Chapter 3

Functional graphs of generalized cyclotomic mappings

Let f be a generalized cyclotomic mapping of $\mathbb{F}_q$ of index d. From the introduction, we recall our notation C_i for $i \in \{0, 1, \ldots, d\}$, where $C_d = \{0_{\mathbb{F}_q}\}$, and $C_i = \omega^i C$ for $i < d$ is a coset of the index d subgroup C of $\mathbb{F}_q^* = \langle \omega \rangle$. Moreover, we recall that for each $i \in \{0, 1, \ldots, d-1\}$ we have a natural bijection $\iota_i : \mathbb{Z}/s\mathbb{Z} \to C_i = \omega^i C$, $x \mapsto \omega^{i+dx}$, by virtue of which we view C_i as a copy of $\mathbb{Z}/s\mathbb{Z}$, where

$$s = (q-1)/d = |C|.$$

As long as f does not map C_i constantly to $C_d = \{0\}$, this allows us to view the restriction $f_{|C_i}$ as an affine function $A_i : \mathbb{Z}/s\mathbb{Z} \to \mathbb{Z}/s\mathbb{Z}$. Finally, we recall the induced function $\bar{f} : \{0, 1, \ldots, d\} \to \{0, 1, \ldots, d\}$ (the unique function such that $f(C_i) \subseteq C_{\bar{f}(i)}$ for each i).

Our goal in this chapter is to describe methods through which the isomorphism type of the functional graph Γ_f can be understood, following the approach outlined in the introduction.

3.1 Periodic points and CRL-lists

Understanding the periodic points and finding a CRL-list of f can be reduced to the corresponding tasks for affine maps of finite cyclic groups. We observe that periodic points of f are necessarily contained in "periodic blocks" (i.e., blocks C_i such that i is periodic under $\bar{f}$). We assume that it is an easy task (due to d being sufficiently small) to find a CRL-list $\bar{\mathcal{L}}$ for $\bar{f}$. We determine the periodic points of f according to the "block cycle" of $\bar{f}$ they lie on, so let $(i, \ell) \in \bar{\mathcal{L}}$.

If $i = d$, then $\ell = 1$, and the only point to consider is the field element 0, which is by definition periodic of cycle length 1 under f. We note the contribution $\mathcal{L}_d := \{(0, 1)\}$ to the CRL-list $\mathcal{L}$ of f we are building.

Now we assume that $i < d$, and that the cycle of i under $\bar{f}$ is $(i_0, i_1, \ldots, i_{\ell-1})$ with $i_0 = i$. A point $x \in C_i$ is periodic under f if and only if it is periodic under the iterate f^ℓ, which stabilizes C_i and acts on the corresponding copy of $\mathbb{Z}/s\mathbb{Z}$ via the affine map $\mathcal{A}_i := A_{i_0} A_{i_1} \cdots A_{i_{\ell-1}}$. In other words, the periodic points of f in C_i are in bijection (via ι_i) to the periodic points of $\mathcal{A}_i$ in $\mathbb{Z}/s\mathbb{Z}$, and are thus characterized by Lemma 2.1.14. We note that the set of periodic points of f in a different coset C_{i_t} of the same $\bar{f}$-cycle is simply the iterated set image $f^t(\mathrm{per}(f_{|C_i}))$. Moreover, the cycle length of a periodic point $x \in C_i$ under f is the ℓ-fold of its cycle length under $\mathcal{A}_i$. It follows that if $\mathcal{L}_i' \subseteq \mathbb{Z}/s\mathbb{Z} \times \mathbb{N}^+$ is a CRL-list of $\mathcal{A}_i$ (which we can determine as

described in Section 2.3), then $\mathcal{L}_i := \{(\iota_i(r), \ell \cdot l) : (r, l) \in \mathcal{L}'_i\}$ is CRL-list of the restriction of f to the entire "coset cycle spanned by C_i" (i.e., to the set $\bigcup_{t=0}^{\ell-1} C_{i_t}$). In summary, we obtain the following proposition.

Proposition 3.1.1. *Let $\bar{\mathcal{L}}$ be a CRL-list for $\bar{f}$. We set $\mathcal{L}_d := \{(0_{\mathbb{F}_q}, 1)\}$. Moreover, for $i < d$ with $(i, \ell) \in \bar{\mathcal{L}}$, we define $\mathcal{L}_i$ as follows. Let $(i_0, i_1, \ldots, i_{\ell-1})$ with $i_0 = i$ be the $\bar{f}$-cycle of i, and let $\mathcal{L}'_i$ be a CRL-list of the affine map $\mathcal{A}_i = A_{i_0} A_{i_1} \cdots A_{i_{\ell-1}}$ of $\mathbb{Z}/s\mathbb{Z}$. Then we set $\mathcal{L}_i := \{(\iota_i(r), \ell \cdot l) : (r, l) \in \mathcal{L}'_i\}$. With this definition of $\mathcal{L}_i$ for each $(i, \ell) \in \bar{\mathcal{L}}$, we have that*

$$\mathcal{L} := \bigcup_{(i,\ell)\in\bar{\mathcal{L}}} \mathcal{L}_i$$

is a CRL-list of f.

3.2 The induced subgraph on the periodic cosets

Our next goal is to understand the trees $\mathrm{Tree}_{\Gamma_f}(x)$ in Γ_f above periodic points x of f, in the sense of Definition 1.4. We remind the reader that $\mathrm{Tree}_{\Gamma_f}(x)$ is defined for arbitrary vertices x of Γ_f, not just periodic ones. In general, it is advantageous to take a recursive approach, understanding $\mathrm{Tree}_{\Gamma_f}(x)$ for vertices x according to their depth in Γ_f, starting with leaves and working toward periodic vertices, which are at the end of the recursion. Before we carry this out, however, we must understand the induced subgraph Γ_{per} of Γ_f on the union of all periodic blocks C_i (i.e., blocks where i is periodic under $\bar{f}$) as a stepping stone.

We observe that Γ_{per} is the functional graph of the restriction f_{per} of f to the union of all periodic blocks. Just like f has the induced function $\bar{f}$ on the index set $\{0, 1, \ldots, d\}$, the restriction f_{per} has the induced function $\overline{f_{\mathrm{per}}}$, which is the restriction of $\bar{f}$ to its set of periodic points. Hence, $\overline{f_{\mathrm{per}}}$ is a permutation of its domain of definition, a fact that is important for our argument.

Similarly to the situation described in Proposition 2.1.8, if we know, for a given periodic vertex x of $\Gamma_{\mathrm{per}} = \Gamma_{f_{\mathrm{per}}}$, that each $\mathrm{Tree}_{\Gamma_{\mathrm{per}}}(y)$, where y is a child of x in Γ^*_{per}, has rigid procreation, and we know the first $h = h(y)$ procreation numbers of each child y, where h is the height of $\mathrm{Tree}_{\Gamma_{\mathrm{per}}}(y)$, then this characterizes the isomorphism type of each $\mathrm{Tree}_{\Gamma_{\mathrm{per}}}(y)$, and thus of $\mathrm{Tree}_{\Gamma_{\mathrm{per}}}(x)$, uniquely. And indeed, while Γ^*_{per} itself need not have rigid procreation in the more general setting we are considering here, the trees we just referred to do have it. More specifically, we have the following result (in which the exclusion of $i = d$ is without loss of generality, because $\mathrm{Tree}_{\Gamma_{\mathrm{per}}}(0_{\mathbb{F}_q})$ is trivial anyway).

Theorem 3.2.1. *Let f_{per} and $\overline{f_{\mathrm{per}}}$ be as above. Moreover, let $i \in \mathrm{dom}(\overline{f_{\mathrm{per}}}) = \mathrm{per}(\bar{f})$ with $i < d$, and let $(i_0, i_1, \ldots, i_{\ell-1})$ be the cycle of $i = i_0$ under $\overline{f_{\mathrm{per}}}$. We extend the*

notation i_t to arbitrary $t \in \mathbb{Z}$ by reducing t modulo ℓ (so that, for instance, $i_\ell = i_0$). For $t = 0, 1, \ldots, \ell - 1$, say $A_{i_t} : z \mapsto \alpha_{i_t} z + \beta_{i_t}$ is the affine map of $\mathbb{Z}/s\mathbb{Z}$ that describes how f_{per} (or, equivalently, f) maps from C_{i_t} to $C_{i_{t+1}}$, and let $\varphi_{i_t} := \mu_{\alpha_{i_t}} : z \mapsto \alpha_{i_t} z$, be the associated group endomorphism of $\mathbb{Z}/s\mathbb{Z}$. Then the following holds for any positive integer k. If $x \in C_i$ has at least k successor generations in Γ^*_{per} (we note that those successor generations need not be entirely contained in C_i), then

$$\mathrm{proc}_k^{(\Gamma^*_{\mathrm{per}})}(x) = \left|\ker\left(\prod_{j=0}^{k-1} \varphi_{i_{-k+j}}\right) : \ker\left(\prod_{j=0}^{k-2} \varphi_{i_{-k+j}}\right)\right| = \frac{\gcd\left(\prod_{j=0}^{k-1} \alpha_{i_{-k+j}}, s\right)}{\gcd\left(\prod_{j=0}^{k-2} \alpha_{i_{-k+j}}, s\right)},$$

independently of x.

Proof. This theorem can be seen as a generalization of Theorem 2.1.21 (which corresponds to the case $\ell = 1$), and likewise, its proof is a generalization of that of Theorem 2.1.21. We proceed by induction on k. For $k = 1$, we observe that $C_{\overline{f_{\mathrm{per}}}^{-1}(i)} = C_{i_{-1}}$ is the unique coset which f_{per} maps to C_i. Hence

$$\mathrm{proc}_1^{(\Gamma^*_{\mathrm{per}})}(x) = \#\text{ children of } x \text{ in } \Gamma^*_{\mathrm{per}} = |\{y \in \mathbb{Z}/s\mathbb{Z} : A_{i_{-1}}(y) = x\}| = |\ker(\varphi_{i_{-1}})|,$$

which implies the statement for $k = 1$ since an empty product of group endomorphisms is by definition the identity function id.

