (3.1.84)–(3.1.85) for the case $\mu < -\beta^{-1/2}$. For the third term we use (3.3.10) and (3.3.11). For the fourth term we use (3.1.3) to obtain

$$\|(\mathcal{L}_\beta^{\mathfrak{N},\mathfrak{D}} - \beta\lambda)^{-1}(U'(x_\nu)\tilde{v}'_{\mathfrak{D}})\|_\infty \le C\, x_\nu[\beta x_\nu]^{-1/2}\|\tilde{v}'_{\mathfrak{D}}\|_2 .$$

Finally, for the fifth term we use (4.6.3) as in the proof of (5.10.22d). Combining the above yields for $x_\nu \ge C\nu_2^{1/2}\beta^{-1/4}$

$$\begin{aligned} \|v_{\mathfrak{D}}\|_\infty \le C\,\big[& \beta^{-3/4}(\|f\|_2 + |\phi''(1)|\,\|\hat{g}\|_2) \\ & + [\beta x_\nu]^{-1/2}(\|\phi\|_{1,2} + \|\phi'' - \phi''(1)\widehat{\psi}\|_2 + \|\tilde{v}_{\mathfrak{D}}\|_2) \\ & + \beta^{-1/2}x_\nu^{1/2}\|\tilde{v}'_{\mathfrak{D}}\|_2 + \beta^{-1}|\phi''(1)|\big]. \end{aligned}$$

Using (5.11.9) we then obtain

$$\begin{aligned} \|v_{\mathfrak{D}}\|_\infty \le C\,\big[& \beta^{-3/4}(\|f\|_2 + |\phi''(1)|\,\|\hat{g}\|_2) \\ & + [\beta x_\nu]^{-1/2}(\|\phi\|_{1,2} + \|\tilde{v}_{\mathfrak{D}}\|_2 + \beta^{-1/12}x_\nu^{-1}\log(\beta^{1/4}x_\nu)\,|\phi(x_\nu)|) \\ & + \beta^{-1/2}x_\nu^{1/2}\|\tilde{v}'_{\mathfrak{D}}\|_2 + \beta^{-1}|\phi''(1)|\big]. \end{aligned} \tag{5.11.12}$$

For the estimate of $\|\tilde{v}_{\mathfrak{D}}\|_2$ we use a combination of (5.9.4) and (5.11.7).

Let $\check{x}_\mu := \check{x}_\mu(\Upsilon,\beta)$ be defined by (5.9.2). Then, there exists $\Upsilon > 0$ such that

$$\|\tilde{v}_{\mathfrak{D}}\|_2 \leq C \begin{cases} \big[\Upsilon^{-5/2}\beta^{-1/4}\|f\|_2+(\Upsilon^{-3/8}\mu_{+,\beta}^{3/4}+\Upsilon^{3/2}\beta^{3/16})|\phi(x_\nu)| \\ \qquad +(\mu_{+,\beta}^{1/2}+\Upsilon^2\beta^{1/8})\|\phi'\|_2\big] & x_\nu < \check{x}_\mu, \\ \big(x_\nu^{-1}\|\phi'\|_2+\beta^{1/6}x_\nu^{-5/6}|\phi(x_\nu)|+[\beta x_\nu]^{-2/3}\|f\|_2\big) & \text{otherwise.} \end{cases} \tag{5.11.13}$$

Thus, we set

$$\gamma(x_\nu,\check{x}_\mu) = \begin{cases} 1 & x_\nu \geq \check{x}_\mu, \\ 0 & x_\nu < \check{x}_\mu. \end{cases}$$

Substituting (5.11.4), (5.11.8), (5.11.13), (4.6.5), and (5.4.18) into (5.11.12) yields

$$\begin{aligned}\|v_{\mathfrak{D}}\|_\infty &\leq C\,\big(\beta^{-5/8}(x_\nu^{1/2}+\Upsilon^{-5/2}\beta^{-1/8}x_\nu^{-1/2})\|f\|_2 \\ &\quad+[\gamma(x_\nu,\check{x}_\mu)\beta^{-1/3}x_\nu^{-4/3}+\beta^{-5/24}x_\nu^{-1/3}+\mu_+^{1/2}\beta^{1/6}x_\nu^{-1/3}]|\phi(x_\nu)| \\ &\quad+[\gamma(x_\nu,\check{x}_\mu)\beta^{-1/2}x_\nu^{-3/2}+\beta^{-3/8}x_\nu^{-1/2}+\beta^{-7/24}x_\nu^{1/4}+\mu_+^{1/2}x_\nu^{-1/2}]\|\phi\|_{1,2}\big).\end{aligned} \tag{5.11.14}$$

We now obtain an estimate of $|\phi(x_\nu)|$ (see (5.11.18) below).

Step 4. With, as in (5.10.25)

$$\phi = \phi_{\mathfrak{D}} + \check{\phi}, \tag{5.11.15}$$

$$\phi_{\mathfrak{D}} = \mathcal{A}_{\lambda,\alpha}^{-1}v_{\mathfrak{D}};\quad \check{\phi} = -\mathcal{A}_{\lambda,\alpha}^{-1}([U+i\lambda]\phi''(1)\widehat{\psi}) \tag{5.11.16}$$

and

$$\begin{aligned}&\mathcal{C}_2(\nu_1,\nu_2,\Upsilon,\alpha_1,\beta_0,\kappa_0) \\ &\quad:= \Big\{(\lambda,\alpha,\beta)\in\mathcal{C}_1\,\Big|\,\beta^{-2}\leq|\mu|\leq\Upsilon\beta^{-1/2} \text{ or } -\frac{|U(0)-\nu|}{\kappa_0}<\mu<-\Upsilon\beta^{-1/2}\Big\},\end{aligned} \tag{5.11.17}$$

we prove that there exist $C>0$, Υ_0, and $\hat{\kappa}_0$ such that, for $0<\Upsilon\leq\Upsilon_0$ and $\kappa_0\geq\hat{\kappa}_0$, there exist $\hat{\nu}_2=\hat{\nu}_2(\Upsilon,\kappa_0)$ and $\beta_0=\beta_0(\Upsilon,\kappa_0)$ so that for $\nu_2\geq\hat{\nu}_2$ and $(\lambda,\alpha,\beta)\in\mathcal{C}_2(\nu_1,\nu_2,\Upsilon,\alpha_1,\beta_0,\kappa_0)$ we have

$$\begin{aligned}&|\check{\phi}(x_\nu)|+|\phi_{\mathfrak{D}}(x_\nu)| \\ &\quad\leq C\,\log(|\mu|^{-1/2}x_\nu)\,\big([\beta^{-1/4}+\gamma(x_\nu,\check{x}_\mu)\beta^{-1/2}x_\nu^{-3/2}+\mu_+^{1/2}x_\nu^{-1/2}]\|\phi'\|_2 \\ &\qquad+\beta^{-5/8}[x_\nu^{1/2}+\Upsilon^{-5/2}\beta^{-1/8}x_\nu^{-1/2}]\|f\|_2\big).\end{aligned} \tag{5.11.18}$$

By (2.8.47) applied to the pair $(\check{\phi},([U+i\lambda]\phi''(1)\widehat{\psi}))$, which holds for $(\lambda,\alpha,\beta)\in\mathcal{C}_2(\nu_1,\nu_2,\Upsilon,\alpha_1,\beta_0,\kappa_0)$ when $\nu_2\geq\Upsilon\kappa_0$, (5.11.4), and (5.2.2) for $s=1/2$, we obtain that

$$\begin{aligned}|\check{\phi}(x_\nu)| &\leq C\, x_\nu^{1/2}|\phi''(1)|\|(1-x)^{1/2}\hat{\psi}\|_1 \\ &\leq \hat{C}\,\big(\beta^{-1/4}\|\phi'\|_2 + \beta^{-1/4}x_\nu^{-1/2}\log(\beta^{1/4}x_\nu)|\phi(x_\nu)| \\ &\qquad + \beta^{-13/12}x_\nu^{-1/3}\|f\|_2\big). \end{aligned} \tag{5.11.19}$$

