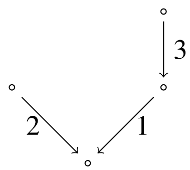
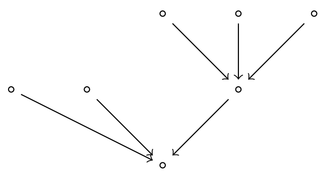
Chapter 4

Computations and examples

The goal of this chapter is to illustrate the theory developed thus far through some concrete computations and examples/special cases. We start by introducing a useful notation that is used in Sections 4.1 and 4.2. We consider finite directed rooted trees (with all arcs oriented toward the root) that have non-negative integers as edge weights. An *isomorphism* of such graphs is one of the underlying non-edge-weighted directed graphs that preserves the weights of arcs. With each isomorphism type $\mathfrak{I}$ of such trees, we associate an isomorphism type $\mathrm{Expand}(\mathfrak{I})$ of non-edge-weighted, finite directed rooted trees as follows. If $y_1, y_2, \ldots, y_n$ are the neighbors of the root x of $\mathfrak{I}$, and they have the (isomorphism types of) edge-weighted rooted trees $\mathfrak{I}_1, \mathfrak{I}_2, \ldots, \mathfrak{I}_n$ attached to them and the edge joining x and y_j has weight $\mathfrak{w}_j \in \mathbb{N}_0$, then $\mathrm{Expand}(\mathfrak{I})$ is defined recursively by taking a new root and attaching $\mathfrak{w}_j$ copies of $\mathrm{Expand}(\mathfrak{I}_j)$ to it for each $j = 1, 2, \ldots, n$. For example, if $\mathfrak{I}$ is



then $\mathrm{Expand}(\mathfrak{I})$ is



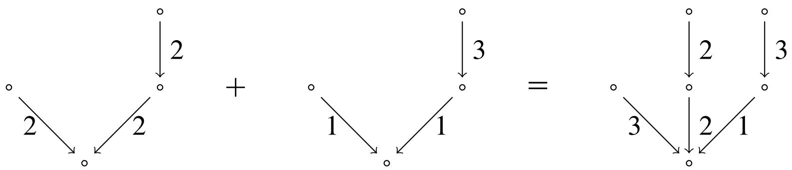
By recursion on the height of $\mathfrak{I}$, we define $\mathfrak{I}$ to be *simplified* as follows. The trivial isomorphism type $\mathfrak{I}$ is simplified, and if $\mathfrak{I}$ is of positive height, then $\mathfrak{I}$ is simplified if all isomorphism types $\mathfrak{I}'$ attached to the root of $\mathfrak{I}$ (which are all of smaller height than $\mathfrak{I}$) are simplified, pairwise distinct, and none of them is attached to the root with edge weight 0. In fact, we could have omitted 0 as an edge weight from the start, but it is more convenient to include it because in the subsequent discussion, the weights sometimes are formulas involving greatest common divisors that may simplify to 0 under certain assumptions.

The simplified isomorphism types of finite edge-weighted directed rooted trees are in bijection with the isomorphism types of finite directed rooted trees via Expand (as can be easily proved by induction on the height). This allows us to define the *simplified form* $\mathrm{SF}(\mathfrak{I})$ of an arbitrary isomorphism type $\mathfrak{I}$ of finite edge-weighted directed rooted trees as the unique simplified isomorphism type such that

$$\mathrm{Expand}(\mathrm{SF}(\mathfrak{I})) = \mathrm{Expand}(\mathfrak{I}).$$

We write $\mathfrak{I} \sim \mathfrak{I}'$ for $\mathrm{SF}(\mathfrak{I}) = \mathrm{SF}(\mathfrak{I}')$ (equivalently, $\mathrm{Expand}(\mathfrak{I}) = \mathrm{Expand}(\mathfrak{I}')$). The simplified form of $\mathfrak{I}$ can be constructed explicitly from $\mathfrak{I}$ in a simple recursion on the tree height (going through the $\mathfrak{I}'$ attached to the root of $\mathfrak{I}$, computing their simplified forms, and adding up edge weights that belong to the same $\mathrm{SF}(\mathfrak{I}')$).

In Section 3.3, we introduced a sum of isomorphism types of non-edge-weighted finite directed rooted trees (turning their class into a class-sized monoid), and there is a unique way to define a *sum of simplified edge-weighted rooted tree isomorphism types* such that Expand becomes a monoid isomorphism (i.e., $\mathrm{Expand}(\mathfrak{I}_1 + \mathfrak{I}_2) = \mathrm{Expand}(\mathfrak{I}_1) + \mathrm{Expand}(\mathfrak{I}_2)$). Explicitly, $\mathfrak{I}_1 + \mathfrak{I}_2$ may be defined as follows. Pick a new root x and consider the edge-weighted rooted trees $\mathfrak{I}'$ that are attached to the root in $\mathfrak{I}_1$ or $\mathfrak{I}_2$. Let $\mathfrak{w}_j$ for $j = 1, 2$ be the weight of the arc that attaches $\mathfrak{I}'$ to the root in $\mathfrak{I}_j$ (treating $\mathfrak{w}_j$ as 0 if such an arc does not exist). For each such $\mathfrak{I}'$, attach a copy of $\mathfrak{I}'$ to x through an arc with weight $\mathfrak{w}_1 + \mathfrak{w}_2$. For example,



This addition can be extended to arbitrary isomorphism types of finite edge-weighted directed rooted trees by setting $\mathfrak{I}_1 + \mathfrak{I}_2 := \mathrm{SF}(\mathfrak{I}_1) + \mathrm{SF}(\mathfrak{I}_2)$. Henceforth, we frequently drop the word “isomorphism type” (thus identifying a finite (edge-weighted) directed rooted tree with its isomorphism type) for the sake of simplicity.

4.1 Rooted trees under rigid procreation

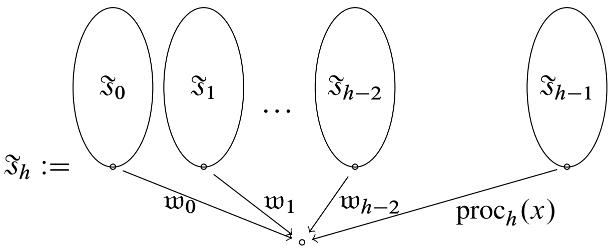
Let $\Gamma = \Gamma_g$ be a finite functional graph such that Γ^* has rigid procreation (see Definition 2.1.7 (4)). Moreover, let $x \in \mathrm{V}(\Gamma)$ be g-periodic, and let $(\mathrm{proc}_k(x))_{k\geq 1}$ be the sequence of procreation numbers of x in Γ^* (which is independent of x due to rigid procreation). Proposition 2.1.8 states that the isomorphism type of the rooted tree $\mathrm{Tree}_\Gamma(x)$, which also does not depend on the choice of g-periodic vertex x, is entirely determined by this sequence of procreation numbers. We would like to understand explicitly how that isomorphism type can be derived from $(\mathrm{proc}_k(x))_{k\geq 1}$.

Since Γ is finite, so is $H := \mathrm{ht}(\mathrm{Tree}_\Gamma(x))$. We observe that x has g-transient children with at least $H - 1$ successor generations in Γ^*, but no such children with at least H successor generations. This means that $\mathrm{proc}_h(x) > 1$ for $h = 1, 2, \ldots, H$, but $\mathrm{proc}_{H+1}(x) = 1$; the unique g-periodic child of x in Γ^* has infinitely many successor generations, thus providing a contribution of 1 to all procreation numbers. This allows us to read off H from the sequence $(\mathrm{proc}_k(x))_{k\geq 1}$ alone.

Now, Lemma 2.1.10 implies that for all g-transient vertices $y \in \mathrm{V}(\Gamma)$ with a fixed tree height $h \in \{0, 1, \ldots, H-1\}$ above them in Γ, the rooted tree isomorphism type $\mathrm{Tree}_\Gamma(y)$ is always the same. We recursively define a (not necessarily simplified) edge-weighted directed rooted tree $\mathfrak{I}_h = \mathfrak{I}_h((\mathrm{proc}_k(x))_{k\geq 1})$ such that the said isomorphism type is $\mathrm{Expand}(\mathfrak{I}_h)$. Clearly, the only choice for $\mathfrak{I}_0$ is a single vertex without arcs. If $h \in \{1, 2, \ldots, H-1\}$, then we define $\mathfrak{I}_h$ as follows. We fix a new root, and

- for $k = 0, 1, \ldots, h-2$, we attach a copy of $\mathfrak{I}_k$ to the new root with edge weight $\mathrm{proc}_{k+1}(x) - \mathrm{proc}_{k+2}(x) =: \mathfrak{w}_k$; and
- we attach a copy of $\mathfrak{I}_{h-1}$ to the new root with edge weight $\mathrm{proc}_h(x)$.

Here is a visual version of this definition.