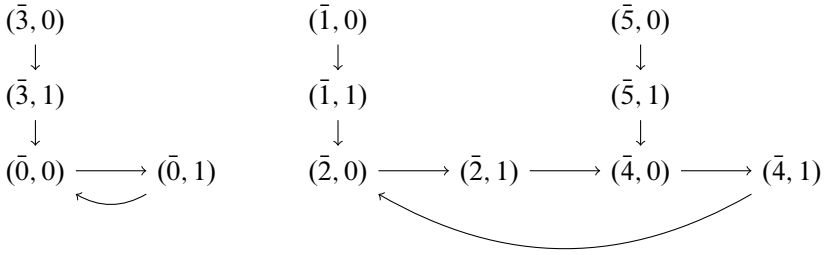
Now we assume that $k \geq 2$ and that the statement holds up to $k - 1$ for points in C_{i_t}, where $t \in \mathbb{Z}$ is arbitrary. For $h = 1, 2, \ldots, k-1$ and $t \in \mathbb{Z}$, we denote by $\mathrm{proc}_{i_t,h}$ the common procreation number $\mathrm{proc}_h^{(\Gamma^*_{\mathrm{per}})}(y)$ for all vertices $y \in C_{i_t}$ with at least h successor generations in Γ^*_{per}. For each $h = 1, 2, \ldots, k-1$, the number of endpoints of paths of length h in Γ^*_{per} starting at a vertex in C_{i_t} with at least h successor generations in Γ^*_{per} is $\prod_{j=0}^{h-1} \mathrm{proc}_{i_{t-j},h-j}$, and an easy induction on h shows that it is also equal to $|\ker(\prod_{j=0}^{h-1} \varphi_{i_{t-h+j}})|$. Using this, it follows that for each $x \in C_i = C_{i_0}$ with at least k successor generations, one has

$$\begin{aligned}
&\mathrm{proc}_k^{(\Gamma^*_{\mathrm{per}})}(x) \cdot \left|\ker\left(\prod_{j=0}^{k-2} \varphi_{i_{-k+j}}\right)\right| \\
&\quad = \mathrm{proc}_k^{(\Gamma^*_{\mathrm{per}})}(x) \cdot \prod_{j=0}^{k-2} \mathrm{proc}_{i_{-j-1},k-1-j} \\
&\quad = (\#\text{ endpoints of paths of length } k \text{ starting at } x) = \left|\ker\left(\prod_{j=0}^{k-1} \varphi_{i_{-k+j}}\right)\right|,
\end{aligned}$$

from which the asserted formula for $\mathrm{proc}_k^{(\Gamma^*_{\mathrm{per}})}(x)$ follows readily. ∎

The following example highlights some properties that do *not* hold in general.

Example 3.2.2. Let $q = 13$, $d = 2$ (thus $s = 6$), and $\bar{f} = (0,1)(2)$, so that $\Gamma_f = \Gamma_{\text{per}}$. Moreover, we assume that $A_0(z) = z$ and $A_1(z) = 2z$. Then $f_{|\mathbb{F}_q^*}$ has the following functional graph, in which we denote the point in C_i corresponding to $z \in \mathbb{Z}/6\mathbb{Z}$ by (z, i):



We note the following.

- The rooted trees above periodic vertices in C_0 are *not* isomorphic to the rooted trees above periodic vertices in C_1.
- Transient vertices in C_1 have strictly larger tree height in Γ_f than periodic vertices in C_1. More specifically, the transient vertices in C_1 are just those with tree height 1, the periodic vertices are those with tree height 0 in Γ_f.
- The set of possible tree heights in Γ_f above vertices in C_0 is $\{0, 2\}$, which is *not* an integer interval.

3.3 The rooted trees

We describe a recursive approach for understanding $\text{Tree}_{\Gamma_f}(x)$ for each vertex $x \in \mathbb{F}_q = \text{V}(\Gamma_f)$. We proceed in three steps, according to the unique $i \in \{0, 1, \ldots, d\}$ such that $x \in C_i$. Unless $i = d$, our goal is to find an arithmetic partition $\mathcal{P}_i$ of $\mathbb{Z}/s\mathbb{Z}$, corresponding to a partition of C_i via the bijection ι_i (that we also call an *arithmetic partition of* C_i), such that for vertices $x \in C_i$ from a common block $\mathcal{B}(\mathcal{P}_i, \vec{v}^{(\mathcal{P}_i)})$ of that partition, the isomorphism type of $\text{Tree}_{\Gamma_f}(x)$ is constant, denoted by $\text{Tree}_i(\mathcal{P}_i, \vec{v}^{(\mathcal{P}_i)})$. We also want to understand $\text{Tree}_i(\mathcal{P}_i, \vec{v}^{(\mathcal{P}_i)})$ in terms of $\vec{v}^{(\mathcal{P}_i)}$ explicitly, and verify that for fixed d, the maximum (arithmetic) complexity of $\mathcal{P}_i$ (in the sense of Definition 1.5 (3)) is in $O(d^2 \,\text{mpe}(q-1)) \subseteq O(d^2 \log q)$, where $\text{mpe}(m) := \max_p \nu_p(m)$ for $m \in \mathbb{N}^+$ (and p ranges over all primes). First, we introduce a few notations.

- For $M \subseteq \mathbb{F}_q$ and $x \in \mathbb{F}_q$, the notation $\text{Tree}_{\Gamma_f}(x, M)$ denotes the digraph isomorphism type of the subgraph of Γ_f that is a rooted tree with root x, obtained by attaching to that root all rooted trees $\text{Tree}_{\Gamma_f}(y)$, where y is an f-transient pre-image of x with $y \in M$. We note that $\text{Tree}_{\Gamma_f}(x, \mathbb{F}_q) = \text{Tree}_{\Gamma_f}(x)$.

- Let $M \subseteq \mathbb{F}_q$, and let us assume that $\mathcal{P}$ is an arithmetic partition of C_i with a fixed sequence of spanning congruences such that for $x \in C_i$, the rooted tree isomorphism type $\mathrm{Tree}_{\Gamma_f}(x, M)$ only depends on the block $\mathcal{B}(\mathcal{P}, \vec{\nu})$ of $\mathcal{P}$ in which x lies (but not on x itself). Then we denote that isomorphism type by $\mathrm{Tree}_i(\mathcal{P}, M, \vec{\nu})$. We also set $\mathrm{Tree}_i(\mathcal{P}, \vec{\nu}) := \mathrm{Tree}_i(\mathcal{P}, \mathbb{F}_q, \vec{\nu})$.
- If $\mathfrak{T}_1, \mathfrak{T}_2, \ldots, \mathfrak{T}_N$ are isomorphism types of rooted trees, then their *sum* $\mathfrak{T}_1 + \mathfrak{T}_2 + \cdots + \mathfrak{T}_N$ is defined as the rooted tree isomorphism type obtained by glueing disjoint copies of the $\mathfrak{T}_j$ together at their roots. This addition turns the class of rooted tree isomorphism types into a class-sized monoid, the neutral element of which is the trivial rooted tree isomorphism type (a single vertex without arcs).
- If $\mathfrak{T}$ is a rooted tree isomorphism type, we denote by $\mathfrak{T}^+$ the rooted tree isomorphism type obtained by connecting a copy of $\mathfrak{T}$ to a new root via an arc from the old to the new root. For example, iterating this operation starting from the trivial rooted tree isomorphism type, one obtains those finite digraphs that are directed paths.
- If $\mathfrak{T}$ is a rooted tree isomorphism type and n is a non-negative integer, we define the *multiple* $n \cdot \mathfrak{T} = n\mathfrak{T}$ as follows recursively. $0\mathfrak{T}$ is the trivial rooted tree isomorphism type, and $(n+1)\mathfrak{T} := n\mathfrak{T} + \mathfrak{T}$.
- In view of the previous two bullet points, non-negative integer linear combinations $n_1\mathfrak{T}_1 + n_2\mathfrak{T}_2 + \cdots + n_N\mathfrak{T}_N$ of rooted tree isomorphism types are well defined.
- If $\mathcal{X}_1, \ldots, \mathcal{X}_n$ are arithmetic partitions of $\mathbb{Z}/m\mathbb{Z}$, then $\bigwedge_{k=1}^{n} \mathcal{X}_k$ denotes the infimum of the $\mathcal{X}_k$ in the lattice of all partitions of $\mathbb{Z}/m\mathbb{Z}$ (i.e., the roughest common refinement of the $\mathcal{X}_k$). Equivalently, if a spanning m-congruence sequence is fixed for each $\mathcal{X}_k$, then $\bigwedge_{k=1}^{n} \mathcal{X}_k$ is the arithmetic partition of $\mathbb{Z}/m\mathbb{Z}$ that is spanned by the concatenation of those sequences.

For the first step of our approach, we consider the case where $i \neq d$ and C_i is *not* a periodic coset (let us call such cosets *transient*). For this case, we proceed by recursion on the height of $\mathrm{Tree}_{\Gamma_{\bar{f}}}(i)$. The base of the recursion is when i is a leaf in $\Gamma_{\bar{f}}$. Then $x \in C_i$ is a leaf in Γ_f, i.e., $x \notin \mathrm{im}(f)$. This means that $\mathrm{Tree}_{\Gamma_f}(x)$ consists of the single vertex x and has no arcs. Therefore, we may choose $\mathcal{P}_i$ as the trivial partition $\mathfrak{P}(\emptyset) = \{\mathbb{Z}/s\mathbb{Z}\}$ of $\mathbb{Z}/s\mathbb{Z}$, and $\mathrm{Tree}_i(\mathcal{P}_i, \emptyset)$ as the trivial rooted tree isomorphism type. We note that $\mathrm{AC}(\mathcal{P}_i)$, the complexity of $\mathcal{P}_i$, is $0 = \mathfrak{v}_i$ where, for a general $j \in \{0, 1, \ldots, d\}$, we set $\mathfrak{v}_j := |\mathrm{V}(\mathrm{Tree}_{\Gamma_{\bar{f}}}(j))| - 1$, the number of vertices that are strictly above j in the corresponding subtree of $\Gamma_{\bar{f}}$.