By (5.11.14) and (2.8.65) applied to the pair $(\phi_{\mathfrak{D}}, v_{\mathfrak{D}})$, we obtain that

$$\begin{aligned}|\phi_{\mathfrak{D}}(x_\nu)| \leq C\, \log\Big(\frac{x_\nu}{|\mu|^{1/2}}\Big)\big(&\beta^{-5/8}(x_\nu^{1/2} + \Upsilon^{-5/2}\beta^{-1/8}x_\nu^{-1/2})\|f\|_2 \\ &+ [\gamma(x_\nu,\check{x}_\mu)\beta^{-1/3}x_\nu^{-4/3} + \beta^{-5/24}x_\nu^{-1/3} + \mu_+^{1/2}\beta^{1/6}x_\nu^{-1/3}]|\phi(x_\nu)| \\ &+ [\gamma(x_\nu,\check{x}_\mu)\beta^{-1/2}x_\nu^{-3/2} + \beta^{-3/8}x_\nu^{-1/2} + \beta^{-7/24}x_\nu^{1/4} \\ &\quad + \mu_+^{1/2}x_\nu^{-1/2}]\|\phi\|_{1,2}\big).\end{aligned}$$

Combining the above yields, using the fact that $\beta^{-1/4} < x_\nu < 1$ and that $|\mu| < \mu_0$

$$\begin{aligned}|\check{\phi}(x_\nu)| &+ |\phi_{\mathfrak{D}}(x_\nu)| \\ &\leq C\, \log(|\mu|^{-1/2}x_\nu)\big([\beta^{-1/4} + \gamma(x_\nu,\check{x}_\mu)\beta^{-1/2}x_\nu^{-3/2} + \mu_+^{1/2}x_\nu^{-1/2}]\|\phi'\|_2 \\ &\qquad + \big[\beta^{-1/4}x_\nu^{-1/2}\log(\beta^{1/4}x_\nu) + \gamma(x_\nu,\check{x}_\mu)\beta^{-1/3}x_\nu^{-4/3} \\ &\qquad + \beta^{-5/24}x_\nu^{-1/3} + \mu_+^{1/2}\beta^{1/6}x_\nu^{-1/3}\big]|\phi(x_\nu)| \\ &\qquad + \beta^{-5/8}[x_\nu^{1/2} + \Upsilon^{-5/2}\beta^{-1/8}x_\nu^{-1/2}]\|f\|_2\big).\end{aligned} \tag{5.11.20}$$

We proceed by showing that the coefficient of $|\phi(x_\nu)|$ on the right-hand side of (5.11.20) can be made arbitrarily small by choosing ν_2 sufficiently large. We separately bound each term in the brackets.

For the second term we first observe that for $x_\nu > \Upsilon\mu_+^{-1/2}\beta^{-1/2}$, we have

$$\begin{aligned}\log(|\mu|^{-1/2}x_\nu)\,\beta^{-1/3}x_\nu^{-4/3} &\leq \log(\Upsilon^{-1}x_\nu^2\beta^{1/2})[\beta^{1/2}x_\nu^2]^{-2/3} \\ &\leq [\log(x_\nu^2\beta^{1/2}) + \log(\Upsilon^{-1})][\beta^{1/2}x_\nu^2]^{-2/3} \leq C[\nu_2^{-1/3} + \log\Upsilon^{-1}\nu_2^{-2/3}].\end{aligned}$$

Furthermore, for $\beta_0(\kappa_0)$ large enough and $(\lambda,\alpha,\beta) \in \mathcal{C}_2(\nu_1,\nu_2,\Upsilon,\alpha_1,\beta_0,\kappa_0)$, it holds that

$$|\mu|^{-1/2}x_\nu \leq \beta. \tag{5.11.21}$$

Hence, for $x_\nu > \beta^{-1/8}$, we have

$$\log(|\mu|^{-1/2}x_\nu)\beta^{-1/3}x_\nu^{-4/3} \leq \beta^{-1/6}\log(|\mu|^{-1/2}x_\nu) \leq C\beta^{-1/6}\log\beta.$$

Combining the pair of estimates, we obtain

$$\log(|\mu|^{-1/2}x_\nu)\gamma(x_\nu,\check{x}_\mu)\beta^{-1/3}x_\nu^{-4/3} \leq \tilde{C}\big(\nu_2^{-1/3} + \log\Upsilon^{-1}\nu_2^{-2/3} + \beta^{-1/6}\log\beta\big). \tag{5.11.22}$$

For the third term, we write, assuming that $\nu_2\Upsilon \geq 1$ (which implies $\mu_+^{-1/2}x_\nu \geq \frac{1}{C}\nu_2^{1/2}\Upsilon^{1/2}$ for $(\lambda,\alpha,\beta) \in \mathcal{C}_2(\nu_1,\nu_2,\Upsilon,\alpha_1,\beta_0,\kappa_0)$),

$$\begin{aligned}\log(|\mu|^{-1/2}x_\nu)\mu_+^{1/2}\beta^{1/6}x_\nu^{-1/3} &= \log(|\mu|^{-1/2}x_\nu)[\mu_+^{-1/2}x_\nu]^{-1/3}\mu_+^{1/3}\beta^{1/6}\\ &\leq C\mu_+^{1/3}\beta^{1/6} \leq \hat{C}\Upsilon^{1/3}.\end{aligned}$$

For the first term we obtain with the aid of (5.11.21)

$$\log(|\mu|^{-1/2}x_\nu)\beta^{-1/4}x_\nu^{-1/2}\log(\beta^{1/4}x_\nu) \leq C\beta^{-1/8}\log^2\beta.$$

We can now conclude (5.11.18).

Step 5. We prove that there exist $C>0$, Υ_0, and $\hat{\kappa}_0$ such that, for $0<\Upsilon\leq\Upsilon_0$ and $\kappa_0 \geq \hat{\kappa}_0$, we have for some $\widehat{\nu}_2 = \widehat{\nu}_2(\Upsilon,\kappa_0)$ and $\beta_0 = \beta_0(\Upsilon,\kappa_0)$ that, for $\nu_2 \geq \widehat{\nu}_2$ and $(\lambda,\alpha,\beta) \in \mathcal{C}_2(\nu_1,\nu_2,\Upsilon,\alpha_1,\beta_0,\kappa_0)$, (5.11.2) holds true

We first use (2.8.1b) (which holds $(\lambda,\alpha,\beta) \in \mathcal{C}_2$) applied to the pair $(\phi_{\mathfrak{D}}, v_{\mathfrak{D}})$ and (5.11.14) to obtain that

$$\begin{aligned}\|\phi'_{\mathfrak{D}}\|_2 \leq C\ \log(|\mu|^{-1/2}x_\nu)\big(&\beta^{-5/8}(1+\Upsilon^{-5/2}\beta^{-1/8}x_\nu^{-1})\|f\|_2\\ &+[\gamma(x_\nu,\check{x}_\mu)\beta^{-1/3}x_\nu^{-11/6}+\beta^{-5/24}x_\nu^{-5/6}+\mu_+^{1/2}\beta^{1/6}x_\nu^{-5/6}]|\phi(x_\nu)|\\ &+[\gamma(x_\nu,\check{x}_\mu)\beta^{-1/2}x_\nu^{-2}+\beta^{-3/8}x_\nu^{-1}+\beta^{-7/24}x_\nu^{-1/4}+\mu_+^{1/2}x_\nu^{-1}]\|\phi\|_{1,2}\big).\end{aligned} \tag{5.11.23}$$

Using (5.11.22) (which holds for $(\lambda,\alpha,\beta) \in \mathcal{C}_2$), and the fact that $\beta^{-1/4} < x_\nu$, we write

$$\begin{aligned}&\log(|\mu|^{-1/2}x_\nu)\gamma(x_\nu,\check{x}_\mu)\beta^{-1/2}x_\nu^{-2}\\ &\quad\leq \widetilde{C}\beta^{-1/6}x_\nu^{-2/3}\big(\nu_2^{-1/3}+\log\Upsilon^{-1}\nu_2^{-2/3}+\beta^{-1/6}\log\beta\big)\\ &\quad\leq \widetilde{C}\big(\nu_2^{-2/3}+\log\Upsilon^{-1}\nu_2^{-1}+\beta^{-1/6}\log\beta\big).\end{aligned} \tag{5.11.24}$$

Substituting (5.11.18) and (5.11.24) into (5.11.23), we obtain the existence of $C>0$ and $\beta_0>0$ such that for all $\beta>\beta_0$ we have

$$\begin{aligned}&\|\phi'_{\mathfrak{D}}\|_2 \leq C\big(\log(|\mu|^{-1/2}x_\nu)\beta^{-1/2}\|f\|_2\\ &+\big[\beta^{-1/6}x_\nu^{-2/3}+\beta^{-1/8}\log\beta+\nu_2^{-1/3}+\log\Upsilon^{-1}\nu_2^{-1}+\Upsilon^{1/4}\big]\,[\|\check{\phi}'\|_2+\|\phi'_{\mathfrak{D}}\|_2]\big).\end{aligned} \tag{5.11.25}$$

Note that in the proof we repeatedly use the inequalities

$$(|\mu|^{-1/2}x_\nu)^{-1/2}\log(|\mu|^{-1/2}x_\nu) \leq 1 \tag{5.11.26}$$

(which holds since $|\mu|^{-1/2}x_\nu > 1$ for sufficiently large ν_2), $\frac{1}{\widehat{C}}\nu_2^{1/2}\beta^{-1/4} < x_\nu < 1$ (which holds for some $\widehat{C} > 0$), $\mu_+ \leq \Upsilon\beta^{-1/2}$, and (5.11.21).