This definition of $\mathfrak{I}_h$ does the job, because by the proof of Lemma 2.1.10, for each $k \in \{0, 1, \ldots, h-1\}$, the number of children z of y in Γ^* such that $\mathrm{Tree}_\Gamma(z)$ has height exactly $k-1$ (and hence is isomorphic to $\mathfrak{I}_{k-1}$ by induction) is equal to

$$\mathrm{proc}_{k+1}(y) - \mathrm{proc}_{k+2}(y) = \begin{cases} \mathrm{proc}_{k+1}(x) - \mathrm{proc}_{k+2}(x) = \mathfrak{w}_k, & \text{if } k < h-1, \\ \mathrm{proc}_h(x) - 0 = \mathrm{proc}_h(x), & \text{if } k = h-1. \end{cases}$$

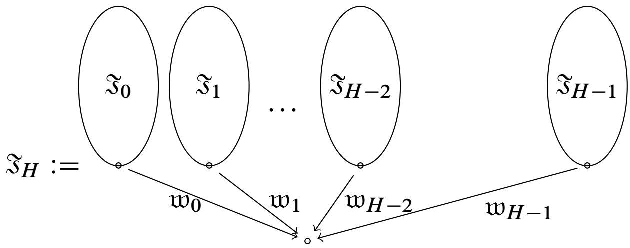
For our fixed periodic vertex x, the determination of $\mathrm{Tree}_\Gamma(x)$ is analogous, but one must take into account that x has a (unique) g-periodic child in Γ^*, which does not appear in $\mathrm{Tree}_\Gamma(x)$. This means that the weight with which $\mathfrak{I}_k$ for $k \in \{0, 1, \ldots, H-2\}$ is attached to the root of $\mathfrak{I}_H$ is

$$(\mathrm{proc}_{k+1}(x) - 1) - (\mathrm{proc}_{k+2}(x) - 1) = \mathfrak{w}_k,$$

whereas the weight with which $\mathfrak{I}_{H-1}$ is attached is

$$\mathrm{proc}_H(x) - 1 = \mathrm{proc}_H(x) - \mathrm{proc}_{H+1}(x) =: \mathfrak{w}_{H-1}.$$

In short, we obtain the following definition of $\mathfrak{I}_H$ such that $\mathrm{Expand}(\mathfrak{I}_H) \cong \mathrm{Tree}_\Gamma(x)$:



We can use similar ideas to describe, for each index d generalized cyclotomic mapping f of $\mathbb{F}_q$, the rooted trees above non-zero periodic vertices in Γ_{per}, the induced subgraph of Γ_f on the union of all periodic blocks C_i (in particular, we can obtain such a description for Γ_f as a whole in case $\bar{f}$ is a permutation). Let $i \in \{0, 1, \ldots, d-1\}$ be $\bar{f}$-periodic, with $\bar{f}$-cycle $(i_0, i_1, \ldots, i_{\ell-1})$, where $i_0 = i$. For $t \in \mathbb{Z}$, we set

$$i_t := i_{t \bmod \ell}.$$

Theorem 3.2.1 states that for fixed $t \in \mathbb{Z}$ and $h \in \mathbb{N}^+$, if $x, y \in C_{i_t}$ each have at least h successor generations in Γ^*_{per} (i.e., if $\min\{\mathrm{proc}_h^{(\Gamma^*_{\mathrm{per}})}(x), \mathrm{proc}_h^{(\Gamma^*_{\mathrm{per}})}(y)\} > 0$), then $\mathrm{proc}_h^{(\Gamma^*_{\mathrm{per}})}(x) = \mathrm{proc}_h^{(\Gamma^*_{\mathrm{per}})}(y)$. This allows us to set $\mathrm{proc}_{i_t,h} := \mathrm{proc}_h^{(\Gamma^*_{\mathrm{per}})}(x)$ for any $x \in C_{i_t}$ with at least h successor generations in Γ^*_{per} (such as an f-periodic x); this notation agrees with the one used in the proof of Theorem 3.2.1.

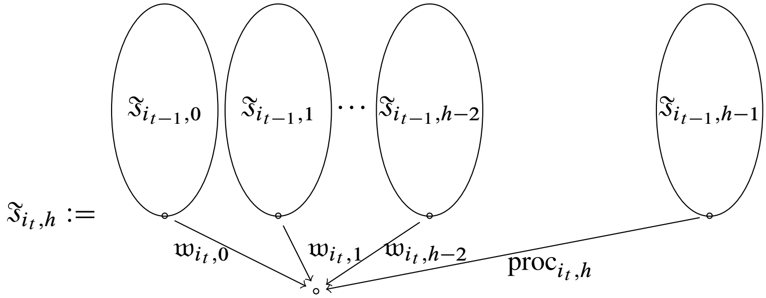
According to the comment before Theorem 3.2.1, for periodic $x \in C_{i_t}$, the isomorphism type of $\mathrm{Tree}_{\Gamma_{\mathrm{per}}}(x)$ only depends on i_t and the numbers $\mathrm{proc}_{i_{t'},h}$ for $h \geq 1$ and $t' \in \mathbb{Z}$ (i.e., it is independent of the choice of x). We describe how to read off this rooted tree from the data it depends on. We set

$$\mathcal{H}_{i_t} := \min\{h \in \mathbb{N}^+ : \mathrm{proc}_{i_t,h} = 1\} - 1,$$

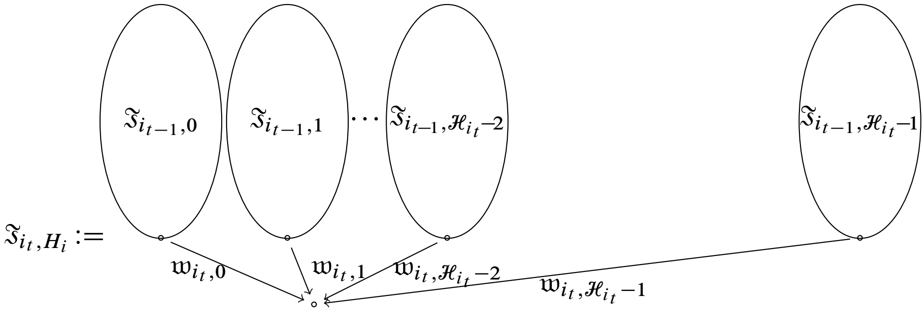
the common height of the rooted trees in Γ_{per} above periodic vertices in C_{i_t}. Moreover, as in Section 3.3, we let $H_i := \max\{\mathcal{H}_{i_t} : t = 0, 1, \ldots, \ell-1\}$, the maximum such tree height along the cycle of i. Then, in generalization of what was stated above, for all vertices $x \in C_{i_t}$, the isomorphism type of $\mathrm{Tree}_{\Gamma_{\mathrm{per}}}(x)$ only depends on i_t, the numbers $\mathrm{proc}_{i_{t'},h}$ and the $\mathfrak{h}$-value of x (see formula (3.1) in Section 3.3 for the definition of $\mathfrak{h}$). It should be noted that $\mathfrak{h}$ does not necessarily assume all of its possible values $0, 1, \ldots, H_i$ on each coset C_{i_t} (see Example 3.2.2), but this is not an issue for our construction.

We recursively define edge-weighted rooted trees $\mathfrak{I}_{i_t,h}$ such that the rooted tree in Γ_{per} above any $x \in C_{i_t}$ with $\mathfrak{h}(x) = h$ is isomorphic to $\mathrm{Expand}(\mathfrak{I}_{i_t,h})$, a property that is certainly satisfied whenever there are no $x \in C_{i_t}$ of that $\mathfrak{h}$-value. For $k \in \mathbb{N}_0$, we set $\mathfrak{w}_{i_t,k} := \mathrm{proc}_{i_t,k+1} - \mathrm{proc}_{i_t,k+2}$. We define $\mathfrak{I}_{i_t,0}$ to be the trivial rooted tree. If $h \in \{1, 2, \ldots, H_i - 1\}$ (we observe that vertices in C_{i_t} of such an $\mathfrak{h}$-value are

f-transient), then we set



Finally, the rooted tree above any vertex in C_{i_t} that is f-periodic (equivalently, which has $\mathfrak{h}$-value H_i) may be constructed as



4.2 An illustrative example

In this section, we follow the approach from Chapter 3 to derive the cyclic sequences of rooted tree isomorphism types that characterize the connected components of the functional graph of the following generalized cyclotomic mapping f of $\mathbb{F}_{2^8}$ of index $d = 5$:

$$f(x) = \begin{cases} 0, & \text{if } x = 0, \\ \omega^5 x^9, & \text{if } x \in C_0, \\ x^3, & \text{if } x \in C_1, \\ x^{17}, & \text{if } x \in C_2, \\ \omega^3 x^{34}, & \text{if } x \in C_3, \\ \omega^4 x^9, & \text{if } x \in C_4, \end{cases}$$

where ω is any fixed primitive element of $\mathbb{F}_{2^8}$ (the minimal polynomial of ω over $\mathbb{F}_2$ is not relevant here). These cyclic sequences were also derived in our introduction from a drawing of Γ_f (see the text passage between Definitions 1.4 and 1.5), but the approach of Chapter 3 is usually more computationally efficient (see Chapter 5, especially Theorem 5.1.9).

We observe that if a generalized cyclotomic mapping of a finite field of known index is not given in the above cyclotomic form, but in polynomial form, then one must first convert it into cyclotomic form before one can apply our methods. An algorithm for doing so is [15, Algorithm 1].

Because $d = 5$, we have $s = (2^8 - 1)/5 = 51 = 3 \cdot 17$. We view each coset C_i as a copy of $\mathbb{Z}/51\mathbb{Z}$ via the bijection

$$\iota_i : \mathbb{Z}/51\mathbb{Z} \to C_i, \quad k \mapsto \omega^{i+5k}.$$

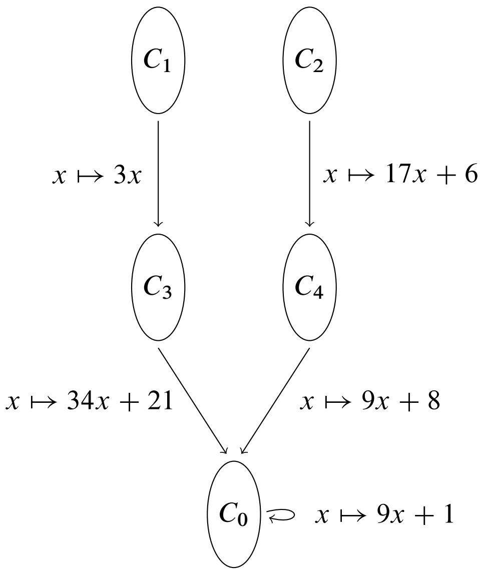
Let us work out what the monomial formulas for the values of f in the different cases become under this identification. For example, if $x \in C_4$, then $x = \omega^{4+5k}$ for some $k \in \mathbb{Z}$, and

$$f(x) = \omega^4 x^9 = \omega^{4+9\cdot 4+9\cdot 5k} = \omega^{0+5\cdot(9k+8)},$$

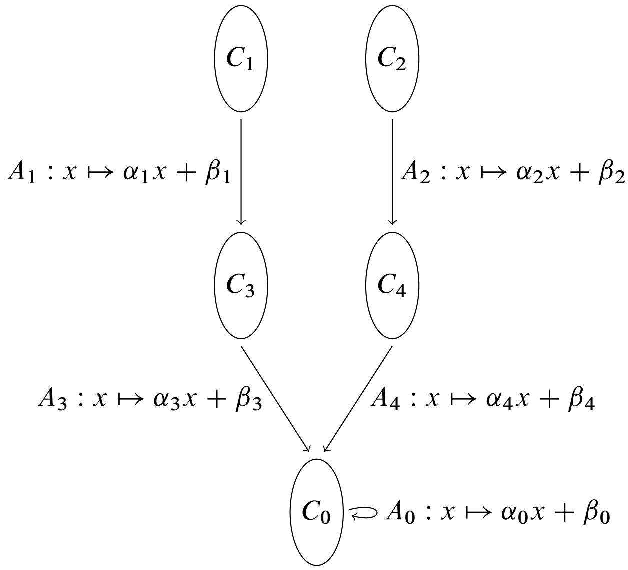
which shows that f maps C_4 to C_0 via the affine map

$$x \mapsto 9x + 8.$$

In total, we obtain the following picture describing the mapping behavior of f between the cosets C_i when viewing them as copies of $\mathbb{Z}/51\mathbb{Z}$.



