Chapter 1

Introduction

Classically explicit reciprocity laws or formulae usually mean an explicit computation of Hilbert symbols or (local) cup products using, e.g., differential forms, (Coleman) power series, etc., and a bunch of manifestations of this idea exists in the literature due to Artin–Hasse, Iwasawa, Wiles, Kolyvagin, Vostokov, Brückner, Coleman, Sen, de Shalit, Fesenko, Bloch–Kato, Benois, … In the same spirit, Perrin-Riou's reciprocity law gives an explicit calculation of the Iwasawa cohomology pairing in terms of big exponential and regulator maps for crystalline representations of $G_{\mathbb{Q}_p}$; more precisely, the latter maps are adjoint to each other when also involving the crystalline duality pairing after base change to the distribution algebra corresponding to the cyclotomic situation. We refer the interested reader to the survey [92] concerning the historical development of explicit reciprocity laws.

The motivation for this article is the question of what happens if one replaces the cyclotomic $\mathbb{Z}_p$-extension by a Lubin–Tate extension L_∞ over some finite extension L over $\mathbb{Q}_p$ with Galois group $\Gamma_L = \mathrm{Gal}(L_\infty/L)$ and Lubin–Tate character $\chi_{\mathrm{LT}} : G_L \to o_L^\times$ which all arise from a Lubin–Tate formal group attached to a prime $\pi_L \in o_L$, the additive group of the ring of integers o_L of L; by q we denote the cardinality of the residue field $o_L/o_L\pi_L$. This situation is considerably more complicated for various reasons. First of all, for $L \neq \mathbb{Q}_p$, the group Γ_L requires more than one topological generator and in order to achieve again a sort of one-dimensional theory one has to work in the "L-analytic case". Secondly, the quotient $\frac{q}{\pi_L}$ related to the two meanings of p in the cyclotomic case as uniformizer and cardinality of the residue field is no longer a unit and appears directly in formulae. We try to extend the above sketched cyclotomic picture to the Lubin–Tate case at least for L-*analytic* crystalline representations of the absolute Galois group G_L of L. As pointed out in [80] already, the character $\tau := \chi_{\mathrm{cyc}} \cdot \chi_{\mathrm{LT}}^{-1}$ plays a crucial role, again related to $\frac{q}{\pi_L}$.

To this aim we study (φ_L, Γ_L)-modules over different Robba rings with coefficients in suitable complete intermediate fields $L \subseteq K \subseteq \mathbb{C}_p$. The starting point is the theory of Schneider and Teitelbaum. The p-adic functional analysis of the additive group of p-adic integers $\mathbb{Z}_p$ is based on the fact that the continuous p-adic characters of $\mathbb{Z}_p$ are parametrized by the points of the p-adic open unit disk. The generalization of this basic feature to the additive group o_L was constructed in [74]: a rigid analytic group variety $\mathfrak{X} = \mathfrak{X}_L$ over L whose points parametrize the locally analytic characters of o_L. The Fourier isomorphism of [74] identifies the ring of holomorphic functions $\mathcal{O}_L(\mathfrak{X})$ with the locally analytic distribution algebra $D(o_L, L)$ of o_L. A connection to the Lubin–Tate setting is also established in loc. cit. Under the assumption that the period Ω of the dual of the fixed Lubin–Tate group belongs to K an isomorphism

$\kappa : \mathbf{B}_K \cong \mathfrak{X}_K$ of rigid analytic varieties over K, called the Lubin–Tate isomorphism, is constructed. Here $\mathbf{B}$ denotes the rigid analytic open unit disk and the index K indicates base change to K. In other words, $\mathfrak{X}_L$ is a form of the open unit disk, which is non-trivial if $L \neq \mathbb{Q}_p$.