We now apply (2.8.46) (which holds for $(\lambda,\alpha,\beta) \in \mathcal{C}_2$ with sufficiently large ν_2) to the pair $(\check{\phi}, ([U+i\lambda]\phi''(1)\widehat{\psi}))$, (5.2.2) for $s=1/2$, and (5.11.4) as in (5.11.19) to obtain that

$$\|\check{\phi}'\|_2 \leq C\left(\beta^{-1/4}x_\nu^{-1/2}\|\phi'\|_2 + \beta^{-1/4}x_\nu^{-1}|\phi(x_\nu)| + \beta^{-13/12}x_\nu^{-5/6}\|f\|_2\right). \tag{5.11.27}$$

By (5.11.21), (5.11.18), (5.11.24), and (5.11.26) we have

$$\begin{aligned}|\phi(x_\nu)| \leq \\ C\Big(\{\beta^{-1/4}\log\beta + x_\nu^{1/2}[\beta^{-1/6}\log\beta + \nu_2^{-2/3} + \nu_2^{-1}\log\Upsilon^{-1}] + \Upsilon^{1/4}\beta^{-1/8}\}\|\phi'\|_2 \\ + \log(|\mu|^{-1/2}x_\nu)\beta^{-5/8}\Upsilon^{-5/2}\|f\|_2\Big).\end{aligned}$$

Hence, for $s \geq 1/2$, we have

$$\begin{aligned}(\beta^{-1/4}x_\nu^{-1})^s|\phi(x_\nu)| \leq C\big[(\nu_2^{-2/3} + \Upsilon^{1/4})\beta^{-1/8}\|\phi'\|_2 \\ + \beta^{-5/8}\log\beta\,\Upsilon^{-5/2}\nu_2^{-1/2}\|f\|_2\big].\end{aligned} \tag{5.11.28}$$

By (5.11.27) and (5.11.28) for $s=1$, we obtain

$$\|\check{\phi}'\|_2 \leq C(\beta^{-1/2}\log\beta\|f\|_2 + \beta^{-1/8}\|\phi'\|_2). \tag{5.11.29}$$

Combining (5.11.29) with (5.11.25) yields if $(\lambda,\alpha,\beta) \in \mathcal{C}_2(\nu_1,\nu_2,\alpha_1,\beta_0,\Upsilon,\kappa_0,)$ that there exist positive β_0 and C such that for all $\beta > \beta_0$ (the reader is referred for a precise description of the parameter space to the statement of step 4)

$$\|\phi'\|_2 \leq \|\check{\phi}'\|_2 + \|\phi'_{\mathfrak{D}}\|_2 \leq C\beta^{-1/2}\log\beta\|f\|_2. \tag{5.11.30}$$

Together with Poincaré's inequality, (5.11.30) implies (5.11.2) under the assumptions of Step 4.

To prove (5.11.2) in the next step for $|\mu| < \beta^{-2}$, we need to obtain an estimate of $\|\phi'' - \alpha^2\phi\|_2$ under the present condition on μ. By (5.11.28) for $s=5/6$ and (5.11.30) we obtain under (5.11.17) that for some $C>0$ (here and in the sequel the explicit dependence of $C = C(\kappa_0,\Upsilon)$ on (κ_0,Υ) is of little concern)

$$x_\nu^{-5/6}|\phi(x_\nu)| \leq C\beta^{-5/12}\log\beta\,\|f\|_2. \tag{5.11.31}$$

Substituting (5.11.30) and (5.11.31) into (5.11.7) yields that

$$\|\tilde{v}_{\mathfrak{D}}\|_2 \leq C\beta^{-1/4}\log\beta\|f\|_2.$$

Consequently, by (5.11.4), (5.4.6), (5.11.30), (5.11.31), and (4.6.5), it holds under the assumptions and conclusions of Steps 4 and 5 that

$$\begin{aligned}\|\phi'' - \alpha^2\phi\|_2 &\leq \|\tilde{v}_{\mathfrak{D}}\|_2 + |\phi''(1)|\,\|\widehat{\psi}\|_2 \\ &\leq C\beta^{-1/8}[x_\nu\beta^{1/4}]^{-1/6}\log(x_\nu\beta^{1/4})\log\beta\|f\|_2 \\ &\leq C\beta^{-1/8}\log\beta\|f\|_2. \end{aligned} \tag{5.11.32}$$

Step 6: We prove (5.11.2) *for* $|\mu| \leq \beta^{-2}$. We begin by writing as in (5.10.47)

$$\mathcal{B}_{\lambda+2\beta^{-2},\alpha}\phi = f + 2\beta^{-1}(\phi'' - \alpha^2\phi), \tag{5.11.33}$$

Hence, by (5.11.32) applied to the pair $(\phi, f + 2\beta^{-1}(\phi'' - \alpha^2\phi))$ with λ replaced by $\lambda + 2\beta^{-2}$, we have

$$\|\phi'' - \alpha^2\phi\|_2 \leq C(\beta^{-9/8}\log\beta\|\phi'' - \alpha^2\phi\|_2 + \beta^{-1/8}\log\beta\|f\|_2).$$

Consequently, we obtain that

$$\|\phi'' - \alpha^2\phi\|_2 \leq C\beta^{-1/8}\log\beta\|f\|_2.$$

We now apply (5.11.2) to (5.11.33) to obtain

$$\|\phi'\|_2 \leq C\beta^{-1/2}\log\beta(\|f\|_2 + \beta^{-1}\|\phi'' - \alpha^2\phi\|_2).$$

Hence,

$$\|\phi'\|_2 \leq C\beta^{-1/2}\log\beta\|f\|_2.$$

Step 7: We prove (5.11.2) *in the case* $-\mu_0 \leq \mu < -\frac{U(0)-\nu}{\kappa_0}$. We note for below that, under the above assumption,

$$|\mu| > \frac{1}{C(\kappa_0)}\nu_2^{1/2}\beta^{-1/2}. \tag{5.11.34}$$

In this case, we can apply (4.4.13), by (5.4.13), to the pair $(\phi'' - \alpha^2\phi, i\beta U''\phi + f)$, so that the L^∞ estimate is applied to $i\beta U''\phi$ and the L^2 estimate to f. We obtain

$$|\phi''(1)| \leq C(\beta^{-1/2}|\mu|^{-3/4}\|f\|_2 + \beta^{1/2}|\mu|^{-1/2}\|\phi\|_\infty).$$

Hence, by (2.9.13) applied to the pair $(\check{\phi}, \phi''(1)(U + i\lambda)\widehat{\psi})$ (see (5.10.27)) together with (5.2.2) for $s = 1/2$ and (5.11.34), we obtain

$$\begin{aligned}\|\check{\phi}'\|_2 &\leq C(|\mu|^{-3/4}\beta^{-5/4}\|f\|_2 + \beta^{-1/4}|\mu|^{-1/2}\|\phi\|_\infty) \\ &\leq \widehat{C}(\beta^{-7/8}\|f\|_2 + \nu_2^{-1/4}\|\phi'\|_2).\end{aligned}$$

From the above we conclude, using Poincaré's inequality, that for ν_2 large enough

$$\|\check{\phi}'\|_2 + |\check{\phi}(x_\nu)| \le C(\beta^{-7/8}\|f\|_2 + \nu_2^{-1/4}\|\phi'_{\mathfrak{D}}\|_2). \tag{5.11.35}$$

We next use (2.9.29) for the pair $(\phi_{\mathfrak{D}}, v_{\mathfrak{D}})$ (see (5.4.31b)) to obtain from (5.11.14)

$$|\phi_{\mathfrak{D}}(x_\nu)| \le C\|v_{\mathfrak{D}}\|_\infty \le \hat{C}\big(\beta^{-5/8}\|f\|_2 + \beta^{-1/4}\|\phi'\|_2 + \beta^{-1/8}|\phi(x_\nu)|\big)$$

which implies for $\beta \ge \beta_0$ with β_0 large enough

$$|\phi_{\mathfrak{D}}(x_\nu)| \le C\big(\beta^{-5/8}\|f\|_2 + \beta^{-1/4}\|\phi'\|_2 + \beta^{-1/8}|\check{\phi}(x_\nu)|\big). \tag{5.11.36}$$

In the sequel the explicit dependence on (κ_0, Υ) is of little concern to us and is therefore omitted.