Until further notice, we put the concrete function f from above aside and assume that, more generally, we have a finite field $\mathbb{F}_q$ with $5 \mid q - 1$ and an index 5 generalized cyclotomic mapping f of $\mathbb{F}_q$ which maps as follows between the five cosets of C in $\mathbb{F}_q^*$, viewed as copies of $\mathbb{Z}/s\mathbb{Z}$ (where $s = (q-1)/5$).



This allows us to describe Γ_f in terms of those general coefficients α_i and β_i, which is more instructive; one can actually see the structure of formulas for relevant parameters, such as the moduli $\mathfrak{a}_{i,j}$ and right-hand sides $\mathfrak{b}_{i,j}$ of the spanning congruences of $\mathcal{P}_i$. Just as for our concrete generalized cyclotomic mapping from above, we assume that $\gcd(\alpha_0, s) = \gcd(\alpha_0^2, s) > 1$, which means that the rooted trees attached to periodic vertices in the induced subgraph of Γ_f on C_0 are of height $H_0 = 1$. We describe the arithmetic partitions $\mathcal{P}_i = \mathfrak{P}(x \equiv \mathfrak{b}_{i,j} \pmod{\mathfrak{a}_{i,j}} : j = 1, 2, \ldots, m_i)$ and the associated rooted tree isomorphism type $\mathrm{Tree}_i(\mathcal{P}_i, \vec{\nu}^{(\mathcal{P}_i)})$ for each block $\mathcal{B}(\mathcal{P}_i, \vec{\nu}^{(\mathcal{P}_i)})$ of $\mathcal{P}_i$ for $\vec{\nu}^{(\mathcal{P}_i)} \in \{\emptyset, \neg\}^{m_i}$.

The partitions $\mathcal{P}_i$ and associated rooted trees are easily determined for $i = 1, 2, 3, 4$.

- For $i \in \{1, 2\}$, every vertex in C_i is a leaf in Γ_f, and so we may choose $\mathcal{P}_1 = \mathcal{P}_2 = \mathfrak{P}(\emptyset)$ (trivial partition with only one block). There is only one isomorphism type of rooted trees here, $\mathrm{Tree}_i(\mathcal{P}_i, \emptyset)$ (with $\emptyset$ representing an empty sequence of logical signs, not the positive logical sign), and it consists of a single vertex without edges.
- For $i \in \{3, 4\}$, since C_i is a transient coset (i.e., it does not lie on a cycle of cosets under f), the discussion in Section 3.3 shows that one can obtain $\mathcal{P}_i$ simply as the lift $\mathfrak{P}'(\mathcal{P}_{i-2}, A_{i-2})$. According to Lemma 2.2.2, that lift is of the form $\mathcal{P}_i = \mathfrak{P}(x \equiv \beta_{i-2} \pmod{\gcd(\alpha_{i-2}, s)})$. The significance of this single congruence is that it characterizes when $x \in C_i$ has at least one pre-image under f in C_{i-2}. We

note that if this is the case, then x has exactly $\gcd(\alpha_{i-2}, s)$ such pre-images, as they form a coset of the kernel of $z \mapsto \alpha_{i-2}z$ in $\mathbb{Z}/s\mathbb{Z}$. Hence, $\mathrm{Tree}_i(\mathcal{P}_i, (\neg))$ is a single vertex without arcs, and $\mathrm{Tree}_i(\mathcal{P}_i, (\emptyset))$ consists of a root with $\gcd(\alpha_{i-2}, s)$ vertices attached to it.

In our discussion for $\mathcal{P}_0$, rather than specify the rooted tree $\mathrm{Tree}_0(\mathcal{P}_0, \vec{\nu}^{(\mathcal{P}_0)})$ associated with a block $\mathcal{B}(\mathcal{P}_0, \vec{\nu}^{(\mathcal{P}_0)})$ of $\mathcal{P}_0$ itself, we specify a (not necessarily simplified isomorphism type of) finite edge-weighted directed rooted tree(s) $\mathfrak{I} = \mathfrak{I}(\mathcal{P}_0, \vec{\nu}^{(\mathcal{P}_0)})$ such that $\mathrm{Tree}_0(\mathcal{P}_0, \vec{\nu}^{(\mathcal{P}_0)}) = \mathrm{Expand}(\mathfrak{I})$. But first, let us determine $\mathcal{P}_0$ itself. We recall that $H_0 = 1$ by assumption. According to the general definition of $\mathcal{P}_i$ for periodic i, which is just before Proposition 3.3.5, we have

$$\mathcal{P}_0 = \mathcal{Q}_{0,1} = \mathcal{P}_{0,1} \wedge \mathcal{U}_0.$$

Moreover, noting that $i = 0$ lies on a cycle of $\bar{f}$ of length 1 (so that $i_t = 0$ for all $t \in \mathbb{Z}$ in the notation of Section 3.3), we conclude that

$$\mathcal{P}_{0,1} = \lambda_0^0(\mathcal{R}_0) \wedge \lambda_0^1(\mathcal{R}_0) = \mathcal{R}_0 \wedge \lambda(\mathcal{R}_0, A_0).$$

Now, $\mathcal{R}_0$ is obtained as the infimum of the $\mathfrak{P}'$-lifts of $\mathcal{P}_3$ and $\mathcal{P}_4$ to C_0. Using the notation (n, m) in place of $\gcd(n, m)$ for simplicity, we conclude that

$$\mathcal{R}_0 = \mathfrak{P}'(\mathcal{P}_3, A_3) \wedge \mathfrak{P}'(\mathcal{P}_4, A_4) = \mathfrak{P}\begin{pmatrix} x \equiv \alpha_3\beta_1 + \beta_3 \pmod{(\alpha_3(\alpha_1, s), s)} \\ x \equiv \beta_3 \pmod{(\alpha_3, s)} \\ x \equiv \alpha_4\beta_2 + \beta_4 \pmod{(\alpha_4(\alpha_2, s), s)} \\ x \equiv \beta_4 \pmod{(\alpha_4, s)} \end{pmatrix}$$

$$= \mathfrak{P}\begin{pmatrix} x \equiv \alpha_3\beta_1 + \beta_3 \pmod{(\alpha_1, s)(\alpha_3, \frac{s}{(\alpha_1,s)})} \\ x \equiv \beta_3 \pmod{(\alpha_3, s)} \\ x \equiv \alpha_4\beta_2 + \beta_4 \pmod{(\alpha_2, s)(\alpha_4, \frac{s}{(\alpha_2,s)})} \\ x \equiv \beta_4 \pmod{(\alpha_4, s)} \end{pmatrix}, \tag{4.1}$$

and thus

$$\lambda(\mathcal{R}_0, A_0) = \mathfrak{P}\begin{pmatrix} x \equiv \alpha_0\alpha_3\beta_1 + \alpha_0\beta_3 + \beta_0 \pmod{(\alpha_0(\alpha_1, s)(\alpha_3, \frac{s}{(\alpha_1,s)}), s)} \\ x \equiv \alpha_0\beta_3 + \beta_0 \pmod{(\alpha_0(\alpha_3, s), s)} \\ x \equiv \alpha_0\alpha_4\beta_2 + \alpha_0\beta_4 + \beta_0 \pmod{(\alpha_0(\alpha_2, s)(\alpha_4, \frac{s}{(\alpha_2,s)}), s)} \\ x \equiv \alpha_0\beta_4 + \beta_0 \pmod{(\alpha_0(\alpha_4, s), s)} \end{pmatrix}$$

$$= \mathfrak{P}\begin{pmatrix} x \equiv \alpha_0\alpha_3\beta_1 + \alpha_0\beta_3 + \beta_0 \pmod{(\alpha_1, s)(\alpha_0(\alpha_3, \frac{s}{(\alpha_1,s)}), \frac{s}{(\alpha_1,s)})} \\ x \equiv \alpha_0\beta_3 + \beta_0 \pmod{(\alpha_0(\alpha_3, s), s)} \\ x \equiv \alpha_0\alpha_4\beta_2 + \alpha_0\beta_4 + \beta_0 \pmod{(\alpha_2, s)(\alpha_0(\alpha_4, \frac{s}{(\alpha_2,s)}), \frac{s}{(\alpha_2,s)})} \\ x \equiv \alpha_0\beta_4 + \beta_0 \pmod{(\alpha_0(\alpha_4, s), s)} \end{pmatrix}.$$

Moreover, by formula (3.3) and the definition of $\mathcal{U}_i$ just after it, we have

$$\mathcal{U}_0 = \mathfrak{P}(\theta_{0,1}) = \mathfrak{P}(x \equiv \beta_0 \pmod{(\alpha_0, s)}).$$

