Chapter 6

Proof of the main theorems

6.1 Proofs of Theorems 1.1.1 and 1.1.2

The proofs of Theorems 1.1.1 and 1.1.2 rely on a combination of the relevant results in Chapter 5.

Proof of Theorem 1.1.1. We present the proof in the following table, which gives the precise range of parameters where each estimate is valid together with the estimate itself.

α	ν	μ	Stated in	Estimate
$\alpha \lesssim \beta^{1/3}$	$-\nu \gg \beta^{-1/3}$	$\mu \leq \beta^{-1/2}$	Prop. 5.12.2	$\beta^{-1}\lvert\nu\rvert^{-1}\log\beta$
$\alpha \ll \beta^{-1/6}$	$\lvert\nu\rvert \leq \nu_0$	$\mu < \beta^{-1/2}$	Prop. 5.8.2	$\beta^{-2/3}$
$\alpha \lesssim \beta^{1/3}$	$\beta^{-1/2} \ll U(0)-\nu$ $\nu \geq \nu_0$	$\mu \ll \beta^{-1/2}$	Prop. 5.11.1	$\beta^{-1/2}\log\beta$
$\alpha \lesssim \beta^{1/3}$	$\lvert U(0)-\nu\rvert \lesssim \beta^{-1/2}$	$\mu \ll \beta^{-1/2}$	Prop. 5.10.1	$\min(\beta^{-5/8}\lvert\mu\rvert^{-1/4}, \beta^{-3/8})$
$\alpha \lesssim \beta^{1/3}$	$\nu - U(0) \gg \beta^{-1/2}$	$\mu \lesssim \beta^{-1/2}$	Prop. 5.12.1	$\beta^{-1/2}$
$\alpha \gtrsim \beta^{1/3}$	$\nu \in \mathbb{R}$	$\mu \leq \mu^*$	Prop. 5.7.1	$\beta^{-1/2}$

In the above, $\mu^* = \min(\Upsilon\beta^{-1/2}, [\mu_m - \hat{\Upsilon}]\beta^{-1/3} - \alpha^2\beta^{-1}/2)$ for some sufficiently small $\Upsilon > 0$ and any $\hat{\Upsilon} > 0$.

From the table we learn that there exist positive α_L, β_0 and Υ such that for all $\beta > \beta_0$,

$$\sup_{\substack{0\leq\alpha\leq\alpha_L\beta^{-1/6}\\ \Re\lambda\leq\Upsilon\beta^{-1/2}}} \left\|(\mathcal{B}^{\mathfrak{D},\mathrm{sym}}_{\lambda,\alpha,\beta})^{-1}\right\| + \left\|\frac{d}{dx}(\mathcal{B}^{\mathfrak{D},\mathrm{sym}}_{\lambda,\alpha,\beta})^{-1}\right\| \leq C\beta^{-3/8}. \tag{6.1.1}$$

Furthermore, for $\mu = \Upsilon\beta^{-1/2}$ it holds for all $\beta > \beta_0$ that

$$\sup_{\substack{0\leq\alpha\leq\alpha_L\beta^{-1/6}\\ \Re\lambda=\Upsilon\beta^{-1/2}}} \left\|(\mathcal{B}^{\mathfrak{D},\mathrm{sym}}_{\lambda,\alpha,\beta})^{-1}\right\| + \left\|\frac{d}{dx}(\mathcal{B}^{\mathfrak{D},\mathrm{sym}}_{\lambda,\alpha,\beta})^{-1}\right\| \leq C\beta^{-1/2}\log\beta. \tag{6.1.2}$$

By (6.1.1) $\mathcal{B}^{\mathfrak{D},\mathrm{sym}}_{\lambda,\alpha}$ depends holomorphically on λ for all $\mu \leq \Upsilon\beta^{-1/2}$, and hence we can use (6.1.2) together with the Phragmén–Lindelöf theorem to obtain (1.1.12). ■

Proof of Theorem 1.1.2. As in the proof of Theorem 1.1.1, we use the following table, which gives the precise range of parameters where each estimate is valid together with the estimate itself.