The multiplicative group $o_L^\times$ acts by multiplication on o_L and hence acts naturally on $\mathfrak{X}$ and $\mathcal{O}_L(\mathfrak{X})$. In [8], the ring $\mathcal{O}_L(\mathfrak{X})$ with its $o_L^\times$-action was enlarged by a geometric construction to the Robba ring $\mathcal{R}_L(\mathfrak{X})$ with $o_L^\times$-action. Some details of this will be recalled in Section 4.1.1. This made it possible to introduce the notion of a (φ_L, Γ_L)-module over $\mathcal{R}_L(\mathfrak{X})$. The main result of [8] relates the category of these new (φ_L, Γ_L)-modules to the Lubin–Tate (φ_L, Γ_L)-modules considered in [44, 73] and therefore to Galois representations. In this paper, we pursue the further systematic development of the theory of (φ_L, Γ_L)-modules over $\mathcal{R}_L(\mathfrak{X})$ in two directions.

First of all, the construction of the variety $\mathfrak{X}_L$ and its corresponding rings is not specific to the additive group o_L. In Sections 4.1.2 and 4.3.4, this will be worked out first for the multiplicative group $o_L^\times$ and then for the Galois group Γ_L (which, by the Lubin–Tate character χ_{LT}, is isomorphic to $o_L^\times$). The corresponding Robba rings will be denoted by $\mathcal{R}_L(\mathfrak{X}^\times)$ and $\mathcal{R}_L(\Gamma_L)$, respectively. We will study the question to which extent the $\Gamma_L \cong o_L^\times$-action on $\mathcal{R}_L(\mathfrak{X})$, resp., on any (φ_L, Γ_L)-module over $\mathcal{R}_L(\mathfrak{X})$, can be extended to an action of the full ring $\mathcal{R}_L(\Gamma_L)$. A first indication of how this works is given by the Mellin transformation in Lemma 4.1.6. The open and closed inclusion of locally analytic manifolds $o_L^\times \hookrightarrow o_L$ induces an isomorphism between the multiplicative distribution algebra $D(o_L^\times, L)$ and the kernel of the ψ-operator on the additive distribution algebra $D(o_L, L)$. This fairly simple observation has a vast conceptual generalization to the following new structural result in Theorem 4.3.23.

Theorem 1 (Theorem 4.3.23). *For any complete intermediate field $L \subseteq K \subseteq \mathbb{C}_p$ and any L-analytic (φ_L, Γ_L)-module M over $\mathcal{R}_K(\mathfrak{X})$, we have the following:*

- *The Γ_L-action on M extends functorially and continuously to an $\mathcal{R}_K(\Gamma_L)$-module structure on $M^{\psi_L=0}$.*
- *Any basis of M over $\mathcal{R}_K(\mathfrak{X})$ can be transformed by a very simple explicit recipe into a basis of $M^{\psi_L=0}$ over $\mathcal{R}_K(\Gamma_L)$ of the same cardinality.*

The proof will be given in Sections 4.3.5–4.3.8. For $\mathbf{B}$ instead of $\mathfrak{X}$ an analogous statement holds if K contains Ω; technically, this is the case we prove first (see Theorem 4.3.21) which then, after involving the Lubin–Tate isomorphism, descends to the above theorem.

Secondly, we will systematically develop the self-duality properties of the Robba rings under consideration in Section 4.2.3. This is based on the Serre duality for the coherent cohomology of rigid analytic spaces as developed in [11, 20, 90]. This is recalled and complemented in Sections 4.2.1 and 4.2.2. In order to apply this to Robba rings these will be related to the cohomology with compact support of the underlying character varieties. This results in residue pairings

$$\Omega^1_{\mathcal{R}_K(-)} \times \mathcal{R}_K(-) \to K$$

for the differentials $\Omega^1_{\mathcal{R}_K(-)}$. The main new results are Corollary 4.2.7 and Proposition 4.2.12. The latter proposition then has to be evaluated for our two cases of main interest $\mathfrak{X}$ and $\mathfrak{X}^\times$ in Section 4.2.3. This leads to two duality pairings,

$$\langle\ ,\ \rangle_{\mathfrak{X}} : \mathcal{R}_L(\mathfrak{X}) \times \mathcal{R}_L(\mathfrak{X}) \to L \quad \text{and} \quad \langle\ ,\ \rangle_{\mathfrak{X}^\times} : \mathcal{R}_L(\mathfrak{X}^\times) \times \mathcal{R}_L(\mathfrak{X}^\times) \to L.$$