Similarly, we obtain, using (2.9.1) with $p = +\infty$ for the pair $(\phi_{\mathfrak{D}}, v_{\mathfrak{D}})$ and (5.11.14)

$$\begin{aligned}\|\phi'_{\mathfrak{D}}\|_2 &\le C|\mu|^{-1/4}\|v_{\mathfrak{D}}\|_\infty \\ &\le \hat{C}\nu_2^{-1/8}\beta^{1/8}\big(\beta^{-5/8}\|f\|_2 + \beta^{-1/4}\|\phi'\|_2 + \beta^{-1/8}|\phi(x_\nu)|\big).\end{aligned}$$

Using Sobolev embedding for $\phi_{\mathfrak{D}}$, we obtain for sufficiently large ν_2

$$\|\phi'_{\mathfrak{D}}\|_2 \le C\nu_2^{-1/8}\big(\beta^{-1/2}\|f\|_2 + \beta^{-1/8}\|\phi'\|_2 + |\check{\phi}(x_\nu)|\big). \tag{5.11.37}$$

We now continue as in the derivation of (5.11.30) to obtain by (5.11.35), (5.11.36), and (5.11.37) for ν_2 and β_0 large enough and $\beta \ge \beta_0$

$$\|\phi'\|_2 \le \|\check{\phi}'\|_2 + \|\phi'_{\mathfrak{D}}\|_2 \le C\beta^{-1/2}\|f\|_2. \tag{5.11.38}$$

Step 8: The case $\mu < -\mu_0$. The proof of Step 7 in Proposition 5.10.1 furnishes (5.10.57) without any modification. Making use of Poincaré's inequality we then establish (5.11.2). ∎

5.12 Large $d(\Im\lambda, [0, U(0)])$

In the following we consider the case where ν lies outside the interval $[0, U(0)]$. We consider two different regimes.

- We begin with case $\nu - U(0) \gg \beta^{-1/2}$.
- We then continue by assuming $\beta^{1/3}(-\nu) \gg 1$.

We begin by introducing for some positive constants α_0, Υ, β_0 and κ_1 the zone

$$\mathcal{F}_1(\alpha_0, \beta_0, \Upsilon, \kappa_1) := \left\{\begin{matrix}(\lambda, \alpha, \beta) \in \mathbb{C} \times \mathbb{R}_+^2,\ \beta \ge \beta_0,\ 0 \le \alpha \le \alpha_0\beta^{1/3}, \\ \mu < \Upsilon\beta^{-1/2}, U(0) + \kappa_1\beta^{-1/2} \le \nu\end{matrix}\right\}.$$

Proposition 5.12.1. *Let $U \in C^4([0,1])$ satisfy (2.1.3) and α_0 and Υ denote positive constants. Then, there exist $\beta_0 > 0$, $\kappa_1 > 0$, and $C > 0$, such that, for $(\lambda, \alpha, \beta) \in \mathcal{F}_1(\alpha_0, \beta_0, \Upsilon, \kappa_1)$, it holds that*

$$\|(\mathcal{B}^{\mathfrak{N},\mathfrak{D}}_{\lambda,\alpha,\beta})^{-1}\| + \Big\|\frac{d}{dx}(\mathcal{B}^{\mathfrak{N},\mathfrak{D}}_{\lambda,\alpha,\beta})^{-1}\Big\| \leq C\beta^{-1/2}. \tag{5.12.1}$$

Proof. Step 1 For positive λ_0 and $\widetilde{\alpha}_0$, we prove that for κ_1 and β_0 large enough, (5.12.1) holds true under the additional conditions that $|\lambda| \leq \lambda_0$ and $\alpha \leq \widetilde{\alpha}_0\beta^{1/4}$.

Let $f \in L^2(0,1)$ and $\phi \in D(\mathcal{B}^{\mathfrak{N},\mathfrak{D}}_{\lambda,\alpha,\beta})$ satisfy

$$\mathcal{B}^{\mathfrak{N},\mathfrak{D}}_{\lambda,\alpha,\beta}\phi = f. \tag{5.12.2}$$

Taking the scalar product of (5.12.2) with $\frac{\phi}{U+i\lambda}$, and integrating by parts yields for the imaginary part

$$-\Im\Big\langle\frac{\phi}{U+i\lambda}, f\Big\rangle = \Im\Big\langle\Big(\frac{\phi}{U+i\lambda}\Big)'', \phi''\Big\rangle + \alpha^2\Im\Big\langle\Big(\frac{\phi}{U+i\lambda}\Big)', \phi'\Big\rangle$$
$$+ \beta\Big(\|\phi'\|_2^2 + \alpha^2\|\phi\|_2^2 + \Re\Big\langle\frac{U''\phi}{U+i\lambda}, \phi\Big\rangle\Big). \tag{5.12.3}$$

By (2.1.3) we obtain the following auxiliary estimate, which we repeatedly use in the sequel

$$\Big\|\frac{(U')^n}{(U+i\lambda)^m}\Big\|_\infty \leq C\Big\|\frac{x^n}{(x^2+\check{\nu})^m}\Big\|_\infty \leq C\,\check{\nu}^{-(m-n/2)} \quad \forall m \in \mathbb{R}_+,\ \forall n \in [0,2m], \tag{5.12.4}$$

where

$$\check{\nu} = \nu - U(0) > 0.$$

As

$$\Big(\frac{\phi}{U+i\lambda}\Big)'' = -2\frac{\phi'U'}{(U+i\lambda)^2} + \frac{\phi''}{U+i\lambda} - \frac{\phi U''}{(U+i\lambda)^2} + 2\frac{\phi(U')^2}{(U+i\lambda)^3}, \tag{5.12.5}$$

we need to estimate four different terms to obtain a bound for the first term on the right-hand side of (5.12.3). We begin by writing

$$\Big|\Big\langle\frac{\phi'U'}{(U+i\lambda)^2}, \phi''\Big\rangle\Big| \leq \Big|\Big\langle\frac{\phi'U'}{(U+i\lambda)^2}, \phi'' - \phi''(1)\widehat{\psi}\Big\rangle\Big| + \Big|\Big\langle\frac{\phi'U'}{(U+i\lambda)^2}, \phi''(1)\widehat{\psi}\Big\rangle\Big|,$$

to obtain by (4.6.5) and (5.12.4)

$$\Big|\Big\langle\frac{\phi'U'}{(U+i\lambda)^2}, \phi''\Big\rangle\Big| \leq C\check{\nu}^{-3/2}\left(\|\phi'' - \phi''(1)\widehat{\psi}\|_2 + \beta^{-1/4}|\phi''(1)|\right)\|\phi'\|_2.$$

The contribution of the second term on the right-hand is obtained as follows:

$$\Big|\Big\langle\frac{\phi''}{U+i\lambda},\phi''\Big\rangle\Big| \leq \check{\nu}^{-1}\|\phi''\|_2^2 \leq C\,\check{\nu}^{-1}\big(\|\phi''-\phi''(1)\widehat{\psi}\|_2^2+\beta^{-1/2}|\phi''(1)|^2\big).$$

To obtain an estimate for the contribution of the third term on the right-hand side of (5.12.5) we write

$$\begin{aligned}\Big|\Big\langle\frac{\phi U''}{(U+i\lambda)^2},\phi''\Big\rangle\Big| &\leq C\,\check{\nu}^{-1}\|(U+i\lambda)^{-1}\phi\|\,\|\phi''\|\\ &\leq \widehat{C}\,\check{\nu}^{-1}\|(U+i\lambda)^{-1}\phi\|[\|\phi''-\phi''(1)\widehat{\psi}\|_2+\beta^{-1/4}|\phi''(1)|].\end{aligned}$$