Now we assume that C_i is transient, that the height of $\mathrm{Tree}_{\Gamma_{\bar{f}}}(i)$ is $h \geq 1$, and that all transient cosets where that height is less than h have been "taken care of" via arithmetic partitions $\mathcal{P}_j$ such that $\mathrm{AC}(\mathcal{P}_j) \leq \mathfrak{v}_j$. In particular, if $\bar{f}^{-1}(\{i\}) = \{j_1, j_2, \ldots, j_K\}$, then for each $t = 1, 2, \ldots, K$, we have an arithmetic partition $\mathcal{P}_{j_t}$

of C_{j_t} with an explicit sequence of spanning s-congruences of length $m_{j_t} \leq \mathfrak{v}_{j_t}$ such that the isomorphism type $\mathrm{Tree}_{\Gamma_f}(y)$ is the same for all vertices $y \in C_{j_t}$ chosen from a common block $\mathcal{B}(\mathcal{P}_{j_t}, \vec{\nu}^{(\mathcal{P}_{j_t})})$ of $\mathcal{P}_{j_t}$, and we understand each such isomorphism type $\mathrm{Tree}_{j_t}(\mathcal{P}_{j_t}, \vec{\nu}^{(\mathcal{P}_{j_t})})$ explicitly. Now, because each pre-image y of $x \in C_i$ under f (i.e., each child of x in Γ_f^*) must lie in one of the cosets C_{j_t} for $t = 1, 2, \ldots, K$, it follows that

$$\mathrm{Tree}_{\Gamma_f}(x) = \sum_{t=1}^{K} \mathrm{Tree}_{\Gamma_f}(x, C_{j_t}).$$

Moreover, for fixed $t \in \{1, 2, \ldots, K\}$, we can write

$$\mathrm{Tree}_{\Gamma_f}(x, C_{j_t}) = \sum_{\vec{\nu}^{(\mathcal{P}_{j_t})} \in \{\emptyset, \neg\}^{m_{j_t}}} (|f^{-1}(\{x\}) \cap \mathcal{B}(\mathcal{P}_{j_t}, \vec{\nu}^{(\mathcal{P}_{j_t})})| \cdot \mathrm{Tree}_{j_t}(\mathcal{P}_{j_t}, \vec{\nu}^{(\mathcal{P}_{j_t})})^+).$$

Let us consider the arithmetic partition $\mathcal{P}'_{j_t} := \mathfrak{P}'(\mathcal{P}_{j_t}, A_{j_t})$ of C_i with its explicit spanning s-congruence sequence of length $m_{j_t} + 1$ from Lemma 2.2.2. If x lies in the block $\mathcal{B}(\mathcal{P}'_{j_t}, \vec{\nu}^{(\mathcal{P}'_{j_t})})$ for some fixed $\vec{\nu}^{(\mathcal{P}'_{j_t})} \in \{\emptyset, \neg\}^{m_{j_t}+1}$, then

$$|f^{-1}(\{x\}) \cap \mathcal{B}(\mathcal{P}_{j_t}, \vec{\nu}^{(\mathcal{P}_{j_t})})| = \sigma_{\mathcal{P}_{j_t}, A_{j_t}}(\vec{\nu}^{(\mathcal{P}_{j_t})}, \vec{\nu}^{(\mathcal{P}'_{j_t})})$$

by Lemma 2.2.2, and thus

$$\mathrm{Tree}_{\Gamma_f}(x, C_{j_t}) = \sum_{\vec{\nu}^{(\mathcal{P}_{j_t})} \in \{\emptyset, \neg\}^{m_{j_t}}} (\sigma_{\mathcal{P}_{j_t}, A_{j_t}}(\vec{\nu}^{(\mathcal{P}_{j_t})}, \vec{\nu}^{(\mathcal{P}'_{j_t})}) \cdot \mathrm{Tree}_{j_t}(\mathcal{P}_{j_t}, \vec{\nu}^{(\mathcal{P}_{j_t})})^+),$$

independently of x itself. Now, let us set $\mathcal{P}_i := \bigwedge_{t=1}^{K} \mathcal{P}'_{j_t}$, viewed as an arithmetic partition of C_i with a spanning sequence of length $m_i := \sum_{t=1}^{K} (m_{j_t} + 1)$. We can view each logical sign tuple $\vec{\nu}^{(\mathcal{P}_i)} \in \{\emptyset, \neg\}^{m_i}$ as a concatenation of logical sign tuples $\vec{\nu}^{(\mathcal{P}'_{j_t})} \in \{\emptyset, \neg\}^{m_{j_t}+1}$ for $t = 1, 2, \ldots, K$, and if $x \in C_i$ lies in the block $\mathcal{B}(\mathcal{P}_i, \vec{\nu}^{(\mathcal{P}_i)})$ of $\mathcal{P}_i$, then x also lies in the block $\mathcal{B}(\mathcal{P}_{j_t}, \vec{\nu}^{(\mathcal{P}'_{j_t})})$ of $\mathcal{P}'_{j_t}$ for each $t = 1, 2, \ldots, K$, whence

$$\mathrm{Tree}_{\Gamma_f}(x) = \sum_{t=1}^{K} \sum_{\vec{\nu}^{(\mathcal{P}_{j_t})} \in \{\emptyset, \neg\}^{m_{j_t}}} (\sigma_{\mathcal{P}_{j_t}, A_{j_t}}(\vec{\nu}^{(\mathcal{P}_{j_t})}, \vec{\nu}^{(\mathcal{P}'_{j_t})}) \cdot \mathrm{Tree}_{j_t}(\mathcal{P}_{j_t}, \vec{\nu}^{(\mathcal{P}_{j_t})})^+),$$

independently of x itself, as required. Moreover, we note that

$$\mathrm{AC}(\mathcal{P}_i) \leq \sum_{t=1}^{K} \mathrm{AC}(\mathcal{P}'_{j_t}) \leq \sum_{t=1}^{K} (\mathfrak{v}_{j_t} + 1) = \mathfrak{v}_i.$$

In summary, we obtain the following result.

Proposition 3.3.1. *For each $\bar{f}$-transient $i \in \{0, 1, \ldots, d-1\}$, the arithmetic partition $\mathcal{P}_i$ of C_i together with an explicit spanning sequence of s-congruences of length m_i and associated rooted tree isomorphism types $\mathrm{Tree}_i(\mathcal{P}_i, \vec{\nu}^{(\mathcal{P}_i)})$ for $\vec{\nu}^{(\mathcal{P}_i)} \in \{\emptyset, \neg\}^{m_i}$ can be defined as follows by recursion on $h_i := \mathrm{ht}(\mathrm{Tree}_{\Gamma_{\bar{f}}}(i))$.*

(1) *If $h_i = 0$, we may set*
   (a) $\mathcal{P}_i := \mathfrak{P}(\emptyset)$,
   (b) $m_i := 0$, *and*
   (c) $\mathrm{Tree}_i(\mathcal{P}_i, \emptyset)$ *to be the trivial rooted tree isomorphism type.*

(2) *If $h_i \geq 1$, we let $\bar{f}^{-1}(\{i\}) = \{j_1, j_2, \ldots, j_K\}$ and set $\mathcal{P}'_{j_t} := \mathfrak{P}'(\mathcal{P}_{j_t}, A_{j_t})$ for $t = 1, 2, \ldots, K$. Then we define*
   (a) $\mathcal{P}_i := \bigwedge_{t=1}^{K} \mathcal{P}'_{j_t}$,
   (b) $m_i := \sum_{t=1}^{K} (m_{j_t} + 1)$, *and*
   (c) *for $\vec{\nu}^{(\mathcal{P}_i)} \in \{\emptyset, \neg\}^{m_i}$, viewed as the concatenation of the logical sign tuples $\vec{\nu}^{(\mathcal{P}'_{j_t})} \in \{\emptyset, \neg\}^{m_{j_t}+1}$ for $t = 1, 2, \ldots, K$,*

$$\begin{aligned}&\mathrm{Tree}_i(\mathcal{P}_i, \vec{\nu}^{(\mathcal{P}_i)})\\ &:= \sum_{t=1}^{K} \sum_{\vec{\nu}^{(\mathcal{P}_{j_t})} \in \{\emptyset,\neg\}^{m_{j_t}}} (\sigma_{\mathcal{P}_{j_t}, A_{j_t}}(\vec{\nu}^{(\mathcal{P}_{j_t})}, \vec{\nu}^{(\mathcal{P}'_{j_t})}) \cdot \mathrm{Tree}_{j_t}(\mathcal{P}_{j_t}, \vec{\nu}^{(\mathcal{P}_{j_t})})^{+}).\end{aligned}$$

With this choice of $\mathcal{P}_i$, we have

$$\mathrm{AC}(\mathcal{P}_i) \leq \mathfrak{v}_i := |\mathrm{V}(\mathrm{Tree}_{\Gamma_{\bar{f}}}(i))| - 1 \leq d - 1 \in O(d) \subseteq O(d^2 \,\mathrm{mpe}(q-1)).$$