The relation between these two pairings will be investigated in Section 4.5.3 and will be the base of our main results in Section 4.5. The right-hand pairing induces topological isomorphisms,

$$\mathrm{Hom}_{K,\mathrm{cont}}(\mathcal{R}_K(\Gamma_L), K) \cong \mathcal{R}_K(\Gamma_L)$$

and

$$\mathrm{Hom}_{K,\mathrm{cont}}(\mathcal{R}_K(\Gamma_L)/D(\Gamma_L, K), K) \cong D(\Gamma_L, K).$$

For an L-analytic (φ_L, Γ_L)-module M over $\mathcal{R} := \mathcal{R}_K(\mathfrak{Y})$ with $\mathfrak{Y}$ equal to either $\mathfrak{X}$ or $\mathbf{B}$, we finally use these isomorphisms to define on the one hand the two Iwasawa pairings

$$\{\ ,\ \}^0_{M,\mathrm{Iw}} : \check{M}^{\psi_L=0} \times M^{\psi_L=0} \to \mathcal{R}_K(\Gamma_L)$$

and

$$\{\ ,\ \}_{M,\mathrm{Iw}} : \check{M}^{\psi_L=\frac{q}{\pi_L}} \times M^{\psi_L=1} \to D(\Gamma_L, K),$$

where $\check{M} := \mathrm{Hom}_{\mathcal{R}}(M, \Omega^1_{\mathcal{R}})$. They are linked by the commutative diagram

$$\begin{array}{ccccc}
\{\ ,\ \}_{M,\mathrm{Iw}} : \check{M}^{\psi_L=\frac{q}{\pi_L}} & \times & M^{\psi_L=1} & \longrightarrow & D(\Gamma_L, K) \\
\downarrow{\scriptstyle \varphi_L - 1} & & \downarrow{\scriptstyle \frac{\pi_L}{q}\varphi_L - 1} & & \downarrow \\
\{\ ,\ \}^0_{M,\mathrm{Iw}} : \check{M}^{\psi_L=0} & \times & M^{\psi_L=0} & \longrightarrow & \mathcal{R}_K(\Gamma_L).
\end{array}$$

Now assume that M arises as $D^\dagger_{\mathrm{rig}}(W)$ under Berger's equivalence of categories, if $\mathfrak{Y} = \mathbf{B}$, and as $D^\dagger_{\mathrm{rig}}(W)_{\mathfrak{X}}$ under the equivalence from [8], if $\mathfrak{Y} = \mathfrak{X}$ (see Theorem 4.5.28), from an L-analytic, crystalline representation W of G_L, whence $\check{M} \cong D^\dagger_{\mathrm{rig}}(W^*(\chi_{\mathrm{LT}}))$ and $\check{M} \cong D^\dagger_{\mathrm{rig}}(W^*(\chi_{\mathrm{LT}}))_{\mathfrak{X}}$, respectively. Then, on the other hand, we obtain the pairing

$$[\ ,\ ]_{D_{\mathrm{cris},L}(W)} : \mathcal{R}^{\psi_L=0} \otimes_L D_{\mathrm{cris},L}\big(W^*(\chi_{\mathrm{LT}})\big) \times \mathcal{R}^{\psi_L=0} \otimes_L D_{\mathrm{cris},L}(W) \to \mathcal{R}_K(\Gamma_L)$$

by base extension of the usual crystalline duality pairing, if $\mathfrak{Y} = \mathbf{B}$ assuming $\Omega \in K$, see (4.94). The work of Kisin–Ren and Berger–Schneider–Xie, respectively, provides comparison isomorphisms