A similar estimate is obtained for the last term on the right-hand side of (5.12.5) by using (5.12.4) with $m=2, n=2$, i.e.,

$$\begin{aligned}&\Big|\Im\Big\langle\Big(\frac{\phi}{U+i\lambda}\Big)'',\phi''\Big\rangle\Big|\\ &\quad\leq C\,\check{\nu}^{-1}\Big(\|\phi''-\phi''(1)\widehat{\psi}\|_2^2+\beta^{-1/2}|\phi''(1)|^2\\ &\qquad\times\Big[\Big\|\frac{\phi}{U+i\lambda}\Big\|_2+\check{\nu}^{-1/2}\|\phi'\|_2\Big][\|\phi''-\phi''(1)\widehat{\psi}\|_2+\beta^{-1/4}|\phi''(1)|]\Big),\end{aligned}$$

from which we conclude that

$$\begin{aligned}\Big|\Im\Big\langle\Big(\frac{\phi}{U+i\lambda}\Big)'',\phi''\Big\rangle\Big| \leq C\,\check{\nu}^{-1}\Big(&\|\phi''-\phi''(1)\widehat{\psi}\|_2^2+\beta^{-1/2}|\phi''(1)|^2\\ &+\Big\|\frac{\phi}{U+i\lambda}\Big\|_2^2+\check{\nu}^{-1}\|\phi'\|_2^2\Big).\end{aligned} \tag{5.12.6}$$

For the second term on the right-hand side (5.12.3), we use (5.12.4) to obtain

$$\alpha^2\Big|\Im\Big\langle\Big(\frac{\phi}{U+i\lambda}\Big)',\phi'\Big\rangle\Big| \leq C\,\alpha^2\check{\nu}^{-1/2}\Big(\Big\|\frac{\phi}{U+i\lambda}\Big\|_2+\check{\nu}^{-1/2}\|\phi'\|_2\Big)\|\phi'\|_2. \tag{5.12.7}$$

Finally, as $U''(x)(U(x)-\nu)>0$, we can deduce that

$$\Re\Big\langle\frac{U''\phi}{U+i\lambda},\phi\Big\rangle \geq \min_{x\in[0,1]}|U''(x)|\check{\nu}\Big\|\frac{\phi}{U+i\lambda}\Big\|_2^2. \tag{5.12.8}$$

Substituting (5.12.6)–(5.12.8) into (5.12.3) yields

$$\begin{aligned}\|\phi'\|_2^2+\alpha^2\|\phi\|_2^2+\check{\nu}\Big\|\frac{\phi}{U+i\lambda}\Big\|_2^2 &\leq C\beta^{-1}\Big[\Big\|\frac{\phi}{U+i\lambda}\Big\|_2\|f\|_2\\ &+\check{\nu}^{-1}\big(\|\phi''-\phi''(1)\widehat{\psi}\|_2^2+\beta^{-1/2}|\phi''(1)|^2+[\check{\nu}^{-1}+\alpha^2]\|\phi'\|_2^2\big)\\ &+(\alpha^2+\check{\nu}^{-1})\Big\|\frac{\phi}{U+i\lambda}\Big\|_2^2\Big].\end{aligned}$$

From the above inequality we conclude, for sufficiently large κ_1 and β_0,

$$\|\phi'\|_2 + \check{\nu}^{1/2}\Big\|\frac{\phi}{U+i\lambda}\Big\|_2 \leq C\beta^{-1/2}\big[\check{\nu}^{1/2}\|f\|_2 + \check{\nu}^{-1/2}(\|\phi'' - \phi''(1)\widehat{\psi}\|_2 + \beta^{-1/4}|\phi''(1)|)\big]. \quad (5.12.9)$$

Clearly,

$$(\mathcal{L}_\beta^{\mathfrak{z}_\alpha} - \beta\lambda)(-\phi'' + \alpha^2\phi) = f + i\beta U''\phi,$$

where $\mathfrak{z}_\alpha$ is given by (4.7.1). For sufficiently large κ_1, (4.2.1) and (4.2.2) hold and since $\mu < \Upsilon\beta^{-1/2}$, we may use (4.2.4) here to obtain

$$|\phi''(1)| \leq C(\beta^{-1/3}\check{\nu}^{-5/12}\|f\|_2 + \beta^{1/2}\check{\nu}^{-1/2}\|\phi'\|_2). \quad (5.12.10)$$

Substituting (5.12.10) into (5.12.9) yields for sufficiently κ_1 and β_0

$$\|\phi'\|_2 + \check{\nu}^{1/2}\Big\|\frac{\phi}{U+i\lambda}\Big\|_2 \leq C\beta^{-1/2}\big[\check{\nu}^{1/2}\|f\|_2 + \check{\nu}^{-1/2}\|\phi'' - \phi''(1)\widehat{\psi}\|_2\big]. \quad (5.12.11)$$

To estimate $\|\phi'' - \phi''(1)\widehat{\psi}\|_2$ we set (see also (5.4.2)–(5.4.7))

$$\hat{v}_{\mathfrak{D}} := -\phi'' + \alpha^2\phi + \frac{U''\phi}{U+i\lambda} + \phi''(1)\widehat{\psi} = \frac{v_{\mathfrak{D}}}{U+i\lambda}. \quad (5.12.12)$$

A simple computation yields

$$(\mathcal{L}_\beta^{\mathfrak{N},\mathfrak{D}} - \beta\lambda)\hat{v}_{\mathfrak{D}} = h, \quad (5.12.13a)$$

where

$$h = -f - \Big(\frac{U''\phi}{U+i\lambda}\Big)'' + \phi''(1)\hat{g}. \quad (5.12.13b)$$

By [16, Theorem 1.3], which can be applied to the even extension of $\hat{v}_{\mathfrak{D}}$ and h to $(-1,1)$, it holds that

$$\|\hat{v}_{\mathfrak{D}}\|_2 \leq C\beta^{-1/2}\|h\|_2.$$

Hence, using (5.12.4) and (4.6.5) yields, as $\|\phi''\|_2 \leq \|\phi'' - \phi''(1)\widehat{\psi}\|_2 + |\phi''(1)|\|\widehat{\psi}\|_2$,

$$\|\hat{v}_{\mathfrak{D}}\|_2 \leq C\beta^{-1/2}\Big(\|f\|_2 + |\phi''(1)|\,\|\hat{g}\|_2 + \check{\nu}^{-1}\|\phi'' - \phi''(1)\widehat{\psi}\|_2 + \beta^{-1/4}|\phi''(1)| + \check{\nu}^{-3/2}\|\phi'\|_2 + \check{\nu}^{-1}\Big\|\frac{\phi}{U+i\lambda}\Big\|_2\Big). \quad (5.12.14)$$

Consequently, by (5.4.19) and (5.12.10) it holds that

$$\|\hat{v}_{\mathfrak{D}}\|_2 \leq C\beta^{-1/2}\Big(\|f\|_2 + \check{\nu}^{-1}\|\phi'' - \phi''(1)\widehat{\psi}\|_2 + (\check{\nu}^{-3/2} + \check{\nu}^{-1/2}\beta^{1/4})\|\phi'\|_2 + \check{\nu}^{-1}\Big\|\frac{\phi}{U+i\lambda}\Big\|_2\Big). \quad (5.12.15)$$

By (5.10.18), it holds that

$$\|\phi'' - \phi''(1)\widehat{\psi}\|_2^2 + \alpha^2\|\phi'\|_2^2 \leq C(\|\tilde{v}_{\mathfrak{D}}\|_2^2 + \alpha^2|\phi''(1)|\,\|\phi'\|_2\|(1-x)^{1/2}\widehat{\psi}\|_1).$$

Here, we recall that

$$\tilde{v}_{\mathfrak{D}} = \hat{v}_{\mathfrak{D}} - \frac{U''\phi}{U + i\lambda}. \tag{5.12.16}$$

Using (5.2.2) and (5.12.10), we may conclude that for sufficiently large κ_1, it holds that

$$\|\phi'' - \phi''(1)\widehat{\psi}\|_2^2 + \alpha^2\|\phi'\|_2^2 \leq \hat{C}\,\left(\|\tilde{v}_{\mathfrak{D}}\|_2^2 + \alpha^2\beta^{-13/6}\check{\nu}^{-5/6}\|f\|_2^2\right).$$

Then, we obtain from (5.12.12) and (5.12.16) that

$$\|\phi'' - \phi''(1)\widehat{\psi}\|_2 \leq C\left(\|\hat{v}_{\mathfrak{D}}\|_2 + \left\|\frac{\phi}{U + i\lambda}\right\|_2 + \beta^{-5/4}\|f\|_2\right).$$

By (5.12.15) we then obtain for sufficiently large κ_1

$$\begin{aligned}&\|\phi'' - \phi''(1)\widehat{\psi}\|_2 \\ &\quad \leq C\left(\beta^{-1/2}\|f\|_2 + (\check{\nu}^{-3/2}\beta^{-1/2} + \check{\nu}^{-1/2}\beta^{-1/4})\|\phi'\|_2 + \left\|\frac{\phi}{U + i\lambda}\right\|_2\right).\end{aligned}$$

Substituting the above into (5.12.11) yields

$$\begin{aligned}\|\phi'\|_2 &+ \check{\nu}^{1/2}\left\|\frac{\phi}{U + i\lambda}\right\|_2 \\ &\leq C\beta^{-1/2}\Big[\check{\nu}^{1/2}\|f\|_2 + (\check{\nu}^{-2}\beta^{-1/2} + \check{\nu}^{-1}\beta^{-1/4})\|\phi'\|_2 \\ &\qquad + \check{\nu}^{-1/2}\left\|\frac{\phi}{U + i\lambda}\right\|_2\Big].\end{aligned}$$

For sufficiently large κ_1 and β_0 we then obtain (5.12.1).