α	ν	μ	Stated in	Estimate
$\alpha \lesssim \beta^{1/3}$	$-\nu \gg \beta^{-1/3}$	$\mu \leq \beta^{-1/2}$	Prop. 5.12.2	$\beta^{-1}\|\nu\|^{-1}\log\beta$
$\alpha \lesssim 1$	$\beta^{-1/5+\delta} \leq \nu < \nu_0$	$\mu < \beta^{-2/5-\delta}$	Prop. 5.4.1	$\beta^{-1/2+\delta}$
$1 \ll \alpha \ll \beta^{1/3}$	$\|\nu\| < \nu_0$	$\mu \ll \beta^{-1/3}$	Prop. 5.6.1	$\beta^{-5/6}$
$\beta^{-1/10+\delta} \leq \alpha \lesssim 1$	$\|\nu\| \leq \beta^{-1/5+\delta}$	$\mu < \beta^{-1/3-\delta}$	Prop. 5.5.1	$\beta^{-1/2+\delta}$
$\alpha \lesssim \beta^{1/3}$	$\beta^{-1/2} \ll U(0)-\nu$, $\nu \geq \nu_0$	$\mu \ll \beta^{-1/2}$	Prop. 5.11.1	$\beta^{-1/2}\log\beta$
$\alpha \lesssim \beta^{1/3}$	$\|U(0)-\nu\| \lesssim \beta^{-1/2}$	$\mu \ll \beta^{-1/2}$	Prop. 5.10.1	$\min\left(\beta^{-5/8}\|\mu\|^{-1/4}, \beta^{-3/8}\right)$
$\alpha \lesssim \beta^{1/3}$	$\nu - U(0) \gg \beta^{-1/2}$	$\mu \lesssim \beta^{-1/2}$	Prop. 5.12.1	$\beta^{-1/2}$
$\alpha \gtrsim \beta^{1/3}$	$\nu \in \mathbb{R}$	$\mu \lesssim \beta^{-1/2}$	Prop. 5.7.1	$\beta^{-1/2}$

From the table we learn that there exist positive β_0 and Υ such that for all $\beta > \beta_0$,

$$\sup_{\substack{\beta^{-1/10+\delta}\leq\alpha \\ \Re\lambda\leq\Upsilon\beta^{-1/2}}} \left\|(\mathcal{B}^{\mathfrak{D},\mathrm{sym}}_{\lambda,\alpha,\beta})^{-1}\right\| + \left\|\frac{d}{dx}(\mathcal{B}^{\mathfrak{D},\mathrm{sym}}_{\lambda,\alpha,\beta})^{-1}\right\| \leq C\beta^{-3/8}. \tag{6.1.3}$$

Furthermore, for $\mu = \Upsilon\beta^{-1/2}$ it holds for any $\delta > 0$ and $\beta > \beta_0$ that

$$\sup_{\substack{\beta^{-1/10+\delta}\leq\alpha \\ \Re\lambda=\Upsilon\beta^{-1/2}}} \left\|(\mathcal{B}^{\mathfrak{D},\mathrm{sym}}_{\lambda,\alpha,\beta})^{-1}\right\| + \left\|\frac{d}{dx}(\mathcal{B}^{\mathfrak{D},\mathrm{sym}}_{\lambda,\alpha,\beta})^{-1}\right\| \leq C\beta^{-1/2+\delta}. \tag{6.1.4}$$

By (6.1.3) $B^{\mathfrak{D},\mathrm{sym}}_{\lambda,\alpha,\beta}$ depends holomorphically on λ for all $\mu \leq \Upsilon\beta^{-1/2}$, and hence we can use (6.1.4) together with the Phragmén–Lindelöf theorem to obtain (1.1.13). ∎