It follows that

$$\mathcal{P}_0 = \mathfrak{P}\left(\begin{array}{l} x \equiv \alpha_3\beta_1 + \beta_3 \pmod{(\alpha_1, s)(\alpha_3, \frac{s}{(\alpha_1,s)})} \\ x \equiv \beta_3 \pmod{(\alpha_3, s)} \\ x \equiv \alpha_4\beta_2 + \beta_4 \pmod{(\alpha_2, s)(\alpha_4, \frac{s}{(\alpha_2,s)})} \\ x \equiv \beta_4 \pmod{(\alpha_4, s)} \\ x \equiv \alpha_0\alpha_3\beta_1 + \alpha_0\beta_3 + \beta_0 \pmod{(\alpha_1, s)(\alpha_0(\alpha_3, \frac{s}{(\alpha_1,s)}), \frac{s}{(\alpha_1,s)})} \\ x \equiv \alpha_0\beta_3 + \beta_0 \pmod{(\alpha_0(\alpha_3, s), s)} \\ x \equiv \alpha_0\alpha_4\beta_2 + \alpha_0\beta_4 + \beta_0 \pmod{(\alpha_2, s)(\alpha_0(\alpha_4, \frac{s}{(\alpha_2,s)}), \frac{s}{(\alpha_2,s)})} \\ x \equiv \alpha_0\beta_4 + \beta_0 \pmod{(\alpha_0(\alpha_4, s), s)} \\ x \equiv \beta_0 \pmod{(\alpha_0, s)} \end{array}\right). \tag{4.2}$$

Now we turn to the determination of the rooted trees above vertices in any given block B of $\mathcal{P}_0$. More specifically, we have

$$B = \mathcal{B}(\mathcal{P}_0, \vec{\nu}^{(\mathcal{P}_0)}),$$

where $\vec{\nu}^{(\mathcal{P}_0)} = (\nu_1, \ldots, \nu_9) \in \{\emptyset, \neg\}^9$ is a tuple of logical signs for the nine spanning congruences of $\mathcal{P}_0$. It is helpful to split $\vec{\nu}^{(\mathcal{P}_0)}$ into segments; namely, in the notation of Proposition 3.3.5, we write $\vec{\nu}^{(\mathcal{P}_0)} = \vec{o'_0} \diamond \vec{o'_1} \diamond \vec{\xi}$, where

- $\vec{o'_0} = (\nu_1, \nu_2, \nu_3, \nu_4)$ controls in which block $\mathcal{B}(\mathcal{R}_0, \vec{o'_0})$ of $\mathcal{R}_0$ the $\mathcal{P}_0$-block $\mathcal{B}(\mathcal{P}_0, \vec{\nu}^{(\mathcal{P}_0)})$ is contained. By Proposition 3.3.3, knowing the logical signs in $\vec{o'_0}$ is enough to understand, uniformly for all $x \in \mathcal{B}(\mathcal{P}_0, \vec{\nu}^{(\mathcal{P}_0)})$, the contribution

  $$\mathrm{Tree}_{\Gamma_f}(x, C_3 \cup C_4) = \mathrm{Tree}_0(\mathcal{R}_0, C_3 \cup C_4, \vec{o'_0})$$

  to $\mathrm{Tree}_{\Gamma_f}(x)$ that comes from those pre-images of x that lie in $C_3 \cup C_4$ (the union of all transient cosets that map to C_0).
- $\vec{o'_1} = (\nu_5, \nu_6, \nu_7, \nu_8)$ controls in which block $\mathcal{B}(\mathcal{S}_{0,1}, \vec{o'_1})$ of $\mathcal{S}_{0,1} = \lambda(\mathcal{R}_0, A_0)$ the $\mathcal{P}_0$-block $\mathcal{B}(\mathcal{P}_0, \vec{\nu}^{(\mathcal{P}_0)})$ is contained. By Proposition 3.3.4, knowing the logical signs in $\vec{o'_1}$ is enough to understand, uniformly for all $x \in \mathcal{B}(\mathcal{P}_0, \vec{\nu}^{(\mathcal{P}_0)})$ *of* $\mathfrak{h}$*-value* 1, i.e., which are f-periodic (or, equivalently here, which are non-leaves in Γ_{per}), the contribution $\mathrm{Tree}_{\Gamma_f}(x, C_0) = \mathrm{Tree}_0^{(1)}(\mathcal{S}_{0,1}, C_0, \vec{o'_1})$ to $\mathrm{Tree}_{\Gamma_f}(x)$ that comes from those pre-images of x that lie in C_0 (the unique periodic coset that maps to C_0). We note that if $x \in C_0$ has $\mathfrak{h}$-value 0, i.e., if x is f-transient (or, equivalently here, if x is a leaf in Γ_{per}), then $\mathrm{Tree}_{\Gamma_f}(x, C_0)$ is trivial, because x has no f-transient pre-images in C_0.
- $\vec{\xi} = (\nu_9)$ controls the $\mathfrak{h}$-value of the vertices in $\mathcal{B}(\mathcal{P}_0, \vec{\nu}^{(\mathcal{P}_0)})$; if $\nu_9 = \neg$, then all of those vertices are leaves in Γ_{per} (i.e., their $\mathfrak{h}$-value is 0), otherwise they all are periodic vertices (i.e., their $\mathfrak{h}$-value is $1 = H_0$).

Let us be more specific about these different contributions to $\mathrm{Tree}_{\Gamma_f}(x)$ for $x \in \mathcal{B}(\mathcal{P}_0, \vec{\nu}^{(\mathcal{P}_0)})$. We recall that by definition,

$$\mathcal{R}_0 = \mathcal{P}_3' \wedge \mathcal{P}_4' = \mathfrak{P}'(\mathcal{P}_3, A_3) \wedge \mathfrak{P}'(\mathcal{P}_4, A_4),$$

and note that $\vec{o_0'}$ can be written as the concatenation $\vec{\nu}^{(\mathcal{P}_3')} \diamond \vec{\nu}^{(\mathcal{P}_4')}$, with $\vec{\nu}^{(\mathcal{P}_3')} = (\nu_1, \nu_2)$, respectively,

$$\vec{\nu}^{(\mathcal{P}_4')} = (\nu_3, \nu_4),$$

controlling the containment of $\mathcal{B}(\mathcal{P}_0, \vec{\nu}^{(\mathcal{P}_0)})$ in $\mathcal{P}_3'$-blocks, respectively, in $\mathcal{P}_4'$-blocks. For $i \in \{3,4\}$, knowing the logical signs in $\vec{\nu}^{(\mathcal{P}_i')}$ is enough to understand, uniformly for all $x \in \mathcal{B}(\mathcal{P}_0, \vec{\nu}^{(\mathcal{P}_0)})$, the contribution

$$\mathrm{Tree}_{\Gamma_f}(x, C_i) = \mathrm{Tree}_0(\mathcal{P}_i', C_i, \vec{\nu}^{(\mathcal{P}_i')})$$

to $\mathrm{Tree}_{\Gamma_f}(x)$ that comes from those pre-images of x that lie in C_i.

Of course, for each given $x \in C_0$, we have

$$\mathrm{Tree}_{\Gamma_f}(x) = \mathrm{Tree}_{\Gamma_f}(x, C_0 \cup C_3 \cup C_4) = \sum_{i \in \{0,3,4\}} \mathrm{Tree}_{\Gamma_f}(x, C_i).$$

In view of what was said above about these three different contributions to $\mathrm{Tree}_{\Gamma_f}(x)$, we have the following formulas (which can also be derived from Propositions 3.3.3 (2) and 3.3.5 as well as the last formula in Proposition 3.3.4):

$$\begin{aligned} &\mathrm{Tree}_0(\mathcal{P}_0, \vec{\nu}^{(\mathcal{P}_0)}) \\ &= \begin{cases} \sum_{i=3}^{4} \mathrm{Tree}_0(\mathcal{P}_i', C_i, \vec{\nu}^{(\mathcal{P}_i')}), & \text{if } \nu_9 = \neg, \\ \sum_{i=3}^{4} \mathrm{Tree}_0(\mathcal{P}_i', C_i, \vec{\nu}^{(\mathcal{P}_i')}) + \mathrm{Tree}_0^{(1)}(\mathcal{S}_{0,1}, C_0, \vec{o_1'}), & \text{if } \nu_9 = \emptyset. \end{cases} \end{aligned} \tag{4.3}$$

In particular, the logical signs $\nu_5, \nu_6, \nu_7, \nu_8$ in $\vec{o_1'}$ are irrelevant for the value of $\mathrm{Tree}_0(\mathcal{P}_0, \vec{\nu}^{(\mathcal{P}_0)})$ if $\nu_9 = \neg$.

Formula (4.3) allows us to split the task of determining $\mathrm{Tree}_0(\mathcal{P}_0, \vec{\nu}^{(\mathcal{P}_0)})$ into subtasks. First, we determine $\mathrm{Tree}_0(\mathcal{P}_i', C_i, \vec{\nu}^{(\mathcal{P}_i')})$ for $i = 3, 4$, which can be done uniformly. We note that

$$\mathcal{P}_i' = \mathfrak{P}\begin{pmatrix} x \equiv \alpha_i \beta_{i-2} + \beta_i \pmod{(\alpha_i(\alpha_{i-2}, s), s)} \\ x \equiv \beta_i \pmod{(\alpha_i, s)} \end{pmatrix}.$$

The two entries of $\vec{\nu}^{(\mathcal{P}_i')} = (\nu_{2i-5}, \nu_{2i-4})$ are logical signs for those two congruences, and we need to distinguish cases according to their truth values. We could just work out $\mathrm{Tree}_0(\mathcal{P}_i', C_i, \vec{\nu}^{(\mathcal{P}_i')})$ in each case "mechanically" following Proposition 3.3.3 (1), using the formula for $\sigma_{\mathcal{P}_i, A_i}(\vec{\nu}^{(\mathcal{P}_i)}, \vec{\nu}^{(\mathcal{P}_i')})$ from Lemma 2.2.2. However, it is more instructive to derive them with direct arguments (inclined readers may still follow the formulaic approach themselves and compare).