$$\mathrm{comp}_M : M\Big[\tfrac{1}{t_{\mathfrak{Y}}}\Big] \cong \mathfrak{R}\Big[\tfrac{1}{t_{\mathfrak{Y}}}\Big] \otimes_L D_{\mathrm{cris},L}(W)$$

and

$$\mathrm{comp}_{\check{M}} : \check{M}\left[\tfrac{1}{t_{\mathfrak{Y}}}\right] \cong \mathfrak{R}\left[\frac{1}{t_{\mathfrak{Y}}}\right] \otimes_L D_{\mathrm{cris},L}\big(W^*(\chi_{\mathrm{LT}})\big).$$

Here $t_{\mathbf{B}} := t_{\mathrm{LT}} := \log_{\mathrm{LT}}(Z) \in \mathcal{R}$ denotes the Lubin–Tate period which stems from the Lubin–Tate logarithm while $t_{\mathfrak{X}} = \log_{\mathfrak{X}}$ as defined before Remark 4.2.9. The Lubin–Tate character χ_{LT} induces isomorphism $\Gamma_L \xrightarrow{\cong} o_L^\times$ as well as $\mathrm{Lie}(\Gamma_L) \xrightarrow{\cong} L$, and we let $\nabla \in \mathrm{Lie}(\Gamma_L)$ be the preimage of 1. Then the abstract reciprocity law we prove is the following statement.

Theorem 2 (Theorem 4.5.32). *For all $x \in \check{M}^{\psi_L=0}$ and $y \in M^{\psi_L=0}$, for which the crystalline pairing is defined via the comparison isomorphism, it holds that*

$$\tfrac{q-1}{q}\{\nabla x, y\}^0_{M,\mathrm{Iw}} = [x, y]_{D_{\mathrm{cris},L}(W)}$$

if $\mathfrak{Y} = \mathfrak{X}$, while the analogous statement for $\mathfrak{Y} = \mathbf{B}$ holds upon assuming $\Omega \in K$.

As explained in more detail at the beginning of Section 4.5, the proof of this abstract reciprocity law is mainly based on the insight, how the residue maps of $\mathfrak{X}$ and $\mathfrak{X}^\times$ and hence their associated pairings $\langle\ ,\ \rangle_{\mathfrak{X}}$ and $\langle\ ,\ \rangle_{\mathfrak{X}^\times}$ are related to each other by Theorem 4.5.12 in Section 4.5.3.

As an application for $\mathfrak{Y} = \mathbf{B}$ we show in Chapter 5 the adjointness of big exponential and regulator maps. Recall that already Fourquaux [32], who initiated the investigation of Perrin-Riou's approach for Lubin–Tate extensions in his thesis in 2005, had achieved a generalization of Colmez's construction of the *Perrin-Riou logarithm*. Moreover, Berger and Fourquaux [7] have constructed for V an L-analytic representation of G_L and an integer $h \geq 1$ such that

- $\mathrm{Fil}^{-h} D_{\mathrm{cris},L}(V) = D_{\mathrm{cris},L}(V)$ and
- $D_{\mathrm{cris},L}(V)^{\varphi_L = \pi_L^{-h}} = 0$,

a *big exponential map* à la Perrin-Riou

$$\Omega_{V,h} : \big(\mathcal{O}_K(\mathbf{B})\big)^{\psi_L=0} \otimes_L D_{\mathrm{cris},L}(V) \to D^\dagger_{\mathrm{rig}}(V)^{\psi_L = \frac{q}{\pi_L}},$$

which up to comparison isomorphism is for $h = 1$ given by $f = (1-\varphi_L)x \mapsto \nabla x$ and which interpolates Bloch–Kato exponential maps $\exp_{L,V(\chi^r_{\mathrm{LT}})}$.

