# Abstract

The functional graph of a function $g : X \to X$ is the directed graph with vertex set $X$ the edges of which are of the form $x \to g(x)$ for $x \in X$. Functional graphs are studied because they allow one to understand the behavior of $g$ under iteration (i.e., to understand the discrete dynamical system $(X, g)$), which has various applications, especially when $X$ is a finite field $\mathbb{F}_q$. This memoir is an extensive study of the functional graphs of so-called index $d$ generalized cyclotomic mappings of $\mathbb{F}_q$, which are a natural and manageable generalization of monomial functions. We provide both theoretical results on the structure of their functional graphs and Las Vegas algorithms for solving fundamental problems, such as parametrizing the connected components of the functional graph by representative vertices, or describing the structure of a connected component given by a representative vertex. The complexity of these algorithms is analyzed in detail, and we make the point that for fixed index $d$ and most prime powers $q$ (in the sense of asymptotic density), suitable implementations of these algorithms have an expected runtime that is polynomial in $\log q$ on quantum computers, whereas their expected runtime is subexponential in $\log q$ on a classical computer. We also discuss four special cases in which one can devise Las Vegas algorithms with this kind of complexity behavior over most finite fields that solve the graph isomorphism problem for functional graphs of generalized cyclotomic mappings.

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