- If $\nu_{2i-4} = \neg$, i.e., if $x \not\equiv \beta_i \pmod{(\alpha_i, s)}$, then x simply has no pre-images in C_i, whence $\mathrm{Tree}_0(\mathcal{P}_i', C_i, \vec{\nu}^{(\mathcal{P}_i')})$ is a single vertex without arcs (the value of ν_{2i-5} is irrelevant here).
- If $\nu_{2i-4} = \emptyset$, i.e., if $x \equiv \beta_i \pmod{(\alpha_i, s)}$, then x has exactly (α_i, s) children in C_i (which form a coset of the kernel of $z \mapsto \alpha_i z$). We need to determine the distribution of those children over the two blocks of

  $$\mathcal{P}_i = \mathfrak{P}(x \equiv \beta_{i-2} \pmod{(\alpha_{i-2}, s)}),$$

  and that distribution is controlled by the truth value ν_{2i-5} of

  $$x \equiv \alpha_i \beta_{i-2} + \beta_i \pmod{(\alpha_i(\alpha_{i-2}, s), s)}. \tag{4.4}$$

  Indeed, following the proof of Lemma 2.2.2, the pre-images y of x in C_i that satisfy

  $$y \equiv \beta_{i-2} \pmod{(\alpha_{i-2}, s)}$$

  (we note that they are exactly those pre-images of x in C_i which are *not* leaves in Γ_f) are characterized by the system of congruences

  $$\begin{aligned} y &\equiv \beta_{i-2} \pmod{(\alpha_{i-2}, s)} \\ \alpha_i y + \beta_i &\equiv x \pmod{s}, \end{aligned}$$

  which is (according to the proof of Lemma 2.2.2) consistent if and only if congruence (4.4) holds, in which case the system is equivalent to a single congruence modulo

  $$\mathrm{lcm}\left((\alpha_{i-2}, s), \frac{s}{(\alpha_i, s)}\right).$$

  Hence, if $\nu_{2i-5} = \emptyset$, i.e., if congruence (4.4) holds, then x has exactly

  $$\frac{s}{\mathrm{lcm}\left((\alpha_{i-2}, s), \frac{s}{(\alpha_i, s)}\right)} = \left(\frac{s}{(\alpha_{i-2}, s)}, (\alpha_i, s)\right)$$

  pre-images $y \in C_i$ with $y \equiv \beta_{i-2} \pmod{(\alpha_{i-2}, s)}$, which are exactly those pre-images that lie in $\mathcal{B}(\mathcal{P}_i, (\emptyset))$. Otherwise, all pre-images of x in C_i are incongruent to β_{i-2} modulo (α_{i-2}, s) and thus lie in $\mathcal{B}(\mathcal{P}_i, (\neg))$. In view of the known value of $\mathrm{Tree}_i(\mathcal{P}_i, (\nu))$ in terms of $\nu \in \{\emptyset, \neg\}$, we find that $\mathrm{Tree}_0(\mathcal{P}_i', C_i, \vec{\nu}^{(\mathcal{P}_i')})$ is the expanded version of the (not necessarily simplified) edge-weighted directed rooted tree specified in Table 4.1.

This settles the first two summands of $\mathrm{Tree}_0(\mathcal{P}_0, \vec{\nu}^{(\mathcal{P}_0)})$ in each of the two cases in formula (4.3). If $\nu_9 = \neg$ (i.e., if the $\mathcal{P}_0$-block in question consists of f-transient points), then these are all the summands in the formula, and one can obtain (an edge-weighted version of) $\mathrm{Tree}_0(\mathcal{P}_0, \vec{\nu}^{(\mathcal{P}_0)})$ simply by adding (the edge-weighted versions

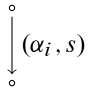
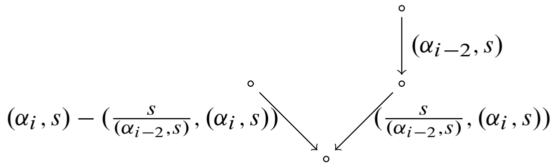
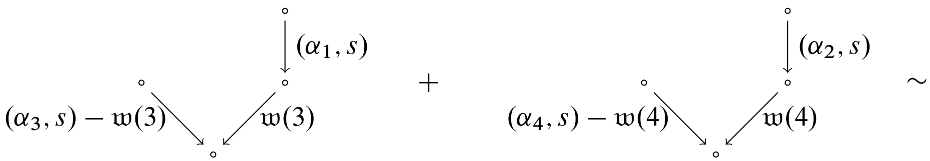
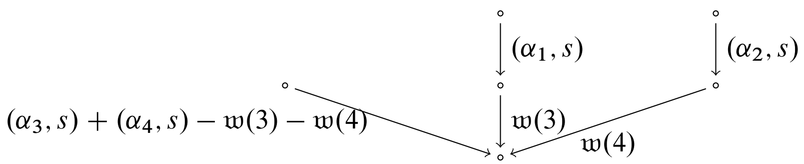
block of $\mathcal{P}'_i = \mathfrak{P}'(\mathcal{P}_i, A_i)$	associated $\mathrm{Tree}_{\Gamma_f}(x, C_i)$
$\nu(x \equiv \alpha_i\beta_{i-2} + \beta_i \pmod{(\alpha_i(\alpha_{i-2}, s), s)})$ $x \not\equiv \beta_i \pmod{(\alpha_i, s)}$ $\nu \in \{\emptyset, \neg\}$	$\circ$
$x \not\equiv \alpha_i\beta_{i-2} + \beta_i \pmod{(\alpha_i(\alpha_{i-2}, s), s)}$ $x \equiv \beta_i \pmod{(\alpha_i, s)}$	
$x \equiv \alpha_i\beta_{i-2} + \beta_i \pmod{(\alpha_i(\alpha_{i-2}, s), s)}$ $x \equiv \beta_i \pmod{(\alpha_i, s)}$	

Table 4.1. Rooted trees using only pre-images in the transient pre-image coset C_i with $i \in \{3, 4\}$.

of) $\mathrm{Tree}_i(\mathcal{P}'_i, \vec{\nu}^{(\mathcal{P}'_i)})$ for $i \in \{3, 4\}$, read off from Table 4.1. For example, if $\nu_j = \emptyset$ for $j = 1, 2, \ldots, 8$ but $\nu_9 = \neg$, then (an edge-weighted version of) $\mathrm{Tree}_0(\mathcal{P}_0, \vec{\nu}^{(\mathcal{P}_0)})$ is as follows, setting $\mathfrak{w}(i) := (\frac{s}{(\alpha_{i-2}, s)}, (\alpha_i, s))$ for $i = 3, 4$:





On the other hand, if $\nu_9 = \emptyset$ (so that all vertices in $\mathcal{B}(\mathcal{P}_0, \vec{\nu}^{(\mathcal{P}_0)})$ are f-periodic), then we also need to compute $\mathrm{Tree}_0^{(1)}(\mathcal{S}_{0,1}, C_0, \vec{o'_1})$, the third summand in formula (4.3), which expresses the contribution coming from f-transient pre-images in C_0. Following Proposition 3.3.4, this can be done by studying the distribution of pre-images of any given point $x \in \mathcal{B}(\mathcal{P}_0, \vec{\nu}^{(\mathcal{P}_0)})$ over certain blocks of the partition $\mathcal{Q}_{0,0} = \mathcal{R}_0 \wedge \mathfrak{P}(x \equiv \beta_0 \pmod{(\alpha_0, s)})$. More specifically, we note that each f-transient pre-image of x is contained in a block of $\mathcal{Q}_{0,0}$ of the form $\mathcal{B}(\mathcal{Q}_{0,0}, \vec{o_0} \diamond (\neg))$ for some $\vec{o_0} \in \{\emptyset, \neg\}^4$ (and we also observe that each block of $\mathcal{Q}_{0,0}$ of this form consists entirely of f-transient points, due to the last logical sign being $\neg$). Being able to count the number of pre-images of x in each such block of $\mathcal{Q}_{0,0}$ is enough to understand $\mathrm{Tree}_0^{(1)}(\mathcal{S}_{0,1}, C_0, \vec{o_1'})$, because $\mathcal{B}(\mathcal{Q}_{0,0}, \vec{o_0} \diamond (\neg)) \subseteq \mathcal{B}(\mathcal{R}_0, \vec{o_0})$ and we already understand the rooted trees above f-transient vertices in a given block $\mathcal{B}(\mathcal{R}_0, \vec{o_0'}) = \mathcal{B}(\mathcal{R}_0, \vec{\nu}^{(\mathcal{P}_3')} \diamond \vec{\nu}^{(\mathcal{P}_4')})$ of $\mathcal{R}_0$.

Now, let us observe that the distribution of the pre-images of any f-periodic point $x \in C_0$ over the blocks of $\mathcal{Q}_{0,0}$ is controlled by the values of ν_j for $j \in \{5,6,7,8\}$, i.e., by the block $\mathcal{B}(\mathcal{S}_{0,1}, \vec{o_1'})$ of $\mathcal{S}_{0,1}$ in which x is contained. This is because $x \in \mathcal{B}(\mathcal{T}_{0,1}, \vec{o_1'} \diamond (\emptyset))$, where $\mathcal{T}_{0,1} = \mathcal{S}_{0,1} \wedge \mathcal{U}_0 = \mathfrak{P}'(\mathcal{Q}_{0,0}, A_0)$, a partition which does indeed control the distribution of pre-images of x over the blocks of $\mathcal{Q}_{0,0}$ according to Lemma 2.2.2. Applying this lemma here leads to the formula for $\mathrm{Tree}_0^{(1)}(\mathcal{S}_{0,1}, C_0, \vec{o_1'})$ from Proposition 3.3.4.