The second step is to describe the isomorphism type of $\mathrm{Tree}_{\Gamma_f}(0_{\mathbb{F}_q})$, which is similar to the recursion step for transient cosets above. Let

$$\bar{f}^{-1}(\{d\}) = \{d, j_1, j_2, \ldots, j_K\}$$

(we note that $K = 0$ unless some coefficient a_i in the cyclotomic form (1.1) of f is 0). The children of 0 in $\mathrm{Tree}_{\Gamma_f}(0)^*$ are just the *non-zero* children of 0 in Γ_f^*, and each such child must lie in C_{j_t} for some $t \in \{1, 2, \ldots, K\}$. We observe that each C_{j_t} is a transient coset, so by Proposition 3.3.1, we already know a suitable arithmetic partition $\mathcal{P}_{j_t}$ of C_{j_t}, with an explicit spanning sequence of length m_{j_t}, and have an explicit understanding of the rooted tree isomorphism types $\mathrm{Tree}_{j_t}(\mathcal{P}_{j_t}, \vec{\nu}^{(\mathcal{P}_{j_t})})$ for $\vec{\nu}^{(\mathcal{P}_{j_t})} \in \{\emptyset, \neg\}^{m_{j_t}}$. Moreover, all vertices in C_{j_t} map to $0_{\mathbb{F}_q}$ under f, so

$$f^{-1}(\{0_{\mathbb{F}_q}\}) \cap \mathcal{B}(\mathcal{P}_{j_t}, \vec{\nu}^{(\mathcal{P}_{j_t})}) = \mathcal{B}(\mathcal{P}_{j_t}, \vec{\nu}^{(\mathcal{P}_{j_t})}) = \mathbf{0}^{-1}(\{0_{\mathbb{Z}/s\mathbb{Z}}\}) \cap \mathcal{B}(\mathcal{P}_{j_t}, \vec{\nu}^{(\mathcal{P}_{j_t})})$$

for all $\vec{\nu}^{(\mathcal{P}_{j_t})} \in \{\emptyset, \neg\}^{m_{j_t}}$, where $\mathbf{0} : \mathbb{Z}/s\mathbb{Z} \to \mathbb{Z}/s\mathbb{Z}, z \mapsto 0 = 0z + 0$. Hence, if we set $\mathcal{P}'_{j_t} := \mathfrak{P}'(\mathcal{P}_{j_t}, \mathbf{0})$, which is, in its standard form from Lemma 2.2.2, spanned by

the single s-congruence $x \equiv 0 \pmod{s}$ repeated $m_{j_t} + 1$ times, then

$$|f^{-1}(\{0_{\mathbb{F}_q}\}) \cap \mathcal{B}(\mathcal{P}_{j_t}, \vec{v}^{(\mathcal{P}_{j_t})})| = \sigma_{\mathcal{P}_{j_t},\mathbf{0}}(\vec{v}^{(\mathcal{P}_{j_t})}, (\emptyset, \dots, \emptyset)).$$

Since $\mathrm{Tree}_{\Gamma_f}(0_{\mathbb{F}_q})$ is obtained by attaching $|f^{-1}(\{0_{\mathbb{F}_q}\}) \cap \mathcal{B}(\mathcal{P}_{j_t}, \vec{v}^{(\mathcal{P}_{j_t})})|$ copies of $\mathrm{Tree}_{j_t}(\mathcal{P}_{j_t}, \vec{v}^{(\mathcal{P}_{j_t})})$ to a common root for each $t = 1, 2, \dots, K$ and each $\vec{v}^{(\mathcal{P}_{j_t})} \in \{\emptyset, \neg\}^{m_{j_t}}$, we obtain the following proposition.

Proposition 3.3.2. *Let $\bar{f}^{-1}(\{d\}) = \{d, j_1, j_2, \dots, j_K\}$. For $t = 1, 2, \dots, K$, let $\mathcal{P}_{j_t}$, m_{j_t} and $\mathrm{Tree}_{j_t}(\mathcal{P}_{j_t}, \vec{v}^{(\mathcal{P}_{j_t})})$ for $t = 1, 2, \dots, K$ be as in Proposition 3.3.1. Moreover, we denote by $\mathbf{0}$ the constant 0 function $\mathbb{Z}/s\mathbb{Z} \to \mathbb{Z}/s\mathbb{Z}$. Then*

$$\mathrm{Tree}_{\Gamma_f}(0_{\mathbb{F}_q}) = \sum_{t=1}^{K} \sum_{\vec{v}^{(\mathcal{P}_{j_t})} \in \{\emptyset,\neg\}^{m_{j_t}}} (\sigma_{\mathcal{P}_{j_t},\mathbf{0}}(\vec{v}^{(\mathcal{P}_{j_t})}, (\emptyset, \dots, \emptyset)) \cdot \mathrm{Tree}_{j_t}(\mathcal{P}_{j_t}, \vec{v}^{(\mathcal{P}_{j_t})})^{+}).$$

In the third and final step, we consider vertices x from a *periodic* coset C_i (with $i < d$). Let us write $\bar{f}^{-1}(\{i\}) = \{i', j_1, j_2, \dots, j_K\}$, where i' is the unique $\bar{f}$-periodic pre-image of i under $\bar{f}$. Hence, C_{j_t} for $t = 1, 2, \dots, K$ is a transient coset; let $\mathcal{P}_{j_t}$ be the arithmetic partition of C_{j_t} defined in Proposition 3.3.1. Moreover, let $(i_0, i_1, \dots, i_{\ell-1})$ be the cycle of $i = i_0$ under $\bar{f}$, and let us define $i_k := i_{k \bmod \ell}$ for $k \in \mathbb{Z}$ (in particular, $i' = i_{-1}$). In addition, let $A_{i_k} : \mathbb{Z}/s\mathbb{Z} \to \mathbb{Z}/s\mathbb{Z}, x \mapsto \alpha_{i_k} x + \beta_{i_k}$, be the affine map that describes how f maps from C_{i_k} to $C_{i_{k+1}}$.

This case is more complicated, and we need to make a recursion by another parameter. As in Section 3.2, we denote by Γ_{per} the induced subgraph of Γ_f on the union of all periodic blocks C_i, i.e., the functional graph of f_{per}. We remind the reader that we explicitly understand the trees above periodic vertices in Γ_{per} thanks to Theorem 3.2.1. The idea is to proceed by recursion on a parameter called $\mathfrak{h}(x)$, which can range from 0 up to the maximum height H_i of the rooted trees in Γ_{per} above periodic vertices in one of the cosets $C_{i_t} = C_{\bar{f}^t(i)}$ for $t = 0, 1, \dots, \ell - 1$. This parameter is defined as follows:

$$\mathfrak{h}(x) = \begin{cases} \mathrm{ht}(\mathrm{Tree}_{\Gamma_{\mathrm{per}}}(x)) \in \{0, 1, \dots, H_i - 1\}, & \text{if } x \text{ is } f\text{-transient}, \\ H_i, & \text{if } x \text{ is } f\text{-periodic}. \end{cases} \tag{3.1}$$

By Example 3.2.2, there is in general no relation between $\mathfrak{h}(x)$ and $\mathrm{ht}(\mathrm{Tree}_{\Gamma_{\mathrm{per}}}(x))$ when $\mathfrak{h}(x) = H_i$ (i.e., when x is f-periodic). Also, $\mathfrak{h}$ need not assume all values in $\{0, 1, \dots, H_i\}$ on a given coset C_{i_t}; in fact, $\mathfrak{h}(C_{i_t})$ need not even be an integer interval. None of this will be an issue for our approach, though.

We observe that $H_i + 1$ is the smallest positive integer k such that for all $t \in \mathbb{Z}$, the common procreation number

$$\mathrm{proc}_k^{(\Gamma_{\mathrm{per}}^*)}(x) = \frac{\gcd(\prod_{j=0}^{k-1} \alpha_{i_{t-k+j}}, s)}{\gcd(\prod_{j=0}^{k-2} \alpha_{i_{t-k+j}}, s)}$$

of all f-periodic vertices $x \in C_{i_t}$ is equal to 1, which is (for each given t) equivalent to

$$\gcd\left(\prod_{j=0}^{k-1} \alpha_{i_{t-k+j}}, s\right) = \gcd\left(\prod_{j=0}^{k-2} \alpha_{i_{t-k+j}}, s\right),$$

and further to

$$\prod_{p \mid \gcd(\alpha_{i_{t-1}}, s)} p^{\nu_p(s)} \mid \prod_{j=0}^{k-2} \alpha_{i_{t-k+j}}.$$

Setting $\overline{\alpha}_i := \prod_{t=0}^{\ell-1} \alpha_{i_t}$ (the linear coefficient of $\mathcal{A}_i$ in the notation of Section 3.1), it is not difficult to see from this that

$$H_i \le \ell \cdot \max_{p \mid \gcd(\overline{\alpha}_i, s)} \left\lceil \frac{\nu_p(s)}{\nu_p(\overline{\alpha}_i)} \right\rceil \le \ell \operatorname{mpe}(s) \le d \operatorname{mpe}(q-1). \tag{3.2}$$

Let us set $\mathcal{P}'_{j_t} := \mathfrak{P}'(\mathcal{P}_{j_t}, A_{j_t})$ for $t = 1, 2, \ldots, K$, and $\mathcal{R}_i := \bigwedge_{t=1}^{K} \mathcal{P}'_{j_t}$. We denote by n_i the length of the spanning congruence sequence for $\mathcal{R}_i$ which we use (in general, $n_i = \sum_{t=1}^{K} (m_{j_t} + 1)$, but in a concrete example there may be repetitions among those congruences, allowing us to delete some of them). A simple observation is that as far as the transient coset contribution $\operatorname{Tree}_{\Gamma_f}(x, \bigcup_{t=1}^{K} C_{j_t})$ to $\operatorname{Tree}_{\Gamma_f}(x)$ is concerned, everything is as in step 1.