On the other hand, based on an extension of the work of Kisin and Ren [44] in the first section, we construct for a lattice $T \subseteq V$, such that $V(\tau^{-1})$ is L-analytic and crystalline and such that V does not have any quotient isomorphic to $L(\tau)$, a *regulator map* à la Loeffler and Zerbes [52]

$$\mathcal{L}^0_V : H^1_{\mathrm{Iw}}(L_\infty/L, T) \cong D_{\mathrm{LT}}\big(T(\tau^{-1})\big)^{\psi_L=1} \to \big(\mathcal{O}_K(\mathbf{B})\big)^{\psi_L=0} \otimes_L D_{\mathrm{cris},L}\big(V(\tau^{-1})\big)$$

as applying the operator

$$1 - \tfrac{\pi_L}{q}\varphi_L$$

up to comparison isomorphism. Then we derive from the abstract version above with $W = V(\tau^{-1})$ the following reciprocity formula.

Theorem 3 (Theorem 5.2.1). *Assume that $V^*(1)$ is L-analytic. We suppose further that* $\mathrm{Fil}^{-1} D_{\mathrm{cris},L}(V^*(1)) = D_{\mathrm{cris},L}(V^*(1))$ *and additionally*

$$D_{\mathrm{cris},L}\big(V^*(1)\big)^{\varphi_L=\pi_L^{-1}} = D_{\mathrm{cris},L}\big(V^*(1)\big)^{\varphi_L=1} = 0.$$

Then the following diagram commutes:

$$\begin{array}{ccccc} D^{\dagger}_{\mathrm{rig}}\big(V^*(1)\big)^{\psi_L=\frac{q}{\pi_L}} & \times & D\big(V(\tau^{-1})\big)^{\psi_L=1} & \xrightarrow{\{\,,\}_{\mathrm{Iw}}} & D(\Gamma_L,\mathbb{C}_p) \\ \uparrow{\scriptstyle \Omega_{V^*(1),1}} & & {\scriptstyle \mathcal{L}^0_V}\downarrow & & \| \\ \big(\mathcal{O}_K(\mathbf{B})\big)^{\psi_L=0}\otimes_L D_{\mathrm{cris},L}\big(V^*(1)\big) & \times & \big(\mathcal{O}_K(\mathbf{B})\big)^{\psi_L=0}\otimes_L D_{\mathrm{cris},L}\big(V(\tau^{-1})\big) & \xrightarrow{[\,,]} & D(\Gamma_L,\mathbb{C}_p). \end{array}$$

While the crystalline pairing satisfies an interpolation property (Proposition 5.2.22) for trivial reasons, the statement that the second Iwasawa pairing interpolates Tate's cup product pairing is more subtle (Corollary 5.2.20). Eventually, the interpolation property of Berger and Fourquaux for $\Omega_{V,h}$ combined with the adjointness of the latter with $\mathcal{L}^0_V$ implies an interpolation formula for the regulator map, which interpolates dual Bloch–Kato exponential maps, see Theorem 5.2.26.

As alluded to at the beginning of this introduction, we expect that the new regulator map $\mathcal{L}^0_V$ will play the same role as the one of Loeffler and Zerbes; i.e., it should be related to ε-isomorphisms and p-adic L-functions. With regard to the first topic, this regulator map should be compared to the one defined in [92, §6.6], which is used in [55] to generalize Nakamura's explicit reciprocity law from the cyclotomic to the Lubin–Tate setting. Moreover, we hope that our regulator map can be used to also generalize [51] to the Lubin–Tate situation.

With respect to the second topic we explain at the end of Section 5.1.1 how in the context of an CM-elliptic curve with supersingular reduction at p the attached p-adic L-function is the image of the Euler system of elliptic units under this regulator map. While this is merely a reinterpretation of the work [74, §5], Manji is working out in his PhD thesis [56] another manifestation of this principle in the context of an ordinary automorphic representation defined over the unitary group $\mathrm{GU}(2,1)$, see also [92, Ex. 7.2] for slightly more details.

In the remaining part of this introduction, we briefly outline the content of the various sections.