Step 2. We prove that there exists $\lambda_0 > 0$ and β_0 such that (5.12.1) holds under the additional condition $|\lambda| \geq \lambda_0$.

Clearly, we must have either $\mu \leq -\lambda_0/2$ or $\nu > \lambda_0/2$.

Consider first the case where $\mu \leq -\lambda_0/2$. As in (5.8.15) we write

$$\Re\langle\phi, \mathcal{B}_{\lambda,\alpha,\beta}\phi\rangle = \|\phi''\|_2^2 + |\mu|\beta\,[\|\phi'\|_2^2 + \alpha^2\|\phi\|_2^2] + \beta\Im\langle U'\phi, \phi'\rangle. \tag{5.12.17}$$

Consequently, using Poincaré's inequality, we obtain

$$\|\phi'\|_2 \leq \frac{C}{\lambda_0}(\beta^{-1}\|f\|_2 + \|\phi'\|_2). \tag{5.12.18}$$

For sufficiently large λ_0 and β_0 we can then conclude (5.12.1).

For $\nu > \lambda_0/2$ we write

$$\Im\langle\phi,\mathcal{B}_{\lambda,\alpha,\beta}\phi\rangle=\beta(-\langle(U-\nu)\phi',\phi'\rangle+\alpha^2\langle(U-\nu)\phi',\phi'\rangle-\Re\langle U'\phi,\phi'\rangle-\langle U''\phi,\phi\rangle). \tag{5.12.19}$$

Using Poincaré's inequality we then obtain

$$\|\phi'\|_2 \leq \frac{C}{\lambda_0 - U(0)}(\beta^{-1}\|f\|_2 + \|\phi'\|_2),$$

which validates (5.12.1) for sufficiently large λ_0 and β_0.

Step 3. We prove that there exist positive β_0, $\widetilde{\alpha}_0$, and α_0 such that (5.12.1) holds for $\widetilde{\alpha}_0\beta^{1/4} \leq \alpha \leq \alpha_0\beta^{1/3}$.

An integration by parts yields, in view of (5.4.6) (see also Step 2 of the proof of Proposition 5.11.1)

$$\langle\phi,\tilde{v}_{\mathfrak{D}}\rangle = \|\phi'\|_2^2 + \alpha^2\|\phi\|_2^2 - \phi''(1)\langle\phi,\widehat{\psi}\rangle.$$

Since by (5.2.2)

$$|\langle\phi,\widehat{\psi}\rangle| \leq \|\phi\|_\infty\|\widehat{\psi}\|_1 \leq C\beta^{-1/2}\|\phi\|_\infty,$$

we obtain

$$\|\phi'\|_2^2 + \alpha^2\|\phi\|_2^2 \leq \|\phi\|_2\|\tilde{v}_{\mathfrak{D}}\|_2 + C\beta^{-1/2}\|\phi\|_\infty|\phi''(1)|.$$

We can now conclude that

$$\|\phi'\|_2^2 + \alpha^2\|\phi\|_2^2 \leq C(\alpha^{-2}\|\tilde{v}_{\mathfrak{D}}\|_2^2 + \beta^{-1/2}|\phi''(1)|\,\|\phi\|_\infty), \tag{5.12.20}$$

yielding

$$\begin{aligned}\|\phi\|_\infty^2 \leq \|\phi'\|_2\|\phi\|_2 &\leq \frac{1}{2\alpha}(\|\phi'\|_2^2 + \alpha^2\|\phi\|_2^2) \\ &\leq C(\alpha^{-3}\|\tilde{v}_{\mathfrak{D}}\|_2^2 + \alpha^{-1}\beta^{-1/2}|\phi''(1)|\,\|\phi\|_\infty).\end{aligned}$$

We may then infer that

$$\|\phi\|_\infty \leq C(\alpha^{-3/2}\|\tilde{v}_{\mathfrak{D}}\|_2 + \alpha^{-1}\beta^{-1/2}|\phi''(1)|).$$

By (5.10.13) and (5.10.15) (both remain valid in the present case) we obtain

$$\|\phi\|_\infty \leq C\widetilde{\alpha}_0^{-1}(\|\phi\|_\infty + \beta^{-1/8}\|\phi'\|_2 + \beta^{-7/8}\|f\|_2).$$

For sufficiently large $\tilde{\alpha}_0$ it follows that

$$\|\phi\|_\infty \leq C\widetilde{\alpha}_0^{-1}(\beta^{-1/8}\|\phi'\|_2 + \beta^{-7/8}\|f\|_2). \tag{5.12.21}$$

Consequently, using (5.10.13) and (5.10.15) once again, it holds that

$$|\phi''(1)| + \beta^{3/8}\|\tilde{v}_{\mathfrak{D}}\|_2 \leq C(\beta^{-1/8}\|f\|_2 + \beta^{5/8}\|\phi'\|_2). \tag{5.12.22}$$

Substituting (5.12.21) into (5.12.20) then leads to

$$\|\phi'\|_2^2 \leq C(\alpha^{-2}\|\tilde{v}_{\mathfrak{D}}\|_2^2 + \tilde{\alpha}_0^{-1}\beta^{-5/8}|\phi''(1)|\,[\|\phi'\|_2 + \beta^{-3/4}\|f\|_2]).$$

Making use of (5.12.22) we then obtain that

$$\|\phi'\|_2 \leq C\tilde{\alpha}_0^{-1/2}(\|\phi'\|_2 + \beta^{-3/4}\|f\|_2).$$

For sufficiently large $\tilde{\alpha}_0$ we can then conclude (5.12.1) under the conditions of this step.

The proposition is proved. ■

We continue by introducing, for some positive constants α_0, Υ, β_0 and κ_2, the zone

$$\begin{aligned}&\mathcal{F}_2(\alpha_0, \beta_0, \Upsilon, \kappa_2)\\ &\quad := \big\{(\lambda, \alpha, \beta) \in \mathbb{C}\times\mathbb{R}_+^2, \beta \geq \beta_0, 0 \leq \alpha \leq \alpha_0\beta^{1/3}, \mu < \Upsilon\beta^{-1/2}, \nu \leq -\kappa_2\beta^{-1/3}\big\}.\end{aligned}$$

Proposition 5.12.2. *Let $U \in C^4([0,1])$ satisfy (2.1.3). Let further α_0 and Υ denote positive constants. Then, there exist $\beta_0 > 0$, $\kappa_2 > 0$, and $C > 0$, such that for all $(\lambda, \alpha, \beta) \in \mathcal{F}_2(\alpha_0, \beta_0, \Upsilon, \kappa_2)$ it holds*

$$\big\|(\mathcal{B}^{\mathfrak{N},\mathfrak{D}}_{\lambda,\alpha,\beta})^{-1}\big\| + \Big\|\frac{d}{dx}(\mathcal{B}^{\mathfrak{N},\mathfrak{D}}_{\lambda,\alpha,\beta})^{-1}\Big\| \leq C\beta^{-1}|\nu|^{-1}\log\beta. \tag{5.12.23}$$