Proposition 3.3.3. *Let $i \in \{0, 1, \ldots, d-1\}$ be $\bar{f}$-periodic, and let $j_1, j_2, \ldots, j_K$ be the $\bar{f}$-transient pre-images of i under $\bar{f}$. Moreover, let $\mathcal{P}'_{j_t} := \mathfrak{P}'(\mathcal{P}_{j_t}, A_{j_t})$ for $t = 1, 2, \ldots, K$ and $\mathcal{R}_i := \bigwedge_{t=1}^{K} \mathcal{P}'_{j_t}$. Then the following hold.*

(1) *For $x \in C_i$ and $t \in \{1, 2, \ldots, K\}$, the isomorphism type $\operatorname{Tree}_{\Gamma_f}(x, C_{j_t})$ only depends on the $\mathcal{P}'_{j_t}$-block $\mathcal{B}(\mathcal{P}'_{j_t}, \vec{\nu}^{(\mathcal{P}'_{j_t})})$ (for some $\vec{\nu}^{(\mathcal{P}'_{j_t})} \in \{\emptyset, \neg\}^{m_{j_t}+1}$) in which x lies. That isomorphism type is denoted by $\operatorname{Tree}_i(\mathcal{P}'_{j_t}, C_{j_t}, \vec{\nu}^{(\mathcal{P}'_{j_t})})$ and can be computed according to the formula*

$$\begin{aligned}&\operatorname{Tree}_i\left(\mathcal{P}'_{j_t}, C_{j_t}, \vec{\nu}^{(\mathcal{P}'_{j_t})}\right)\\ &\quad= \sum_{\vec{\nu}^{(\mathcal{P}_{j_t})} \in \{\emptyset, \neg\}^{m_{j_t}}} \sigma_{\mathcal{P}_{j_t}, A_{j_t}}\left(\vec{\nu}^{(\mathcal{P}_{j_t})}, \vec{\nu}^{(\mathcal{P}'_{j_t})}\right) \operatorname{Tree}_{j_t}\left(\mathcal{P}_{j_t}, \vec{\nu}^{(\mathcal{P}_{j_t})}\right)^{+}.\end{aligned}$$

(2) *For $x \in C_i$, the rooted tree isomorphism type of $\operatorname{Tree}_{\Gamma_f}(x, \bigcup_{t=1}^{K} C_{j_t})$ only depends on the $\mathcal{R}_i$-block $\mathcal{B}(\mathcal{R}_i, \vec{\nu}^{(\mathcal{R}_i)})$ (for some $\vec{\nu}^{(\mathcal{R}_i)} \in \{\emptyset, \neg\}^{n_i}$) in which x lies. That isomorphism type is denoted by $\operatorname{Tree}_i(\mathcal{R}_i, \bigcup_{t=1}^{K} C_{j_t}, \vec{\nu}^{(\mathcal{R}_i)})$ and can be computed according to the formula*

$$\operatorname{Tree}_i\left(\mathcal{R}_i, \bigcup_{t=1}^{K} C_{j_t}, \vec{\nu}^{(\mathcal{R}_i)}\right) = \sum_{t=1}^{K} \operatorname{Tree}_i\left(\mathcal{P}'_{j_t}, C_{j_t}, \vec{\nu}^{(\mathcal{P}'_{j_t})}\right),$$

where $\vec{\nu}^{(\mathcal{P}'_{j_t})} \in \{\emptyset, \neg\}^{m_{j_t}+1}$*, for* $t = 1, 2, \ldots, K$*, is the unique logical sign tuple such that* $\mathcal{B}(\mathcal{R}_i, \vec{\nu}^{(\mathcal{R}_i)}) \subseteq \mathcal{B}(\mathcal{P}'_{j_t}, \vec{\nu}^{(\mathcal{P}'_{j_t})})$*; in the standard situation, where* $n_i = \sum_{t=1}^{K} (m_{j_t} + 1)$*, the tuple* $\vec{\nu}^{(\mathcal{R}_i)}$ *is simply the concatenation of the* $\vec{\nu}^{(\mathcal{P}'_{j_t})}$ *for* $t = 1, 2, \ldots, K$*.*

We remind the reader that we wish to proceed by recursion on the parameter $\mathfrak{h}(x)$ defined in (3.1). This is motivated by Proposition 3.3.3, because if $\mathfrak{h}(x) = 0$, then x has no transient children in $C_{i'}$, whence

$$\mathrm{Tree}_{\Gamma_f}(x) = \mathrm{Tree}_{\Gamma_f}\left(x, \bigcup_{t=1}^{K} C_{j_t}\right).$$

In general, we wish to construct an arithmetic partition $\mathcal{S}_{i,h}$ of C_i such that for vertices $x \in C_i$ with $\mathfrak{h}(x) = h$, the isomorphism type of $\mathrm{Tree}_{\Gamma_f}(x, C_{i'})$ only depends on the $\mathcal{S}_{i,h}$-block containing x and is explicitly understood. Then we are basically done, because

$$\mathrm{Tree}_{\Gamma_f}(x) = \mathrm{Tree}_{\Gamma_f}\left(x, \bigcup_{t=1}^{K} C_{j_t}\right) + \mathrm{Tree}_{\Gamma_f}(x, C_{i'}).$$

In order to construct $\mathcal{S}_{i,h}$ and prove that it has the desired property, we need to introduce quite a few notations.

- For $h \in \{0, 1, \ldots, H_i\}$, a vertex $x \in C_i = C_{i_0}$ has at least h successor generations in Γ^*_{per} if and only if x lies in the image of $A_{i_{-h}} A_{i_{-h+1}} \cdots A_{i_{-1}}$, which is the affine map $\mathcal{A}_{i,h} : z \mapsto \overline{\alpha}_{i,h} z + \overline{\beta}_{i,h}$ of $\mathbb{Z}/s\mathbb{Z}$, where

$$\overline{\alpha}_{i,h} = \prod_{t=1}^{h} \alpha_{i_{-t}} \quad \text{and} \quad \overline{\beta}_{i,h} = \sum_{t=1}^{h} \beta_{i_{-t}} \prod_{k=1}^{t-1} \alpha_{i_{-k}}.$$

  That is, x has at least h successor generations in Γ^*_{per} if and only if it satisfies the following s-congruence, which we denote by $\theta_{i,h}(x)$:

$$x \equiv \overline{\beta}_{i,h} \pmod{\gcd(\overline{\alpha}_{i,h}, s)}.$$

  We observe that the modulus in $\theta_{i,0}(x)$ is 1, so that congruence is trivial.

- Next, we describe, for each $h \in \{0, 1, \ldots, H_i\}$, a simple system $\Theta_{i,h}(x)$ of at most two s-CCs such that for all $x \in C_i$, the equality $\mathfrak{h}(x) = h$ holds if and only if $\Theta_{i,h}(x)$ holds. If $h < H_i$, then the vertices $x \in C_i$ with $\mathfrak{h}(x) = h$ are just those f-transient $x \in C_i$ that have exactly h successor generations in Γ^*_{per}. It follows that for such h, one has $\mathfrak{h}(x) = h$ if and only if $\theta_{i,h}(x)$ and $\neg\theta_{i,h+1}(x)$ both hold. Now we assume that $h = H_i$. By definition of $\mathfrak{h}$, this happens if and only if x is f-periodic, which is (by definition of H_i) equivalent to x having at least H_i

successor generations in Γ^*_{per}. Therefore, the condition $\theta_{i,H_i}(x)$ alone provides the desired characterization in this case.
Now, noting once more that $\theta_{i,0}(x)$ is trivial and may thus be omitted from any system of conditions containing it, we may define $\Theta_{i,h}(x)$ as follows:

$$\Theta_{i,h}(x) := \begin{cases} \emptyset, & \text{if } H_i = (h =)0, \\ \neg\theta_{i,1}(x), & \text{if } h = 0 < H_i, \\ \theta_{i,h}(x) \wedge (\neg\theta_{i,h+1}(x)), & \text{if } 0 < h < H_i, \\ \theta_{i,H_i}(x), & \text{if } h = H_i > 0. \end{cases}$$

- If $\Theta(x)$ is a system of m-CCs, then $\mathfrak{P}(\Theta(x))$ denotes the arithmetic partition of $\mathbb{Z}/m\mathbb{Z}$ spanned by the non-negated versions of the conditions in $\Theta(x)$. For example,

$$\mathfrak{P}\begin{pmatrix} x \equiv 4 \pmod 6 \\ x \not\equiv 3 \pmod 9 \\ x \equiv 0 \pmod 2 \end{pmatrix} = \mathfrak{P}\begin{pmatrix} x \equiv 4 \pmod 6 \\ x \equiv 3 \pmod 9 \\ x \equiv 0 \pmod 2 \end{pmatrix}.$$

We note that if $x \in \mathbb{Z}/m\mathbb{Z}$ is chosen from a fixed block of $\mathfrak{P}(\Theta(x))$, then the truth value of each condition in $\Theta(x)$ is independent of x, and so is the truth value of $\Theta(x)$ itself. We also observe that for each $h \in \{0, 1, \ldots, H_i\}$, the following equality holds for the systems $\Theta_{i,k}(x)$ of s-CCs defined in the previous bullet point:

$$\bigwedge_{k=0}^{h} \mathfrak{P}(\Theta_{i,k}(x)) = \mathfrak{P}(\theta_{i,j}(x) : j = 1, 2, \ldots, \min\{h+1, H_i\}). \tag{3.3}$$

We set $\mathcal{U}_i := \mathfrak{P}(\theta_{i,k}(x) : k = 1, 2, \ldots, H_i)$.