For example, the logical sign tuple $\vec{o_0} = (\neg, \neg, \emptyset, \emptyset)$ corresponds to the block $B := \mathcal{B}(\mathcal{R}_0, \vec{o_0})$ of $\mathcal{R}_0$. If we wish to count how many transient pre-images a vertex $x \in C_i$ with $x \equiv \alpha_0 \pmod{(\alpha_0, s)}$ stemming from, say, the block $B' := \mathcal{B}(\mathcal{S}_{0,1}, \vec{o_1'})$ of $\mathcal{S}_{0,1}$ with $\vec{o_1'} = (\emptyset, \neg, \emptyset, \emptyset)$ has, then we need to compute

$$\sigma_{\mathcal{Q}_{0,0}, A_0}(\vec{o_0} \diamond (\neg), \vec{o_1'} \diamond (\emptyset)),$$

which we do now to illustrate the method. To avoid confusion among readers, we note that the above expression does not perfectly match the notation used in Lemma 2.2.2. Indeed, here we use spanning congruence sequences of length 5 both for $\mathcal{Q}_{0,0} = \mathcal{R}_0 \wedge \mathcal{T}_{0,0} = \mathcal{R}_0 \wedge \mathcal{U}_0$ and for $\mathcal{T}_{0,1} = \mathfrak{P}'(\mathcal{Q}_{0,0}, A_0)$. However, in Lemma 2.2.2, it is assumed that we use the "standard format" of the spanning congruence sequence for $\mathfrak{P}'(\mathcal{Q}_{0,0}, A_0)$, which contains one congruence more than the sequence for $\mathcal{Q}_{0,0}$. This discrepancy occurs because we write $\mathcal{T}_{0,1}$ as $\lambda(\mathcal{R}_0, A_0) \wedge \mathcal{U}_0$ – in the "standard format", it would instead be

$$\begin{aligned}&\lambda(\mathcal{Q}_{0,0}, A_0) \wedge \mathfrak{P}(x \equiv \beta_0 \pmod{(\alpha_0, s)}) \\ &\quad = \lambda(\mathcal{R}_0, A_0) \wedge \mathfrak{P}(x \equiv \beta_0(1+\alpha_0) \pmod{(\alpha_0^2, s)}) \wedge \mathcal{U}_0,\end{aligned}$$

but we can omit the congruence $x \equiv \beta_0(1+\alpha_0) \pmod{(\alpha_0^2, s)}$, which is $\theta_{0,2}(x)$ in the notation of Section 3.3, because (using that $H_0 = 1$) it is equivalent to $x \equiv \beta_0 \pmod{(\alpha_0, s)}$, the unique spanning congruence $\theta_{0,1}(x)$ of $\mathcal{U}_0$. In order to apply Lemma 2.2.2, we put $\mathcal{T}_{0,1}$ into the less concise standard format, which requires us to replace the logical sign sequence $\vec{o_1'} \diamond (\emptyset) = (\emptyset, \neg, \emptyset, \emptyset, \emptyset)$ for the block of $\mathcal{T}_{0,1}$ by $\vec{\nu'} := \vec{o_1'} \diamond (\emptyset, \emptyset) = (\emptyset, \neg, \emptyset, \emptyset, \emptyset, \emptyset)$ (i.e., we double the $\emptyset$ at the end), the j-th entry of which we denote by ν_j'. The logical sign sequence for the block of $\mathcal{Q}_{0,0}$ remains $\vec{\nu} := \vec{o_0} \diamond (\neg) = (\neg, \neg, \emptyset, \emptyset, \neg)$. Our goal now is to compute $\sigma_{\mathcal{Q}_{0,0}, A_0}(\vec{\nu}, \vec{\nu'})$ strictly following Lemma 2.2.2. For $j = 1, \ldots, 5$, we denote by $\bar{\mathfrak{a}}_j$, respectively, $\bar{\mathfrak{b}}_j$, the modulus, respectively, right-hand side, of the j-th spanning congruence of $\mathcal{Q}_{0,0} = \mathcal{R}_0 \wedge \mathcal{U}_0$. That is, for $j \in \{1,2,3,4\}$, the congruence $x \equiv \bar{\mathfrak{b}}_j \pmod{\bar{\mathfrak{a}}_j}$ is the

j-th displayed congruence in the formula for $\mathcal{P}_0$, (4.2). Moreover, $\bar{\mathfrak{a}}_5 = (\alpha_0, s)$ and $\bar{\mathfrak{b}}_5 = \beta_0$. Using the notation from Lemma 2.2.2, we observe that

- $J_-(\vec{\nu}) = \{1, 2, 5\}$ (the set of indices $j \in \{1, \dots, 5\}$ such that the j-th entry of $\vec{o_0} \diamond (\neg)$ is $\neg$);
- $J_+(\vec{\nu}) = \{1, 2, 3, 4, 5\} \setminus J_-(\vec{\nu}) = \{3, 4\}$;
- $J_-(\vec{\nu'}) = \{2\}$;
- for $J \subseteq J_-(\vec{\nu}) = \{1, 2, 5\}$, the condition $E(\vec{\nu}, J)$ demands: "For all $j_1, j_2 \in \{3, 4\} \cup J$: $\gcd(\bar{\mathfrak{a}}_{j_1}, \bar{\mathfrak{a}}_{j_2}) \mid \bar{\mathfrak{b}}_{j_1} - \bar{\mathfrak{b}}_{j_2}$".

According to Lemma 2.2.2, we have

$$\sigma_{\mathcal{Q}_{0,0},A_0}(\vec{\nu}, \vec{\nu'}) = \sum_{J \subseteq J_-(\vec{\nu})} (-1)^{|J|} \kappa_{\mathcal{Q}_{0,0},A_0}(\vec{\nu}, \vec{\nu'}, J),$$

where

$$\kappa_{\mathcal{Q}_{0,0},A_0}(\vec{\nu}, \vec{\nu'}, J) = \delta_{\nu'_6=\emptyset} \cdot \delta_{E(\vec{\nu},J)} \cdot \delta_{(J_+(\vec{\nu}) \cup J) \cap J_-(\vec{\nu'})=\emptyset} \cdot \frac{s}{\operatorname{lcm}(\frac{s}{\gcd(\alpha_0,s)}, \bar{\mathfrak{a}}_{0,j} : j \in J_+(\vec{\nu}) \cup J)}.$$

In this formula, the first Kronecker delta checks whether the last entry of $\vec{\nu'}$ is $\emptyset$, which is the case. The third Kronecker delta is 1 if and only if $2 \notin J$, which leaves the four possibilities $\emptyset, \{1\}, \{5\}, \{1, 5\}$ for $J \subseteq J_-(\vec{\nu}) = \{1, 2, 5\}$ for which $\kappa_{\mathcal{Q}_{0,0},A_0}(\vec{\nu}, \vec{\nu'}, J)$ is potentially non-zero. We conclude that

$$\begin{aligned}\sigma_{\mathcal{Q}_{0,0},A_0}(\vec{\nu}, \vec{\nu'}) = {} & \delta_{E(\vec{\nu},\emptyset)} \frac{s}{\operatorname{lcm}(\frac{s}{(\alpha_0,s)}, \bar{\mathfrak{a}}_3, \bar{\mathfrak{a}}_4)} - \delta_{E(\vec{\nu},\{1\})} \frac{s}{\operatorname{lcm}(\frac{s}{(\alpha_0,s)}, \bar{\mathfrak{a}}_1, \bar{\mathfrak{a}}_3, \bar{\mathfrak{a}}_4)} \\ & - \delta_{E(\vec{\nu},\{5\})} \frac{s}{\operatorname{lcm}(\frac{s}{(\alpha_0,s)}, \bar{\mathfrak{a}}_3, \bar{\mathfrak{a}}_4, \bar{\mathfrak{a}}_5)} \\ & + \delta_{E(\vec{\nu},\{1,5\})} \frac{s}{\operatorname{lcm}(\frac{s}{(\alpha_0,s)}, \bar{\mathfrak{a}}_1, \bar{\mathfrak{a}}_3, \bar{\mathfrak{a}}_4, \bar{\mathfrak{a}}_5)}\end{aligned}$$

is the number of f-transient children in $B = \mathcal{B}(\mathcal{R}_0, \vec{o_0})$ of each given $x \in B' = \mathcal{B}(\mathcal{S}_{0,1}, \vec{o'_1})$. Each of these children provides a copy of

$$\mathrm{Tree}_0^{(0)}(\mathcal{P}_{0,0}, \vec{o_0}) = \mathrm{Tree}_0(\mathcal{R}_0, C_3 \cup C_4, \vec{o_0})$$

that is attached to the root of $\mathrm{Tree}_0^{(1)}(\mathcal{S}_{0,1}, C_0, \vec{o'_1})$. If we carry this computation out for fixed $\vec{o'_1}$ and all possible values of $\vec{o_0} \in \{\emptyset, \neg\}^4$, then we obtain a "complete picture" of $\mathrm{Tree}_0^{(1)}(\mathcal{S}_{0,1}, C_0, \vec{o'_1})$.

It is time to particularize the gained explicit understanding of the rooted trees back to the concrete example we started from. First, we deal with the rooted trees

$\text{Tree}_{\Gamma_f}(x, C_3 \cup C_4)$ in terms of the blocks of $\mathcal{R}_0$. By substituting $s = 51$, $\alpha_0 = 9$, $\alpha_1 = 3$, $\alpha_2 = 17$, $\alpha_3 = 34$, $\alpha_4 = 9$, $\beta_0 = 1$, $\beta_1 = 0$, $\beta_2 = 6$, $\beta_3 = 21$, $\beta_4 = 8$ into formula (4.1), we get

$$\mathcal{R}_0 = \mathfrak{P}\begin{pmatrix} x \equiv 21 \pmod{51} \\ x \equiv 4 \pmod{17} \\ x \equiv 11 \pmod{51} \\ x \equiv 2 \pmod{3} \end{pmatrix}.$$

There are dependencies between these congruences. For example, the first implies the second as well as the negations of the third and fourth. Table 4.2 lists all $\vec{o_0'} \in \{\emptyset, \neg\}^4$ such that $\mathcal{B}(\mathcal{R}_0, \vec{o_0'})$ is nonempty, along with a description of the set $\mathcal{B}(\mathcal{R}_0, \vec{o_0'})$ and the (simplified edge-weighted form of the) corresponding rooted tree $\text{Tree}_0(\mathcal{R}_0, C_3 \cup C_4, \vec{o_0'})$ above each point in $\mathcal{B}(\mathcal{R}_0, \vec{o_0'})$, obtained by adding the (edge-weighted forms of the) rooted trees $\text{Tree}_0(\mathcal{P}_i', C_i, \vec{\nu_i'})$ for $i = 3, 4$ read off from Table 4.1.

Now we turn to the description of $\text{Tree}_{\Gamma_f}(x, C_0)$ for f-periodic $x \in C_0$ in terms of the $\mathcal{S}_{0,1}$-block $\mathcal{B}(\mathcal{S}_{0,1}, \vec{o_1'})$ in which x lies. First, we substitute our concrete values of s and the α_i and β_i into the four spanning congruences for $\mathcal{S}_{0,1}$ to get that

$$\mathcal{S}_{0,1} = \mathfrak{P}\begin{pmatrix} x \equiv 37 \pmod{51} \\ x \equiv 37 \pmod{51} \\ x \equiv 49 \pmod{51} \\ x \equiv 1 \pmod{3} \end{pmatrix}.$$

It is not necessary to strictly follow the computations described in Lemma 2.2.2 (as outlined above) to gain a complete understanding of the trees here – we give a conceptual argument instead.