Proof. Taking the scalar product of (5.12.2) with $w = (U-\nu)^{-1}\phi$, and integrating by parts yields for the imaginary part (see (5.12.3))

$$\begin{aligned}-\Im\langle w, f\rangle &= \Im\langle w'', \phi''\rangle + \alpha^2\Im\langle w', \phi'\rangle\\ &\quad + \beta\Big(\|(U-\nu)w'\|_2^2 + \alpha^2\|\phi\|_2^2\alpha^2\|\phi\|_2^2 - |\mu|^2\Big\langle\frac{\phi}{U-\nu}, \frac{U''\phi}{|U+i\lambda|^2}\Big\rangle\Big).\end{aligned}$$

Hence, since $(U-\nu)^{-1}U'' < 0$ we obtain that for any $\delta > 0$ there exists $C > 0$ such that

$$\begin{aligned}&\frac{1}{2}(\|(U-\nu)w'\|_2^2 + \nu^2\|w'\|_2^2) + \alpha^2\|\phi\|_2^2\\ &\quad \leq C\big(\delta^{-1}\beta^{-2}\|f\|_2^2 + \delta\|w\|_2^2 + \alpha^2[\beta^{-4/3}\|w'\|_2^2 + \beta^{-2/3}\|\phi'\|_2^2] + \beta^{-1}|\Im\langle w'', \phi''\rangle|\big).\end{aligned}$$

Step 1. With $\lambda_0 > 0$, we prove (5.12.23) for $(\lambda, \alpha, \beta) \in \mathcal{F}_2(\alpha_0, \beta_0, \Upsilon, \kappa_2)$ satisfying $|\lambda| \leq \lambda_0$.

Step 1a. We estimate $\hat{v}_{\mathfrak{D}} = \frac{v_{\mathfrak{D}}}{U+i\lambda}$ (as defined by (5.12.12)). By (5.12.13) and (3.1.3) it holds that

$$\|\hat{v}_{\mathfrak{D}}\|_2 \leq C\,\beta^{-2/3}\,\|h\|_2, \tag{5.12.24}$$

where h is given by (5.12.13b). Hence,

$$\|h\|_2 \le C\Big(\|f\|_2 + |\phi''(1)|\,\|\hat g\|_2 + \Big\|\frac{\phi}{(U+i\lambda)^3}\Big\|_2 + \Big\|\frac{\phi'}{(U+i\lambda)^2}\Big\|_2 + \Big\|\frac{\phi''}{(U+i\lambda)}\Big\|_2\Big)$$
$$\le C\Big(\|f\|_2 + |\phi''(1)|\,\|\hat g\|_2 + |\lambda|^{-1}\|\phi''\|_2 + \Big\|\frac{\phi}{(U+i\lambda)^3}\Big\|_2 + \Big\|\frac{\phi'}{(U+i\lambda)^2}\Big\|_2\Big). \tag{5.12.25}$$

To estimate the last two terms we use a decomposition of the interval of integration. (Note that for $\nu \le -1$ we have $\|\cdot\|_{L^2(0,1-|\nu|^{1/2})} = 0$ and $\|\cdot\|_{L^2(1-|\nu|^{1/2},1)} = \|\cdot\|_2$.) In addition, we need the bounds

$$\|(U+i\lambda)^{-m}\|_{L^\infty(0,1-|\nu|^{1/2})} \le C|\nu|^{-m/2}$$

and

$$\|(U+i\lambda)^{-m}\|_{L^\infty(0,1)} \le C|\lambda|^{-m}.$$

We now write

$$\|h\|_2 \le C\Big(\|f\|_2 + |\phi''(1)|\,\|\hat g\|_2 + |\lambda|^{-1}\|\phi''\|_2 + |\lambda|^{-2}\|\phi'\|_{L^2(1-|\nu|^{1/2},1)}$$
$$+ |\nu|^{-1}\|\phi'\|_{L^2(0,1-|\nu|^{1/2})} + |\lambda|^{-2}\Big\|\frac{\phi}{U+i\lambda}\Big\|_{L^2(1-|\nu|^{1/2},1)}$$
$$+ |\nu|^{-1}\Big\|\frac{\phi}{U+i\lambda}\Big\|_{L^2(0,1-|\nu|^{1/2})}\Big).$$

By Hardy's inequality (2.2.8) it holds that

$$\Big\|\frac{\phi}{U+i\lambda}\Big\|_{L^2(0,1-|\nu|^{1/2})} \le \Big\|\frac{\phi}{U}\Big\|_{L^2(0,1-|\nu|^{1/2})}$$
$$\le C\Big\|\frac{\phi}{1-x}\Big\|_{L^2(0,1-|\nu|^{1/2})} \le \widehat{C}\,\|\phi'\|_{L^2(0,1-|\nu|^{1/2})}.$$

On the interval $(1-|\nu|^{1/2},1)$, we have again by (2.2.8)

$$\Big\|\frac{\phi}{U+i\lambda}\Big\|_{L^2(1-|\nu|^{1/2},1)} \le C\|\phi'\|_2.$$

Combining the above yields

$$\|h\|_2 \le C\big(\|f\|_2 + |\phi''(1)|\,\|\hat g\|_2 + |\lambda|^{-1}\|\phi''\|_2 + |\lambda|^{-2}\|\phi'\|_{L^2(1-|\nu|^{1/2},1)} + |\nu|^{-1}\|\phi'\|_2\big). \tag{5.12.26}$$

By (5.4.15) and (5.4.19) it holds that

$$|\phi''(1)|\,\|\hat g\|_2 \le C|\lambda|^{-3/4}(\beta^{1/4}\|\phi'\|_2 + \beta^{-7/12}\|f\|_2). \tag{5.12.27}$$

Substituting (5.12.26) and (5.12.27) into (5.12.25) we obtain

$$\|h\|_2 \leq C\big(\|f\|_2 + |\lambda|^{-1}\|\phi''\|_2 + [|\nu|^{-1} + \beta^{1/4}|\lambda|^{-3/4}]\|\phi'\|_2 + |\lambda|^{-2}\|\phi'\|_{L^2(1-|\nu|^{1/2},1)}\big). \quad (5.12.28)$$

To bound $\|\phi''\|_2$ we use (5.10.54)–(5.10.55) and (5.12.16) to obtain

$$\|\phi''\|_2 \leq \|\phi'' - \alpha^2\phi\|_2 \leq \|\hat{v}_{\mathfrak{D}}\|_2 + \|U''\|_\infty \Big\|\frac{\phi}{U + i\lambda}\Big\|_2 + |\phi''(1)|\,\|\widehat{\psi}\|_2.$$

By (4.6.5), (5.12.24), (5.12.28), (2.2.8), and (5.4.15) (repeatedly using the lower bound $|\nu| \geq \kappa_2\beta^{-1/3}$) it holds that

$$\|\phi''\|_2 \leq C(|\lambda|^{1/4}\beta^{-7/12}\|f\|_2 + \beta^{-2/3}|\lambda|^{-1}\|\phi''\|_2 + |\lambda\beta|^{1/4}\|\phi'\|_2).$$

For sufficiently large β_0 we then obtain

$$\|\phi''\|_2 \leq C(|\lambda|^{1/4}\beta^{-7/12}\|f\|_2 + |\lambda\beta|^{1/4}\|\phi'\|_2). \quad (5.12.29)$$

Substituting (5.12.29) into (5.12.28) yields

$$\|h\|_2 \leq C\big(\|f\|_2 + [|\nu|^{-1} + \beta^{1/4}|\lambda|^{-3/4}]\|\phi'\|_2 + |\lambda|^{-2}\|\phi'\|_{L^2(1-|\nu|^{1/2},1)}\big). \quad (5.12.30)$$

To use (2.7.1b) we must provide an estimate for

$$N(v_{\mathfrak{D}}, \lambda) = \|(1-x)^{1/2}(U + i\lambda)^{-1}v_{\mathfrak{D}}\|_1 = \|(1-x)^{1/2}\hat{v}_{\mathfrak{D}}\|_1.$$

Thus, we use (3.1.75) and (5.12.13) to obtain

$$\|(1-x)^{1/2}\hat{v}_{\mathfrak{D}}\|_1 \leq C\|(U-\nu)^{1/2}\hat{v}_{\mathfrak{D}}\|_1 \leq C\,\|(U-\nu)^{-1/2}\|_2\,\|(U-\nu)\hat{v}_{\mathfrak{D}}\|_2 \leq \hat{C}\,\frac{\log\beta}{\beta}\|h\|_2.$$