- For $k \in \{0, 1, \ldots, H_i\}$, we denote by $\vec{\xi}_{i,k}$ the logical sign tuple $(\nu_1, \nu_2, \ldots, \nu_{H_i})$ with $\nu_t = \emptyset$ if and only if $t \leq k$. With this definition, if we view $\mathcal{U}_i$ as an arithmetic partition of C_i, then the block $\mathcal{B}(\mathcal{U}_i, \vec{\xi}_{i,k})$ consists precisely of those $x \in C_i$ such that $\mathfrak{h}(x) = k$ (and every block of $\mathcal{U}_i$ is of this form for some k).
- Let $\mathcal{P} = \mathfrak{P}(x \equiv \mathfrak{b}_j \pmod{\mathfrak{a}_j} : j = 1, 2, \ldots, K)$ be an arithmetic partition of $\mathbb{Z}/m\mathbb{Z}$, and let $A : z \mapsto az + b$, be an affine map of $\mathbb{Z}/m\mathbb{Z}$. In dependency of $\mathcal{P}$ (actually, of the fixed sequence of spanning congruences for $\mathcal{P}$, rather than $\mathcal{P}$ itself) and A, we define another arithmetic partition $\lambda(\mathcal{P}, A)$ of $\mathbb{Z}/m\mathbb{Z}$ as follows: $\lambda(\mathcal{P}, A) := \mathfrak{P}(x \equiv a\mathfrak{b}_j + b \pmod{\gcd(a\mathfrak{a}_j, m)} : j = 1, 2, \ldots, K)$. Here is a list of important facts concerning this notation (for which we have $m = s$ throughout):
  - If $\mathcal{P}$ is an arithmetic partition of $C_{i'}$, then

$$\mathfrak{P}'(\mathcal{P}, A_{i'}) = \lambda(\mathcal{P}, A_{i'}) \wedge \mathfrak{P}(\theta_{i,1}(x)).$$

– For $h = 0, 1, \ldots, H_{i'} = H_i$, we have

$$\lambda(\mathfrak{P}(\theta_{i',h}(x)), A_{i'}) = \begin{cases} \mathfrak{P}(\theta_{i,h+1}(x)), & \text{if } h < H_i, \\ \mathfrak{P}(\theta_{i,H_i}(x)), & \text{if } h = H_i. \end{cases}$$

Indeed, for $h < H_i$, this is immediate by the definition of $\theta_{i,h}(x)$ and the facts that $A_{i'}(\overline{\beta}_{i',h}) = \overline{\beta}_{i,h+1}$ and $\gcd(a\gcd(a', s), s) = \gcd(aa', s)$ for all $a, a' \in \mathbb{Z}$. Moreover, noting that our definition of $\theta_{i,h}(x)$ also makes sense if $h > H_i$, we have $\lambda(\mathfrak{P}(\theta_{i',H_{i'}}(x))) = \mathfrak{P}(\theta_{i,H_i+1}(x))$. However, for $x \in C_i$, the condition $\theta_{i,H_i+1}(x)$ holds if and only if x has at least $H_i + 1$ successor generations in Γ^*_{per}, which is the case if and only if x is periodic, i.e., if and only if $\theta_{i,H_i}(x)$ holds. Hence, the congruences $\theta_{i,H_i+1}(x)$ and $\theta_{i,H_i}(x)$ have the same solution set in $\mathbb{Z}/s\mathbb{Z}$, whence $\mathfrak{P}(\theta_{i,H_i+1}(x)) = \mathfrak{P}(\theta_{i,H_i}(x))$. This concludes the proof of the above formulas for $\lambda(\mathfrak{P}(\theta_{i',h}(x)), A_{i'})$.

– By the previous two bullet points and equality (3.3), applied with i' in place of i, we have that

$$\mathfrak{P}'\Big(\bigwedge_{k=0}^{h} \mathfrak{P}(\Theta_{i',k}(x)), A_{i'}\Big) = \bigwedge_{k=0}^{\min\{h+1,H_i\}} \mathfrak{P}(\Theta_{i,k}(x)),$$

and, in particular, $\mathfrak{P}'(\mathcal{U}_{i'}, A_{i'}) = \mathcal{U}_i$.

- Let $\mathcal{P}$ be an arithmetic partition of C_i. We define the notation $\lambda_i^t(\mathcal{P})$, where t is a non-negative integer, as follows recursively: $\lambda_i^0(\mathcal{P}) := \mathcal{P}$, and for $t \geq 1$, we set $\lambda_i^t(\mathcal{P}) := \lambda(\lambda_i^{t-1}(\mathcal{P}), A_{i_{t-1}})$. In other words, $\lambda_i^t(\mathcal{P})$ is the arithmetic partition of C_{i_t} obtained by pushing $\mathcal{P}$ forward t times along the $\bar{f}$-cycle of i via the operation λ, using the appropriate affine function A_{i_k} in each step.
- For $h \in \{0, 1, \ldots, H_i\}$, we introduce the following arithmetic partitions of C_i:
  - $\mathcal{S}_{i,h} := \bigwedge_{t=1}^{h} \lambda_{i_{-t}}^t(\mathcal{R}_{i_{-t}})$;
  - $\mathcal{P}_{i,h} := \mathcal{R}_i \wedge \mathcal{S}_{i,h} = \bigwedge_{t=0}^{h} \lambda_{i_{-t}}^t(\mathcal{R}_{i_{-t}})$;
  - $\mathcal{T}_{i,h} := \mathcal{S}_{i,h} \wedge \mathcal{U}_i = \bigwedge_{t=1}^{h} \lambda_{i_{-t}}^t(\mathcal{R}_{i_{-t}}) \wedge \mathcal{U}_i$;
  - $\mathcal{Q}_{i,h} := \mathcal{R}_i \wedge \mathcal{T}_{i,h} = \bigwedge_{t=0}^{h} \lambda_{i_{-t}}^t(\mathcal{R}_{i_{-t}}) \wedge \mathcal{U}_i$.

  The motivation for considering $\mathcal{S}_{i,h}$ and $\mathcal{P}_{i,h}$ is that their blocks control the rooted tree isomorphism type of $\mathrm{Tree}_{\Gamma_f}(x, C_{i'})$ and $\mathrm{Tree}_{\Gamma_f}(x)$, respectively, for vertices $x \in C_i$ *of* $\mathfrak{h}$*-value* h contained in them – see Proposition 3.3.4 below. An explicit formula for $\mathrm{Tree}_{\Gamma_f}(x, C_{i'})$ in terms of the $\mathcal{S}_{i,h}$-block containing x is also given in Proposition 3.3.4, and that formula involves distribution numbers (as in Lemma 2.2.2 – see the sentence after that lemma) of the partitions $\mathcal{Q}_{i',k}$ for $0 \leq k < h$. The partitions of the form $\mathcal{T}_{i,h}$ are not mentioned in the statement of

Proposition 3.3.4, but they play an important role in its proof due to the fact that for $h \geq 1$, one has $\mathcal{T}_{i,h} = \mathfrak{P}'(\mathcal{Q}_{i',h-1}, A_{i'})$.

- We denote the concatenation of logical sign tuples $\vec{\nu}$ and $\vec{\nu}'$ by $\vec{\nu} \diamond \vec{\nu}'$.

We are now in a position to formulate in detail how the blocks B of $\mathcal{S}_{i,h}$ affect the rooted trees above vertices $x \in B$ with $\mathfrak{h}(x) = h$.

Proposition 3.3.4. *Let $i \in \{0, 1, \ldots, d-1\}$ be $\bar{f}$-periodic, with $\bar{f}$-pre-images $i', j_1, j_2, \ldots, j_K$, where i' is $\bar{f}$-periodic. Moreover, let $h \in \{0, 1, \ldots, H_i\}$, and let $x \in C_i$ with $\mathfrak{h}(x) = h$. Then the following hold.*

(1) *The isomorphism type $\mathrm{Tree}_{\Gamma_f}(x, C_{i'})$ only depends on the $\mathcal{S}_{i,h}$-block $\mathcal{B}(\mathcal{S}_{i,h}, \vec{\nu}^{(\mathcal{S}_{i,h})})$ in which x lies and is denoted by $\mathrm{Tree}_i^{(h)}(\mathcal{S}_{i,h}, C_{i'}, \vec{\nu}^{(\mathcal{S}_{i,h})})$.*

(2) *The isomorphism type $\mathrm{Tree}_{\Gamma_f}(x)$ depends on $\mathcal{B}(\mathcal{P}_{i,h}, \vec{\nu}^{(\mathcal{P}_{i,h})})$, the $\mathcal{P}_{i,h}$-block in which x lies and is denoted by $\mathrm{Tree}_i^{(h)}(\mathcal{P}_{i,h}, \vec{\nu}^{(\mathcal{P}_{i,h})})$.*

More specifically, for $h = 0$, where $\mathcal{S}_{i,h} = \mathcal{P}(\emptyset)$ and $\mathcal{P}_{i,h} = \mathcal{R}_i$, the rooted tree $\mathrm{Tree}_i^{(0)}(\mathcal{S}_{i,0}, C_{i'}, \emptyset)$ is trivial, and

$$\mathrm{Tree}_i^{(0)}(\mathcal{P}_{i,0}, \vec{\nu}^{(\mathcal{P}_{i,0})}) = \mathrm{Tree}_i\left(\mathcal{R}_i, \bigcup_{t=1}^{K} C_{j_t}, \vec{\nu}^{(\mathcal{P}_{i,0})}\right).$$

For $h \geq 1$, writing $\vec{\nu}^{(\mathcal{S}_{i,h})} = \diamond_{j=1}^{h} \vec{o}'_t$ with $\vec{o}'_t \in \{\emptyset, \neg\}^{n_{i-t}}$, we have the following, where $\vec{o}_t$ for $t = 0, 1, \ldots, h-1$ is a variable ranging over $\{\emptyset, \neg\}^{n_{i-t-1}}$:

$$\begin{aligned}&\mathrm{Tree}_i^{(h)}(\mathcal{S}_{i,h}, C_{i'}, \vec{\nu}^{(\mathcal{S}_{i,h})})\\ &= \sum_{k=0}^{h-1} \sum_{\vec{o}_0, \ldots, \vec{o}_k} \sigma_{\mathcal{Q}_{i',k}, A_{i'}}(\diamond_{t=0}^{k} \vec{o}_t \diamond \vec{\xi}_{i',k}, \diamond_{t=1}^{k+1} \vec{o}'_t \diamond \vec{\xi}_{i,h})\, \mathrm{Tree}_{i'}^{(k)}(\mathcal{P}_{i',k}, \diamond_{t=0}^{k} \vec{o}_t)^{+},\end{aligned}$$

and, viewing $\vec{\nu}^{(\mathcal{P}_{i,h})}$ as the concatenation of $\vec{o}'_0 \in \{\emptyset, \neg\}^{n_i}$ and $\vec{\nu}^{(\mathcal{S}_{i,h})}$, we have

$$\mathrm{Tree}_i^{(h)}(\mathcal{P}_{i,h}, \vec{\nu}^{(\mathcal{P}_{i,h})}) = \mathrm{Tree}_i\left(\mathcal{R}_i, \bigcup_{t=1}^{K} C_{j_t}, \vec{o}'_0\right) + \mathrm{Tree}_i^{(h)}(\mathcal{S}_{i,h}, C_{i'}, \vec{\nu}^{(\mathcal{S}_{i,h})}).$$