After fixing some general notation in Chapter 2, we dedicate Chapter 3 to the generalization of Wach modules to the Lubin–Tate situation. First of all, we recall in Section 3.1 the work [44] of Kisin and Ren, who already accomplished most of this task. Then we check a missing compatibility of their constructions in Lemma 3.1.5, i.e., the commutativity of diagram (3.2), and add Corollary 3.1.15 to the picture to identify how to recover $D_{\mathrm{cris},L}(V)$ from the Kisin–Ren module attached to an L-analytic presentation V. Moreover, we compare in Proposition 3.1.10 their approach to the construction in [5, Prop. II.1.1], in particular, applying the concept of *positive* representations. After calculating the determinant of the crystalline comparison isomorphism in Section 3.2, we generalize in Section 3.3 for non-negative Hodge–Tate weights Appendix A of [4] to the Lubin–Tate situation culminating in the statement in Lemma 3.3.6 that under certain conditions the part of the (φ_L, Γ_L)-module fixed under the ψ-operator is contained in the associated Kisin–Ren module.

The main section of our article is Chapter 4 concerning the theory of (φ_L, Γ_L)-modules over different types of Robba rings as explained above. In particular, following the ideas of [42] in the cyclotomic setting, we deduce from the residue pairing in Section 4.5.1 for an L-analytic (φ_L, Γ_L)-module M and its dual $\check{M}$ the corresponding Iwasawa pairing in Section 4.5.4. In Section 4.5.5, we recall as Theorem 4.5.28 the equivalences of categories between L-analytic Galois representation and L-analytic, étale (φ_L, Γ_L)-modules over the Robba rings $\mathcal{R}_L(\mathbf{B})$ and $\mathcal{R}_L(\mathcal{X})$, respectively. The study of the compatibility of the Iwasawa pairings under the Kisin–Ren comparison isomorphisms (4.91) and also using in the proof of Lemma 4.5.26 the consequence (4.86) from Theorem 4.5.12 leads to our main result, the abstract reciprocity law in Theorem 4.5.32.

In Chapter 5, we apply first the results from Chapter 3 concerning the Wach modules à la Kisin–Ren to construct under certain technical assumptions the regulator map in Section 5.1. As a reality check and for further motivation, we study how previous work [74, §5] by the first named author and Teitelbaum fits into this picture. In Section 5.2, we recall Berger's and Fourquaux's construction of their big exponential map. From this and the definition of our regulator map the adjointness in Theorem 5.2.1 of the big exponential map and the regulator map is an immediate consequence of the abstract reciprocity law in Theorem 4.5.32. After fixing some notation of homological algebra in Section 5.2.1 and recalling the self-duality of Koszul complexes in Section 5.2.2, we study in Section 5.2.3 continuous and analytic cohomology in order to consider various generalized Herr complexes of (φ_L, Γ_L)-modules. The version using Koszul complexes being the most explicit one is then used to prove Proposition 5.2.18 and Corollary 5.2.20, i.e., the statement that Iwasawa pairing interpolates the local cup product. Based on this compatibility of the dualities, we shall derive in Section 5.2.4 the interpolation property for the regulator map from that for the big exponential map, see Theorem 5.2.26.

In Appendix A, we first show how to construct a quasi-isomorphism between the two Herr complexes built of continuous cocycle and an appropriate Koszul complex, respectively. In order to compare the cup product pairings between (φ_L, Γ_L)-modules over different coefficient rings, we have to generalize some technical lemmata from [19, 42] to our setting, especially concerning the cokernel of the operator $\psi - 1$. In particular, the derived finite dimensionality of some cohomology group h^2 is needed for the interchange of taking continuous duals with forming cohomology h^1 in the sense of Remark 5.2.6.

Finally, Appendix B recalls the quasi-isomorphism (B.5) between the complex defining Iwasawa cohomology in the sense of Nekovar and the complex given by the operator $\psi - 1$ by [45] and calculates its descent, which is needed for the interpolation property of the Iwasawa pairing.