We note that by Lemma 2.1.14, the points in $\mathbb{Z}/51\mathbb{Z}$ that are periodic under $A_0 : x \mapsto 9x + 1$ are just those that are congruent to 1 modulo 3 (the unique fixed point of A_0 modulo $3 = \gcd(\alpha_0^L, s)$, where L is as in Lemma 2.1.14). Hence, the last spanning congruence of $\mathcal{S}_{0,1}$ is always true for those x we are considering. We observe that it is just a coincidence that the spanning sequence for $\mathcal{S}_{0,1}$ contains the characterizing congruence for periodic vertices – in general, this information needs to be added "externally" if one wants to control it via the partition blocks, using the partition $\mathcal{T}_{0,1} = \mathcal{S}_{0,1} \wedge \mathcal{U}_0$ instead.

Since the children of x in C_0 form a coset of the kernel $17\mathbb{Z}/51\mathbb{Z}$ of $z \mapsto 9z$, it follows that x has precisely three pre-images in C_0, one in each congruence class modulo 3. But the pre-image y of x in C_0 with $y \equiv 1 \pmod{3}$ is f-periodic and hence does not occur in $\text{Tree}_{\Gamma_f}(x, C_0)$. We also note the following.

- For $x = 37$, the remaining two, f-transient pre-images are 21 and 38.
- For $x = 49$ the f-transient pre-images are 11 and 45.

$\vec{o}_0'$	$B := \mathcal{B}(\mathcal{R}_0, \vec{o}_0')$	$\mathrm{Tree}_{\Gamma_f}(x, C_3 \cup C_4)$ for $x \in B$
$\begin{pmatrix}\neg\\ \neg\\ \neg\\ \neg\end{pmatrix}$	$\{x \in \mathbb{Z}/51\mathbb{Z} : x \not\equiv 2 \pmod 3\} \setminus \{4, 21\}$	$\circ$
$\begin{pmatrix}\neg\\ \emptyset\\ \neg\\ \neg\end{pmatrix}$	$\{4\}$	$\circ \xrightarrow{17} \circ$
$\begin{pmatrix}\emptyset\\ \emptyset\\ \neg\\ \neg\end{pmatrix}$	$\{21\}$	$\circ \xrightarrow{3} \circ \xrightarrow{17} \circ$
$\begin{pmatrix}\neg\\ \neg\\ \neg\\ \emptyset\end{pmatrix}$	$\{x \in \mathbb{Z}/51\mathbb{Z} : x \equiv 2 \pmod 3\} \setminus \{11, 38\}$	$\circ \xrightarrow{3} \circ$
$\begin{pmatrix}\neg\\ \neg\\ \emptyset\\ \emptyset\end{pmatrix}$	$\{11\}$	$\circ \xrightarrow{17} \circ \xrightarrow{3} \circ$
$\begin{pmatrix}\neg\\ \emptyset\\ \neg\\ \emptyset\end{pmatrix}$	$\{38\}$	$\circ \xrightarrow{20} \circ$

Table 4.2. Rooted trees using only pre-images in transient pre-image cosets.

- For all other $x \equiv 1 \pmod 3$, there is one f-transient pre-image each in the two "generic" blocks $\mathcal{B}(\mathcal{R}_0, (\neg, \neg, \neg, \neg))$ and $\mathcal{B}(\mathcal{R}_0, (\neg, \neg, \neg, \emptyset))$ of $\mathcal{R}_0$; this is because all other blocks in Table 4.2 except $\{4\}$ have already been "used up", and $4 \equiv 1 \pmod 3$.

From Table 4.2, we can read off $\mathrm{Tree}_{\Gamma_f}(y, C_3 \cup C_4) = \mathrm{Tree}_{\Gamma_f}(y)$ for each of the two f-transient pre-images y of x in C_0, thus obtaining the shape of $\mathrm{Tree}_{\Gamma_f}(x, C_0)$ specified in Table 4.3.

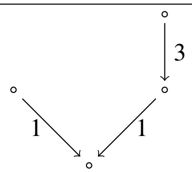
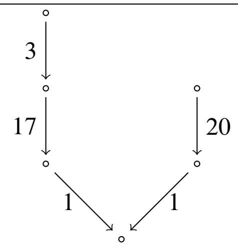
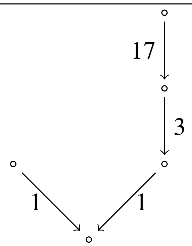
$\vec{o}'_1$	$B' := \mathcal{B}(\mathcal{S}_{0,1}, \vec{o}'_1)$	$\mathrm{Tree}_{\Gamma_f}(x, C_0)$ for periodic $x \in B'$
$\begin{pmatrix} \neg \\ \neg \\ \neg \\ \neg \end{pmatrix}$	$\{x \in \mathbb{Z}/51\mathbb{Z} : x \not\equiv 1 \pmod 3\}$	n/a (no periodic x in this block)
$\begin{pmatrix} \neg \\ \neg \\ \neg \\ \emptyset \end{pmatrix}$	$\{x \in \mathbb{Z}/51\mathbb{Z} : x \equiv 1 \pmod 3\} \setminus \{37, 49\}$	
$\begin{pmatrix} \emptyset \\ \emptyset \\ \neg \\ \emptyset \end{pmatrix}$	$\{37\}$	
$\begin{pmatrix} \neg \\ \neg \\ \emptyset \\ \emptyset \end{pmatrix}$	$\{49\}$	

Table 4.3. Rooted trees using only pre-images in C_0.

Finally, our formula (4.3) leads us to a tabular list of $\mathrm{Tree}_{\Gamma_f}(x)$, which we specify in Table 4.4, where we also introduce the notation $\mathfrak{I}_n$ for $n = 1, 2, 3, 4$ to denote the different isomorphism types of $\mathrm{Tree}_{\Gamma_f}(x)$ for periodic x. Additionally, we define $\mathfrak{I}_0$ to denote the isomorphism type of the trivial rooted tree, consisting of a single vertex without arcs.

Now that we have a full understanding of the rooted trees above vertices in Γ_f, let us turn to the determination of the f-periodic points and to the construction of a CRL-list for f. Following the discussion in Section 3.1, the periodic points of f are the field element 0 as well as all periodic points of f in the unique periodic coset C_0, which we already identified above (using Lemma 2.1.14) to be precisely those $x \in \mathbb{Z}/51\mathbb{Z}$ with $x \equiv 1 \pmod 3$, so there are $51/3 = 17$ periodic points in C_0.

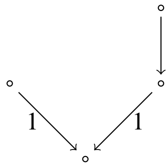
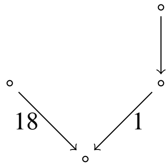
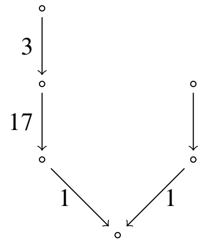
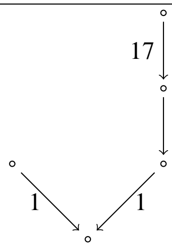
block B of $\mathcal{P}_0$	$\mathrm{Tree}_{\Gamma_f}(x)$ for $x \in B$
$\{x \in \mathbb{Z}/51\mathbb{Z} : x \equiv 1 \pmod 3\} \setminus \{4, 37, 49\}$	 $=: \mathfrak{T}_1$
$\{4\}$	 $=: \mathfrak{T}_2$
$\{37\}$	 $=: \mathfrak{T}_3$
$\{49\}$	 $=: \mathfrak{T}_4$
$B \subseteq \{x \in \mathbb{Z}/51\mathbb{Z} : x \not\equiv 1 \pmod 3\}$	$\mathrm{Tree}_{\Gamma_f}(x, C_3 \cup C_4)$ (see Table 4.2)

Table 4.4. The rooted trees $\mathrm{Tree}_{\Gamma_f}(x)$ for $x \in C_0$.

With such a small number of periodic points, it would be easy to just determine the cycle structure and a CRL-list by brute force, but we would still like to proceed as described in Section 3.1 (and Section 2.3, on which Section 3.1 builds) to illustrate the method.

First, we observe that the nontrivial prime powers of the form $p^{\nu_p(51)}$ are just 3 and 17. By the approach from Section 2.3, we need to determine a CRL-list of each *bijective* reduction of A_0 modulo $p^{\nu_p(51)}$, i.e., here only for the reduction of A_0 modulo 17. We can read off such a CRL-list from Table 2.2. More specifically, since

$$\nu_{17}^{(1)}(1) = 0 \geq 0 = \nu_{17}^{(1)}(9-1),$$

and since ord(9), the multiplicative order of 9 modulo 17, is 8, case 1 in that table with

$$\mathrm{r} := 3 \quad \text{and} \quad \mathfrak{f} := -\frac{1}{17^0} \cdot \mathrm{inv}_{17}\left(\frac{8}{17^0}\right) = -15 = 2$$

tells us that

$$\{(3^0 17^0 + 2, 8), (3^1 17^0 + 2, 8), (3^0 17^1 + 2, 1)\} = \{(3, 8), (5, 8), (2, 1)\}$$

is a CRL-list of A_0 modulo 17. Modulo 3, the only periodic point of A_0 is 1, so in order to get a CRL-list for A_0 modulo 51, we just map the first entries of the above CRL-list modulo 17 under the function $\Lambda : \mathbb{Z}/17\mathbb{Z} \to \mathbb{Z}/51\mathbb{Z}$ with $\Lambda(x) \equiv x \pmod{17}$ and $\Lambda(x) \equiv 1 \pmod{3}$, which leads to the following CRL-list of A_0 modulo 51:

$$\{(37, 8), (22, 8), (19, 1)\}.$$

We can now describe the isomorphism types of the four connected components of Γ_f as cyclic sequences (necklaces, isomorphism types of necklace graphs) of finite directed rooted trees simply by enumerating the elements on the cycles of f by iteration, then looking up the associated rooted tree isomorphism types in Table 4.4. We note that this is a brute-force approach that is not viable when the number of f-periodic points is large and should then be replaced by the approach described in Section 3.4 instead.