Hence, we may conclude from (5.12.30) that

$$\|(1-x)^{1/2}\hat{v}_{\mathfrak{D}}\|_1 \leq C\,\frac{\log\beta}{\beta}\big(\|f\|_2 + [|\nu|^{-1} + \beta^{1/4}|\lambda|^{-3/4}]\|\phi'\|_2 + |\lambda|^{-2}\|\phi'\|_{L^2(1-|\nu|^{1/2},1)}\big). \quad (5.12.31)$$

Step 1b: We prove (5.12.23). As in (5.4.31), we let $\phi = \phi_{\mathfrak{D}} + \check{\phi}$, where

$$\phi_{\mathfrak{D}} = \mathcal{A}_{\lambda,\alpha}^{-1}([U + i\lambda]\hat{v}_{\mathfrak{D}}) = \mathcal{A}_{\lambda,\alpha}^{-1}v_{\mathfrak{D}}$$

and

$$\check{\phi} = \mathcal{A}_{\lambda,\alpha}^{-1}(\phi''(1)[U + i\lambda]\widehat{\psi}).$$

By (2.7.2) applied to the pair $(\check{\phi}, \phi''(1)[U + i\lambda]\widehat{\psi})$ it holds that

$$|\check{\phi}(x)| \leq C(1-x)^{1/2}[1 + \nu^{-1/2}(1-x)^{1/2}]|\phi''(1)\langle\check{\phi}, \widehat{\psi}\rangle|^{1/2}.$$

Hence, integrating over (0, 1),

$$|\langle\check{\phi}, \widehat{\psi}\rangle| \leq C(\|(1-x)^{1/2}\widehat{\psi}\|_1 + |\nu|^{-1/2}\|(1-x)\widehat{\psi}\|_1)|\langle\check{\phi}, \widehat{\psi}\rangle|^{1/2}|\phi''(1)|^{1/2},$$

which implies

$$|\langle\check{\phi}, \widehat{\psi}\rangle| \leq C(\|(1-x)^{1/2}\widehat{\psi}\|_1 + |\nu|^{-1/2}\|(1-x)\widehat{\psi}\|_1)^2|\phi''(1)|.$$

By (5.2.2) and (5.4.15) we then obtain

$$|\langle\check{\phi}, \widehat{\psi}\rangle| \leq C\,|\lambda\beta|^{-1}(\|\phi'\|_2 + \beta^{-5/6}\|f\|_2). \tag{5.12.32}$$

Using (2.7.3) applied to the pair $(\check{\phi}, \phi''(1)[U+i\lambda]\widehat{\psi})$, together with (2.7.6), (5.12.32), and (5.4.15) yields

$$\|\check{\phi}'\|_2^2 \leq C\,|\nu|^{-1}|\lambda|^{-1/2}\beta^{-1/2}(\|\phi'\|_2 + \beta^{-5/6}\|f\|_2)^2.$$

For sufficiently large κ_2 (and $(\lambda, \alpha, \beta) \in \mathcal{F}_2(\alpha_0, \beta_0, \Upsilon, \kappa_2)$) we then conclude that

$$\|\check{\phi}'\|_2 \leq C\,|\nu|^{-3/4}\beta^{-1/4}(\|\phi_{\mathfrak{D}}'\|_2 + \beta^{-5/6}\|f\|_2). \tag{5.12.33}$$

By (2.7.8) and (2.7.3), applied again to the pair $(\check{\phi}, \phi''(1)[U + i\lambda]\widehat{\psi})$, we then obtain that

$$\|\check{\phi}'\|^2_{L^2(1-|\nu|^{1/2},1)} \leq C\,|\nu|^{-1/2}\,|\phi''(1)|\,|\langle\check{\phi}, \widehat{\psi}\rangle|.$$

Using (5.12.32), we then conclude that

$$\|\check{\phi}'\|_{L^2(1-|\nu|^{1/2},1)} \leq C\,|\nu|^{-1/2}\beta^{-1/4}(\|\phi_{\mathfrak{D}}'\|_2 + \beta^{-5/6}\|f\|_2). \tag{5.12.34}$$

By (2.7.1a) applied to the pair $(\phi_{\mathfrak{D}}, (U+i\lambda)\hat{v}_{\mathfrak{D}})$, (5.12.31), (5.12.33), and (5.12.34), it holds, as $\phi = \phi_{\mathfrak{D}} + \check{\phi}$, that

$$\begin{aligned}\|\phi_{\mathfrak{D}}\|_{1,2} &\leq C|\nu|^{-1}\|(1-x)^{1/2}\hat{v}_{\mathfrak{D}}\|_1\\ &\leq \hat{C}\frac{\log\beta}{\beta}|\nu|^{-1}\big(\|f\|_2 + [|\nu|^{-1} + \beta^{1/4}|\nu|^{-3/4} + |\nu|^{-5/2}\beta^{-1/4}]\|\phi_{\mathfrak{D}}'\|_2\\ &\qquad + |\lambda|^{-2}\|\phi_{\mathfrak{D}}'\|_{L^2(1-|\nu|^{1/2},1)}\big).\end{aligned} \tag{5.12.35}$$

For sufficiently large β_0 we obtain from (5.12.35)

$$\|\phi_{\mathfrak{D}}\|_{1,2} \leq C\frac{\log\beta}{\beta}|\nu|^{-1}\big(\|f\|_2 + |\lambda|^{-2}\|\phi_{\mathfrak{D}}'\|_{L^2(1-|\nu|^{1/2},1)}\big). \tag{5.12.36}$$

For $\nu < -1/2$, we immediately obtain from (5.12.36) that for sufficiently large β_0

$$\|\phi_{\mathfrak{D}}\|_{1,2} \leq C\frac{\log\beta}{\beta}|\nu|^{-1}\|f\|_2. \tag{5.12.37}$$

For $\nu \geq -1/2$, we substitute (5.12.36) into (5.12.31) to conclude, with the aid of (5.12.33) and (5.12.34)

$$\|(1-x)^{1/2}\hat{v}_{\mathfrak{D}}\|_1 \leq C\frac{\log\beta}{\beta}\big(\|f\|_2 + |\lambda|^{-2}\|\phi'_{\mathfrak{D}}\|_{L^2(1-|\nu|^{1/2},1)}\big).$$

We can now use (2.7.1b) applied to the pair $(\phi_{\mathfrak{D}}, (U+i\lambda)\hat{v}_{\mathfrak{D}})$ to obtain

$$\|\phi'_{\mathfrak{D}}\|_{L^2(1-|\nu|^{1/2},1)} \leq C\frac{\log\beta}{\beta}|\nu|^{-3/4}\big(\|f\|_2 + |\lambda|^{-2}\|\phi'_{\mathfrak{D}}\|_{L^2(1-|\nu|^{1/2},1)}\big).$$

For sufficiently large β_0 we obtain that

$$\|\phi'_{\mathfrak{D}}\|_{L^2(1-|\nu|^{1/2},1)} \leq C\frac{\log\beta}{\beta}|\nu|^{-3/4}\|f\|_2,$$

which, when substituted into (5.12.36), yields in the case $-1/2 \leq \nu$

$$\|\phi_{\mathfrak{D}}\|_{1,2} \leq C\frac{\log\beta}{\beta}|\nu|^{-1}\,\|f\|_2.$$

The above inequality, together with inequalities (5.12.33), (5.12.37) yields (5.12.23) for $|\lambda| < \lambda_0$.

Step 2. We prove that there exists $\lambda_0 > 0$ such that (5.12.23) holds true for any $(\lambda, \alpha, \beta) \in \mathcal{F}_2(\alpha_0, \beta_0, \Upsilon, \kappa_2)$ satisfying $|\lambda| > \lambda_0$.

The proof is almost identical with Step 2 of Proposition 5.12.1. If $\mu < -\lambda_0/2$ we obtain (5.12.18) from (5.12.17) and then (5.12.23) for sufficiently large λ_0. If $\nu < -\lambda_0/2$ we use (5.12.19) to obtain (5.12.18) once again. ■

Remark 5.12.3. Note that there exists $\mu_1 > 0$ such that for all $\mu < -\mu_1\beta^{-1/3}$ equation (5.12.33) remains valid even in the case where κ_2 is not necessarily large. Thus, we may conclude that under the conditions of Proposition 5.12.2 for all $\kappa_2 > 0$, there exist $\beta_0 > 0$, $\mu_1 > 0$, and $C > 0$, such that for all $(\lambda, \alpha, \beta) \in \mathcal{F}_2(\alpha_0, \beta_0, \Upsilon, \kappa_2)$ satisfying $\mu < -\mu_1\beta^{-1/3}$ (5.12.23) holds true.