Proof. The formulas for $h = 0$ are clear because vertices $x \in C_i$ with $\mathfrak{h}(x) = 0$ have no f-transient children in $C_{i'}$. We may thus assume that $h \geq 1$. With regard to $\mathrm{Tree}_i^{(h)}(\mathcal{S}_{i,h}, C_{i'}, \vec{\nu}^{(\mathcal{S}_{i,h})})$, we note that if $x \in C_i$ lies in the block $\mathcal{B}(\mathcal{S}_{i,h}, \vec{\nu}^{(\mathcal{S}_{i,h})})$ of $\mathcal{S}_{i,h}$ and satisfies $\mathfrak{h}(x) = h$, then for each $k \in \{0, 1, \ldots, h-1\}$, we have that x lies in the block $\mathcal{B}(\mathcal{T}_{i,k+1}, \diamond_{j=1}^{k+1} \vec{o}'_j \diamond \vec{\xi}_{i,h})$ of $\mathcal{T}_{i,k+1}$. By definition, $\mathcal{T}_{i,k+1} = \mathcal{P}'(\mathcal{Q}_{i',k}, A_{i'})$. Each ($f$-transient) pre-image y of x in $C_{i'}$ with $\mathfrak{h}(y) = k$ lies in some block of $\mathcal{Q}_{i',k}$ of the form $\mathcal{B}(\mathcal{Q}_{i',k}, \diamond_{j=0}^{k} \vec{o}_j \diamond \vec{\xi}_{i',k})$, where $\vec{o}_j \in \{\emptyset, \neg\}^{n_{i-j-1}}$ for $j = 0, 1, \ldots, k$.

Moreover, by Lemma 2.2.2, the number of such pre-images in that block is

$$\sigma_{\mathcal{Q}_{i',k},A_{i'}}(\diamond_{j=0}^{k}\vec{o}_j \diamond \vec{\xi}_{i',k}, \diamond_{j=1}^{k+1}\vec{o}'_j \diamond \vec{\xi}_{i,h}).$$

Each pre-image y of x in $C_{i'}$ which is contained in $\mathcal{B}(\mathcal{Q}_{i',k}, \diamond_{j=0}^{k}\vec{o}_j \diamond \vec{\xi}_{i',k})$ is also contained in $\mathcal{B}(\mathcal{P}_{i',k}, \diamond_{j=0}^{k}\vec{o}_j)$, whence

$$\mathrm{Tree}_{\Gamma_f}(y) \cong \mathrm{Tree}_{i'}^{(k)}(\mathcal{P}_{i',k}, \diamond_{j=0}^{k}\vec{o}_j).$$

This concludes the proof of the formula for $\mathrm{Tree}_i^{(h)}(\mathcal{S}_{i,h}, C_{i'}, \vec{v}^{(\mathcal{S}_{i,h})})$.

The formula for $\mathrm{Tree}_i^{(h)}(\mathcal{P}_{i,h}, \vec{v}^{(\mathcal{P}_{i,h})})$ is clear because

$$\mathrm{Tree}_{\Gamma_f}(x) = \mathrm{Tree}_{\Gamma_f}\left(x, \bigcup_{t=1}^{K} C_{j_t}\right) + \mathrm{Tree}_{\Gamma_f}(x, C_{i'})$$

and, by Proposition 3.3.3 (2),

$$\mathrm{Tree}_{\Gamma_f}\left(x, \bigcup_{t=1}^{K} C_{j_t}\right) = \mathrm{Tree}_i\left(\mathcal{R}_i, \bigcup_{t=1}^{K} C_{j_t}, \vec{o}'_0\right)$$

for all $x \in C_i$. ∎

Now, let us set

$$\mathcal{P}_i := \mathcal{Q}_{i,H_i} = \mathcal{P}_{i,H_i} \wedge \mathcal{U}_i.$$

Putting everything together, we obtain the following concluding result for this section.

Proposition 3.3.5. *Let $i \in \{0, 1, \ldots, d-1\}$ be $\bar{f}$-periodic, and let $\mathcal{B}(\mathcal{P}_i, \vec{v}^{(\mathcal{P}_i)})$ be a block of $\mathcal{P}_i$. We can view $\vec{v}^{(\mathcal{P}_i)}$ as the concatenation $\diamond_{t=0}^{H_i}\vec{o}'_t \diamond \vec{\xi}$, where $\vec{o}'_t \in \{\emptyset, \neg\}^{n_{i-t}}$ for $t = 0, 1, \ldots, H_i$, and $\vec{\xi} = \vec{\xi}_{i,h}$ for a unique $h \in \{0, 1, \ldots, H_i\}$. Then for $x \in \mathcal{B}(\mathcal{P}_i, \vec{v}^{(\mathcal{P}_i)})$, the isomorphism type of $\mathrm{Tree}_{\Gamma_f}(x)$ does not depend on x, is denoted by $\mathrm{Tree}_i(\mathcal{P}_i, \vec{v}^{(\mathcal{P}_i)})$ and given by the formula*

$$\mathrm{Tree}_i(\mathcal{P}_i, \vec{v}^{(\mathcal{P}_i)}) = \mathrm{Tree}_i^{(h)}(\mathcal{P}_{i,h}, \diamond_{t=0}^{h}\vec{o}'_t).$$

Moreover, we have

$$\mathrm{AC}(\mathcal{P}_i) \leq d^2\,\mathrm{mpe}(q-1) + d - 1 \in O(d^2\,\mathrm{mpe}(q-1)).$$

Proof. The block $\mathcal{B}(\mathcal{P}_i, \vec{v}^{(\mathcal{P}_i)})$ is contained in $\mathcal{B}(\mathcal{U}_i, \xi_{i,h})$, whence $\mathfrak{h}(v) = h$. Additionally, $\mathcal{B}(\mathcal{P}_i, \vec{v}^{(\mathcal{P}_i)})$ is contained in $\mathcal{B}(\mathcal{P}_{i,h}, \diamond_{t=0}^{h}\vec{o}'_t)$, so the asserted formula for $\mathrm{Tree}_i(\mathcal{P}_i, \vec{v}^{(\mathcal{P}_i)})$ is clear by Proposition 3.3.4 (2). For the complexity bound, we note

that by definition of $\mathcal{P}_i = \mathcal{P}_{i,H_i} \wedge \mathcal{U}_i$ and the bound (3.2), we have

$$\begin{aligned}\mathrm{AC}(\mathcal{P}_i) &\le \mathrm{AC}(\mathcal{P}_{i,H_i}) + \mathrm{AC}(\mathcal{U}_i)\\ &\le \sum_{t=0}^{H_i} \mathrm{AC}(\lambda_{i-t}^{t}(\mathcal{R}_{i-t})) + H_i \le (H_i+1)(d-1) + H_i\\ &= H_i d + d - 1 \le d^2 \operatorname{mpe}(q-1) + d - 1,\end{aligned}$$

as required. ∎

Remark 3.3.6. Omitting all explicit details which we worked out in this section, we basically proved that on each coset C_i, there is an arithmetic partition $\mathcal{P}_i$ with $\mathrm{AC}(\mathcal{P}_i) \in O(d^2 \operatorname{mpe}(q-1)) \subseteq O(d^2 \log q)$ which "controls" the rooted trees above vertices in C_i. Now, the trivial partition $\mathcal{T}_s$ of $C_i \cong \mathbb{Z}/s\mathbb{Z}$ which consists entirely of singleton blocks also "controls" the trees above its blocks, for trivial reasons. While it is preferable for our effective purposes to subsume as many isomorphic rooted trees under a common block as possible (and thus $\mathcal{P}_i$ is in general preferable over $\mathcal{T}_s$), it is an interesting question whether $\mathcal{T}_s$ could "beat" $\mathcal{P}_i$ at least as far as arithmetic complexity is concerned.

Let us discuss this problem for a general modulus $m \in \mathbb{N}^+$ (not just $s=(q-1)/d$). We consider the factorization $m = p_1^{v_1} p_2^{v_2} \cdots p_K^{v_K}$ of m into pairwise coprime prime powers. By adding logical signs to the m-CCs in the system consisting of

$$x \equiv b \pmod{p_j^{v_j}}$$

for $j = 1, 2, \ldots, K$ and $b \in \{0, 1, \ldots, p_j^{v_j} - 2\}$, one can obtain each singleton subset of $\mathbb{Z}/m\mathbb{Z}$ as a block of the corresponding arithmetic partition. This shows that $\mathrm{AC}(\mathcal{T}_m) \le p_1^{v_1} + \cdots + p_K^{v_K} - K$. Of course, if m is a prime power, then this bound is just $m-1$. On the other hand, if $m = p_1 p_2 \cdots p_K = p_K\#$ is a primorial, then $p_K\# = \exp((1+o(1))K \log K)$, and thus $\log m \sim K \log K$, i.e., recalling from Remark 1.6 that W denotes the Lambert W function, we have

$$K \sim \frac{\log m}{W(\log m)} \sim \frac{\log m}{\log\log m}.$$

Our bound implies that $\mathrm{AC}(\mathcal{T}_m)$ is at most

$$\begin{aligned}p_1 + \cdots + p_K - K &\sim p_1 + \cdots + p_K \sim \frac{1}{2} K^2 \log K\\ &\sim \frac{1}{2} \frac{\log^2 m}{\log\log^2 m} (\log\log m - \log\log\log m) \sim \frac{1}{2} \frac{\log^2 m}{\log\log m},\end{aligned}$$

which does not beat $O(d^2 \log m)$ for fixed d, and even less so $O(d^2 \operatorname{mpe}(m)) = O(d^2)$, noting that $\operatorname{mpe}(m) = \operatorname{mpe}(p_K\#) = 1$. It is an interesting open question

whether

$$\liminf_{m\to\infty} \frac{\mathrm{AC}(\mathcal{T}_m)}{\log^2(m)/\log\log m} > 0,$$

see also Question 6.3.1.