- The connected component of the field element 0 is a single vertex with a loop, corresponding to the following cyclic sequence of rooted tree isomorphism types: $[\mathfrak{I}_0]$.
- Because the cycle of $22 \in \mathbb{Z}/51\mathbb{Z}$ under A_0 is $(22, 46, 7, 13, 16, 43, 31, 25)$, the connected component of the field element $\iota_0(22) = \omega^{5\cdot 22} = \omega^{110}$ is represented by the cyclic sequence $[\mathfrak{I}_1, \mathfrak{I}_1, \mathfrak{I}_1, \mathfrak{I}_1, \mathfrak{I}_1, \mathfrak{I}_1, \mathfrak{I}_1, \mathfrak{I}_1]$.
- Because the cycle of $37 \in \mathbb{Z}/51\mathbb{Z}$ under A_0 is $(37, 28, 49, 34, 1, 10, 40, 4)$, the connected component of the field element ω^{185} is represented by the cyclic sequence

$$[\mathfrak{I}_3, \mathfrak{I}_1, \mathfrak{I}_4, \mathfrak{I}_1, \mathfrak{I}_1, \mathfrak{I}_1, \mathfrak{I}_1, \mathfrak{I}_2].$$

- Finally, because the cycle of 19 is simply (19), the connected component of the field element ω^{95} is represented by $[\mathfrak{I}_1]$.

As a quick sanity check, we note that $|\mathrm{V}(\mathfrak{I}_1)| = 6$, $|\mathrm{V}(\mathfrak{I}_2)| = 23$, $|\mathrm{V}(\mathfrak{I}_3)| = 91$, and $|\mathrm{V}(\mathfrak{I}_4)| = 57$, so the vertex numbers of the three connected components in $\mathbb{F}_{2^8}^*$ add up to

$$14 \cdot 6 + 23 + 91 + 57 = 255,$$

as they should. One may also verify that these are the same cyclic sequences that were given in our introduction.

4.3 Special case: All A_i are permutations

Let f be an index d generalized cyclotomic mapping of $\mathbb{F}_q$, given in cyclotomic form (1.1). Let us assume that for each $i = 0, 1, \ldots, d-1$, we have

$$\gcd(r_i, s) = \gcd\left(r_i, \frac{q-1}{d}\right) = 1.$$

An important class of functions to which this applies are the index d cyclotomic mappings of $\mathbb{F}_q$ *of first order* (i.e., those generalized cyclotomic mappings for which all r_i are equal to 1).

By our comments between Remark 1.3 and Definition 1.4, the affine map A_i of $\mathbb{Z}/s\mathbb{Z}$, which encodes the restriction $f_{|C_i}$ in case $a_i \neq 0$, is of the form $z \mapsto r_i z + \text{const}$. Our assumption on the r_i is therefore equivalent to demanding that for each i such that $a_i \neq 0$ (and thus A_i is well defined), the function A_i is an affine *permutation* of $\mathbb{Z}/s\mathbb{Z}$.

Our goal is to describe the functional graph Γ_f, which turns out to be particularly easy. Let us start with the rooted trees.

Lemma 4.3.1. *Let $x \in \mathbb{F}_q = \mathrm{V}(\Gamma_f)$.*

(1) *If $x \neq 0$, and if i denotes the unique index in $\{0, 1, \ldots, d-1\}$ such that $x \in C_i$, then* $\mathrm{Tree}_{\Gamma_f}(x)$ *is isomorphic to* $\mathfrak{I}_i := \mathrm{Tree}_{\Gamma_{\bar{f}}}(i)$.

(2) *If $x = 0$, then* $\mathrm{Tree}_{\Gamma_f}(x) = \sum_{i \in \bar{f}^{-1}(\{d\}) \setminus \{d\}} s\mathfrak{I}_i^+$.

Proof. Statement (1) can be proved by induction on $h(x) := \mathrm{ht}(\mathrm{Tree}_{\Gamma_{\bar{f}}}(i))$. If $h(x) = 0$, then all $\bar{f}$-pre-images of i (if any) are $\bar{f}$-periodic. In particular, x has no f-transient pre-images under f, because each such pre-image would need to lie in a coset C_j, where j is an $\bar{f}$-transient pre-image of i under $\bar{f}$. Indeed, otherwise, i, having an $\bar{f}$-periodic pre-image under $\bar{f}$, is $\bar{f}$-periodic itself. By assumption, we can pick an f-transient pre-image y of x under f in $C_{i'}$, where i' is the unique $\bar{f}$-periodic pre-image of i under $\bar{f}$. If ℓ denotes the cycle length of i under $\bar{f}$, then $\mathcal{A}_i = A_{i_0} A_{i_1} \cdots A_{i_{\ell-1}}$ represents the restriction of f^{ℓ} to C_i. Because each A_{i_t} is bijective, so is $\mathcal{A}_i$; in other words, every point in C_i is periodic under $\mathcal{A}_i$ and thus under f (following the discussion in Section 3.1). In particular, x is f-periodic, say with cycle length l. Therefore, $f^{l-1}(x)$ is an f-pre-image of x in $C_{i'}$, as is y. Because $A_{i'}$, which represents the restriction $f_{|C_{i'}} : C_{i'} \to C_i$, is injective, it follows that $f^{l-1}(x) = y$, whence y is f-periodic, contradicting our assumption. The upshot of this discussion is that if $h(x) = 0$, then $\mathrm{Tree}_{\Gamma_f}(x)$ is trivial, as is $\mathfrak{I}_i$.

Now we assume that $h(x) \geq 1$. Let $j_1, j_2, \ldots, j_K$ be the distinct $\bar{f}$-transient pre-images of i under $\bar{f}$. By the argument from the previous paragraph, each f-transient pre-image of x under f must lie in one of the cosets C_{j_t} for $t = 1, 2, \ldots, K$, and since A_{j_t} is bijective for each t, it follows that x has precisely one (transient) preimage $c_{j_t} \in C_{j_t}$ for each $t = 1, 2, \ldots, K$. Therefore, using the induction hypothesis,

$$\mathrm{Tree}_{\Gamma_f}(x) = \sum_{t=1}^{K} \mathrm{Tree}_{\Gamma_f}(c_{j_t})^+ = \sum_{t=1}^{K} \mathfrak{I}_{j_t}^+ = \sum_{t=1}^{K} \mathrm{Tree}_{\Gamma_{\bar{f}}}(j_t)^+ = \mathrm{Tree}_{\Gamma_{\bar{f}}}(i).$$

For statement (2), let $j_1, j_2, \ldots, j_K$ be the distinct $\bar{f}$-transient children of d in $\Gamma_{\bar{f}}^*$. Equivalently, the j_t are the distinct elements of $\bar{f}^{-1}(\{d\}) \setminus \{d\}$. The f-transient children of $0_{\mathbb{F}_q}$ in Γ_f^* are precisely the points in $\bigcup_{t=1}^{K} C_{j_t}$. Using statement (1), it follows that

$$\mathrm{Tree}_{\Gamma_f}(0_{\mathbb{F}_q}) = \sum_{t=1}^{K} \sum_{y \in C_{j_t}} \mathrm{Tree}_{\Gamma_f}(y)^+ \cong \sum_{t=1}^{K} \sum_{y \in C_{j_t}} \mathfrak{I}_{j_t}^+ = \sum_{t=1}^{K} s\mathfrak{I}_{j_t}^+,$$

as required. ■

Because $\mathrm{Tree}_{\Gamma_f}(x)$ for $x \neq 0$ only depends on the coset C_i in which x lies and can be read off directly from $\Gamma_{\bar{f}}$, we only need to know $\Gamma_{\bar{f}}$ and the cycle structure of f on each coset union $U_i := \bigcup_{t=0}^{\ell-1} C_{i_t}$, where $(i_0, i_1, \ldots, i_{\ell-1})$ with $i = i_0$ is the $\bar{f}$-cycle of i, in order to understand the isomorphism type of Γ_f. This can be achieved using analogous ideas to the ones for the determination of CRL-lists in Section 3.1.

Let us set $\mathcal{A}_i := A_{i_0} A_{i_1} \cdots A_{i_{\ell-1}}$. Then $\mathcal{A}_i$ is an affine permutation of $\mathbb{Z}/s\mathbb{Z}$, and its cycle type $\mathrm{CT}(\mathcal{A}_i)$ can be read off from [15, Tables 3 and 4]. Moreover,

$$\mathrm{CT}(f_{|U_i}) = \mathrm{BU}_\ell(\mathrm{CT}(\mathcal{A}_i)),$$

where BU_ℓ, the so-called *ℓ-blow-up function*, is the unique $\mathbb{Q}$-algebra endomorphism of $\mathbb{Q}[x_n : n \in \mathbb{N}^+]$ with $\mathrm{BU}_\ell(x_n) = x_{\ell n}$ for all $n \in \mathbb{N}^+$. Say

$$\mathrm{CT}(f_{|U_i}) = x_1^{e_1} x_2^{e_2} \cdots x_{\ell s}^{e_{\ell s}}.$$

Then, viewing isomorphism types of functional graphs as multisets of cyclic sequences (necklaces) of isomorphism types of finite directed rooted trees (with each such sequence encoding the isomorphism type of one connected component), we have the following:

$$\Gamma_f^{(i)} := \Gamma_{f|U_i} = \bigsqcup_{1 \leq l \leq \ell s, \ell | l} \bigsqcup_{n=1}^{e_l} \{\diamond_{m=1}^{l/\ell} [\mathfrak{I}_{i_0}, \mathfrak{I}_{i_1}, \ldots, \mathfrak{I}_{i_{\ell-1}}]\},$$

and, if $\bar{\mathcal{L}}$ is a CRL-list for $\bar{f}$, then

$$\Gamma_f = \bigsqcup_{(i,\ell) \in \bar{\mathcal{L}}} \Gamma_f^{(i)} \sqcup \Big\{\Big[\sum_{j \in \bar{f}^{-1}(\{d\}) \setminus \{d\}} s\mathfrak{I}_j^+ \Big]\Big\}.$$