3.4 Understanding the connected components

Now we want to combine the theory developed thus far to understand the connected components of Γ_f in their entirety. From the introduction, we recall our approach of associating a necklace of rooted tree isomorphism types with each connected component of Γ_f, which characterizes the digraph isomorphism type of that component.

Let $\mathcal{L}$ be a CRL-list of f (see Section 3.1 on how to construct $\mathcal{L}$). We remind the reader that the first entries of the pairs in $\mathcal{L}$ are representatives not only for the cycles of f, but also for the connected components of Γ_f. Let us fix $(r, l) \in \mathcal{L}$. We aim to give a neat description of the cyclic sequence of rooted tree isomorphism types associated with the connected component of Γ_f containing r.

We note that by our construction of $\mathcal{L}$, if $r = 0_{\mathbb{F}_q}$, then $l = 1$, and the connected component consists of a single rooted tree attached to the looped vertex $0_{\mathbb{F}_q}$. We can determine that tree, $\mathrm{Tree}_{\Gamma_f}(0_{\mathbb{F}_q})$, as described in Section 3.3. The length 1 cyclic sequence $[\mathrm{Tree}_{\Gamma_f}(0_{\mathbb{F}_q})]$ determines the connected component of $0_{\mathbb{F}_q}$ as a whole, and so we may henceforth assume that $r \neq 0_{\mathbb{F}_q}$, contained in a unique coset C_i of C in $\mathbb{F}_q^*$.

We recall that i is necessarily a periodic point of $\bar{f}$, and let $(i_0, i_1, \ldots, i_{\ell-1})$ with $i_0 = i$ be its cycle. For general $t \in \mathbb{Z}$, we set $i_t := i_{t \bmod \ell}$. By Section 3.3, on each coset C_j of C in $\mathbb{F}_q^*$, we have an arithmetic partition $\mathcal{P}_j$ such that the isomorphism type of $\mathrm{Tree}_{\Gamma_f}(x)$ is constant and explicitly understood for vertices $x \in C_j$ chosen from a common block $\mathcal{B}(\mathcal{P}_j, \vec{v})$ of $\mathcal{P}_j$, and we denote the said isomorphism type by $\mathrm{Tree}_j(\mathcal{P}_j, \vec{v})$.

Let us fix $t \in \{0, 1, \ldots, \ell - 1\}$ and recall (from the previous section) the notation $\mathcal{A}_{i_t} = A_{i_t} A_{i_{t+1}} \cdots A_{i_{t+\ell-1}}$ for the product of all affine maps along the cycle of i_t. Also, we recall that f^ℓ stabilizes C_{i_t}, and that the restriction $(f^\ell)_{|C_{i_t}}$ corresponds to the affine map $\mathcal{A}_{i_t}$ of $\mathbb{Z}/s\mathbb{Z}$. Let us set $r'_{i_t} := f^t(r)$. We have $\ell \mid l$, and the cycle of r'_{i_t} under $\mathcal{A}_{i_t}$ is

$$\begin{aligned}&(r'_{i_t}, \mathcal{A}_{i_t}(r'_{i_t}), \mathcal{A}_{i_t}^2(r'_{i_t}), \ldots, \mathcal{A}_{i_t}^{l/\ell-1}(r'_{i_t}))\\ &\quad = (f^t(r), f^{t+\ell}(r), f^{t+2\ell}(r), \ldots, f^{t+((l/\ell)-1)\ell}(r)).\end{aligned}$$

We denote by $\mathcal{B}_{i_t}$ the function defined on C_{i_t} that maps $x \in C_{i_t}$ to the unique block of $\mathcal{P}_{i_t}$ containing x. Our next goal is to understand the block sequence

$$(\mathcal{B}_{i_t}(r'_{i_t}), \mathcal{B}_{i_t}(\mathcal{A}_{i_t}(r'_{i_t})), \ldots, \mathcal{B}_{i_t}(\mathcal{A}_{i_t}^{l/\ell-1}(r'_{i_t})))$$

along the $\mathcal{A}_{i_t}$-cycle of r'_{i_t} for each $t = 0, 1, \ldots, \ell - 1$, because those sequences combine to the block sequence $(\mathcal{B}_{i_n}(f^n(r)))_{n=0,1,\ldots,l-1}$, from which one can read off the cyclic sequence of rooted tree isomorphism types for the connected component of Γ_f containing r.

Now, let us assume that $\mathcal{P}_{i_t} = \mathfrak{P}(x \equiv \mathfrak{b}_{i_t,j} \pmod{\mathfrak{a}_{i_t,j}} : j = 1, 2, \ldots, m_{i_t})$. Understanding the block sequence $(\mathcal{B}_{i_t}(\mathcal{A}_{i_t}^m(r'_{i_t})))_{m=0,1,\ldots,l/\ell-1}$ means understanding the truth values of the congruences $x \equiv \mathfrak{b}_{i_t,j} \pmod{\mathfrak{a}_{i_t,j}}$ as x ranges over the cycle of r'_{i_t} under $\mathcal{A}_{i_t}$. We recall from the previous section that $\overline{\alpha}_{i_t} := \prod_{t=0}^{\ell-1} \alpha_{i_t}$ is the linear coefficient of $\mathcal{A}_{i_t}$. Lemma 2.1.14 implies that all periodic points of $\mathcal{A}_{i_t}$ (in particular all points on the cycle of r'_{i_t} under $\mathcal{A}_{i_t}$) have one particular, explicitly known value modulo $\prod_{p|\gcd(\overline{\alpha}_{i_t},s)} p^{v_p(s)}$. This may cause some of the congruences $x \equiv \mathfrak{b}_{i_t,j} \pmod{\mathfrak{a}_{i_t,j}}$ for $j \in \{1, 2, \ldots, m_{i_t}\}$ to have constant truth value on all periodic points of $\mathcal{A}_{i_t}$. The remaining congruences can be dealt with as follows.

We compute $\log_{\mathcal{A}_{i_t}}^{(\mathfrak{a}_{i_t,j})}(r'_{i_t}, \mathfrak{b}_{i_t,j}) =: \mathfrak{l}_{i_t,j}$ (see Section 2.4 for this discrete log notation) and the cycle length $l_{i_t,j}$ of r'_{i_t} under $\mathcal{A}_{i_t}$ modulo $\mathfrak{a}_{i_t,j}$. If $\mathfrak{l}_{i_t,j} = \infty$ (i.e., $\mathfrak{b}_{i_t,j}$ does not lie on the cycle of r'_{i_t} under $\mathcal{A}_{i_t}$ modulo $\mathfrak{a}_{i_t,j}$), then the congruence $x \equiv \mathfrak{b}_{i_t,j} \pmod{\mathfrak{a}_{i_t,j}}$ is false for all x on the $\mathcal{A}_{i_t}$-cycle of r'_{i_t} modulo s. Otherwise, that congruence is true precisely for those $x = \mathcal{A}_{i_t}^y(r'_{i_t})$ for which $y \equiv \mathfrak{l}_{i_t,j} \pmod{l_{i_t,j}}$. Let us denote by I_{i_t} the set of those $j \in \{1, 2, \ldots, m_{i_t}\}$ for which the truth value of $x \equiv \mathfrak{b}_{i_t,j} \pmod{\mathfrak{a}_{i_t,j}}$ is constant along the $\mathcal{A}_{i_t}$-cycle of r'_{i_t} modulo s. The block sequence $(\mathcal{B}_{i_t}(\mathcal{A}_{i_t}^m(r'_{i_t})))_{m=0,1,\ldots,l/\ell-1}$ is determined by

- the information about the constant truth values along the $\mathcal{A}_{i_t}$-cycle of r'_{i_t} modulo s of the congruences $x \equiv \mathfrak{b}_{i_t,j} \pmod{\mathfrak{a}_{i_t,j}}$ for $j \in I_{i_t}$, and
- the arithmetic partition $\mathcal{P}^{(i_t)} := \mathfrak{P}(y \equiv \mathfrak{l}_{i_t,j} \pmod{l_{i_t,j}} : j \notin I_{i_t})$ of $\mathbb{Z}/(l/\ell)\mathbb{Z}$, which encodes the behavior of the truth values of the remaining congruences $x \equiv \mathfrak{b}_{i_t,j} \pmod{\mathfrak{a}_{i_t,j}}$, for $j \notin I_{i_t}$, along the cycle.

Once the block sequence $(\mathcal{B}_{i_t}(\mathcal{A}_{i_t}^m(r'_{i_t})))_{m=0,1,\ldots,l/\ell-1}$ has been understood that way for each t, the actual necklace that encodes the isomorphism type of the connected component of Γ_f containing r is given by the cyclic sequence

$$[\mathrm{Tree}_{i_n}(\mathcal{P}_{i_n}, \mathcal{B}_{i_n}(\mathcal{A}_{i_n}^{(n-(n \bmod \ell))/\ell}(r'_{i_n})))]_{n=0,1,\ldots,l-1}, \tag{3.4}$$

where, by abuse of notation, $\mathrm{Tree}_i(\mathcal{P}_i, B)$ is to be understood as $\mathrm{Tree}_i(\mathcal{P}_i, \vec{\nu})$ for $B = \mathcal{B}(\mathcal{P}_i, \vec{\nu})$.

The connected components of Γ_f associated with two different choices for r are isomorphic if and only if the corresponding cyclic sequences (3.4) are equal. We do note, however, that it does not appear obvious how to check this efficiently, just as it does not seem clear how to check efficiently whether two given arithmetic partitions are equal – see Problems 6.2.3 and 6.2.4.